Research Article

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Some anisotropic and perfect fluid plane symmetric solutions of Einstein's field equations using killing symmetries

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Abstract: The Einstein's field equations (EFEs), central to the theory of general relativity, often require spacetime symmetries such as those defined by Killing vector fields to simplify their solutions and derive physically meaningful results. Killing vector fields preserve the metric of spacetime and yield vital conservation laws. This article presents a comprehensive study of Killing vector fields in the background of nonstatic plane symmetric spacetime using a novel method, the Rif tree approach. Using a Maple algorithm, this approach provides conditions on the metric coefficients that lead to additional Killing vector fields than the minimum ones. A detailed analysis yields a variety of spacetime metrics that admit different dimensional Killing algebras. The physical implications of the obtained metrics are discussed by finding the associated energy-momentum tensors. Several metrics are found to describe physically realistic models, including anisotropic and perfect fluid solutions of EFEs.

Keywords: spacetime symmetries, killing vector fields, plane symmetric spacetime, Rif tree approach

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1 Introduction

The Einstein's field equations (EFEs) relate the spacetime curvature with energy, momentum, and stress within that spacetime. These equations are ten nonlinear partial differential equations, given by [1]:

$$R_{ab} - \frac{1}{2}Rg_{ab} = kT_{ab}, \tag{1.1}$$

where R_{ab}, g_{ab}, and T_{ab} are the Ricci, metric, and energy-momentum tensors, respectively, R is the Ricci scalar and k is a coupling constant. The nonlinearity of EFEs poses problems in solving these equations, and it is not possible to obtain a solution of these equations without some assumption. However, in literature, different approaches are used to find many solutions of these equations [1–6].

Symmetries of spacetimes have been a central theme in the study of general relativity, providing profound insights into both the mathematical structure of EFEs and the physical properties of gravitational fields. Among the various symmetries, vector fields associated with isometries, conformal transformations, and homotheties are particularly significant. These symmetries not only simplify the EFEs but also serve as a tool for generating their exact solutions, revealing underlying conservation laws, and offering insights into the causal and geometric structure of spacetimes. A Killing vector field (KVF) $\eta = (\eta^0, \eta^1, \eta^2, \eta^3)$ is defined by the relation [7]:

$$\mathcal{L}_\eta g_{ab} = 0; \tag{1.2}$$

$\mathcal{L}$ being the Lie derivative operator and g_{ab} is the spacetime metric. This condition corresponds to the preservation of the spacetime metric along the flow of the vector field, indicating an isometry or symmetry of the spacetime. KVFs are fundamental in describing conserved quantities associated with spacetime symmetries, such as energy, and linear and angular momentum.

In some cases, the metric is preserved up to a local scaling factor, reflecting a conformal symmetry that is defined in terms of a vector field η satisfying the condition [7]:

$$\mathcal{L}_\eta g_{ab} = 2\psi(x^a) g_{ab}, \tag{1.3}$$

where $\psi(x^a)$ is a conformal factor that depends on the spacetime coordinates x^a. Conformal symmetries are crucial in a variety of physical contexts, including the study of asymptotic structures in spacetimes and scale-invariant field theories.

In between the aforementioned defined two symmetries is the notion of homothetic vector field (HVF), defined by the condition:

$$\mathcal{L}_\eta g_{ab} = 2\lambda g_{ab}, \tag{1.4}$$

where λ is a constant. A HVF generates a scaling symmetry of the spacetime, where the metric is preserved up to a constant factor. These symmetries, also referred to as self-similar symmetries of first kind, are particularly relevant in cosmological models, self-similar gravitational collapse, and other scenarios where scale invariance plays a fundamental role.

The symmetries of other tensors, like Ricci, stress-energy and curvature tensors are defined in a similar way by replacing the metric tensor in Eqs. (1.2)–(1.4) by R_{ab}, T_{ab}, and R^a_{bcd}, respectively.

