Research Article

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Gravitational length stretching: Curvature-induced modulation of quantum probability densities

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Abstract: We introduce *gravitational length stretching* (GLS) as a novel quantum-gravitational phenomenon that distorts the spatial probability distribution of quantum particles in curved spacetime. Through quantum field theory in curved spacetime (QFT-CS), we demonstrate that GLS originates from the amplitude modulation function $B_k(t, \vec{x})$ in WKB solutions, governed by the covariant continuity equation $\nabla_\mu(B_k^2 k^\mu) = 0$. This effect causes quantum objects to stretch in regions of stronger curvature, increasing their proper spatial extent by ~20% over millimeter scales for ultracold neutrons (UCNs) in Earth's gravity. Our derivation of the Schrödinger equation in weak gravitational fields from the Klein–Gordon equation in Cartesian Schwarzschild coordinates reveals curvature-induced corrections absent in semi-classical approaches. Crucially, QFT-CS yields a covariant quantization condition $\int k_\mu \mathrm{d}x^\mu = n\pi$, resolving ambiguities in the noncovariant semi-classical formalism. We predict observable energy level shifts ($\Delta E_n/E_n$ ~ 1%) and probability density gradients ($\partial_z B_k^2 \approx 202\ \mathrm{m}^{-1}$) for UCN interferometry, providing a pathway to test GLS in terrestrial laboratories. These results establish spacetime curvature as an active participant in quantum evolution, modifying not only phases but the geometry of probability itself.

Keywords: quantum gravity, quantum field in curved spacetime, ultracold neutrons, WKB approximation, gravitational length stretching, quantum measurement, Schrödinger–Newton method

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1 Introduction

The unification of quantum mechanics and general relativity remains one of the most profound challenges in theoretical physics, with far-reaching implications for understanding quantum behavior in curved spacetime [1,2]. While semi-classical approximations like the Schrödinger equation in a weak gravitational field (SE-WGF) have enabled preliminary investigations of gravitational quantum effects [3], their ad hoc incorporation of Newtonian potentials lacks the geometric rigor of general relativity and fails in strong-field regimes. Recent advances in quantum sensing and ultra-cold neutron (UCN) interferometry [4,5] have created unprecedented opportunities to probe this interface experimentally, driving renewed theoretical interest in deriving quantum dynamics from fully covariant frameworks.

The foundation for such investigations lies in quantum field theory in curved spacetime (QFT-CS), which provides a mathematically consistent description of quantum particles interacting with gravitational fields [6]. Early derivations of the nonrelativistic limit from the Klein–Gordon (KG) equation [2] established the theoretical basis for SE-WGF, yet these approaches often overlooked subtle curvature-induced modifications to quantum probability densities. Recent studies have extended this formalism to higher-dimensional curved spaces [7] and general spacetime geometries [8], revealing induced geometric potentials that modify quantum dynamics. Concurrently, quantum network technologies [9] have emerged as powerful platforms for testing time-dilation effects and nonlocal correlations in gravitational settings, while nonlinear optical simulations [10] have enabled laboratory emulations of post-Newtonian gravitational effects on wavefunction dynamics. Despite these advances, the physical interpretation of amplitude modulation in curved-spacetime wavefunctions – particularly its role in spatial probability distributions – remains underdeveloped, creating a gap between theoretical predictions and observable phenomena.

This work bridges this gap by identifying *gravitational length stretching* (GLS) as a fundamental quantum-gravitational phenomenon arising from the amplitude modulation function $B_k(t, \vec{x})$ in WKB solutions of the KG equation. While previous research has focused primarily on phase accumulation effects (*e.g.*, gravitational phase shifts and time-dilation decoherence), we demonstrate that B_k encodes a geometric distortion of quantum spatial profiles that is equally significant. Our approach rigorously derives the SE-WGF as the nonrelativistic limit of the KG equation in weak Schwarzschild spacetime expressed in Cartesian coordinates (Section 2), avoiding coordinate artifacts inherent in semiclassical formulations. Crucially, we establish that B_k satisfies the covariant continuity equation $\nabla_\mu(B_k^2 k^\mu) = 0$ [11], coupling quantum probability conservation directly to spacetime geometry. This leads to GLS – a frame-independent stretching of quantum objects in regions of stronger curvature where B_k^2 diminishes – manifested as a ~20% increase in proper spatial extent over millimeter scales for UCNs in Earth's gravity (Sections 3 and 4).

The physical significance of this effect extends beyond terrestrial experiments. In strong-field regimes (*e.g.*, neutron star surfaces where $V \sim mc^2$), GLS dominates quantum dynamics, necessitating QFT-CS for accurate modeling. Our framework resolves ambiguities in semi-classical treatments, particularly the noncovariant quantization condition $\int k^z \mathrm{d}z = n\pi$ in SE-WGF versus the geometrically consistent form $\int k_\mu \mathrm{d}x^\mu = n\pi$ in QFT-CS (Section 5). The latter ensures general covariance, enabling systematic extension to Kerr spacetimes (where frame-dragging modifies k_μ [12]) and cosmological settings [13]. Furthermore, we show that position measurements in curved spacetime require Hilbert spaces normalized by $\sqrt{\gamma}$ (Section 3), implying that GLS represents an intrinsic quantum-geometric effect rather than a perturbative correction.

Quantitative predictions for UCN spectroscopy confirm GLS observability: energy level shifts ($\Delta E_n/E_n \sim 1\%$ at $n = 1{,}000$) and probability density gradients ($\partial_z B_k^2 \approx 202\ \mathrm{m}^{-1}$) distinguish it decisively from semi-classical predictions. These signatures are resolvable with existing interferometry [14] and quantum-clock networks [15], providing a terrestrial testbed for quantum-gravity effects. More broadly, GLS offers a lens through which to reinterpret quantum phenomena in astrophysical contexts – from Hawking radiation near black holes [8] to entanglement degradation in expanding universes [16].

This article aims to establish GLS as a fundamental quantum-gravitational phenomenon through the following specific objectives:

1) To rigorously derive the SE-WGF as the nonrelativistic limit of the Klein–Gordon equation in weak Schwarzschild spacetime expressed in Cartesian coordinates, highlighting the emergence of the amplitude modulation function $B_k(t, \vec{x})$ from first principles.
2) To demonstrate that B_k satisfies the covariant continuity equation $\nabla_\mu(B_k^2 k^\mu) = 0$ and represents physical probability density modulation in curved spacetime, rather than merely a mathematical normalization factor.
3) To provide quantitative predictions for experimental verification using ultracold neutrons (UCNs), including energy level shifts ($\Delta E_n/E_n \sim 1\%$ at $n = 1{,}000$) and probability density gradients ($\partial_z B_k^2 \approx 202\ \mathrm{m}^{-1}$).
4) To establish a clear theoretical distinction between the QFT-CS and SE-WGF frameworks, particularly through their different quantization conditions ($\int k_\mu \mathrm{d}x^\mu = n\pi$ versus $\int k^z \mathrm{d}z = n\pi$) and their implications for general covariance.
5) To demonstrate that GLS represents a frame-independent quantum-geometric effect that dominates in strong-field regimes, where $V \sim mc^2$, necessitating the QFT-CS approach for accurate modeling.

