Research Article

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Laplace transform technique and probabilistic analysis-based hypothesis testing in medical and engineering applications

https://doi.org/10.1515/phys-2025-0221
received January 07, 2025; accepted September 28, 2025

Abstract: This study introduces a novel nonparametric hypothesis test designed to evaluate exponentiality against the New Better than Renewal Used in Moment Generating Function ($\mathrm{NBRU}_{\mathrm{mgf}}$) class. The test employs the Laplace transform to construct a scale-invariant statistic, facilitating the efficient analysis of both complete and right-censored data. We derive the asymptotic distribution of the test statistic and calculate Pitman's asymptotic efficiency under common alternatives, including Weibull, Gamma, and Makeham distributions. Extensive Monte Carlo simulations are conducted to determine the critical values and to demonstrate the test's power properties across a wide range of scenarios. The application of this test to real datasets of physical, engineering, and medical relevance corroborates the validity of our method. The test demonstrates superior efficiency and stability compared to existing methodologies, thereby providing a valuable tool for reliability estimation and statistical inference.

Keywords: reliability theory, life distribution classes, nonparametric hypothesis testing, censored data, real world data

1 Introduction

Life distributions are simple instruments used in statistical modeling across a wide range of disciplines, including reliability engineering, biomedical science, and industrial systems. Life distributions provide critical information regarding the failure behavior and performance of systems over time. The most extensively investigated classes are Increasing Failure Rate (IFR), New Better than Used (NBU), New Better than Used in Expectation (NBUE), and Harmonic NBUE (HNBUE). All these classes of aging capture some temporal characteristics and failure patterns. Models based on renewal such as New Renewal Better than Used (NRBU), Renewal New Better than Used (RNBU), and Renewal NBUE (NRBUE) extend these concepts to systems that are replaced or repaired at regular intervals. These models are particularly useful in modeling actual systems where maintenance policies affect the effective lifetime distribution of components, for additional details, see Al-Ruzaiza *et al.* [1], Ross [2], Ahmad [3], Lai and Xie [4], Mahmoud *et al.* [5], Ghosh and Mitra [6], Wasim *et al.* [7], Abbas *et al.* [8], Nazir *et al.* [9] and references therein. Furthermore, the study of life distributions is essential for comprehending time-to-event results in medical research, such as the beginning of sickness or patient recovery. Classes such as IFR, which show an IFR over time, have consequences for treatment planning and medical prognosis. The interaction between these courses and practical medical applications highlights the importance of understanding the temporal dynamics in health contexts.

Renewal classes add even more functionality to the analytical toolbox by providing a framework for modeling systems that require replacement or replenishment on a regular basis. When components are periodically upgraded or replaced, the NRBU, NRBUE, and RNBU classes offer insightful information on renewal processes and enable the characterization of system behavior.

1.1 Renewal classes

In statistical modeling, renewal classes are essential because they offer a flexible framework for comprehending systems that are periodically replenished or replaced. The NRBU, RNBRU, and RNBU classes offer significant insights into the

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renewal processes. These insights facilitate the characterization of system behavior in situations involving periodic upgrades or replacements. Their unique role in statistical modeling highlights their significance for reliability analysis by providing a full toolbox for evaluating the dynamics of periodically renewing components. In this subsection, we explore the key terminologies and approaches related to renewal classes, emphasizing their importance in describing temporal behavior and advancing our knowledge of dependability dynamics. Abouammoh *et al.* [10], Mahmoud *et al.* [11], Mugdadi and Ahmad [12] and EL-Arishy *et al.* [13] can be consulted for further information.

The renewal survival function, denoted by $\bar{W}_F(t) = \frac{1}{\mu}\int_t^{\infty}\bar{F}(u)\mathrm{d}u$, plays a central role in the renewal theory. Here t represents the lifetime of a device with a finite mean $\mu = \int_0^{\infty}\bar{F}(u)\mathrm{d}u$, and captures the cumulative distribution of the device failure times.

The following definitions encapsulate key properties of renewal classes:

- New Better than Renewal Used (NBRU): X is NBRU if

$$\bar{W}_F(t)\bar{F}(x) \geq \bar{W}_F(x+t); t, x > 0. \tag{1}$$

- RNBU: X is RNBU if

$$\bar{W}_F(x)\bar{F}(t) \geq \bar{F}(x+t); t, x > 0. \tag{2}$$

- Renewal new Better than Renewal Used (RNBRU): X is RNBRU if

$$\bar{W}_F(t)\bar{W}_F(x) \geq \bar{W}_F(x+t); t, x > 0. \tag{3}$$

These definitions capture different aspects of renewal processes, highlighting the relationship between survival functions and renewal properties.

Furthermore, we introduce the concept of New Better than Renewal Used in Moment Generating Function order ($\mathrm{NBRU_{mgf}}$)

Definition (1): X is $\mathrm{NBRU_{mgf}}$ if

$$\bar{W}_F(t)\int_0^{\infty}e^{sx}\bar{F}(x)\mathrm{d}x \geq \int_0^{\infty}e^{sx}\bar{W}_F(x+t)\mathrm{d}x, \text{ for all} x, t \geq 0, s \geq 0. \tag{4}$$

Eq. (4) can be written as follows:

$$\int_0^{\infty}\int_t^{\infty}e^{sx}\bar{F}(x)\bar{F}(y)\mathrm{d}y\mathrm{d}x \geq \int_0^{\infty}\int_t^{\infty}e^{sx}\bar{F}(x+y)\mathrm{d}y\mathrm{d}x. \tag{5}$$

The $\mathrm{NBRU_{mgf}}$ class offers a state-of-the-art framework that generalizes and extends the renewal-based aging models. Though of theoretical interest, hypothesis testing for the $\mathrm{NBRU_{mgf}}$ class has so far been examined only partially in the literature, especially when censored data are involved.

In this study, we obtain a new test statistic based on the characterization of the $\mathrm{NBRU_{mgf}}$ class through the Laplace transform and determine its asymptotic distribution, providing expressions for the variance under the null hypothesis. The test efficiency is assessed in terms of Pitman asymptotic efficiencies, whose values are found for commonly considered alternatives, such as Weibull, Makeham, and Linear Failure Rate (LFR) distributions. To facilitate practical implementation, Monte Carlo simulations were used to find critical values across different sample sizes and conduct a power analysis. In addition, the method was validated against real datasets found in engineering and medical studies, with great success and effectiveness. Overall, this study enhances the art of nonparametric hypothesis testing for reliability analysis through the provision of a statistically justifiable and computationally efficient process that is favorably suited for diverse applications.