The focus of the current study is only on KVFs. The significance of KVFs lies in their ability to reduce the complexity of the EFEs, making the search for their exact solutions more tractable. In terms of their physical significance, KVFs are closely connected to conservation laws in spacetime, which are fundamental not only in general relativity but also in other fields of science. In the literature, KVFs have been explored for various spacetime metrics [8–13], where it is shown that ten independent KVFs exist for a spacetime that is flat or it has constant curvature, but non-flat geometries generally admit fewer KVFs. For example, in cylindrically symmetric spacetimes, the dimension of Killing algebra ranges from 3 to 10 [8]. In addition to the KVFs, numerous studies have classified spacetimes based on other symmetries including HVFs, conformal vector fields (CVFs) and Ricci collineations [14–20]. In recent literature, these symmetries are also explored for different spacetimes in modified theories of gravity. The CVFs of locally rotationally symmetric Bianchi type I spacetime in $f(Q)$ gravity were found by Ayyoub *et al.* [21]. Qazi *et al.* [22] investigated CVFs of Bianchi type II spacetime in $f(T)$ theory of gravity. Mehmood *et al.* [23] worked on CVFs of Bianchi type I spacetime in $f(R, T)$ gravity.

These previous studies on Killing and other symmetries have focused on specific cases such as static plane, static spherically and static cylindrically symmetric spacetimes, and certain other simple cosmological models. However, non-static spacetimes remain a fertile area of exploration, with relatively fewer comprehensive studies because of the difficulties one faces while solving the highly nonlinear equations defining spacetime symmetries.

A plane symmetric spacetime is a class of spacetime models characterized by symmetry along two spatial dimensions, resembling the geometry of a plane. This spacetime is useful for studying systems that exhibit uniform properties in two directions, such as gravitational waves, cosmological models, or certain exact solutions of EFEs. The metric of this spacetime remains invariant under translations and rotations in the plane, which simplifies the study of gravitational fields and their effects. Due to its symmetry, plane symmetric spacetime is an ideal framework for investigating the properties of gravitational fields, conservation laws, and the behavior of matter in general relativity.

The study of KVFs in non-static spacetimes, particularly those with plane symmetry, is of considerable interest due to the rich geometric structure these spacetimes exhibit. Nonstatic plane symmetric spacetimes describe scenarios where the spacetime evolves with time while maintaining planar symmetry, making them suitable models for investigating gravitational waves, cosmological phenomena, and anisotropic gravitational fields. Solutions to EFEs in these spacetimes are often complex, and finding explicit KVFs can aid in simplifying their structure and providing insight into their solutions.

To explore the symmetries of a spacetime, one always needs to solve a system of determining equations representing these symmetries. The conventional method used to solve these determining equations is known as direct integration technique. In this method, the determining equations are decoupled and integrated directly to find the explicit form of symmetry vector fields. The process usually gives rise to a number of cases depending upon the conditions on the metric functions under which the spacetime under consideration admits the desired symmetries. It is a quite lengthy and cumbersome technique which may result in lack of potential spacetime metrics admitting the required symmetries.

In recent literature, the Rif tree approach has emerged as a powerful computational tool for analyzing systems of

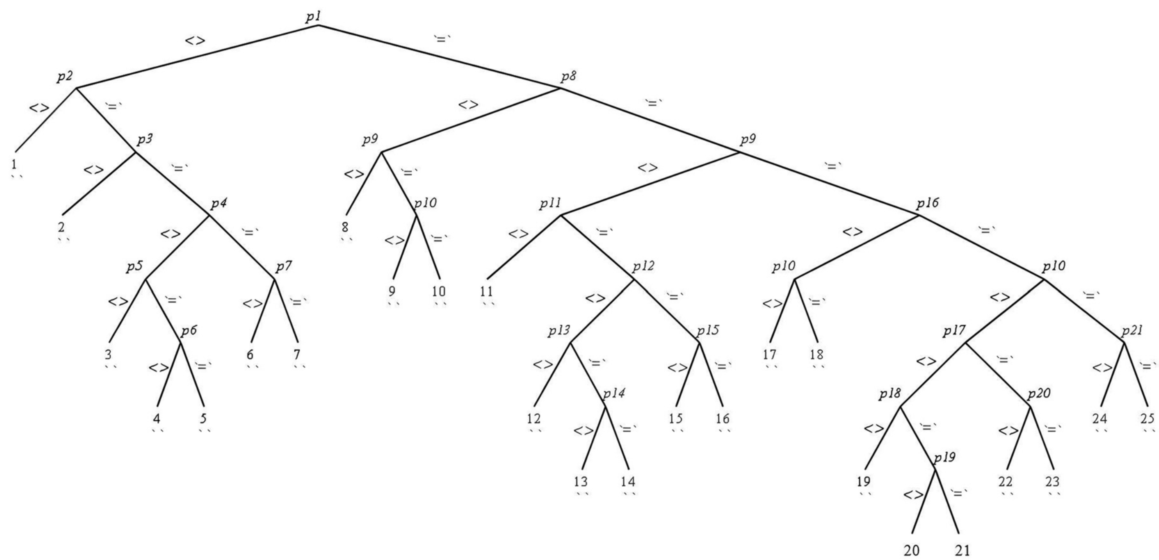

Figure 1: Rif Tree.