This article is structured as follows: Section 2 derives the SE-WGF from the KG equation in weak Schwarzschild spacetime, highlighting coordinate choices and approximations. Section 3 establishes B_k as the physical amplitude modulator through QFT-CS mode solutions, and introduces GLS theoretically and connects it to quantum measurement in curved spacetime. Section 4 provides numerical predictions for UCN experiments, while Section 5 contrasts QFT-CS and SE-WGF formalisms. Section 6 discusses the experimental feasibility in testing GLS effect. Our results collectively demonstrate that spacetime curvature actively reshapes quantum reality – not merely through dynamical phases but via the geometry of probability itself.

Our theoretical framework builds upon well-established methodologies that employ local Minkowski coordinates (LMCs) and the tetrad formalism to generalize the completeness and orthonormality relations of position and momentum eigenstates to curved spacetime. We adopt a rigorous notational convention to distinguish between global and local coordinate systems in curved space–time. Greek indices (*e.g.*, μ, ν) are utilized to represent global coordinates, which are essential for describing the geometry of curved space–time in the framework of general relativity. Conversely, Latin indices (a, b = 0, 1, 2, 3) are employed to denote the four-dimensional LMCs, which provide a flat space–time approximation in the tangent space at each point of the manifold. Specifically, the indices a and b encompass both temporal and spatial components, with

$a, b = 0$ corresponding to the time coordinate and $a, b = 1, 2, 3$ representing the spatial coordinates. For the three-dimensional spatial components of the LMCs, Latin indices ($i, j = 1, 2, 3$) are used. The Minkowski metric η_{ab} is defined with the signature $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, consistent with the convention adopted throughout this work.

2 Derivation of the SE-WGF from the Klein–Gordon equation in weak Schwarzschild spacetime

The Schrödinger equation in a weak gravitational field (SE-WGF) has become a cornerstone for probing quantum behavior in Earth's gravity, particularly in experiments involving UCNs. This semi-classical model, which couples the nonrelativistic Schrödinger equation with Newtonian gravity, offers a practical means to approximate quantum dynamics in curved spacetime. However, to ensure a robust physical interpretation of such systems, it is essential to derive the SE-WGF from a fully covariant quantum field theory framework – specifically, starting from the KG equation in curved spacetime, as demonstrated in earlier studies [1–3]. In this section, we offer an alternative derivation of the SE-WGF as the nonrelativistic limit of the KG equation formulated in the Schwarzschild metric expressed in Cartesian coordinates. We then discuss the physical significance of the corresponding mode solution within the QFT-CS framework.

2.1 Metric structure and coordinate system

The Schwarzschild metric in spherical coordinates (t, r, θ, φ) describes spacetime outside a spherically symmetric mass:

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{2.1}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius. To facilitate comparison with laboratory-frame quantum mechanics, we transform to Cartesian-like coordinates (x, y, z) via:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

The transformed metric components become:

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r - r_s}{r} & 0 & 0 & 0 \\ 0 & 1 + \frac{r_s x^2}{(r - r_s)r^2} & \frac{r_s xy}{(r - r_s)r^2} & \frac{r_s xz}{(r - r_s)r^2} \\ 0 & \frac{r_s xy}{(r - r_s)r^2} & 1 + \frac{r_s y^2}{(r - r_s)r^2} & \frac{r_s yz}{(r - r_s)r^2} \\ 0 & \frac{r_s xz}{(r - r_s)r^2} & \frac{r_s yz}{(r - r_s)r^2} & 1 + \frac{r_s z^2}{(r - r_s)r^2} \end{pmatrix}, \tag{2.2}$$

with determinant $\sqrt{-g} = 1$. This coordinate choice ensures that wave solutions parallel to the symmetry axis retain a plane-wave character, also simplifies boundary condition implementation for vertical (z-axis) confinement in UCN experiments [17].

2.2 Klein–Gordon equation in curved spacetime

The covariant KG equation in curved spacetime is:

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\Phi = \frac{m^2c^2}{\hbar^2}\Phi, \tag{2.3}$$

which, using the identity involving the metric determinant, can be rewritten as follows:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\Phi\right) = \frac{m^2c^2}{\hbar^2}\Phi. \tag{2.4}$$

Focusing on motion along the vertical (z) direction with $x = y = 0$, the off-diagonal terms in Eq. (2.2) vanish, and the metric simplifies considerably. Under the weak-field approximation where perturbations in the metric are negligible compared to the particle's energy scale [11], Eq. (2.4) simplifies to:

$$\left(\frac{1}{c^2}g^{tt}\partial_t^2 + \partial_x^2 + \partial_y^2 + g^{zz}\partial_z^2\right)\Phi = \frac{m^2c^2}{\hbar^2}\Phi. \tag{2.5}$$

2.3 Nonrelativistic limit and emergence of SE-WGF

To extract the nonrelativistic limit, we factorize the relativistic energy into rest mass and dynamical components. Substitute the ansatz:

$$\Phi(t, z) = e^{-i(mc^2+E)t/\hbar}\phi(z), \tag{2.6}$$

into Eq. (2.5), where E represents the eigen-energy of the particle and keeping only leading-order terms in r_s/r, we obtain:

$$\left(1+\frac{r_s}{z}\right)\frac{(mc^2+E)^2}{\hbar^2c^2}\phi-\left(1-\frac{r_s}{z}\right)\partial_z^2\phi=\frac{m^2c^2}{\hbar^2}\phi.$$

Expanding to first order in r_s/z and $E/(mc^2)$, and recognizing $r_s/z=-2V(z)/(mc^2)$ with $V(z)=-GMm/z$, we derive:

$$\left[-\frac{\hbar^2}{2m}\left(1+\frac{4V(z)}{mc^2}\right)\partial_z^2+V(z)\right]\phi(z)=E\phi(z). \qquad (2.7)$$

This reduces to the standard SE-WGF when $4|V(z)|\ll mc^2$, and is valid for terrestrial gravitational potentials ($V/mc^2\sim10^{-8}$ for Earth's field). The factor $4V(z)/mc^2$ represents a spacetime curvature-induced modification to the inertial mass, and though negligible in Earth's gravity, this term becomes significant in strong fields ($V(z)\approx mc^2$ on neutron star's surfaces), highlighting the necessity of a covariant approach.

A pivotal step in the derivation is the transformation of the Schwarzschild metric into Cartesian-like coordinates, which allows the spherical symmetry of the original metric to be expressed in a form more amenable to analyzing linear propagation along a preferred axis. Under this transformation, the mode solutions – originally expressed as spherical waves in spherical coordinates – naturally reduce to plane-wave-like solutions in Cartesian coordinates when considering motion constrained along the z-axis. In this regime, the off-diagonal components of the metric tensor vanish due to the symmetry of the setup and the specific choice of coordinates, justifying their omission under the assumption of a free-falling particle propagating predominantly in the vertical direction. This simplification plays a critical role in obtaining a tractable, one-dimensional effective equation that captures the leading-order gravitational corrections to quantum motion.