The remainder of this study is organized as follows. In Section 2, a test statistic utilizing the Laplace Transform technique is presented to test whether the alternative hypothesis is not exponential and belongs to the $\mathrm{NBRU_{mgf}}$ class, against the null hypothesis that has an exponential distribution. In Section 3, Mathematica 13.3 is used to depict the critical points of the Monte Carlo null distribution for sample sizes ranging from 5 to 100. In Section 4, the Pitman asymptotic efficiency of widely used alternatives is presented. The power estimates for the proposed test are presented in Section 5. In Section 6, we present a testing procedure using right-censored data. To demonstrate the significance and applicability of the test presented in this study, sets of real data are analyzed in Section 7.

2 Testing exponentiality

Based on the Laplace transform, statisticians and reliability analysts have created novel techniques in recent years for assessing exponentiality over a range of life distribution classes. Studies by Atallah *et al.* [14], Mahmoud *et al.* [15], Abu-Youssef *et al.* [16], Gadallah [17], EL-Sagheer *et al.* [18], and Bakr *et al.* [19] demonstrate this progress.

The theoretical foundation for building a departure measure from exponentiality is the following inequality, which is a consequence of the definitions of $\mathrm{NBRU_{mgf}}$ class in Eq. (4) and Laplace transform properties. The distribution is exponential if equality occurs; it is in the $\mathrm{NBRU_{mgf}}$ class if inequality occurs strictly.

$$\int_0^\infty\int_0^\infty e^{-bt}e^{sx}\bar{W}_F(t)\bar{F}(x)\mathrm{d}x\mathrm{d}t \geq \int_0^\infty\int_0^\infty e^{-bt}e^{sx}\bar{W}_F(x+t)\mathrm{d}x\mathrm{d}t. \quad (6)$$

After several calculations, assuming $\varphi(s) = E[e^{sx}]$, we obtain

$$I_1 = \int_0^\infty e^{sx}\bar{F}(x)\mathrm{d}x, = \frac{1}{s}(\varphi(s)-1), \quad (7)$$

$$I_2 = \int_0^\infty e^{-bt}\bar{W}_F(t)\mathrm{d}t = \frac{1}{\mu b^2}(b\mu + \varphi(-b) - 1), \quad (8)$$

$$\begin{aligned} I_3 &= \int_0^\infty\int_0^\infty e^{-bt}e^{sx}\bar{W}_F(x+t)\mathrm{d}x\mathrm{d}t \\ &= \frac{1}{sb\mu(s+b)}\left(\frac{b}{s}[\varphi(s)-1] - \frac{s}{b}[\varphi(-b)-1] - (s+b)\mu\right). \end{aligned} \quad (9)$$

Substituting Eqs (7)–(9) in Eq. (6), we obtain

$$\begin{aligned} &\frac{1}{b}(\varphi(s)-1)(b\mu+\varphi(-b)-1) \\ &\geq \frac{1}{sb\mu(s+b)}\left(\frac{b}{s}[\varphi(s)-1] - \frac{s}{b}[\varphi(-b)-1] - (s+b)\mu\right). \end{aligned} \quad (10)$$

Then, we formulate the following exponential departure measure:

$$\begin{aligned} \Delta(s,b) = &\left(\frac{s+b}{b}\right)(\varphi(s)-1)(b\mu+\varphi(-b)-1) \\ &- \frac{b}{s}(\varphi(s)-1) + (b+s)\mu + \frac{s}{b}(\varphi(-b)-1). \end{aligned} \quad (11)$$

Thus, in order to make the test scale invariant

$$\hat{\delta}(s,b) = \frac{\Delta(s,b)}{\bar{X}^2}. \quad (12)$$

The empirical estimate of $\hat{\delta}(s,b)$, as defined in Eq. (12), can be expressed as follows:

$$\begin{aligned} &\hat{\delta}_n(s,b) \\ &= \frac{1}{n^2\ \bar{X}^2} \\ &\sum_{i=1}^n\sum_{j=1}^n\left[\begin{array}{l} \left(\frac{s+b}{b}\right)(e^{s\ X_j}-1)(bX_i + e^{-b\ X_i}-1) \\ - \frac{b}{s}(e^{s\ X_i}-1) + (b+s)X_i + \frac{s}{b}(e^{-bX_i}-1) \end{array}\right]. \end{aligned} \quad (13)$$

Theorem 2.1. *As* $n \to \infty$, $\sqrt{n}(\hat{\delta}_n(s,b) - \delta_n(s,b))$ *converges to an asymptotically normal distribution with a mean of zero and a variance of* $\sigma^2(s,b)$. *Under the null hypothesis* H_0, *the variance becomes* $\sigma_0^2(s,\beta)$ *as expressed in* Eq. (18).

Proof:

Let,

$$\begin{aligned} \omega(X_1,X_2) = &\left[\left(\frac{s+b}{b}\right)(e^{s\ X_2}-1)(bX_1 + e^{-b\ X_1}-1)\right. \\ &\left.- \frac{b}{s}(e^{s\ X_1}-1) + (b+s)X_1 + \frac{s}{b}(e^{-bX_1}-1)\right], \end{aligned} \quad (14)$$

$$\begin{aligned} &E(\omega(X_1,X_2)|\ X_1) \\ &= \frac{s^2(1+b)(1-e^{-bX_1}) - sb(s+b)X_1 - b^2(s-1)(e^{sX_1}-1)}{bs(s-1)}, \end{aligned} \quad (15)$$

$$E(\omega(X_1,X_2)|\ X_2) = \frac{b(b+s)[(s-1)e^{sX_2}+1]}{(b+1)(s-1)}. \quad (16)$$

Let us define $\beta(X)$ such that

$$\begin{aligned} \beta(X) &= E(\omega(X_1,X_2)|\ X_1) + E(\omega(X_1,X_2)|\ X_2) \\ &= \frac{s - e^{-bx}s + \frac{b^2(1+e^{sx}(-1+s))(b+s)}{1+b} - \frac{b^2(1+e^{sx}(-1+s)+s(-1+x))}{s} + bs(1-e^{-bx}-x)}{b(-1+s)}. \end{aligned} \quad (17)$$

After computation, it is easy to discover that under H_0, $\mu_0 = E(\beta(X)) = 0$ and the variance is

$$\sigma_0{}^2 = -\frac{b^2(5+b(7+2b-2s)-s)s^2(b+s)^2}{(1+b)^2(1+2b)(1+b-s)(-1+s)^2(-1+2s)}. \quad (18)$$

To assess the proposed test, we study finite-sample performance through large-scale Monte Carlo simulations, tabulating critical values and tabulated percentiles for sample sizes from 5 to 100. We also calculate Pitman's asymptotic efficiency (PAE) for some common alternatives, the Makeham, Weibull, and LFR distributions, providing a theoretical benchmark for sensitivity and calculating empirical power under various alternatives to test robustness. All of the calculations and simulations were coded in Mathematica 13.3.