partial differential equations that govern the existence of symmetries in spacetimes. This algorithmic method transforms the system of determining equations into an involutive form, systematically solving them and allowing for the classification of vector fields such as KVFs. This method relies on a Maple algorithm (Rif algorithm), which is implemented using the Exterior package in Maple. The process begins by loading the "Exterior" package. Next, the system

Table 1: Metrics with 4-dimensional Killing algebra

Metric no./branch no.	Metric coefficients	Additional KVFs
4a	f = const., $g = a_1 t + a_2$, $h = \frac{1}{a_2}(a_3 e^{a_1 x} + a_4 e^{-a_1 x})(a_1 t + a_2)$,	$V_{(4)} = -\frac{a_2}{a_1 a_4}\frac{h_{,x}}{g}\partial_t + \frac{a_2}{a_4}\frac{h}{g^2}\partial_x$
1	where $a_i \neq 0$; $i = 1, \ldots, 4$	
4b	f = const., $g = g(x)$, $h = a_1 t + a_2$,	$V_{(4)} = \frac{1}{g(x)}\partial_x$
1	where $g_{,x}(x) \neq 0$ and $a_1 \neq 0$	
4c	f = const., $g = g(x)$, $h = \frac{(a_1 t + a_2)^{1+\frac{a_3}{a_1}}}{a_1 + a_3}$,	$V_{(4)} = \frac{1}{g(x)}\partial_x$
1	where $g_{,x}(x) \neq 0$, $a_1 \neq 0$ and $a_1 \neq -a_3$	
4d	$f = f(t)$, g = const., $h = a_1 x + a_2 \int f(t) \mathrm{d}t$,	$V_{(4)} = \frac{1}{f(t)}\partial_t - \frac{a_2}{a_1}\partial_x$
2	where $f_{,t}(t) \neq 0$, $a_1 \neq a_2$, $a_1 \neq 0$, and $a_2 \neq 0$.	
4e	$f = \sqrt{2a_1 x + 2a_2}$, g = const. $h = e^{a_3 t}$,	$V_{(4)} = -\frac{1}{a_3}\partial_t + y\partial_y + z\partial_z$,
2	where $a_1 \neq 0$ and $a_3 \neq 0$	
4f	f = const., g = const., $h = \frac{a_1}{2}(t^2 - x^2) + a_2 x + a_3 t + a_4$,	$V_{(4)} = -\frac{-a_1 x + a_2}{a_1}\partial_t + \frac{a_1 t + a_3}{a_1}\partial_x$,
3	where a_1, a_2, and a_3 are nonzero	
4g	f = const., $g = a_2 t$, $h = (a_1 t)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)} = \partial_x$
7	where $a_1 \neq 0$, $a_2 \neq 0$ and $a_1 \neq 2a_3$	
4h	f = const., $g = (a_1 t + 2a_2)^{1-\frac{2a_3}{a_1}}$, $h = (a_1 t + 2a_2)^{1-\frac{2a_4}{a_1}}$,	$V_{(4)} = \partial_x$
7	where $a_1 \neq 0$, $a_1 \neq 2a_3$ and $a_1 \neq 2a_4$	
4i	f = const., g = const., $h = (a_1 t + 2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)} = \partial_x$
7	where $a_1 \neq 0$, $a_1 \neq 2a_3$	
4j	f = const., g = const., $h = a_1 t + a_2$,	$V_{(4)} = \partial_x$
7	where $a_1 \neq 0$	

