

Research Article

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On the derivation of solitary wave solutions for the time-fractional Rosenau equation through two analytical techniques

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Abstract: An analytical study of the Rosenau equation, which is crucial for analyzing wave phenomena in a variety of physical systems where nonlinear dynamics and dispersion are significant, includes fluid dynamics, plasma physics, and materials science. This equation was proposed to explain the dynamic behavior of dense discrete systems. We employed the Sardar subequation approach and the generalized Riccati equation mapping method to derive analytical solutions. By applying the fractional wave transformation and the conformal fractional derivative, we were able to obtain these solutions. Using these techniques, we obtained a variety of solutions in the form of exponential, trigonometric, mixed hyperbolic, rational, and hyperbolic functions. The solutions include numerous solitary wave solutions, as well as bright and dark soliton solutions. By varying the parameter values, the analytical soliton solutions are further visualized through 2D, 3D, and contour plots using Mathematica 13.0.

Keywords: Sardar subequation approach, generalized Riccati equation mapping method, Rosenau equation, exact solutions

1 Introduction

Nonlinear partial differential equations (NLPDEs) are powerful mathematical tools for modeling complex and uncertain natural processes. Many systems in fields, such as biology [1], engineering [2], and physics [3], rely on nonlinear effects and variables that cannot be adequately described by linear differential equations. NLPDEs are widely used in numerous scientific and industrial disciplines [4], including materials science [5], astronautics [6], reaction–diffusion systems [7], chaotic systems [8], medicine [9], fluid dynamics [10], and wave propagation equilibria [11]. Although NLPDEs model a vast number of physical phenomena, these phenomena can only be fully understood by obtaining their exact solutions [12]. Several analytical methods have been proposed in the literature, including the unified algebraic method [13], the modified subequation extended method [14], the modified Exp-function method [15], variational principles [16], the tanh–coth method [17], the generalized Kudryashov method [18], the new auxiliary equation method [19], the amplitude ansatz technique [20], the sinh-Gordon expansion method [21], the first integral method [22], and the Jacobi elliptic function method [23]. These dynamic approaches [24] have been developed to effectively solve NLPDEs.

On the other hand, ordinary fractional differential equations and, more generally, fractional partial differential equations (FPDEs) have gained significant attention due to their broad applications in fields such as fluid mechanics, biology, physics, and engineering [25]. Consequently, solving fractional ordinary differential equations [26], FPDEs, and integral equations [27] in real-world physical contexts has become a major focus. Research has shown that FPDEs are essential for describing a wide range of nonlinear natural and physical systems. In this context, the present study focuses on the nonlinear Rosenau equation [28]:

$$D_t^\gamma P + D_x P + D_{xxxx} D_t^\gamma P - 30P^2 D_x P + 60P^4 D_x P = 0, \quad (1.1)$$

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where x represents the one-dimensional spatial position and t is the time. The operators D_x and D_{xxx} are standard integer-order spatial derivatives, and γ is a free parameter. In addition, D_t^γ represents the fractional derivative operators, where $\gamma \in (0, 1)$. The precise definition is provided next.

Definition 1.1. Assume $b : [0, \infty] \rightarrow \mathbb{R}$, and $\gamma \in (0, 1)$. The conformable functional derivative is defined as

$$D^\gamma(b)(t) = \frac{\lim_{\varpi \rightarrow 0} (b(\varpi t^{1-\gamma} + t) - bt)}{\varpi}, \quad (1.2)$$

for all $t > 0$ and $\gamma \in (0, 1]$.

The key properties of the conformable fractional derivative are summarized in the following.

Theorem 1.2. Let η and ζ be functions, and let $\gamma \in (0, 1]$. If η and ζ are γ -differentiable at $t > 0$, then the conformable fractional derivative satisfies the following properties:

1. $D^\gamma(\eta * p + \zeta * q) = p * D^\gamma(\eta) + q * D^\gamma(\zeta)$, for all $p, q \in \mathbb{R}$.
2. $D^\gamma(t^u) = u * t^{u-\gamma}$, for all $u \in \mathbb{R}$.
3. $D^\gamma(\eta \cdot \zeta) = \eta * D^\gamma(\zeta) + \zeta * D^\gamma(\eta)$.
4. $D^\gamma\left(\frac{\eta}{\zeta}\right) = \frac{\zeta * D^\gamma(\eta) - \eta * D^\gamma(\zeta)}{\zeta^2}$.
5. $D^\gamma(p) = 0$, for any constant function $q(t) = p$.
6. If q is differentiable, then $D^\gamma(\eta)(t) = t^{1-\gamma} * \frac{d\eta(t)}{dt}$.

Various numerical and analytical methods have been used to investigate the dynamics of different Rosenau equations. Park [28] investigated whether the Rosenau equation can be solved, and he definitely gave the exact answer. Chung and Ha [29] studied existence and uniqueness of the exact solutions, as well as the error estimates, using Galerkin finite element methods for a Rosenau equation similar to KdV. Zuo [30] employed sine-cosine and tanh techniques to obtain solitonic and periodic solutions of the Rosenau-KdV equation. Wang and Dai [31] presented a numerical solution to the Rosenau-KdV-RLW problem using quintic B-splines and finite element techniques.

A widely used analytical method for a variety of differential equations, especially nonlinear equations, is the generalized Riccati equation mapping (GREM) method [32], valued for its flexibility and simplicity [33], generality [34], nonlinearity [35], and the ability to simplify and solve challenges. GREM allows for systematic treatment of nonlinear PDEs, but its effectiveness depends on familiarity with the method, the characteristics of the problem, and an appropriate level of analytical precision. Nonlinear PDEs have numerous real-world applications. Numerical models based on complex nonlinear polynomial structures

are used to generate nightly TV weather reports. It is important to note that the main advantage of the Sardar subequation method (SSM) [36] is its ability to generate a wide variety of soliton solutions, including periodic singular solitons, combined dark-bright solitons, and mixed dark-singular solitons. The method is notable for its simplicity, reliability, and adaptability to nonlinear equations.