3 Monte Carlo null distribution critical points

Herein, we report the simulation results used to evaluate the performance of the proposed test. Simulations were performed with 10,000 random samples for each sample size $n \in \{5, \quad 10, \ldots, \quad 100\}$. A null distribution was estimated using the standard exponentiality assumption. For each, we computed critical values at commonly used significance levels (1, 5, 10, 90, 95, and 99%). The results are tabulated and graphed in supporting figures.

The critical values are shown in Tables 1 and 2, respectively, at $s = 0.01$, $b = 5$, and $s = 0.1$, $b = 5$.

Tables 1 and 2, along with Figures 1 and 2, clearly demonstrate that as the sample size increases, the critical values stabilize and converge, thereby illustrating the consistency of the proposed test. Furthermore, the percentile values decrease with the increase in sample size, as evidenced by the enhanced information derived from larger samples.

Table 1: Percentile points $\hat{\delta}(0.01, 5)$

n	1%	5%	10%	90%	95%	99%
5	−0.02837	−0.01405	−0.0069	0.01760	0.01923	0.02142
10	−0.03753	−0.01669	−0.00919	0.01480	0.016386	0.018392
15	−0.026188	−0.017189	−0.011236	0.01267	0.0142075	0.016274
20	−0.02765	−0.01519	−0.00938	0.011639	0.01292	0.015314
25	−0.0298	−0.01327	−0.007645	0.010744	0.01201	0.014396
30	−0.02633	−0.01309	−0.00887	0.00966	0.011023	0.012956
35	−0.027	−0.0134	−0.00908	0.00892	0.01041	0.013171
40	−0.02507	−0.01078	−0.00762	0.008658	0.010174	0.01213
45	−0.01874	−0.011165	−0.007456	0.00819	0.00967	0.012215
50	−0.01929	−0.010914	−0.00737	0.008029	0.00944	0.011278
55	−0.0175	−0.01090	−0.00727	0.007649	0.008827	0.011389
60	−0.01524	−0.00943	−0.00620	0.00765	0.0086	0.01059
65	−0.0174	−0.010014	−0.00624	0.00738	0.008396	0.01053
70	−0.01949	−0.00924	−0.00674	0.00697	0.008136	0.01014
75	−0.01709	−0.00855	−0.00538	0.00654	0.00824	0.00963
80	−0.01644	−0.00876	−0.00595	0.00616	0.0073	0.00896
85	−0.01393	−0.00868	−0.006598	0.006138	0.007449	0.00916
90	−0.01595	−0.00849	−0.00555	0.00637	0.007464	0.00941
95	−0.013102	−0.007598	−0.005367	0.006184	0.00723	0.00906
100	−0.01273	−0.00792	−0.005891	0.0059180	0.0071235	0.00895

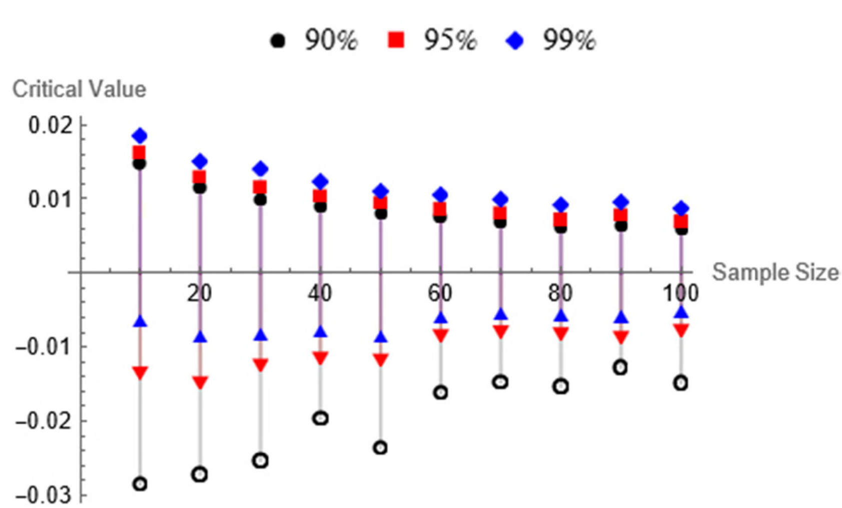

Figure 1: Relation between n and percentile points at $s = 0.01$, $b = 5$.

Table 2: Percentile points $\hat{\delta}(0.1, 5)$

n	**1%**	**5%**	**10%**	**90%**	**95%**	**99%**
5	−0.277746	−0.133843	−0.0573371	0.200031	0.217643	0.248175
10	−0.384055	−0.16892	−0.0900234	0.163156	0.176799	0.210931
15	−0.419416	−0.17994	−0.100138	0.147477	0.161652	0.189299
20	−0.346888	−0.182371	−0.0999247	0.134256	0.147959	0.175123
25	−0.300169	−0.143848	−0.0865586	0.124446	0.142333	0.165532
30	−0.381979	−0.180721	−0.112804	0.112581	0.129599	0.149253
35	−0.282259	−0.136437	−0.0805488	0.107287	0.123885	0.143892
40	−0.26349	−0.134984	−0.0887312	0.0962538	0.114215	0.137399
45	−0.234946	−0.142924	−0.0926471	0.0952688	0.107379	0.128674
50	−0.290997	−0.112527	−0.0727774	0.0926585	0.108896	0.134277
55	−0.271661	−0.126957	−0.0844715	0.0863861	0.0990114	0.116895
60	−0.253524	−0.127403	−0.0865372	0.0873587	0.101944	0.123945
65	−0.218246	−0.120645	−0.0830349	0.0824862	0.0949735	0.120214
70	−0.22568	−0.110913	−0.0737533	0.0810302	0.0937786	0.112777
75	−0.182346	−0.106899	−0.0725559	0.0804655	0.0943715	0.115878
80	−0.218004	−0.104037	−0.0701519	0.0742153	0.0889227	0.115672
85	−0.159681	−0.0896711	−0.0623137	0.07312	0.0864806	0.102854
90	−0.176648	−0.100136	−0.0699739	0.0744652	0.0849296	0.106745
95	−0.194096	−0.103761	−0.0723404	0.0695266	0.0832672	0.102215
100	−0.197167	−0.0887166	−0.062909	0.069655	0.0799656	0.1003