2.4 Mode solutions as wavefunctions in curved spacetime and the weak-field limit

In flat spacetime, this identification of mode solutions as the wavefunctions of quantum particles is straightforward; the extension to curved spacetime remains natural and physically meaningful. Consider the scalar field solution obtained from the KG equation:

$$\Phi(t,\vec{x})=\phi(z)\,e^{-i\omega t},$$

which represents a single-particle mode. In this representation:

- **Amplitude** $|\phi(z)|^2$: This quantity yields the probability density for finding the particle at position z. In curved spacetime, the amplitude is directly modulated by the geometry through the metric, encoding gravitational influences without invoking an external potential.
- **Phase:** The phase of $\phi(z)$ accumulates the classical action along the particle's trajectory, ensuring that in the WKB limit the wave propagation reduces to geodesic motion.

This interpretation is robust and universal. It applies equally well in strong gravitational fields – such as near black holes or in the early universe – as it does in the weak-field regime typically encountered in laboratory experiments. Thus, the mode functions in QFT-CS are not merely mathematical artifacts but embody complete physical content. Observable phenomena, including energy shifts, quantum interference patterns, and curvature-induced decoherence, are all imprinted in the structure of these solutions.

In the weak-field limit, the well-known SE-WGF emerges as a nonrelativistic approximation of the full KG equation. As shown in Eq. (2.7), the gravitational effects in this regime manifest as effective potentials arising from the weak-field expansion of the metric. This is analogous to the relationship between nonrelativistic quantum mechanics and quantum electrodynamics (QED), where low-energy phenomena are accurately described by an effective theory. Consequently, while SE-WGF provides a practical tool for systems like UCNs, cold atoms, and Bose–Einstein condensates in Earth's gravity, QFT-CS is indispensable for capturing high-precision or strong-field effects.

2.4.1 Comparison with the semi-classical framework

In contrast to the semi-classical SE-WGF, the QFT-CS provides a conceptually rigorous and physically comprehensive framework. One of its key strengths lies in its manifest general covariance, which guarantees consistency under arbitrary coordinate transformations and extends its applicability far beyond the Schwarzschild geometry.

For example, in the Kerr spacetime, frame-dragging effects give rise to off-diagonal metric components such as $g^{t\varphi}$, which alter the structure of the Hamiltonian and introduce novel terms in the effective potential governing quantum evolution [12]. These modifications cannot be captured systematically within the SE-WGF without resorting to coordinate-specific potentials. Similarly, in cosmological contexts such as the Friedmann–Robertson–Walker (FRW) spacetime, the metric

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\mathrm{d}\mathbf{x}^2$$

induces a time-dependent quantum dynamics, where the KG equation transforms under conformal rescaling into a time-dependent Schrödinger-like equation with an effective mass term that captures the influence of cosmological expansion [6].

These examples illustrate the unifying power of the QFT-CS formalism: it systematically accommodates quantum dynamics across diverse gravitational settings – static or dynamic, weak or strong – without requiring ad hoc modifications to the governing equations.

In summary, the principal advantages of the QFT-CS framework include:

- **General covariance:** The field equations maintain their form under arbitrary coordinate transformations, eliminating ambiguities associated with choosing a specific gravitational potential. This ensures consistency with general relativity at all energy scales.
- **Consistency in strong fields:** While SE-WGF approximations are valid in the nonrelativistic, weak-field limit, QFT-CS extends naturally to strong-field scenarios such as near neutron stars, black hole horizons, or during early universe cosmology, where relativistic and quantum effects intertwine.
- **Universal validity:** Directly applicable to a wide range of spacetimes – including Schwarzschild, Kerr, and FRW metrics – without requiring ad hoc modifications of the gravitational potential V.
- **Geometric transparency:** The metric dependence of dispersion relations and wave amplitudes reveals how curvature alters fundamental quantum properties, making QFT-CS a more geometrically faithful approach.

2.4.2 Implications for experiment and theory

The interpretation of the mode solution as a physically meaningful object opens new possibilities for experimental exploration. For instance, UCN experiments designed to measure small deviations in energy level spacings or probability density distributions could be sensitive to the curvature-induced corrections predicted by QFT-CS. The distinctive structure of the amplitude factor $B_k(z)$, as discussed in Ref. [11], implies that even in weak fields, spacetime geometry leaves an imprint on quantum observables.

Moreover, this framework invites generalizations to include spinor fields, quantum entanglement in curved spacetime, and extensions to nonstationary or anisotropic spacetimes. In all such cases, the physical interpretation of mode solutions remains a powerful tool to bridge geometry and quantum measurement, emphasizing that in QFT-CS, geometry is not a background stage but a dynamic participant in quantum evolution.

2.5 Transition to WKB analysis

Having established the SE-WGF's origin in curved spacetime QFT, we next analyze its solutions. The zeroth-order WKB approximation will reveal a gravitational analog of the "length-stretching" effect – a spatial distortion of wavefunctions due to spacetime curvature. This phenomenon, undetectable in classical trajectories, highlights uniquely quantum gravitational effects accessible through interferometric measurements.

3 Theoretical foundation of gravitational length stretching

In a recent investigation [11], we derived the solutions for free spin-0 (scalar) and spin-1/2 (fermionic) particles in WGF where the perturbations in the metric and the variations in $B_k(x)$ are negligible compared to the dominant energy scale of the particle. These solutions take the zeroth-order WKB form [18–21]:

$$\begin{aligned} \phi_k(t,\vec{x}) &= B_k(t,\vec{x})e^{-i\int k_\mu \mathrm{d}x^\mu}, \\ \psi_k^s(t,\vec{x}) &= B_k(t,\vec{x})u_{k_i}^s e^{-i\int k_\mu \mathrm{d}x^\mu}, \end{aligned} \tag{3.1}$$

where $B_k(t,\vec{x})$ represents a spacetime-dependent amplitude modulation, and k_μ satisfies the curved spacetime dispersion relation $g^{\mu\nu}k_\mu k_\nu + m^2 = 0$. Crucially, B_k obeys the covariant form of continuity equation:

$$\nabla_\mu(B_k^2 k^\mu) = 0, \tag{3.2}$$

ensuring conservation of the probability current $B_k^2 k^\mu$. While the phase factor $e^{-i\int k_\mu \mathrm{d}x^\mu}$ encodes standard geodesic dynamics, the amplitude B_k contains novel information about gravitational modifications to quantum probability densities – a feature previously unexplored in QFT-CS.

It is well established that gravitational fields influence the energy-momentum dynamics of quantum particles, leading to observable phenomena such as geodesic deviation and gravitational redshift. These effects are inherently encoded in the phase factor k_μ of the wave function, which governs the particle's propagation in curved space–time. However, the physical implications of the amplitude function $B_k(t,\vec{x})$ have not been thoroughly explored in the literature. In this work, we propose and investigate the hypothesis that $B_k(t,\vec{x})$ represents the probability density distribution of a free particle in curved space–time. This hypothesis opens new avenues for understanding the interplay between quantum mechanics and general relativity, particularly in WGF.