Before closing this stage, it is worth pointing out that there are more analytical techniques which have been used to derive soliton solutions for NLPDEs. For example, recently, Chatziafratis *et al.* [37] discovered a novel long-range instability phenomenon which is a previously unknown type for the inhomogeneous linear Schrödinger equation on the vacuum spacetime quarter-plane by developing the linear Fokas' unified transform method. Additionally, by developing the Dbar-steepest descent method, with respect to deriving the solutions of Wadati-Konno-Ichikawa equation, the authors of [38,39] have performed some interesting work. They solved the long-time asymptotic behavior of the solutions of the equation, and proved the soliton resolution conjecture and the asymptotic stability of solutions of the equation. Besides, Wu *et al.* [40] employed the recursion operator from the bi-Hamiltonian structure and diagonalized the non-diagonal matrix form in the second variation of the Lyapunov functional, thereby proving the stability of exact smooth multi-solitons for the two-component Camassa-Holm system.

2 Description of methods

In this section, we describe two analytical techniques – SSM and GREM – for solving the nonlinear Rosenau equation. The general form of the NLPDE is

$$A(P, P_t^\gamma P, D_x P, D_t^{2\gamma} P, P_{xx} P, \dots) = 0, \quad (2.1)$$

where A is a polynomial in P and its derivatives. We apply the transformation:

$$P(x, t) = S(\phi), \quad (2.2)$$

with $\phi = mx + \frac{ct^\gamma}{\gamma}$, and γ is the fractional order. Substituting (2.2) into (2.1), we obtain

$$B(S, S', S'', S''', \dots) = 0, \quad (2.3)$$

where B is a polynomial in S and its derivatives.

2.1 SSM

The steps of the SSM are as follows:

Step 1. Let us assume that Eq. (2.3) has a solution

$$S(\phi) = \sum_{j=0}^m a_j \psi^j(\phi), \quad a_j \neq 0, \quad (2.4)$$

where the coefficients a_j ($j = 0, 1, 2, \dots, m$) are to be determined later, and $\psi(\phi)$ satisfies the ordinary differential equation of the form

$$\psi'(\phi) = \sqrt{\kappa + \Omega \psi^2(\phi) + \psi^4(\phi)}. \quad (2.5)$$

Eq. (2.5) has the following solutions:

- **Case 1:** If $\kappa = 0$ and $\Omega > 0$, then

$$\psi_1^\pm(\phi) = \pm \sqrt{-db\Omega} \operatorname{sech}_{db}(\sqrt{\Omega}\phi),$$

$$\psi_2^\pm(\phi) = \pm \sqrt{-db\Omega} \operatorname{csch}_{db}(\sqrt{\Omega}\phi),$$

where $\operatorname{sech}_{db}\phi = 2(de^\phi + be^{-\phi})^{-1}$ and $\operatorname{csch}_{db}\phi = 2(de^\phi - be^{-\phi})^{-1}$.

- **Case 2:** If $\kappa = 0$ and $\Omega < 0$, then

$$\psi_3^\pm(\phi) = \pm \sqrt{-db\Omega} \operatorname{sec}_{db}(\sqrt{-\Omega}\phi),$$

$$\psi_4^\pm(\phi) = \pm \sqrt{-db\Omega} \operatorname{csc}_{db}(\sqrt{-\Omega}\phi),$$

where $\operatorname{sec}_{db}\phi = 2(de^{i\phi} + be^{-i\phi})^{-1}$ and $\operatorname{csc}_{db}\phi = 2i(de^{i\phi} - be^{-i\phi})^{-1}$.

- **Case 3:** If $\Omega < 0$ and $\kappa = \frac{\Omega^2}{4}$, then

$$\psi_5^\pm(\phi) = \pm \sqrt{-\frac{\Omega}{2}} \tanh_{db}\left(\sqrt{-\frac{\Omega}{2}}\phi\right),$$

$$\psi_6^\pm(\phi) = \pm \sqrt{-\frac{\Omega}{2}} \coth_{db}\left(\sqrt{-\frac{\Omega}{2}}\phi\right),$$

$$\psi_7^\pm(\phi) = \pm \sqrt{-\frac{\Omega}{2}} (\tanh_{db}(\sqrt{-2\Omega}\phi) + \sqrt{db} \operatorname{sech}_{de}(\sqrt{-2\Omega}\phi)),$$

$$\psi_8^\pm(\phi) = \pm \sqrt{-\frac{\Omega}{2}} (\coth_{db}(\sqrt{-2\Omega}\phi) + \sqrt{db} \operatorname{csch}_{de}(\sqrt{-2\Omega}\phi)),$$

$$\psi_9^\pm(\phi) = \pm \sqrt{-\frac{\Omega}{2}} \left(\tanh_{db}\left(\sqrt{-\frac{\Omega}{2}}\phi\right) + \coth_{db}\left(\sqrt{-\frac{\Omega}{2}}\phi\right) \right),$$

where

$$\tanh_{db}\phi = \frac{de^\phi - be^{-\phi}}{be^\phi + be^{-\phi}},$$

$$\coth_{db}\phi = \frac{de^\phi + be^{-\phi}}{de^\phi - be^{-\phi}}.$$

- **Case 4:** If $\Omega > 0$ and $\kappa = \frac{\Omega^2}{4}$, then

$$\psi_{10}^\pm(\phi) = \pm \sqrt{\frac{\Omega}{2}} \tan_{db}\left(\sqrt{\frac{\Omega}{2}}\phi\right),$$

$$\psi_{11}^\pm(\phi) = \pm \sqrt{\frac{\Omega}{2}} \cot_{db}\left(\sqrt{\frac{\Omega}{2}}\phi\right),$$