Table 2: Metrics with 4- and 6-dimensional Killing algebra

Metric no./branch no.	Metric coefficients	Additional KVFs
4k	f = const., g = const., $h = a_1x + a_2t$,	$V_{(4)} = -\frac{a_1}{a_2}\partial_t + \partial_x$
7	where $a_1 \neq a_2 \neq 0$	
4l	f = const., $g = \sqrt{2a_1t + 2a_2}$, $h = e^{a_3x}$,	$V_{(4)} = -\frac{1}{a_3}\partial_x + y\partial_y + z\partial_z$,
8	where $a_1 \neq 0$, $a_3 \neq 0$	
4m	$f = f(t)$, g = const., $h = (a_1x + 2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)} = \frac{1}{f(t)}\partial_t$
9	where $a_1 \neq 0$, $a_1 \neq 2a_3$ and $f_{,t}(t) \neq 0$,	
4n	$f = f(t)$, g = const., $h = a_1x + a_2$,	$V_{(4)} = \frac{1}{f(t)}\partial_t$
9	where $f_{,t}(t) \neq 0$ and $a_1 \neq 0$,	
4o	$f = (a_1x + 2a_2)^{1-\frac{2a_3}{a_1}}$, g = const., $h = (a_1x + 2a_2)^{1-\frac{2a_4}{a_1}}$,	$V_{(4)} = \partial_t$
10	where $a_1 \neq 0$, $a_1 \neq 2a_3$ and $a_1 \neq 2a_4$	
4p	f = const., g = const., $h = (a_1x + 2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)} = \partial_t$
10	where $a_1 \neq 0$, $a_1 \neq 2a_3$	
4q	f = const., g = const., $h = a_1x + a_2$,	$V_{(4)} = \partial_t$
10	where $a_1 \neq 0$,	
4r	$f = (a_1\int g(x)\mathrm{d}x + 2a_2)^{1-\frac{2a_3}{a_1}}$, $g = g(x)$, h= const.	$V_{(4)} = \partial_t$
18	where $g_{,x}(x) \neq 0$, $a_1 \neq 2a_3$, and $a_1 \neq 0$.	
4s	$f = (a_1x + 2a_2)^{1-\frac{2a_3}{a_1}}$, g=const., h= const.,	$V_{(4)} = \partial_t$
24	where $a_1 \neq 2a_3$ and $a_1 \neq 0$.	
6a	f = const., $g = a_1t + a_2$, $h = \frac{1}{a_2}(a_1t + a_2)e^{a_3x}$,	$V_{(4)} = \partial_x - a_3y\partial_y - a_3z\partial_z$
1	where a_1, a_2, a_3 are nonzero	$V_{(5)} = \frac{y}{a_3}\partial_x + (\frac{z^2-y^2}{2} + \frac{a_2^2}{2a_3^2e^{2a_3x}})\partial_y - yz\partial_z$
		$V_{(6)} = -\frac{z}{a_3}\partial_x + yz\partial_y + (\frac{z^2-y^2}{2} - \frac{a_2^2}{2a_3^2e^{2a_3x}})\partial_z$
6b	f = const., $g = h = (a_1t + 2a_2)^{1-\frac{2a_3}{a_1}}$,	$V_{(4)} = x\partial_y - y\partial_x$
7	where $a_1 \neq 2a_3$, $a_1 \neq 0$,	$V_{(5)} = x\partial_z - z\partial_x$,
		$V_{(6)} = \partial_x$
6c	f = const., $g = a_1t$, $h = \frac{1}{a_2}t$,	$V_{(4)} = \partial_x$
7	where $a_1 \neq 0$, and $a_2 \neq 0$.	$V_{(5)} = x\partial_y - \frac{y}{a_1^2a_2^2}\partial_x$
		$V_{(6)} = x\partial_z - \frac{z}{a_1^2a_2^2}\partial_x$
6d	$f = h = (a_1x + 2a_2)^{1-\frac{2a_3}{a_1}}$, g = const.,	$V_{(4)} = t\partial_y + y\partial_t$
10	where $a_1 \neq 2a_3$ and a_1 is nonzero	$V_{(5)} = t\partial_z + z\partial_t$,
		$V_{(6)} = \partial_t$
6e	$f = h = a_1x + a_2$, g = const.,	$V_{(4)} = \partial_t$
10	where $a_1 \neq 0$.	$V_{(5)} = t\partial_y + y\partial_t$
		$V_{(6)} = t\partial_z + z\partial_t$
6f	$f = a_1e^{a_2x} + a_3e^{-a_2x}$,	$V_{(4)} = \frac{f_{,x}(x)}{mf(x)}\cos(2a_2\sqrt{a_1a_3}\,t)\partial_t + \sin(2a_2\sqrt{a_1a_3}\,t)\partial_x$,
24	g and h are constant functions	$V_{(5)} = -\frac{f_{,x}(x)}{mf(x)}\sin(2a_2\sqrt{a_1a_3}\,t)\partial_t + \cos(2a_2\sqrt{a_1a_3}\,t)\partial_x$,
	and a_is are nonzero	$V_{(6)} = \partial_t$,
		where $m = 2a_2\sqrt{a_1a_3}$,