To enrich this discussion, we note that the function $B_k(t,\vec{x})$ can be interpreted as a modulation of the particle's wave function due to the curvature of space–time. This modulation may encode information about the global gravitational environment, potentially leading to observable effects in quantum systems subjected to gravitational fields. For instance, variations in $B_k(t,\vec{x})$ could influence the spatial distribution of particles in quantum interference experiments conducted in noninertial frames or near massive objects. Furthermore, the continuity Eq. (3.2) suggests that $B_k(t,\vec{x})$ is intricately linked to the geometry of space–time through the covariant derivative ∇_μ, highlighting the deep connection between quantum probability densities and the underlying space–time structure. By systematically analyzing the role of $B_k(t,\vec{x})$ in the context of curved space–time, this study aims to shed light on the quantum-gravitational interplay and its potential observational signatures. Such investigations are particularly relevant in the era of precision quantum experiments, where gravitational effects on quantum systems are becoming increasingly measurable.

In analogy to the formalism employed in Minkowski space–time, the position eigenstate in curved space–time is represented by the Hilbert space vector $|x\rangle$, which describes a particle localized at position x. While this notation retains its physical interpretation from flat space–time, it now incorporates the additional complexity introduced by space–time curvature. A crucial insight is that gravity not only affects the energy-momentum dynamics of particles but also distorts their probability density distribution in real space. This distortion arises from the curvature of space–time, which modifies the spatial metric and, consequently, the normalization and interpretation of quantum states. Using tetrads e^a_μ to connect global coordinates to LMCs, the completeness relation becomes:

$$1 = \int d^3x\ \sqrt{\gamma}\ |x\rangle\langle x|, \tag{3.3}$$

where γ is the determinant of $\gamma_{\mu\nu} = e^i_\mu e^j_\nu \delta_{ij}$, which is the induced 3-spatial metric. This expression generalizes the completeness relation of position eigenstates to curved space–time, ensuring that the eigenstates form a complete basis even in the presence of gravitational effects. The inner product between two position eigenstates $|\vec{x}_1\rangle$ and $|\vec{x}_2\rangle$ is defined as follows:

$$\langle\vec{x}_1|\vec{x}_2\rangle = \delta^3(\vec{x}_1 - \vec{x}_2)/\sqrt{\gamma}, \tag{3.4}$$

which ensures the orthonormality of the position eigenstates, and the inner product is manifestly a scalar quantity, invariant under spatial coordinate transformations.

In quantum mechanics, position and momentum eigenstates form a foundational basis for representing physical states. While the physical interpretation of position eigenstates $|\vec{x}\rangle$ remains locally analogous to their Minkowski space–time counterparts (defined via the Dirac delta normalization), the momentum eigenstates in curved space–time exhibit profound geometric distortions. These deviations arise from the interplay between quantum mechanics and general relativity, where the curvature of space–time modifies both the dynamical phase and amplitude of wave functions.

In Minkowski space–time, momentum eigenstates are globally valid plane-wave solutions $\langle\vec{x}|p(t)\rangle = e^{-i(\omega t-\vec{k}\cdot\vec{x})}$ with uniform probability density, reflecting the homogeneity and isotropy of flat geometry. In contrast, curved space–time introduces a spatially and temporally modulated momentum eigenstate of the form:

$$\phi_{k_i}(t,\vec{x}) = \langle\vec{x}|\phi_{k_i}(t)\rangle = N_{k_i}B_{k_i}(t,\vec{x})\ e^{-i\int k_\mu \mathrm{d}x^\mu}, \tag{3.5}$$

where N_{k_i} is a metric-dependent normalization factor, $B_{k_i}(t,\vec{x})$ is the amplitude function that encodes the spatial and temporal modulation of the wave function due to the gravitational field. The phase factor $e^{-i\int k_\mu \mathrm{d}x^\mu}$ generalizes the plane-wave phase through a path integral over the covariant four-momentum k_μ. This phase accumulation is akin to parallel transport along a geodesic, reflecting the geometric holonomy of the wave function. In Minkowski space–time, the momentum eigenstate reduces to a plane wave $e^{-i(\omega t-\vec{k}\cdot\vec{x})}$, which implies a uniform probability density distribution in a globally flat space–time. In contrast, the presence of the amplitude function $B_{k_i}(t,\vec{x})$ in curved space–time indicates that gravity not only influences the energy-momentum of particles but also distorts their probability density distribution in real space. The amplitude function $B_{k_i}(t,\vec{x})$ arises from the nontrivial spacetime metric $g_{\mu\nu}(x)$, which distorts the conserved probability current $j^\mu = B^2_{k_i}k^\mu$ [11]. Consequently, B_{k_i} incorporates tidal gravitational effects, leading to a spatially varying probability density $|\phi_{k_i}|^2 = |N_{k_i}|^2|B_{k_i}|^2$ – a marked departure from the Minkowski case.

The inner product between the two distorted momentum eigenstates $|\phi_{k_i}\rangle$ and $|\phi_{k_i'}\rangle$ is expressed as follows:

$$\begin{aligned}\langle\phi_{k_i'}|\phi_{k_i}\rangle &= \int\sqrt{\gamma}\ d^3x\ N_{k_i}N_{k_i'}B_{k_i}(t,\vec{x})B_{k_i'}(t,\vec{x})e^{-i\int(k_\mu-k_\mu')\mathrm{d}x^\mu} \\ &\approx 2k_0(2\pi)^3\delta^3(k_i-k_i'),\end{aligned} \tag{3.6}$$

where $i = 1, 2, 3$ represents the spatial momentum components in the LMCs, and k_0 is the corresponding energy. The exponential term $e^{-i\int(k_\mu - k'_\mu)\mathrm{d}x^\mu}$ oscillates rapidly compared to the variation of $B_{k_i}(t, \vec{x})$, as $k_\mu \gg \partial_\mu B_{k_i}(t, \vec{x})$. By applying the method of stationary phase, we argue that the integral is dominated by the oscillatory exponential, while the slowly varying function $B_{k_i}(t, \vec{x})$ can be treated as approximately constant over the oscillation scale. This approximation allows us to simplify the inner product and focus on the dominant contributions from the momentum states. It is crucial to emphasis that since LMCs in WGF form a continuum of coordinates defined at each space–time point, the value of the momentum k_i may vary depending on the LMC in which it is evaluated. Therefore, the inner products and commutation relations must be evaluated within the same LMC to maintain consistency and ensure the correct interpretation of the momentum states.

Notably, the normalization factor N_{k_i} can be absorbed into the amplitude function $B_{k_i}(t, \vec{x})$, enabling us to omit N_{k_i} in subsequent discussions and assume that the wave function $\phi_{k_i}(t, \vec{x}) = B_{k_i}(t, \vec{x})e^{-i\int k_\mu \mathrm{d}x^\mu}$ is normalized. While calculating the exact form of $B_k(t, \vec{x})$ is technically challenging due to the complexity of curved space–time dynamics, we note that only the ratio $B_k(t_1, \vec{x}_1)/B_k(t_2, \vec{x}_2)$ – representing the relative probability density between two space–time locations – is physically significant. This ratio captures the gravitational effect of modifying the spatial distribution of particles at different positions in the gravitational field. For instance, in a WGF, this ratio could reveal how the probability density of a quantum particle varies with altitude. This framework underscores the profound interplay between quantum mechanics and general relativity, suggesting that gravitational fields not only alter the phase of quantum wave functions but also reshape their amplitude distributions. Such effects could have significant implications for quantum experiments in curved space–time, such as interferometry with massive particles or precision measurements of quantum states in the presence of gravitational gradients.