$$\psi_{12}^\pm(\phi) = \pm \sqrt{\frac{\Omega}{2}} (\tan_{db}(\sqrt{2\Omega}\phi) + \sqrt{db} \operatorname{sec}_{de}(\sqrt{2\Omega}\phi)),$$

$$\psi_{13}^\pm(\phi) = \pm \sqrt{\frac{\Omega}{2}} (\cot_{db}(\sqrt{2\Omega}\phi) + \sqrt{db} \operatorname{csc}_{de}(\sqrt{2\Omega}\phi)),$$

$$\psi_{14}^\pm(\phi) = \pm \sqrt{\frac{\Omega}{2}} \left(\tan_{db}\left(\sqrt{\frac{\Omega}{2}}\phi\right) + \cot_{db}\left(\sqrt{\frac{\Omega}{2}}\phi\right) \right),$$

where

$$\tanh_{de}\phi = -\frac{i(de^\phi - be^{-\phi})}{de^\phi + be^{-\phi}},$$

$$\coth_{de}\phi = -\frac{i(de^\phi + be^{-\phi})}{de^\phi - be^{-\phi}}.$$

Step 2. Determine the degree m in Eq. (2.4) using the balancing rule.

Step 3. Substitute (2.4) and (2.5) into (2.3), and expand the result in powers of $\psi(\phi)$.

Step 4. Set the coefficients of like powers of $\psi(\phi)$ to zero to form a system of algebraic equations. Solve for the unknown parameters ensuring non-trivial solutions.

2.2 GREM method

We propose that the solution to Eq. (2.3) takes the form

$$S(\phi) = \sum_{j=0}^m a_j \psi(\phi)^j,$$

where a_j are the coefficients to be determined. The following simplified Riccati equation requires $\psi(\phi)$ to balance the highest order linear term with the nonlinear term in order to present the solution, which defines the value of m :

$$\psi'(\phi) = r + p\psi(\phi) + s\psi^2(\phi), \quad (2.6)$$

with $r, p, s \in \mathbb{R}$. The parameters a_j ($j = 1, \dots, m$) and the function ψ are determined by substituting Eqs. (2.4) and (2.5) into the relevant ordinary differential Eq. (2.3) and equating all coefficients of like powers of ψ to zero. This

procedure yields a system of algebraic equations. Solving this system yields non-traveling wave solutions to the NPDE.

The 27 solutions of Eq. (2.6) can subsequently be obtained.

Type 1. When $d^2 - 4ef > 0$ and either $de \neq 0$ or $ef \neq 0$, we define $\sqrt{p^2 - 4sr} = \alpha_1$. The solutions of Eq. (2.6) are given by

$$\begin{aligned}\psi_1(\phi) &= -\frac{1}{2s} \left[p + \alpha_1 \tanh\left(\frac{\alpha_1}{2}\phi \right) \right], \\ \psi_2(\phi) &= -\frac{1}{2s} \left[p + \alpha_1 \coth\left(\frac{\alpha_1}{2}\phi \right) \right], \\ \psi_3(\phi) &= -\frac{1}{2s} [p + \alpha_1 (\tanh(\alpha_1\phi) \pm \operatorname{sech}(\alpha_1\phi))], \\ \psi_4(\phi) &= -\frac{1}{2s} [p + \alpha_1 (\coth(\alpha_1\phi) \operatorname{csch}(\alpha_1\phi))], \\ \psi_5(\phi) &= -\frac{1}{4s} \left[2p + \alpha_1 \left[\tanh\left(\frac{\alpha_1}{4}\psi \right) \pm \coth\left(\frac{\alpha_1}{4}\psi \right) \right] \right], \\ \psi_6(\phi) &= \frac{1}{2s} \left[-p + \frac{\sqrt{(A^2 + B^2)} \alpha_1 - B \alpha_1 \cosh(\alpha_1\psi)}{A \sinh(\alpha_1\psi) + B} \right], \\ \psi_7(\phi) &= \frac{1}{2s} \left[-p - \frac{\sqrt{(B^2 - A^2)} \alpha_1 + A \alpha_1 \cosh(\alpha_1\psi)}{A \sinh(\alpha_1\psi) + B} \right],\end{aligned}$$

where A and B are two non-zero real constants such that $B^2 - A^2 > 0$, and

$$\begin{aligned}\phi_8(\phi) &= \frac{2r \cosh\left(\frac{\alpha_1}{2}\psi \right)}{\alpha_1 \sinh(\alpha_1 2\psi) - p \cosh(\alpha_1 2\psi)}, \\ \psi_9(\phi) &= \frac{-2r \sinh\left(\frac{\alpha_1}{2}\psi \right)}{p \sinh(\alpha_1 2\psi) - \alpha_1 \cosh(\alpha_1 2\psi)}, \\ \psi_{10}(\phi) &= \frac{2r \cosh\left(\frac{\alpha_1}{2}\psi \right)}{\alpha_1 \sinh(\alpha_1\psi) - p \cosh(\alpha_1\psi) \pm i\alpha_1}, \\ \psi_{11}(\phi) &= \frac{2r \sinh\left(\frac{\alpha_1}{2}\psi \right)}{-p \sinh(\alpha_1\psi) + \alpha_1 \cosh(\alpha_1\psi) \pm \alpha_1}, \\ \psi_{12}(\phi) &= \frac{4r \sinh\left(\frac{\alpha_1}{2}\psi \right) \cosh\left(\frac{\alpha_1}{2}\psi \right)}{-2p \sinh(\alpha_1 4\psi) \cosh(\alpha_1 4\psi) + 2\alpha_1 \cosh^2(\alpha_1 4\psi) - \alpha_1}.\end{aligned}$$