of differential equations defining KVFs is inserted using the command "sysDEs." The third step involves applying the "symmetry, eq ≔ findsymmetry" command, which analyzes the symmetry equations and identifies the conditions on the metric functions that allow for KVFs. The algorithm then displays these conditions. A graphical representation

Table 3: Metrics with 7 and 10-dimensional Killing algebra

Metric no./branch no.	Metric coefficients	Additional symmetries
7a 1	f=const., $g = g(x)$, $h = e^{a_1 t}$, where $g_{,x}(x) \neq 0$, and $a_1 \neq 0$.	$V_{(4)} = \frac{1}{g(x)}\partial_x$ $V_{(5)} = \frac{y}{a_1}\partial_t + (\frac{z^2-y^2}{2} - \frac{e^{-2a_1 t}}{2a_1^2})\partial_y - yz\partial_z$ $V_{(6)} = -\frac{z}{a_1}\partial_t + yz\partial_y + (\frac{z^2-y^2}{2} + \frac{e^{-2a_1 t}}{2a_1^2})\partial_z$ $V_{(7)} = \partial_t - a_1 y\partial_y - a_1 z\partial_z$
7b 7	f = const., g = const., $h = e^{a_1 t}$, where $a_1 \neq 0$.	$V_{(4)} = \partial_x$ $V_{(5)} = \frac{y}{a_1}\partial_t + (\frac{z^2-y^2}{2} - \frac{e^{-2a_1 t}}{2a})\partial_y - yz\partial_z$ $V_{(6)} = -\frac{z}{a_1}\partial_t + yz\partial_y + (\frac{z^2-y^2}{2} + \frac{e^{-2a_1 t}}{2a_1})\partial_z$ $V_{(7)} = \partial_t - a_1 y\partial_y - a_1 z\partial_z$
7c 9	$f = f(t)$, g = const., $h = e^{a_1 x}$, where $f_{,t}(t) \neq 0$ and $a_1 \neq 0$	$V_{(4)} = \frac{1}{f(t)}\partial_t$ $V_{(5)} = \frac{y}{a_1}\partial_x + (\frac{z^2-y^2}{2} + \frac{e^{-2a_1 x}}{2a_1^2})\partial_y - yz\partial_z$ $V_{(6)} = -\frac{z}{a_1}\partial_x + yz\partial_y + (\frac{z^2-y^2}{2} - \frac{e^{-2a_1 x}}{2a_1^2})\partial_z$ $V_{(7)} = -\frac{1}{a_1}\partial_x + y\partial_y + z\partial_z$
7d 10	f = const., g = const., $h = e^{a_1 x}$, where $a_1 \neq 0$,	$V_{(4)} = \partial_t$ $V_{(5)} = \frac{y}{a_1}\partial_x + (\frac{z^2-y^2}{2} + \frac{e^{-2a_1 x}}{2a_1^2})\partial_y - yz\partial_z$ $V_{(6)} = -\frac{z}{a_1}\partial_x + yz\partial_y + (\frac{z^2-y^2}{2} - \frac{e^{-2a_1 x}}{2a_1^2})\partial_z$ $V_{(7)} = \partial_x - a_1 y\partial_y - a_1 z\partial_z$