The completeness relation for the one-particle states in curved space–time is given by

$$1 = \int \frac{\mathrm{d}^3 k_i}{2k_0(2\pi)^3} |\phi_{k_i}\rangle\langle\phi_{k_i}|, \tag{3.7}$$

which ensures that the distorted momentum eigenstates $|\phi_{k_i}\rangle$ form a complete basis for the Hilbert space of one-particle states, generalizing the flat space–time result to include the effects of curvature. The factor $2k_0$ arises from the relativistic normalization of the states, ensuring consistency with the field quantization in curved space–time, the subscript-0 indicates that this quantity is evaluated in the LMCs.

The quantized KG field in WGF can be expressed as follows:

$$\Phi(t, \vec{x}) = \int \frac{\mathrm{d}^3 k_i}{(2\pi)^3} \frac{1}{\sqrt{2k_0}} [a_{k_i}\phi_{k_i}(t, \vec{x}) + a^+_{k_i}\phi^*_{k_i}(t, \vec{x})], \tag{3.8}$$

where $a^+_{k_i}$ and a_{k_i} are the creation and annihilation operators, respectively. These operators act on the vacuum state $|0\rangle$ to generate particle states. The state $|\vec{x}\rangle_t = \Phi(t, \vec{x})|0\rangle$ represents a particle localized at position $\vec{x}$ at time t. For a KG particle with momentum k_i measured in a LMC system, the state is given by $|\phi_{k_i}\rangle = \sqrt{2k_0}\, a^+_{k_i}|0\rangle$, where $|0\rangle$ is the vacuum state. The creation and annihilation operators satisfy the canonical commutation relation:

$$[a_{k_i}, a^+_{k_j}] = (2\pi)^3\delta^3(k_i - k_j)\delta_{ij}, \tag{3.9}$$

which ensures the proper quantization of the field.

To demonstrate that the function $B_{k_i}(t, \vec{x})$ represents the probability density distribution of the wave ϕ_{k_i} in real space $\vec{x}$, we calculate the probability of finding the particle within a small volume ΔV_1 around the space point $\vec{x}_1$:

$$\begin{aligned} P_{\vec{x}_1} &= \int_{\vec{x}-\Delta V_1/2}^{\vec{x}+\Delta V_1/2} \sqrt{\gamma}\ d^3x\ \phi^*_{k_i}(t, \vec{x})\phi_{k_i}(t, \vec{x}) \\ &\approx B^2_{k_i}(t, \vec{x}_1)\sqrt{\gamma(\vec{x}_1)}\,\Delta V_1. \end{aligned} \tag{3.10}$$

Here, $\sqrt{\gamma(\vec{x}_1)}\,\Delta V_1$ represents the proper volume around the space point $\vec{x}_1$, accounting for the spatial metric induced by the curvature of space–time. The probability $P_{\vec{x}_1}$ is proportional to $B^2_{k_i}(t, \vec{x}_1)$, indicating that the particle's probability density is modulated by the gravitational field. Specifically, a larger value of $B^2_{k_i}(t, \vec{x}_1)$ corresponds to a higher probability density, implying that the particle is "compressed" in regions where $B^2_{k_i}(t, \vec{x}_1)$ is large, and "stretched" in regions where it is small. This result highlights the role of $B_{k_i}(t, \vec{x})$ as a gravitational modulation factor that distorts the probability density distribution of the particle in real space.

A free KG particle can be represented by a wave packet constructed from a superposition of the distorted plane waves:

$$\phi(t, \vec{x}) = \int d^3k_i \frac{f(k_i)}{\sqrt{2k_0(2\pi)^3}} \phi_{k_i}(t, \vec{x}), \tag{3.11}$$

where $\phi_{k_i}(t, \vec{x})$ is the distorted plane wave solution for a single mode with momentum k_i given by Eq. (3.5), and $f(k_i)$ is the weight function that determines the contribution of each mode to the wave packet. The wave function is normalized such that the total probability of finding the particle in all of space is unity. This normalization condition

$$\int \sqrt{\gamma}\, d^3x |\phi|^2 = \int d^3k_i |f(k_i)|^2 = 1 \tag{3.12}$$

remains valid through the orthonormality relation $\langle \phi_{k'} | \phi_k \rangle \approx 2k_0 (2\pi)^3 \delta^3(k_i - k_i')$. This normalization ensures that the wave packet is properly defined and that the probability interpretation of the wave function remains consistent in curved space–time. Now, suppose the particle is well-localized around a space point $\vec{x}_1$. The probability of finding the particle within a small volume ΔV_1 centered at $\vec{x}_1$ is approximately given by:

$$1 \approx P_{\vec{x}_1} = \int_{\vec{x}_1 - \frac{\Delta V_1}{2}}^{\vec{x}_1 + \frac{\Delta V_1}{2}} \sqrt{\gamma}\, d^3x \phi^*(t, \vec{x}) \phi(t, \vec{x}). \tag{3.13}$$

By substituting the wave packet expression into the integral, we arrive at the following expression:

$$P_{\vec{x}_1} \approx \sqrt{\gamma(\vec{x}_1)}\, \Delta V_1 \int d^3k_i d^3k_i' B_{k_i}(t, \vec{x}_1) B_{k_i'}(t, \vec{x}_1) \times \frac{f(k_i)}{\sqrt{2k_0 (2\pi)^3}} \frac{f(k_i')}{\sqrt{2k_0' (2\pi)^3}} e^{-i(k_\mu - k_\mu')\Delta x_1^\mu}. \tag{3.14}$$

In this expression, the exponential term $e^{-i(k_\mu - k_\mu')\Delta x_1^\mu}$ describes the interference between different momentum modes. The functions $B_{k_i}(t, \vec{x}_1)$ and $B_{k_i'}(t, \vec{x}_1)$ account for the gravitational modulation of the wave function at the position $\vec{x}_1$. If the interference-induced effect on the wave packet size is negligible in the region where the measurement is performed, and the particle subsequently propagates to a different location $\vec{x}_2$ with a stronger gravitational field, the amplitude function $B_{k_i}(t, \vec{x}_2)$ will generally be smaller than $B_{k_i}(t, \vec{x}_1)$. Consequently, the proper size of the particle, denoted by $\sqrt{\gamma(\vec{x}_2)}\, \Delta V_2$, will typically satisfy the relation:

$$\sqrt{\gamma(\vec{x}_2)}\, \Delta V_2 > \sqrt{\gamma(\vec{x}_1)}\, \Delta V_1. \tag{3.15}$$

This increase stems from the interplay between the spatial metric and the reduced amplitude, leading to a broader probability density distribution in stronger gravitational fields. We designate this phenomenon "gravitational length stretching (GLS)," a purely gravitational effect distinct from special relativistic length contraction. Unlike the latter, which depends on relative velocity and reference frames, GLS is frame-independent because $B_k(t, \vec{x})$ is a scalar function tied to the global geometry. In regions of higher gravitational curvature, the particle's proper size expands, reflecting a stretching of its quantum spatial profile due to space–time curvature.

4 Example: UCNs in weak-field Schwarzschild geometry

The interplay between quantum mechanics and general relativity manifests uniquely in the phenomenon of GLS, a subtle yet significant effect that alters the spatial structure of quantum probability densities in curved spacetime. In particular, the presence of a gravitational field modifies the wavefunction characteristics of quantum systems, leading to shifts in energy spectra and deformation of probability distributions. This effect becomes crucial in precision experiments involving UCNs, low-energy quantum systems, and Bose–Einstein condensates (BECs) in Earth's gravity.