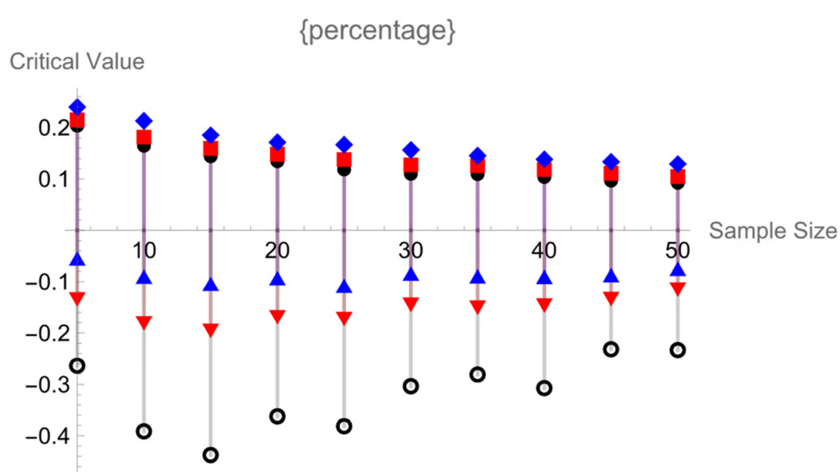

Figure 2: Relation between n and percentile points at $s = 0.1$, $b = 5$.

4 PAEs

To investigate the sensitivity of the proposed test to different types of departures from exponentiality, we computed PAEs for three rival distributions: Makeham, Weibull, and LFR. The PAE values are a measure of how well the test separates the null and alternative hypotheses. The result show that the test performs particularly well for the Weibull and LFR families with high asymptotic relative efficiency compared to benchmark tests.

$$\mathrm{PAE}(\Delta(s,b)) = \frac{1}{\sigma_0}\left|\frac{\mathrm{d}}{\mathrm{d}\theta}\Delta(s,b)\right|_{\theta\to\theta_0},$$

where

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\Delta(s,b) = {} & \left(\frac{s+b}{b}\right)[(\varphi(s)-1)(b\mu' + \varphi'(-b)) \\ & + \varphi'(s)(b\mu + \varphi(-b) - 1)] - \frac{b}{s}\varphi'(s) \\ & + \frac{s}{b}\varphi'(-b) + (s+b)\mu',\end{aligned} \tag{19}$$

where $\mu' = \int_0^\infty x\,\mathrm{d}\grave{F}_\theta(x)$, $\varphi'(s) = \int_0^\infty \mathrm{e}^{sx}\mathrm{d}\grave{F}_\theta(x)$.

Here we obtain

Table 3: $\Delta(0.01, b)$ efficiencies at various b values

Distribution	$b = 2$	$b = 3$	$b = 4$	$b = 5$
Makeham	0.23156	0.23799	0.24137	0.24337
Weibull	0.90820	0.93570	0.95133	0.96122
LFR	0.99301	0.99665	0.99810	0.99880

Table 4: Asymptotic relative efficiencies

Distribution	$(\Delta(0.01, 5),\ \hat{\delta}(0.8))$	$(\Delta(0.01, 5),\ \Delta(0.01))$
LFR	1.14288	1.12125
Makeham	1.31558	1.11×10^{-4}
Weibull	88.0964	1.44520

$$\mathrm{PAE}(\Delta(s,b),\ \mathrm{Makeham}) = \frac{1}{\sigma_0}\left|\frac{bs(3+b)(b+s)}{2(1+b)(2+b)(s-2)(s-1)}\right|, \tag{20}$$

$$\mathrm{PAE}(\Delta(s,b),\ \mathrm{Weibull}) = \frac{1}{\sigma_0}\left|\frac{s\ln[1+b] + b(1+b+s)\ln[1-s]}{(1+b)(s-1)}\right|, \tag{21}$$

$$\mathrm{PAE}(\Delta(s,b),\ \mathrm{LFR}) = \frac{1}{\sigma_0}\left|\frac{bs(2+b)(b+s)}{(1+b)^2(s-1)^2}\right|. \tag{22}$$

As indicated from Table 3 and Figure 3, the PAE's of $\Delta(s, b)$ increased as s decreased, and b increased.

A comparison is made between the PAE of our proposed test $\Delta(s, b)$ and $\hat{\delta}(s)$, which Hassan and Said [20] depicted for $\mathrm{NBRU_{mgf}}$ based on moment inequalities, and $\Delta(s)$, which Bakr *et al.* [21] showed for $\mathrm{NBRU_{mgf}}$ based on measure of deviation given in Table 4.

From Table 4, our test clearly has the highest efficiency in Weibull and LFR families.

5 Power estimates

At the significance level $\alpha = 0.05$, Table 5 will be used to conduct the power estimation of the suggested test $\hat{\delta}(0.01,5)$. Based on 10,000 simulated samples for sizes $n = 10$, 20, and 30, these powers were determined for the Gamma and Weibull distributions.