10a 1	f = const., $g = a_1 t + a_2$, $h = \frac{a_1 t + a_2}{a_2}e^{a_1 x}$, where a_1 and a_2 are nonzero	$V_{(4)} = \partial_x - a_1 y\partial_y - a_1 z\partial_z$ $V_{(5)} = e^{a_1 x}\partial_t - \frac{e^{a_1 x}}{a_1 t + a_2}\partial_x$ $V_{(6)} = \left[\frac{a_1^2}{a_2^2}(y^2 + z^2)e^{a_1 x} + e^{-a_1 x}\right]\partial_t + \left[-\frac{a_1^2}{a_2^2(a_1 t + a_2)}(y^2 + z^2)e^{a_1 x} + \frac{e^{-a_1 x}}{a_1 t + a_2}\right]\partial_x$ $-\frac{2a_1 y e^{-a_1 x}}{a_1 t + a_2}\partial_y - \frac{2a_1 z e^{-a_1 x}}{a_1 t + a_2}\partial_z,$ $V_{(7)} = ye^{a_1 x}\partial_t - \frac{ye^{a_1 x}}{(a_1 t + a_2)}\partial_x - \frac{a_2^2}{a_1(a_1 t + a_2)}e^{-a_1 x}\partial_y,$ $V_{(8)} = ze^{a_1 x}\partial_t - \frac{ze^{a_1 x}}{(a_1 t + a_2)}\partial_x - \frac{a_2^2}{a_1(a_1 t + a_2)}e^{-a_1 x}\partial_z,$ $V_{(9)} = \frac{y}{a_1}\partial_x + (\frac{z^2-y^2}{2} + \frac{a_2^2}{2a_1^2}e^{-2a_1 x})\partial_y - yz\partial_z$ $V_{(10)} = -\frac{z}{a_1}\partial_x + yz\partial_y + (\frac{z^2-y^2}{2} - \frac{a_2^2}{2a_1^2}e^{-2a_1 x})\partial_z$
10b 2	$f = f(t)$, g = const., $h = a_1 x + a_1\int f(t)\mathrm{d}t$, where $f_{,t}(t) \neq 0$ and $a_1 \neq 0$	$V_{(4)} = -\frac{x}{f(t)}\partial_t - \int f(t)dt\partial_x + y\partial_y + z\partial_z$ $V_{(5)} = \left[\frac{a_1^2(y^2+z^2)}{2f(t)} + \frac{1}{f(t)}\right]\partial_t - a_1^2(\frac{y^2+z^2}{2})\partial_x - \frac{a_1 y}{h(t,x)}\partial_y - \frac{a_1 z}{h(t,x)}\partial_z$ $V_{(6)} = (\frac{y^2+z^2}{2})\frac{a_1^2}{f(t)}\partial_t + \left[\frac{-a_1^2(y^2+z^2)}{2} + 1\right]\partial_x - \frac{a_1 y}{h(t,x)}\partial_y - \frac{a_1 z}{h(t,x)}\partial_z$ $V_{(7)} = \frac{y}{f(t)}\left[-\int f(t)dt + \frac{C}{a_1}\right]\partial_t + y\int f(t)dt\partial_x + \left[\frac{z^2-y^2}{2} + \frac{\int f(t)\mathrm{d}t}{a_1 h(t,x)}\right]\partial_y - yz\partial_z$ $V_{(8)} = \frac{z}{f(t)}\left[\int f(t)dt - \frac{C}{a_1}\right]\partial_t + z\int f(t)\mathrm{d}t\partial_x + yz\partial_y + \left[\frac{z^2-y^2}{2} - \frac{\int f(t)\mathrm{d}t}{a_1 h(t,x)}\right]\partial_z$ $V_{(9)} = \frac{y}{f(t)}\partial_t - y\partial_x - \frac{1}{a_1 h(t,x)}\partial_y$ $V_{(10)} = \frac{z}{f(t)}\partial_t - z\partial_x - \frac{1}{a_1 h(t,x)}\partial_z$
10c 7	f = const.,	$V_{(4)} = y\partial_t - yx\partial_x + (\frac{z^2-y^2}{2} - \frac{e^{-2t}}{2} + \frac{x^2}{2})\partial_y - yz\partial_z$

(*Continued*)