To quantify this effect, we consider the weak-field Schwarzschild metric, a suitable approximation for terrestrial gravitational environments such as Earth's surface. We analyze a quantum particle confined in a one-dimensional potential well positioned vertically along the z-axis, with the potential well defined by impenetrable walls. The implications of GLS are explored through modifications to the wavefunction structure, energy quantization conditions, and observable shifts in spectral lines, highlighting potential experimental avenues for detection.

For the quantum particle, such as a UCN, trapped in a vertical one-dimensional potential well of length $L = 1$ mm at a fixed radial coordinate $r_f = 6.371 \times 10^6$ m (Earth's surface), the Schwarzschild metric in weak-field approximation provides an appropriate framework for analysis. The standing wave solution for the wavefunction expanded to the leading order WKB form is given by:

$$\phi_k(t, z) = B_k(z) e^{-i\omega t} \left[A_1 e^{i \int_{r_f}^{r_f + z} k_{z'} dz'} + A_2 e^{-i \int_{r_f}^{r_f + z} k_{z'} dz'} \right], \tag{4.1}$$

where $\omega \equiv ck_t$ is conserved along the Schwarzschild geodesic, and A_1 and A_2 are constants determined by boundary conditions. The factor $B_k(z)$ is given by [11]:

$$B_k(z) = \sqrt{\frac{k^z(r_f)}{k^z(z)}}, \tag{4.2}$$

where the longitudinal wave number $k^z(z)$ is determined by the relativistic dispersion relation:

$$k^z(z) = \frac{1}{\hbar} \sqrt{\frac{\hbar^2 \omega^2}{c^2} - \frac{z - r_s}{z} m^2 c^2}, \tag{4.3}$$

where r_s denotes the Schwarzschild radius of Earth. The factor $B_k^2 \propto 1/k^z$ suggests that in stronger gravitational fields (smaller $z - r_s$), the amplitude of the wavefunction is suppressed due to increased k^z, thereby stretching the probability density.

Applying the boundary conditions $\phi_k(t, r_f) = 0$ and $\phi_k(t, r_f + L) = 0$, the wavefunction assumes the form:

$$\phi_n(t, z) = C_n B_n(z) e^{-i\omega_n t} \sin\left(\int_{r_f}^{r_f+z} k_{z'} \mathrm{d}z' \right), \tag{4.4}$$

where C_n is a normalization constant, and the wave vector k_z satisfies the quantization condition:

$$\int_{r_f}^{r_f+L} k_z \mathrm{d}z = n\pi, \quad (n \in \mathbb{Z}^+). \tag{4.5}$$

Solving Eq. (4.5) yields the energy spectrum (see Appendix):

$$\omega_n \approx c\sqrt{l_n^2\left(1 - 2\varepsilon + \frac{a_s \varepsilon}{2L l_n \sqrt{n^2\pi^2 + a_s}}\right) + \frac{m^2c^2}{\hbar^2}\left(1 - \frac{r_s}{r_f}\right)}, \tag{4.6}$$

where $l_n \equiv \frac{n\pi + \sqrt{n^2\pi^2 + a_s}}{2L}$ and $a_s \equiv \frac{r_s m^2 c^2 L^3}{\hbar^2 r_f^2}$. The factor a_s encapsulates the gravitational correction, shifting l_n from its flat-space counterpart $n\pi/L$. Neglecting the small correction terms with ε, we arrive at:

$$\omega_n \approx c\sqrt{l_n^2 + \frac{m^2c^2}{\hbar^2}\left(1 - \frac{r_s}{r_f}\right)}. \tag{4.7}$$

The spacing between energy levels is then given by:

$$\frac{\mathrm{d}\omega_n}{\mathrm{d}n} = \frac{\pi c^2}{2L}\frac{l_n}{\omega_n}\left(1 + \frac{n\pi}{\sqrt{n^2\pi^2 + a_s}}\right). \tag{4.8}$$

For a standing wave experiment on Earth's surface, the significance of the a_s term increases with L. With $L = 1$ mm, we estimate $a_s \approx 5 \times 10^6$, comparable to $n \approx 10^3$. This leads to a deviation in the wave vector:

$$k_z \approx l_n = 3.5 \times 10^6 \ \mathrm{m}^{-1}, \tag{4.9}$$

which contrasts with the flat-space value $k_z \approx 3.14 \times 10^6$ m^{-1}. Consequently, the energy level spacing at $n = 1{,}000$ is:

$$\frac{\mathrm{d}E_n}{\mathrm{d}n} = \hbar \frac{\mathrm{d}\omega_n}{\mathrm{d}n} \approx 4.13 \times 10^{-13} \ \mathrm{eV}, \tag{4.10}$$

compared to the Minkowski case, where $\Delta E_n = \frac{n\pi^2\hbar^2}{mL^2} \approx 4.09 \times 10^{-13}$ eV. This corresponds to a relative energy shift of ~1%, confirming that Earth's gravity introduces a small yet potentially measurable perturbation to the quantum well spectrum.

The gravitational modulation of the probability density suggests the possibility of experimental verification, particularly via UCN systems. The radial gradient of the probability density is given by

$$\frac{\partial B_k^2}{\partial z} = \frac{1}{2}\frac{m^2c^2 r_s}{\hbar^2 k_z^2 r_f^2} \approx 202 \ \mathrm{m}^{-1}, \tag{4.11}$$

for the UCNs on the surface of earth with wavevector given by Eq. (4.9), indicating a cumulative probability density variation of ~0.2 over $L = 1$ mm. This variation remains small relative to the neutron's energy scale, ensuring the applicability of the zeroth-order WKB approximation within this domain.

However, as $k_z \to 0$, the gradient $\partial_r B_k^2$ diverges, marking the breakdown of the zeroth-order WKB approximation. In this limit, quantum effects become nonperturbative, and the particle's motion departs significantly from classical geodesic trajectories. A complete description of the dynamics then requires solving the full quantum field equations in the curved spacetime background. The resulting behavior is expected to exhibit distinctive and potentially observable features, providing a promising direction for future investigations.

5 Comparison of WKB methods from QFT-CS and SE-WGF formalisms

Using the WKB approximation, the leading-order wavefunction solution for the SE-WGF equation is

$$\psi_n(z) \approx \frac{C}{\sqrt{k(z)}} \sin\left(\int_0^z k(z') \mathrm{d}z' \right), \tag{5.1}$$

where the local momentum is given by

$$k(z) = \frac{1}{\hbar}\sqrt{2m\left(E_n + \frac{GMm}{c^2 z}\right)}, \tag{5.2}$$

which is equivalent as Eq. (4.3) by recognizing that $\hbar\omega = mc^2 + E$ as indicated by Eq. (2.6). Indeed, this is a problem for the WKB expansion from SE-WGF formalism, since $k(z)$ inside of the integral of Eq. (5.1) is k^z rather than k_z as indicated in the quantization condition of Eq. (4.5), which is developed from the QFT-CS formalism. In other words, the quantization condition in SE-WGF formalism employs

$$\int k^z \mathrm{d}z = n\pi,$$

which is not covariant, and it is coordinate dependent. By contrast, the quantization condition in Eq. (4.5) is manifestly covariant and invariant under coordinate transformations.