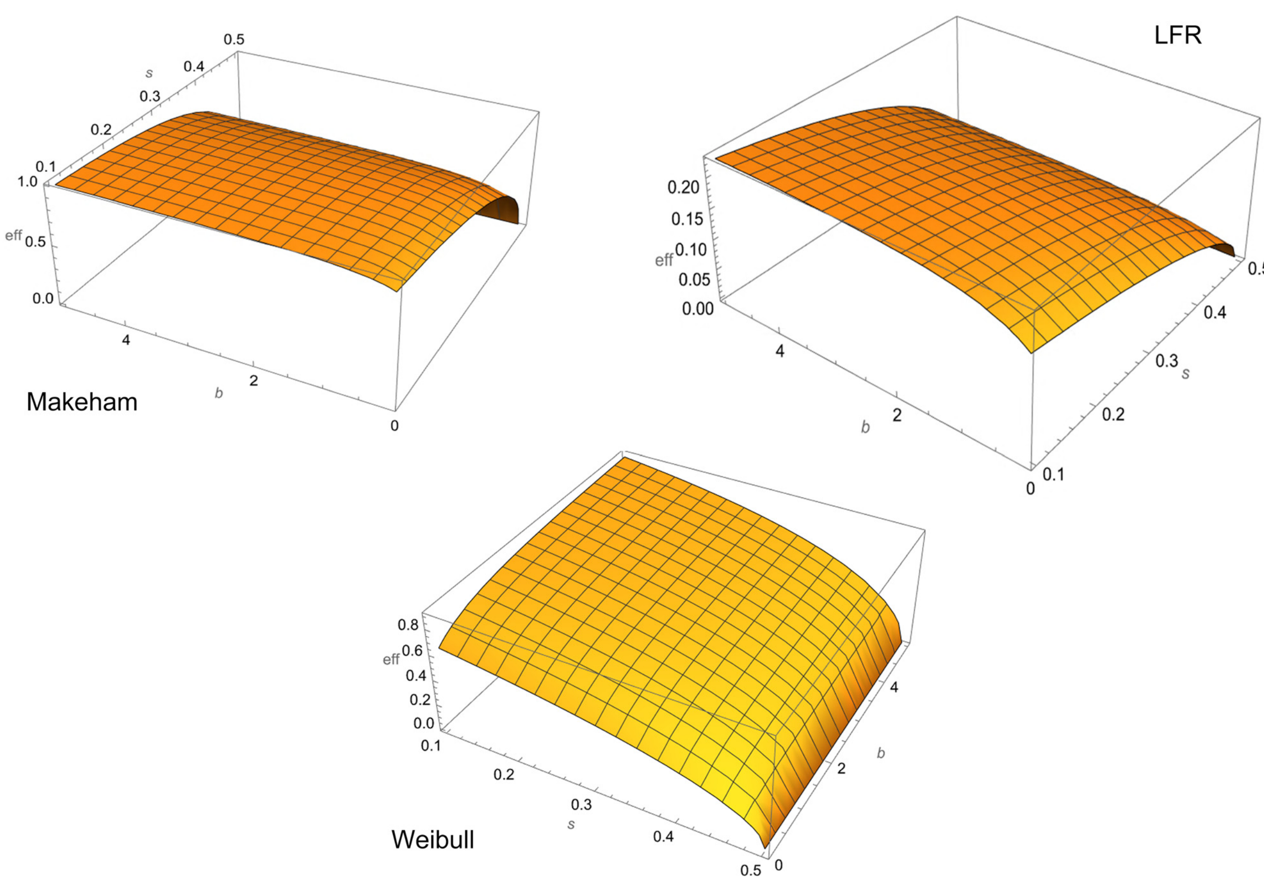

Figure 3: The relationship between PAE and the values of s and b for Makeham, Weibull, and LFR.

Table 5: Powers estimates at $\alpha = 0.05$

Distribution	n	$\theta = 2$	$\theta = 3$	$\theta = 4$
Gamma	10	0.5802	0.9121	0.9913
	20	0.701	0.9725	0.9988
	30	0.8082	0.9922	0.9997
Weibull	10	0.6947	0.9848	0.9998
	20	0.9456	0.9999	1
	30	0.9934	1	1

Table 5 clearly demonstrates the suitability of our test for the Weibull and Gamma distributions, as the test's power consistently increases with larger sample sizes (n) and higher parameter values (θ).

Table 6: Percentile points $\hat{\delta}^{c}_{U_{\text{mgf}}}$ (0.1, 5)

N	1%	5%	10%	90%	95%	99%
4	3.49871	3.71034	3.82311	4.61405	4.72682	4.93845
8	3.70422	3.85387	3.93361	4.49289	4.57263	4.72228
12	3.78985	3.91204	3.97715	4.43379	4.4989	4.62109
16	3.83276	3.93858	3.99497	4.39043	4.44682	4.55264
18	3.84298	3.94275	3.99591	4.36876	4.42192	4.52168
20	3.84396	3.93861	3.98904	4.34276	4.39319	4.48784
30	3.98714	4.06442	4.1056	4.3944	4.43558	4.51286
40	4.02236	4.08928	4.12494	4.37506	4.41072	4.47764
50	4.16285	4.22271	4.25461	4.47832	4.51021	4.57007
60	4.10866	4.16331	4.19242	4.39664	4.42576	4.4804
70	4.10725	4.15784	4.18479	4.37386	4.40082	4.45141
80	4.11113	4.15845	4.18367	4.36053	4.38574	4.43306
90	4.11602	4.16064	4.18442	4.35116	4.37493	4.41955
100	4.12093	4.16326	4.18581	4.344	4.36655	4.40888

6 Test for NBRU$_{\text{mgf}}$ in case for right censored data

Using the estimator of Kaplan and Meier, the measure of departure provided in Eq. (11) can be expressed as follows:

$$\hat{\delta}^{c}_{U_{\text{mgf}}} = \left(\frac{s+b}{b}\right)(\eta-1)(b\zeta+\tau-1) - \frac{b}{s}(\eta-1) + (b+s)\zeta + \frac{s}{b}(\zeta-1), \tag{23}$$

where

$$\zeta = \sum_{k=1}^{n}\left[\prod_{m=1}^{k-1} C_m^{\delta(m)}\left(Z_{(k)} - Z_{(k-1)}\right)\right];$$

$$\eta = \sum_{j=1}^{n} e^{sZ_{(j)}}\left[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}\right];$$

$$\tau = \sum_{j=1}^{n} e^{-bZ_{(j)}}\left[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}\right];$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_{j-2}),$$

$$c_k = \frac{n-k}{n-k+1}.$$

The critical value percentiles of $\hat{\delta}^{c}_{U_m}$ for sample size n = 2(4)20(10)100 are displayed in Table 6.

7 Applications

To demonstrate the practical utility of the test, we used it on actual datasets from physical, engineering, and medical sciences. These examples demonstrate the potential of the test to detect departures from exponentiality and its usefulness for complete and censored data (Figure 4).

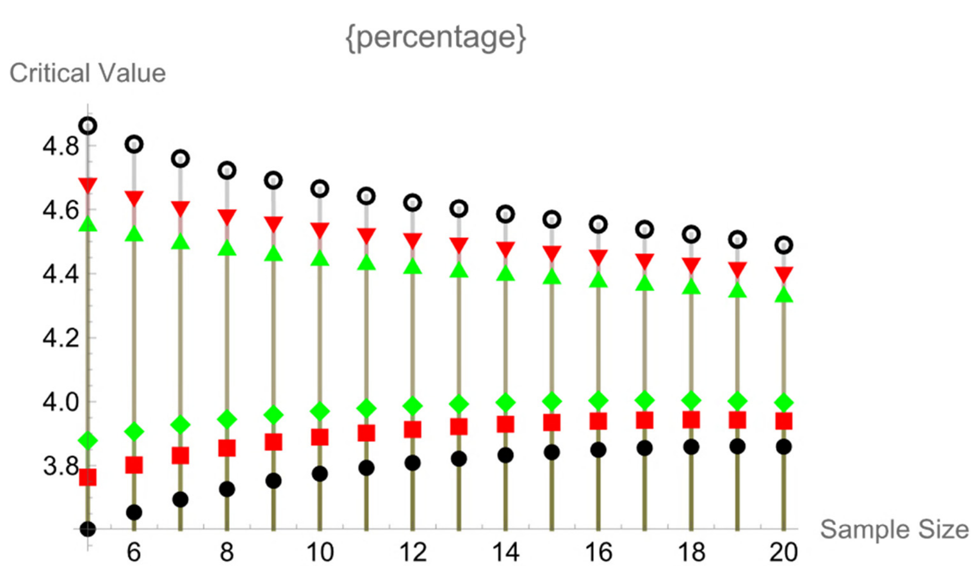

Figure 4: Relation between n and percentile points.