Type 2. When $p^2 - 4sr < 0$ and $ps \neq 0$ (or $sr \neq 0$), we define $\sqrt{4ef - d^2} = \alpha_2$. The solutions of Eq. (2.6) are

$$\begin{aligned}\psi_{13}(\phi) &= \frac{1}{2s} \left[-p + \alpha_2 \tan\left(\frac{\alpha_2}{2}\psi \right) \right], \\ \psi_{14}(\phi) &= -\frac{1}{2s} \left[p + \alpha_2 \cot\left(\frac{\alpha_2}{2}\psi \right) \right], \\ \psi_{15}(\phi) &= \frac{1}{2s} [-p + \alpha_2 (\tan(\alpha_2\psi) \pm \sec(\alpha_2\psi))], \\ \psi_{16}(\phi) &= -\frac{1}{2s} [p + \alpha_2 (\cot(\alpha_2\psi) \pm \csc(\alpha_2\psi))], \\ \psi_{17}(\phi) &= \frac{1}{4s} \left[-2p + \alpha_2 \left[\tan\left(\frac{\alpha_2}{4}\psi \right) - \coth\left(\frac{\alpha_2}{4}\psi \right) \right] \right], \\ \psi_{18}(\phi) &= \frac{1}{2s} \left[-p + \frac{\pm \sqrt{(A^2 - B^2)} \alpha_2 - A \alpha_2 \cos(\alpha_2\psi)}{A \sinh(\alpha_2\psi) + B} \right], \\ \psi_{19}(\phi) &= \frac{1}{2s} \left[-p - \frac{\pm \sqrt{(A^2 - B^2)} \alpha_2 + A \alpha_2 \cos(\alpha_2\psi)}{A \sin(\alpha_2\psi) + B} \right],\end{aligned}$$

where A and B are two non-zero real constants such that $A^2 - B^2 > 0$. Also,

$$\begin{aligned}\psi_{20}(\phi) &= -\frac{2r \cos\left(\frac{\alpha_2}{2}\psi \right)}{\alpha_2 \sin(\alpha_2 2\psi) + p \cos(\alpha_2 2\psi)}, \\ \psi_{21}(\phi) &= \frac{2r \sin\left(\frac{\alpha_2}{2}\psi \right)}{-p \sin(\alpha_2 2\psi) + \alpha_2 \cos(\alpha_2 2\psi)}, \\ \psi_{22}(\phi) &= -\frac{2r \cos\left(\frac{\alpha_2}{2}\psi \right)}{\alpha_2 \sin(\alpha_2\psi) + p \cos(\alpha_2\psi) \pm \alpha_2}, \\ \psi_{23}(\phi) &= -\frac{2r \sin\left(\frac{\alpha_2}{2}\psi \right)}{-p \sin(\alpha_2\psi) + \alpha_2 \cos(\alpha_2\psi) \pm \alpha_2}, \\ \psi_{24}(\phi) &= \frac{4r \sin\left(\frac{\alpha_2}{4}\psi \right) \cos\left(\frac{\alpha_2}{4}\psi \right)}{-2p \sin(\alpha_2 4\psi) \cos(\alpha_2 4\psi) + 2\alpha_2 \cos^2(\alpha_2 4\psi) - \alpha_2}.\end{aligned}$$

Type 3. When $r = 0$ and $ps \neq 0$, the solutions of Eq. (2.6) are

$$\begin{aligned}\psi_{25}(\phi) &= \frac{-pg}{s[g + \cosh(p\psi) - \sinh(p\psi)]}, \\ \psi_{26}(\phi) &= -\frac{[\cosh(p\psi) + \sinh(p\psi)]}{s[g + \cosh(p\psi) + \sinh(p\psi)]},\end{aligned}$$

where g is an arbitrary constant.

Type 4. When $s \neq 0$ and $p = r = 0$, the solutions of Eq. (2.6) are

$$\psi_{27}(\phi) = -\frac{1}{s\psi + h_1},$$

where h_1 is an arbitrary constant.

3 Solutions via SSM

Now, substituting Eq. (2.2) into Eq. (1.1), we obtain the ordinary differential equation

$$cm^4 S'''' + cS + 12mS^5 - 10mS^3 + mS = 0. \quad (3.1)$$

Applying the homogeneous balance principle to Eq. (3.1), we find $m = 1$. Substituting this into Eq. (2.4) gives

$$S(\phi) = a_0 + a_1\psi(\phi), \quad (3.2)$$

where a_0 and a_1 are the constants. Substituting Eq. (3.2) and its derivative into Eq. (3.1) results in a system of algebraic equations of the form:

$$\begin{aligned} a_0c + 12a_0^5m - 10a_0^3m + a_0m &= 0, \\ 12a_1cm^4 + a_1cm^4\Omega^2 + a_1c + 60a_0^4a_1m - 30a_0^2a_1m + a_1m &= 0, \\ 120a_0^3a_1^2m - 30a_0a_1^2m &= 0, \\ 20a_1cm^4\Omega + 120a_0^2a_1^3m - 10a_1^3m &= 0, \\ 60a_0a_1^4m &= 0, \\ 24a_1cm^4 + 12a_1^5m &= 0. \end{aligned}$$

The system of equations is solved numerically using Mathematica to find the values of the parameters:

$$\begin{aligned} a_0 &= 0, & a_1 &= \frac{i}{\sqrt{\Omega}}, & m &= \frac{1}{\sqrt[4]{\Omega^2 - 12\kappa}}, \\ c &= -\frac{(\Omega^2 - 12\kappa)^{3/4}}{2\Omega^2}, \end{aligned}$$

Case 1: If $\kappa = 0$ and $\Omega > 0$, then

$$\begin{aligned} P_1(x, t) &= \frac{i\sqrt{-deq}}{\sqrt{\Omega}} \left[\operatorname{sech} \left[\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right], \\ P_2(x, t) &= \frac{i\sqrt{-deq}}{\sqrt{\Omega}} \left[\operatorname{csch} \left[\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right]. \end{aligned}$$

Case 2: If $\kappa = 0$ and $\Omega < 0$, then

$$\begin{aligned} P_3(x, t) &= \frac{i\sqrt{-deq}}{\sqrt{\Omega}} \left[\sec \left[\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right], \\ P_4(x, t) &= \frac{i\sqrt{-deq}}{\sqrt{\Omega}} \left[\csc \left[\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right]. \end{aligned} \quad (3.4)$$