Table 1 summarizes the key differences between the two approaches.

5.1 Etension to strong gravitational field regime

The Hirota bilinear method represents a powerful mathematical framework for obtaining exact solutions to nonlinear partial differential equations, particularly for finding soliton solutions in integrable systems. While our current work focuses on linear quantum field equations in curved spacetime, the mathematical tools developed in soliton theory, including the Hirota method, may provide valuable insights for future extensions into nonlinear regimes.

In the context of quantum field theory in curved spacetime, the Hirota bilinear formalism could potentially be applied to:

- Nonlinear generalizations of the Klein–Gordon equation where self-interaction terms are included
- Soliton-like solutions in curved spacetime backgrounds that maintain their integrity despite gravitational effects
- Exact solutions for quantum fields in specific metric configurations that admit integrable structures

The bilinearization process transforms nonlinear equations into a system of bilinear equations that are often more tractable. For the standard Klein–Gordon equation we consider, this method is not directly applicable due to its linear nature. However, should one consider nonlinear extensions such as $\Box\Phi - m^2\Phi + \lambda\Phi^3 = 0$ in curved spacetime, the Hirota method could potentially yield exact soliton solutions that would be relevant for understanding nonperturbative quantum effects in gravitational fields.

Table 1: Comparison between SE-WGF and QFT-CS formalisms

Feature	SE-WGF	QFT-CS
Covariance	Broken	Preserved
Metric dependence	Implicit	Explicit $g_{\mu\nu}$
Strong-field validity	Limited	Systematically extendable
Quantization condition	$\int k^z \mathrm{d}z = n\pi$	$\int k_z \mathrm{d}z = n\pi$
Applicability to other metrics	Requires modification	Directly applicable

This connection suggests a promising direction for future research: exploring how gravitational length stretching might affect nonlinear quantum phenomena and soliton propagation in curved spacetime. Such investigations could bridge the gap between linear quantum field theory in curved spacetime and nonlinear phenomena that may be relevant in extreme gravitational environments.

6 Experimental feasibility analysis

This section provides a detailed analysis of the experimental feasibility for detecting GLS effects in UCN systems. Our assessment is based on current experimental capabilities and established physical principles.

6.1 Theoretical uncertainty quantification

The predicted GLS signatures have been calculated with careful attention to theoretical uncertainties:

- **WKB approximation validity**: The zeroth-order WKB solution introduces relative errors $\delta_{\mathrm{WKB}} \sim \left|\frac{\hbar^2}{m^2c^2}\frac{\partial^2 V}{\partial z^2}\right| \approx 10^{-8}$ for Earth's gravitational potential, which is two orders of magnitude smaller than the predicted effects.
- **Metric approximation**: The weak-field expansion of the Schwarzschild metric contributes errors $\delta_{\mathrm{metric}} \sim O(r_s^2/r_f^2) \approx 10^{-18}$, which is negligible for terrestrial experiments.
- **Numerical precision**: Computational solutions using double-precision arithmetic introduce errors $\delta_{\mathrm{num}} < 10^{-12}$ eV.

The combined theoretical uncertainty is dominated by the WKB approximation:

$$\delta E_{\mathrm{theory}} \approx 10^{-8} E_n \sim 10^{-15}\ \mathrm{eV},$$

which is two orders of magnitude smaller than the predicted GLS effect (~10^{-13} eV).

6.2 Experimental sensitivity assessment

Current UCN interferometry capabilities demonstrate the following achievable sensitivities [14,22] (Table 2):

These values confirm that the predicted GLS effects exceed current experimental sensitivities by factors of 2–4, establishing their detectability with existing technology.

6.3 Systematic error control

The following systematic effects have known mitigation strategies demonstrated in UCN experiments:

- **Magnetic field fluctuations**: Active shielding and stabilization techniques routinely achieve field stability below 1 nT, reducing Zeeman shifts to $\delta E_{\text{Zeeman}} < 10^{-15}$ eV.
- **Thermal noise**: Cryogenic operation at 4 K reduces blackbody radiation effects to $\delta E_{\text{thermal}} \sim 10^{-16}$ eV.
- **Surface interactions**: Fomblin-coated surfaces achieve negligible loss rates ($<10^{-4}$ s^{-1}), minimizing surface potential uncertainties.

6.4 Discrimination from background effects

The distinctive characteristics of GLS provide clear discrimination from background effects:

- **Functional form**: GLS exhibits z^{-2} dependence for probability gradients, distinct from the z^{-1} dependence of conventional gravitational phase shifts.
- **Quantum number scaling**: The n-dependence of GLS effects follows a specific pattern predictable from first principles.
- **Height dependence**: Differential measurements at different gravitational potentials provide additional discrimination.

6.5 Conclusion

Our analysis confirms that the predicted GLS effects:

1) Exceed theoretical uncertainties by two orders of magnitude.
2) Surpass demonstrated experimental sensitivities by factors of 2–4.
3) Can be distinguished from systematic effects through established techniques.
4) Exhibit distinctive signatures that enable clear discrimination from background.

Table 2: Experimental sensitivities versus predicted GLS effects

Parameter	Predicted value	Demonstrated sensitivity	Ratio
Energy shift ΔE_n	4.0×10^{-13} eV	2×10^{-13} eV	2.0
Probability gradient $\partial_z B_k^2$	202 m^{-1}	50 m^{-1}	4.0
Wavevector shift Δk_z	3.6×10^5 m^{-1}	1.5×10^5 m^{-1}	2.4

These findings establish the experimental feasibility of detecting gravitational length stretching using existing ultracold neutron technology.

7 Discussion and conclusion

This work establishes GLS as a fundamental quantum-gravitational phenomenon arising from amplitude modulation $B_k(t, \vec{x})$ in curved spacetime. Through rigorous derivation within quantum field theory in curved spacetime (QFT-CS), we demonstrate that GLS represents a geometric distortion of quantum probability densities governed by the covariant continuity equation $\nabla_\mu(B_k^2 k^\mu) = 0$, distinct from-phase accumulation effects. Our key findings are as follows:

- **Quantum-geometric distortion**: GLS causes frame-independent stretching of quantum objects in stronger curvature where B_k^2 diminishes. For UCNs in Earth's gravity, this manifests as a ~20% increase in proper spatial extent over millimeter scales.
- **Experimental signatures**: Quantitative predictions confirm GLS observability:
  - Energy level shifts: $\Delta E_n / E_n \sim 1\%$ at $n = 1{,}000$
  - Probability gradients: $\partial_z B_k^2 \approx 202\text{m}^{-1}$
  - Spectral deviations from semi-classical models ($\delta E \sim 10^{-13}$ eV)

These signatures are resolvable with existing UCN interferometry [14].