7.1 Complete data applications

7.1.1 Dataset #1: Data about carbon fibers' breaking stress

Examine the information studied by Marzouk *et al.* [22], which shows the breaking stress of carbon fibers (measured in Gba). Figure 5 displays the data visualization graphs.

We find that $\hat{\delta}(0.01, 5) = 0.02149$ and $\hat{\delta}(0.1, 5) = 0.26381$. These results fall within the rejection region of H_0. Therefore, we can reject the exponential property of these data.

7.1.2 Dataset #2: UK COVID-19 data

Suppose the dataset, as presented in Almetwally *et al.* [23], which depicts a 36-day period of COVID-19 data in Canada, spanning from April 10 to May 15, 2020 (Figure 6). This dataset is based on the daily death rate due to COVID-19.

After calculation, we find $\hat{\delta}(0.01, 5) = 0.02315$ and $\hat{\delta}(0.1, 5) = 0.2933$, both of which exceed the critical values from Tables 1 and 2. Therefore, H_0 is rejected, which conclude that the data exhibit the $NBRU_{mgf}$ property.

7.1.3 Dataset #3: Dataset in Keating

We consider a classical real data in Keating *et al.* [24] set on the times, in operating days, between successive failures of air conditioning equipment in an aircraft (Figure 7).

It is clear that the values of the test statistic $\hat{\delta}(0.01, 5) = 0.00785$ and $\hat{\delta}(0.1, 5) = 0.126$. This indicates that the data collection doesn't exhibit the $NBRU_{mgf}$ property.

7.2 Censored data application

7.2.1 Dataset #4: Cyclophosphamide-treated lung cancer patients' survival data

We consider a classical real data in Kamran Abbas *et al.* [25] dataset, which had data on the life durations of some patients with terminal lung cancer undergoing cyclophosphamide treatment. Twenty-eight edited and 33 unedited observations each feature a patient whose condition worsened and therapy was stopped (Figure 8).

At $\alpha = 0.05$, $\hat{\delta}^{c}_{U_{mgf}} = 4.77 \times 10^{-9}$, this value results in the acceptance area of H_0. Therefore, the null hypothesis,

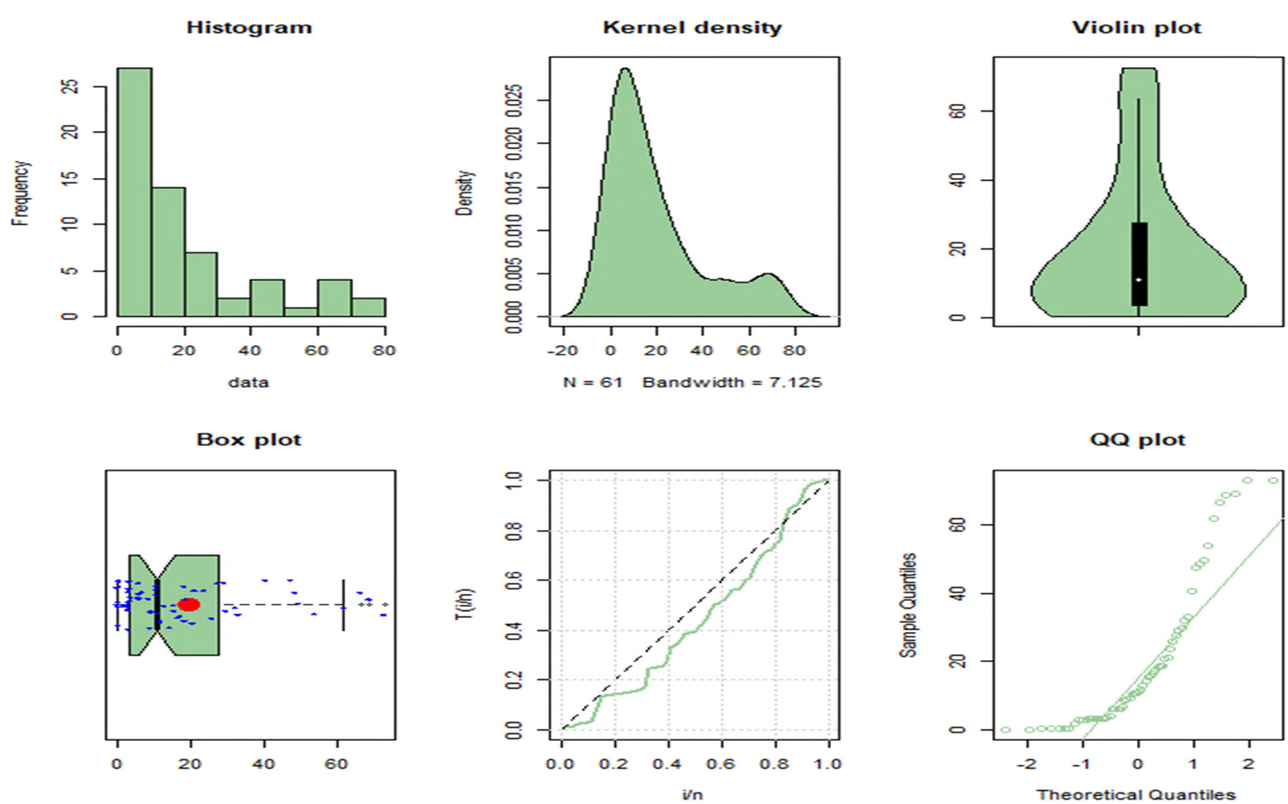

Figure 5: Representation of fibers' breaking stress dataset.