Case 3: If $\Omega < 0$ and $\kappa = \frac{\Omega^2}{4}$, then

$$\begin{aligned} P_5(x, t) &= \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\tanh \left[\frac{\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right] \right], \end{aligned} \quad (3.5)$$

$$\begin{aligned} P_6(x, t) &= \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\coth \left[\frac{\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right] \right], \end{aligned} \quad (3.6)$$

$$\begin{aligned} P_7(x, t) &= \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\sqrt{de} \operatorname{sech} \left[\sqrt{2}\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right. \\ &\quad \left. - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right] + \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \\ &\quad \times \left[\tanh \left[\sqrt{2}\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right], \end{aligned}$$

$$\begin{aligned} P_8(x, t) &= \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\sqrt{de} \operatorname{csch} \left[\sqrt{2}\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right. \\ &\quad \left. - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right] + \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \\ &\quad \times \left[\coth \left[\sqrt{2}\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right) \right] \right], \end{aligned}$$

$$\begin{aligned} P_9(x, t) &= \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\tanh \left[\frac{\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right] \right] \\ &\quad + \frac{i\sqrt{-q}}{\sqrt{2}\sqrt{\Omega}} \left[\coth \left[\frac{\sqrt{-q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4}t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right] \right]. \end{aligned} \quad (3.3)$$

Case 4: If $\Omega > 0$ and $\kappa = \frac{\Omega^2}{4}$, then

$$\begin{aligned}
 P_{10}(x, t) &= \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\tan \left(\frac{\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right) \right], \\
 P_{11}(x, t) &= \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\cot \left(\frac{\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right) \right], \\
 P_{12}(x, t) &= \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\sqrt{de} \sec \left(\sqrt{2}\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right) \right) \right. \\
 &\quad \left. - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right] + \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \\
 &\quad \times \left[\tan \left(\sqrt{2}\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right) \right) \right], \\
 P_{13}(x, t) &= \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\sqrt{de} \csc \left(\sqrt{2}\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right) \right) \right. \\
 &\quad \left. - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right] + \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \\
 &\quad \times \left[\cot \left(\sqrt{2}\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right) \right) \right], \\
 P_{14}(x, t) &= \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\tan \left(\frac{\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right) \right] \\
 &\quad + \frac{i\sqrt{q}}{\sqrt{2}\sqrt{\Omega}} \left[\cot \left(\frac{\sqrt{q} \left(\frac{x}{\sqrt[4]{\Omega^2 - 12\kappa}} - \frac{(\Omega^2 - 12\kappa)^{3/4} t^\gamma}{2\gamma\Omega^2} \right)}{\sqrt{2}} \right) \right].
 \end{aligned} \tag{3.7}$$

4 Solutions via the GREM method

Applying the balancing principle to Eq. (1.2), we obtain $m = 1$. Substituting this value into (2.4) gives

$$S(\phi) = a_0 + a_1 \psi(\phi),$$

where a_0 and a_1 are the constants. Substituting this expression and its derivative into Eq. (1.2) leads to a system of algebraic equations of the form:

$$\begin{aligned}
 a_1 c m^4 p^3 r + 8 a_1 c m^4 p r^2 s + a_0 c + 12 a_0^5 m - 10 a_0^3 m + a_0 m &= 0, \\
 a_1 c m^4 p^4 + 22 a_1 c m^4 p^2 r s + 16 a_1 c m^4 r^2 s^2 + a_1 c + 60 a_0^4 a_1 m \\
 - 30 a_0^2 a_1 m + a_1 m &= 0, \\
 15 a_1 c m^4 p^3 s + 60 a_1 c m^4 p r s^2 + 120 a_0^3 a_1^2 m - 30 a_0 a_1^2 m &= 0, \\
 50 a_1 c m^4 p^2 s^2 + 40 a_1 c m^4 r s^3 + 120 a_0^2 a_1^3 m - 10 a_1^3 m &= 0, \\
 60 a_1 c m^4 p s^3 + 60 a_0 a_1^4 m &= 0, \\
 24 a_1 c m^4 s^4 + 12 a_1^5 m &= 0.
 \end{aligned}$$

The system of algebraic equations was solved numerically using Mathematica, yielding the following values for the parameters a_0 , a_1 , m , and c :

$$\begin{aligned}
 a_0 &= \frac{p}{\sqrt{2p^2 - 8rs}}, \\
 a_1 &= \frac{2s}{\sqrt{2p^2 - 8rs}}, \\
 m &= \frac{\sqrt[4]{2}}{\sqrt[4]{-(p^2 - 4rs)^2}}, \\
 c &= \frac{\sqrt[4]{2}}{\sqrt[4]{-(p^2 - 4rs)^2}}.
 \end{aligned}$$

We define

$$\alpha_3 = \frac{\sqrt[4]{2} t^\gamma}{\gamma \sqrt[4]{-(p^2 - 4rs)^2}} + \frac{\sqrt[4]{2} x}{\sqrt[4]{-(p^2 - 4rs)^2}}.$$

Next, we analyze each of the possible cases.