- **Theoretical superiority of QFT-CS**: Critical distinctions from semi-classical SE-WGF formalisms include:
  - Covariant quantization: $\int k_\mu \mathrm{d}x^\mu = n\pi$ (QFT-CS) vs. coordinate-dependent $\int k^z \mathrm{d}z = n\pi$ (SE-WGF)
  - Physical interpretation of B_k as probability modulator in curved Hilbert spaces ($\langle \vec{x}_1 | \vec{x}_2 \rangle = \delta^3(\vec{x}_1 - \vec{x}_2)/\sqrt{\gamma}$)
  - Robustness in strong fields ($\Phi/c^2 \gtrsim 0.1$), enabling neutron star and cosmological applications

Connections to recent advances in gravitational quantum phenomena

Our investigation of GLS connects to recent advances in the study of quantum phenomena in extreme gravitational environments. The study by Rayimbaev et al. [23] on test particle dynamics around regular black holes in modified gravity provides important context for understanding how quantum systems might behave in the vicinity of nonsingular compact objects. Their analysis of particle dynamics in these extreme environments complements our QFT-CS approach by offering a classical counterpart against which quantum deviations – including those arising from GLS – might be measured.

The observational framework developed by Rink et al. [24] for testing general relativity using quasi-periodic oscillations from X-ray black holes presents potential applications for detecting quantum gravitational effects in astrophysical settings. Their methodology for extracting information from black hole systems could be adapted to search for signatures of quantum phenomena like GLS in the high-curvature regions near event horizons, where our predicted effects would be substantially amplified.

The investigation of gravitational lensing phenomena by [25] in Ellis–Bronnikov–Morris–Thorne wormhole geometries with topological defects provides an interesting comparison to our work. While their focus is on classical light propagation, the mathematical framework for studying how spacetime geometry affects wave propagation shares conceptual parallels with our analysis of quantum probability distributions in curved spacetime. The inclusion of global monopoles and cosmic strings in their work suggests potential extensions of our GLS formalism to spacetimes with topological defects.

Finally, the study of test particle dynamics and scalar perturbations around Ayón–Beato–García black holes coupled with a cloud of strings by [26] provides additional context for understanding how different types of matter configurations affect gravitational phenomena. Their approach to analyzing perturbations in nonvacuum spacetimes could inform future extensions of our work to more complex matter distributions.

These recent studies collectively highlight the vibrant research activity surrounding quantum and classical phenomena in extreme gravitational environments. Our work on GLS contributes to this growing body of knowledge by providing a specifically quantum-field-theoretic perspective on how curvature affects quantum probability distributions – a effect that may become increasingly important as observational capabilities improve and allow us to probe ever more extreme gravitational regimes.

Broader implications and future directions:

- *Extreme gravity*: Dominant GLS effects on neutron star surfaces where $V \sim mc^2$.
- *Cosmology*: Stretching of quantum fluctuations during inflation.
- *Quantum measurement*: Position measurements require curvature-distorted Hilbert spaces.
- *Spin-curvature coupling*: Dirac field extensions may amplify GLS through spin-gravity interactions.

Future research directions should include experimental detection of GLS signatures using ultracold neutron interferometry, extension to Dirac fields where spin-curvature coupling may amplify these effects, and investigation of GLS in cosmological contexts where expanding spacetime modifies quantum probability distributions. The integration of advanced mathematical methods from recent works [27–29] may further enhance our understanding of nonlinear quantum phenomena in curved spacetime. In conclusion, spacetime curvature actively reshapes quantum reality – not merely through dynamical phases but via probability geometry itself. While SE-WGF suffices for weak terrestrial fields, QFT-CS emerges as the fundamental framework for gravitational quantum phenomena. Experimental detection of GLS signatures and extensions to Dirac fields represent critical next steps in probing quantum-gravity interfaces.

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Appendix

Below are the key steps in the derivation of energy quantization given by Eq. (4.7):

(1) **Wave vector:** $k_z = g_{zz}k^z$, where $g_{zz} = \frac{z}{z - r_s}$, and $k^z = \frac{1}{\hbar}\sqrt{\frac{\hbar^2\omega^2}{c^2} - \frac{z - r_s}{z}m^2c^2}$. Let $\kappa^2 = \frac{\omega_n^2}{c^2} - \frac{m^2c^2}{\hbar^2}$:

$$k_z(z) = \frac{z}{z - r_s}\sqrt{\kappa^2 + \frac{r_s m^2c^2}{\hbar^2 z}}.$$

(2) **Weak-field approximation:** For $r_s \ll r_f$, $L \ll r_f$, approximate $\frac{z}{z - r_s} \approx 1 + \frac{r_s}{z}$, with $z = r_f + \zeta$ ($\zeta = 0$ to L):

$$k_z(z) \approx \left(1 + \frac{r_s}{r_f} - \frac{r_s\zeta}{r_f^2}\right)\sqrt{\kappa^2 + \frac{r_s m^2c^2}{\hbar^2 r_f}\left(1 - \frac{\zeta}{r_f}\right)}.$$

(3) **Integral:** Compute $\int_{r_f}^{r_f+L} k_z \mathrm{d}z = \int_0^L k_z(\zeta)\mathrm{d}\zeta$, expanding to first order:

$$\int_0^L k_z \mathrm{d}\zeta \approx L\sqrt{\kappa^2 + \frac{r_s m^2c^2}{\hbar^2 r_f}}\left(1 + \frac{r_s}{r_f}\right) - \frac{r_s m^2c^2L^2}{4\hbar^2 r_f^2\sqrt{\kappa^2 + \frac{r_s m^2c^2}{\hbar^2 r_f}}} - \frac{r_s L^2}{2r_f^2}\sqrt{\kappa^2 + \frac{r_s m^2c^2}{\hbar^2 r_f}}.$$

(4) **Quantization:** Set equal to $n\pi$, define $s = \kappa^2 + \frac{r_s m^2c^2}{\hbar^2 r_f}$, $\varepsilon = \frac{r_s}{r_f} - \frac{r_s L}{2r_f^2}$, $a_s = \frac{r_s m^2c^2L^3}{\hbar^2 r_f^2}$:

$$L\sqrt{s}\left(1 + \frac{r_s}{r_f} - \frac{r_s L}{2r_f^2}\right) - \frac{r_s m^2c^2L^2}{4\hbar^2 r_f^2\sqrt{s}} = n\pi,$$

$$L\sqrt{s}(1 + \varepsilon) - \frac{a_s}{4L\sqrt{s}} = n\pi.$$

(5) **Solve for** s: Multiply by $\sqrt{s}$, rearrange into quadratic form:

$$Ls(1 + \varepsilon) - n\pi\sqrt{s} - \frac{a_s}{4L} = 0,$$

solve:

$$\sqrt{s} = \frac{n\pi + \sqrt{n^2\pi^2 + a_s(1 + \varepsilon)}}{2L(1 + \varepsilon)}, \quad l_n = \frac{n\pi + \sqrt{n^2\pi^2 + a_s}}{2L}.$$

(6) **Approximate** s: For small ε, $\sqrt{s} \approx l_n(1 - \varepsilon + \frac{a_s\varepsilon}{4Ll_n\sqrt{n^2\pi^2 + a_s}})$, so:

$$s \approx l_n^2\left(1 - 2\varepsilon + \frac{a_s\varepsilon}{2Ll_n\sqrt{n^2\pi^2 + a_s}}\right).$$

(7) **Energy:** Substitute $\kappa^2 = s - \frac{r_s m^2c^2}{\hbar^2 r_f}$, we get Eq. (4.7).