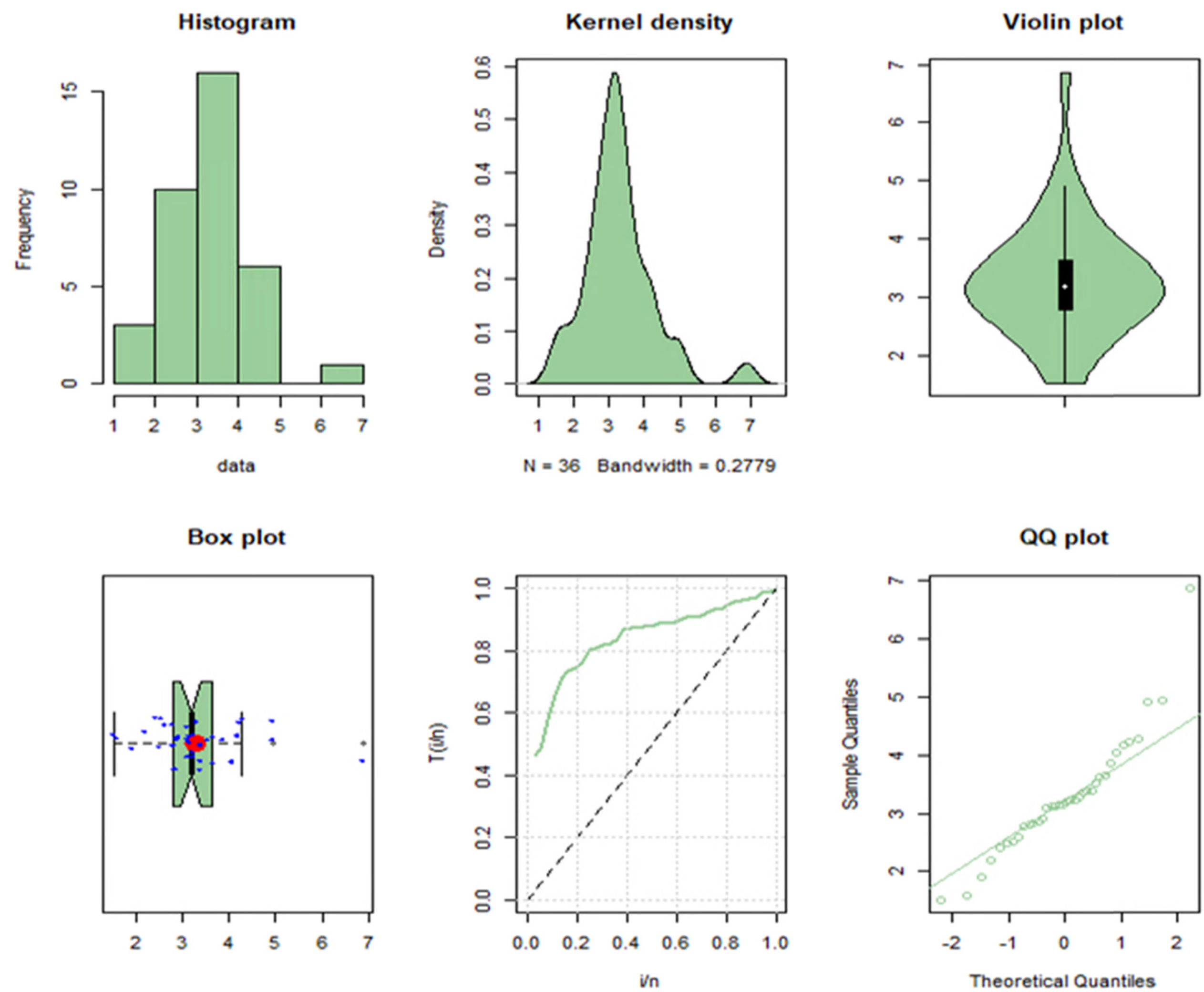

Figure 6: Representation of (COVID-19) dataset.

according to which the data exhibit exponential properties, cannot be rejected.

8 Conclusion

In this study, we introduce a novel nonparametric hypothesis testing method to assess exponentiality against the $NBRU_{mgf}$ class of life distributions. The test is grounded in solid theoretical foundations, including the derivation of its variance and asymptotic distribution under the null hypothesis. It shows competitive PAE compared to established methods, particularly under Gamma and Weibull alternatives. Extensive Monte Carlo simulations confirm the empirical reliability of the test, demonstrating sensitivity to deviations from exponentiality even with small sample sizes. Its practical utility is further highlighted through applications to both complete and right-censored real-world datasets from engineering, medical, and physical domains.

Despite these strengths, the current study is limited to univariate, independent observations. Future work could explore extending the test to multivariate or dependent data scenarios. Additionally, integrating alternative kernel functions or bootstrap-based methods may further improve its robustness and flexibility. We also suggest investigating broader classes of life distributions and different censoring schemes, which would enhance the applicability of the proposed test in diverse real-world reliability and survival analysis contexts.

In conclusion, the proposed Laplace transform-based test offers a valuable addition to the statistical tools used in reliability analysis. It provides a theoretically robust and practically flexible approach for identifying non-exponential behavior within the $NBRU_{mgf}$ class. This method shows