Case 1: When $d^2 - 4ef > 0$ and $de \neq 0$ or $ef \neq 0$,

$$\begin{aligned}
 P_{15}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(a_1 \tanh \left(\frac{a_1 \alpha_3}{2} \right) + p \right), \\
 P_{16}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(a_1 \coth \left(\frac{a_1 \alpha_3}{2} \right) + p \right), \\
 P_{17}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times (a_1 (\tanh(a_1 \alpha_3) + \operatorname{sech}(a_1 \alpha_3)) + p), \\
 P_{18}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times (a_1 (\coth(a_1 \alpha_3) + \operatorname{csch}(a_1 \alpha_3)) + p), \\
 P_{19}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{2\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(a_1 \tanh \left(\frac{a_1 \alpha_3}{4} \right) + \coth \left(\frac{a_1 \alpha_3}{4} \right) + 2p \right), \\
 P_{20}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(\frac{a_1 \sqrt{A^2 + B^2} - a_1 A \cosh(a_1 \alpha_3) - p}{A \sinh(a_1 \alpha_3) + B} \right), \\
 P_{21}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(\frac{-a_1 \sqrt{B^2 - A^2} + a_1 A \cosh(a_1 \alpha_3)}{A \sinh(a_1 \alpha_3) + B} - p \right), \\
 P_{22}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
 &\quad \times \left(\frac{\cosh \left(\frac{a_1 \alpha_3}{2} \right)}{a_1 \sinh(a_1 \alpha_3/2) - p \cosh(a_1 \alpha_3/2)} \right),
 \end{aligned} \tag{4.1}$$

$$\begin{aligned}
P_{23}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sinh\left(\frac{a_1 a_3}{2}\right)}{p \sinh(a_1 a_3 2) - a_1 \cosh(a_1 a_3 2)} \right], \\
P_{24}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\cosh\left(\frac{a_1 a_3}{2}\right)}{a_1 a_3 + a_1 \sinh(a_1 a_3) - p \cosh(a_1 a_3)} \right], \\
P_{25}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sinh\left(\frac{a_1 a_3}{2}\right)}{a_1 + a_1 \cosh(a_1 a_3) - p \sinh(a_1 a_3)} \right], \\
P_{26}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{8rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sinh\left(\frac{a_1 a_3}{4}\right) \cosh\left(\frac{a_1 a_3}{4}\right)}{-a_1 + 2a_1 \cosh^2(a_1 a_3 4) - 2p \sinh(a_1 a_3 4) \cosh(a_1 a_3 4)} \right].
\end{aligned}
\tag{4.2}$$

Case 2: When $p^2 - 4sr < 0$ and $ps \neq 0$ or $sr \neq 0$, we have

$$\begin{aligned}
P_{27}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \left(a_2 \tan\left(\frac{a_2 a_3}{2}\right) + p \right), \\
P_{28}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} \left(a_2 \cot\left(\frac{a_2 a_3}{2}\right) + p \right), \\
P_{29}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{\sqrt{2p^2 - 8rs}} (a_2 \tan(a_2 a_3) + \sec(a_2 a_3) - p), \\
P_{30}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{1}{\sqrt{2p^2 - 8rs}} (a_2 (\cot(a_2 a_3) + \csc(a_2 a_3)) + p), \\
P_{31}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{2\sqrt{2p^2 - 8rs}} \left(a_2 \left(\tan\left(\frac{a_2 a_3}{4}\right) - \cot\left(\frac{a_2 a_3}{4}\right) \right) - 2p \right), \\
P_{32}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sqrt{A^2 - B^2} - a_2 a_2 A \cos(a_2 a_3)}{A \sin(a_2 a_3) + B} - p \right], \\
P_{33}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{1}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[-\frac{a_2 \sqrt{A^2 - B^2} + a_2 A \cos(a_2 a_3)}{B \sin(a_2 a_3) + B} - p \right], \\
P_{34}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\cos\left(\frac{a_2 a_3}{2}\right)}{a_2 \sin(a_2 a_3 2) + p \cos(a_2 a_3 2)} \right],
\end{aligned}$$

$$\begin{aligned}
P_{35}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sin\left(\frac{a_2 a_3}{2}\right)}{a_2 \cos(a_2 a_3 2) - p \sin(a_2 a_3 2)} \right], \\
P_{36}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\cos\left(\frac{a_2 a_3}{2}\right)}{a_2 a_3 + a_2 \sin(a_2 a_3) + p \cos(a_2 a_3)} \right], \\
P_{37}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{4rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sin\left(\frac{a_2 a_3}{2}\right)}{a_2 a_3 + a_2 \cos(a_2 a_3) - p \sin(a_2 a_3)} \right], \\
P_{38}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} + \frac{8rs}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sin\left(\frac{a_2 a_3}{4}\right) \cos\left(\frac{a_2 a_3}{4}\right)}{-a_2 + 2a_2 \cos^2(a_2 a_3 4) - 2p \sin(a_2 a_3 4) \cos(a_2 a_3 4)} \right].
\end{aligned}$$

Case 3: When $r = 0$ and $ps \neq 0$, then

$$\begin{aligned}
P_{39}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{2gp}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{1}{g + \sinh(a_3 p) + \cosh(a_3 p)} \right], \\
P_{40}(x, t) &= \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{2p}{\sqrt{2p^2 - 8rs}} \\
&\quad \times \left[\frac{\sinh(a_3 p) + \cosh(a_3 p)}{\sinh(a_3 p) + \cosh(a_3 p) + p} \right].
\end{aligned}
\tag{4.3}$$

Case 4: When $s \neq 0$ and $p = r = 0$, then

$$P_{41}(x, t) = \frac{p}{\sqrt{2p^2 - 8rs}} - \frac{2s}{(h_1 + a_3 s) \sqrt{2p^2 - 8rs}}.$$

The solution analysis reveals distinct characteristics for each method: the SSM generated 14 exact solutions, while the GREM method produced 27 exact solutions. Moreover, their solution profiles differ significantly – SSM yields solutions exclusively in trigonometric and hyperbolic forms, whereas GREM provides a broader spectrum, including trigonometric, hyperbolic, and rational solutions (Figure 1).

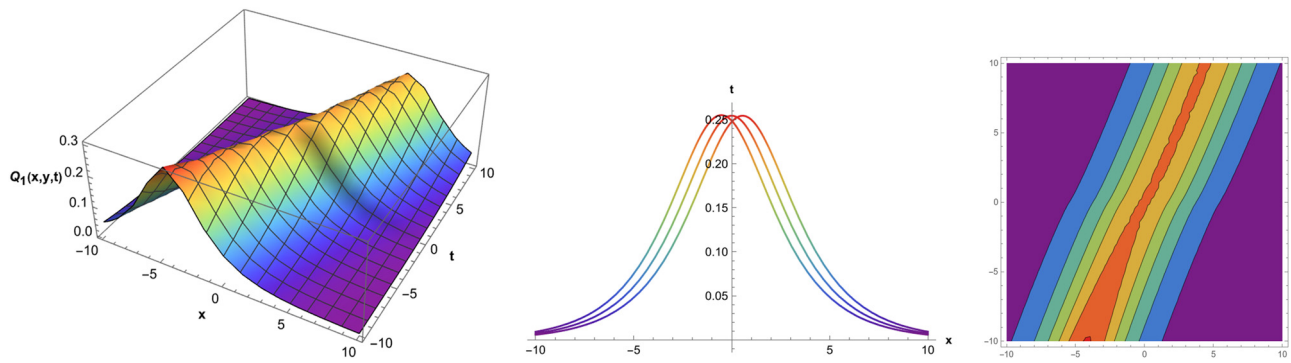


Figure 1: Graphical behavior of $P_1(x, t)$ in Eq. (3.3) for the parameters $c = -1$, $\gamma = 0.9$, $d = -0.9$, $e = -0.4$, $\kappa = 0$, $q = 0.09$, and $\Omega = -0.5$.

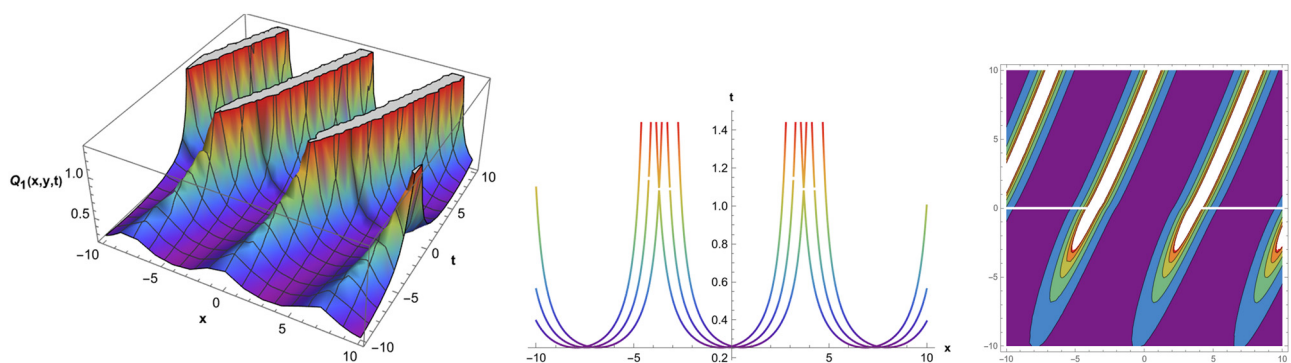


Figure 2: Graphical behavior of $P_3(x, t)$ in Eq. (3.4) for the parameters $c = 1$, $\gamma = 0.9$, $d = 0.9$, $e = 0.4$, $\kappa = 0$, $q = -0.09$, and $\Omega = 0.5$.

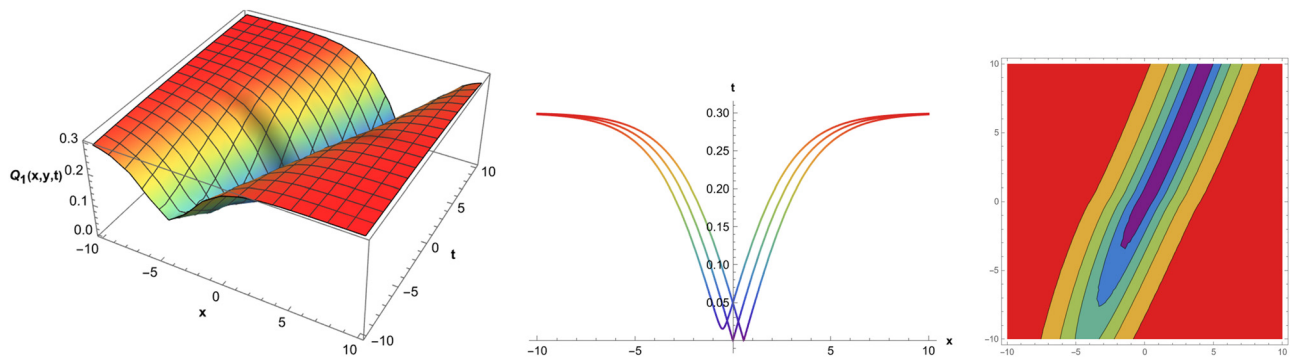


Figure 3: Graphical behavior of $P_5(x, t)$ in Eq. (3.5) for the parameters $c = -1$, $\gamma = 0.9$, $d = -0.9$, $e = -0.4$, $\kappa = 0$, $q = -0.09$, and $\Omega = -0.5$.

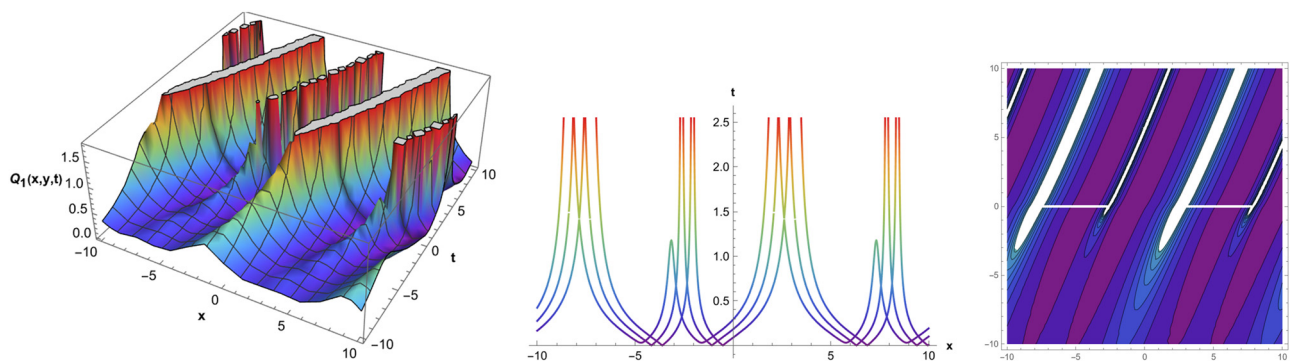


Figure 4: Graphical behavior of $P_{12}(x, t)$ in Eq. (3.7) for the parameters $c = -1$, $\gamma = 0.9$, $d = -0.9$, $e = -0.4$, $\kappa = 0$, $q = 0.09$, and $\Omega = -0.5$.