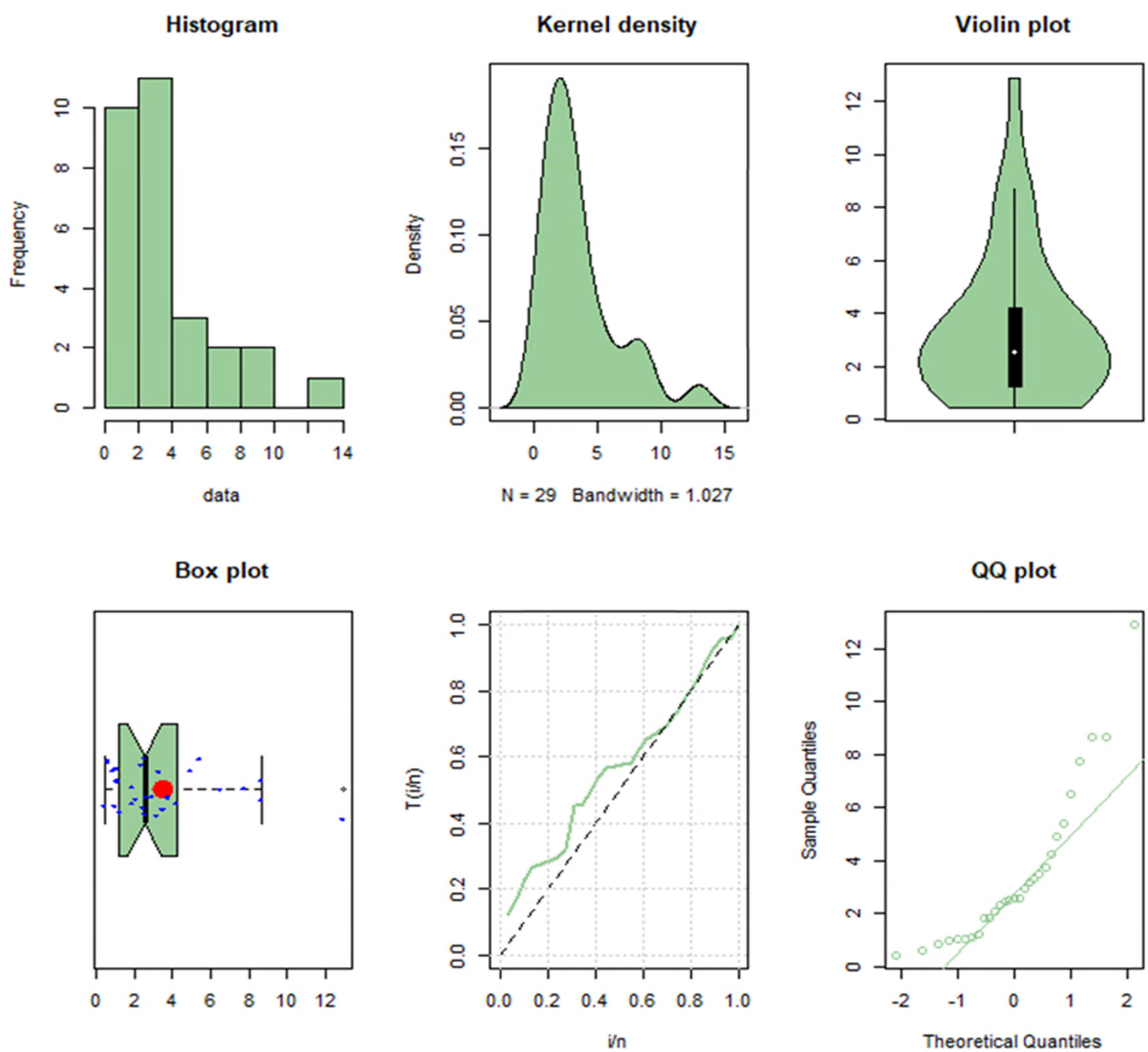

Figure 7: Representation of Keating dataset.

strong potential for application across various scientific and industrial fields where understanding life distribution patterns is essential.

Acknowledgments: The authors acknowledge the funding by Ongoing Research Funding program, (ORF-2025-1004), King Saud University, Riyadh, Saudi Arabia.

Funding information: This study was funded by Ongoing Research Funding program, (ORF-2025-1004), King Saud University, Riyadh, Saudi Arabia.

Author contributions: The authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: All data generated or analyzed during this study are included in this published article.

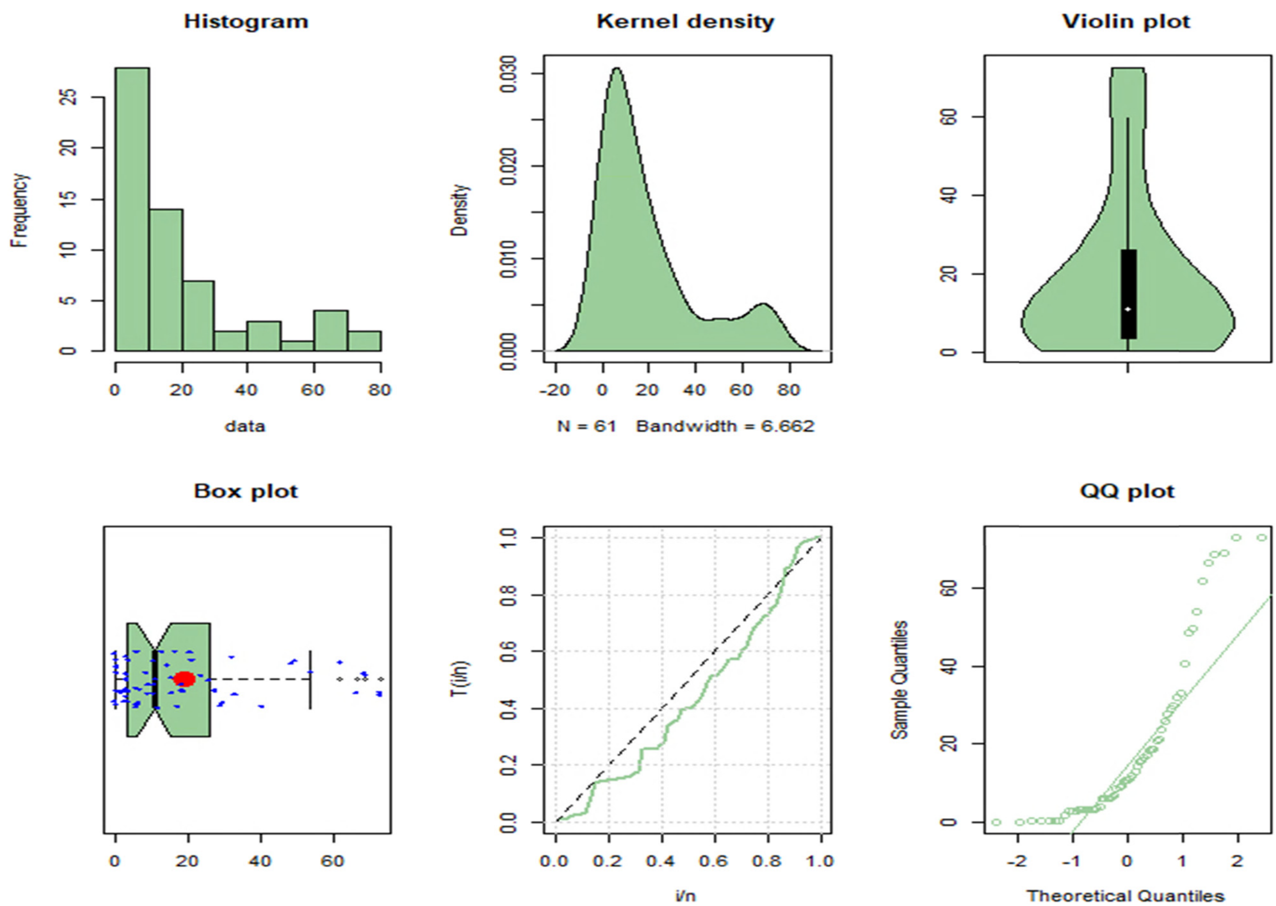

Figure 8: Cyclophosphamide-treated lung cancer patients' survival data.

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