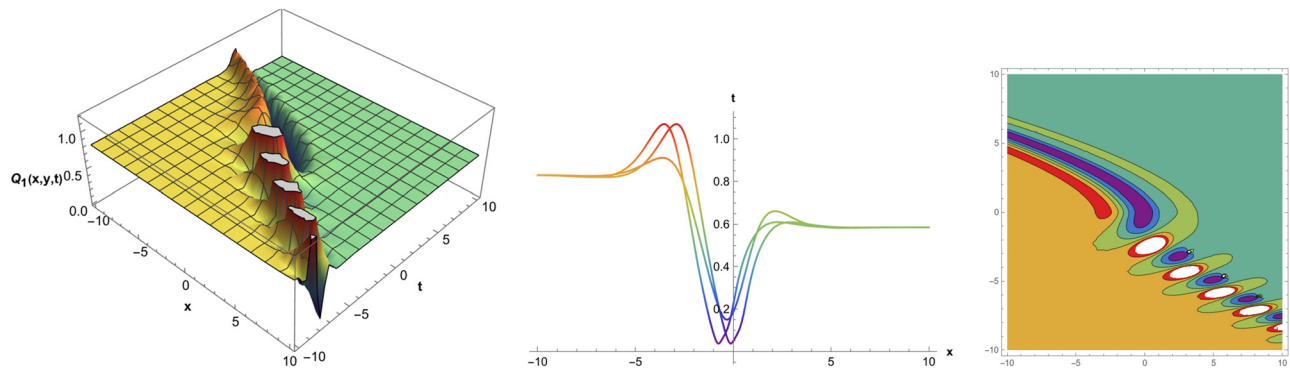


Figure 5: Graphical behavior of $P_6(x, t)$ in Eq. (3.6) for the parameters $A = 1$, $B = -1$, $\gamma = 1.6$, $p = 0.4$, $r = -1$, and $s = 1.3$.

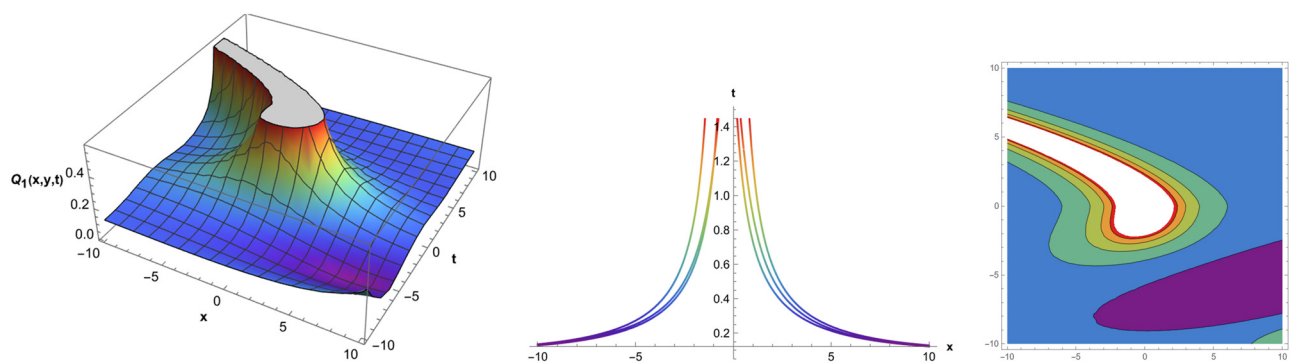


Figure 6: Graphical behavior of $P_{25}(x, t)$ in Eq. (4.2) with parameters $\gamma = 1.6$, $g = -1$, $p = 0.4$, $r = -1$, and $s = 1.3$.

The graphical representations provide valuable insights into the dynamics of wave propagation. By selecting appropriate parameters, we visualize several solution profiles that exhibit different wave behaviors. Several remarkable patterns emerge: the graphical behavior of $P_3(x, t)$ and $P_{12}(x, t)$ in Figures 2 and 4 displays the characteristic solitary wave profiles. In Figures 3 and 5, illustrating solutions $P_5(x, t)$ and $P_6(x, t)$, respectively, the presence of dark solitons becomes evident. Finally, Figure 6, showing the solution of $P_{39}(x, t)$, exhibits a bright soliton profile.

5 Conclusion

In this study, analytical solutions of the Rosenau model were obtained using two methods: SSM and GREM method. The mathematical model is a fractional extension [41,42] of the classical Rosenau equation, and several solutions representing solitary waves and solitons were successfully derived in the form of trigonometric, mixed trigonometric, hyperbolic, mixed hyperbolic, rational, and exponential functions. The analytical results include various solitary wave solutions, as well as bright and dark soliton profiles. Fractional derivatives and fractional wave transformations were employed to

obtain these solutions. By selecting appropriate parameter values, 3D, 2D, and corresponding contour plots are presented to illustrate the physical behavior of the analytical results. This approach provides a versatile tool for research and experimentation, allowing researchers to adjust parameters to explore a wide range of wave phenomena.

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Data availability statement: The code/datasets generated and/or analysed during the current study are available from the corresponding author on reasonable request.

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