Table 3: *Continued*

Metric no./branch no.	Metric coefficients	Additional symmetries
	$g = h = e^t$.	$V_{(5)} = -z\partial_t + xz\partial_x + yz\partial_y + (\frac{z^2-y^2}{2} - \frac{e^{-2t}}{2} - \frac{x^2}{2})\partial_z$, $V_{(6)} = -x\partial_t - (\frac{y^2+z^2}{2} - \frac{e^{-2t}}{2} - \frac{x^2}{2})\partial_x + xy\partial_y + xz\partial_z$ $V_{(7)} = -\partial_t + x\partial_x + y\partial_y + z\partial_z$ $V_{(8)} = -y\partial_x + x\partial_y$, $V_{(9)} = -z\partial_x + x\partial_z$, $V_{(10)} = \partial_x$

of these conditions can be viewed using the "caseplot(eq, pivots)" command, resulting in a tree shape, called Rif tree. The branches of the Rif tree illustrate the conditions under which the spacetime may admit KVFs. Finally, the symmetry equations are solved under these branch-specific conditions, yielding the explicit form of the KVFs. The novelty of this approach is that it gives a better classification of the spacetime through its symmetries. Recently, the Rif tree approach has been applied in the classification of KVFs and other symmetries, leading to the discovery of additional spacetime metrics that were previously unidentified using the conventional direct integration technique [24–29]. In this article, we focus on finding the KVFs of nonstatic plane symmetric spacetimes, using the Rif tree approach. We aim to identify all the nonstatic plane symmetric metrics for which Killing symmetries exist and to derive the corresponding solutions to the EFEs for anisotropic or perfect fluid sources.

2 Killing symmetries

The metric of nonstatic plane symmetric spacetime is given by [1]:

$$ds^2 = -f^2(t,x)dt^2 + g^2(t,x)dx^2 + h^2(t,x)[dy^2 + dz^2], \quad (2.1)$$

and the set of its minimum KVFs is given by $M_3 = \{\partial_y, \partial_z, z\partial_y - y\partial_z\}$. Several well-known and significant classes of solutions to EFEs are included in the metric (2.1). For example, this metric defines a static plane symmetric spacetime with an extra KVF, ∂_t, when f, g, and h depend only on x. Such metrics are crucial for the derivation of the famous Taub solution and Kasner's spatially homogenous solutions of EFEs, which makes them significant in a variety of physical contexts. Similarly, the metric (2.1) turns into the Bianchi type I metric possessing the extra KVF,∂_x, for $f = f(t)$, $g = g(t)$, and $h = h(t)$. It is commonly known that Bianchi type metrics are homogeneous, while not necessarily isotropic, cosmological models that solve the EFEs; the kind of metrics varies depending on the scale factor selection. Moreover, if $f = g = h = f(t)$, then the metric (2.1) simplifies to the Friedmann metric, which is commonly used in cosmology.

We apply the definition of KVFs, given in Eq. (1.2), to the metric (2.1) to derive the set of symmetry equations. The explicit form of Eq. (1.2) is given by:

$$g_{ab,c}\eta^c + g_{ac}\eta^c_{,b} + g_{bc}\eta^c_{,a} = 0. \quad (2.2)$$

The commas in the subscript denote partial derivatives with respect to spacetime coordinates. From Eq. (2.1), we can see that the components of the metric tensor are $g_{00} = f^2(t,x)$, $g_{11} = g^2(t,x)$, and $g_{22} = g_{33} = h^2(t,x)$. For $a = b = 0$, Eq. (2.2) gives $g_{00,c}\eta^c + 2g_{0c}\eta^c_{,0} = 0$. Here, c is the dummy index. As $g_{00} = f^2(t,x)$ that depends on t and x only and the nondiagonal components of the metric tensor are zero, thus using the summation convention for c, the last equation becomes $g_{00,t}\eta^0 + g_{00,x}\eta^1 + 2g_{00}\eta^0_{,t} = 0$. Consequently, the first symmetry equation is derived as follows:

$$f_{,t}\eta^0 + f_{,x}\eta^1 + f\eta^0_{,t} = 0. \quad (2.3)$$

Similarly, giving different values to a and b, and applying summation convention on the dummy index c, Eq. (2.2) gives rise to the following nine more symmetry equations.

$$g_{,t}\eta^0 + g_{,x}\eta^1 + g\eta^1_{,x} = 0, \quad (2.4)$$

$$h_{,t}\eta^0 + h_{,x}\eta^1 + h\eta^2_{,y} = 0, \quad (2.5)$$

$$h_{,t}\eta^0 + h_{,x}\eta^1 + h\eta^3_{,z} = 0, \quad (2.6)$$

$$f^2\eta^0_{,x} - g^2\eta^1_{,t} = 0, \quad (2.7)$$

$$f^2\eta^0_{,y} - h^2\eta^2_{,t} = 0, \quad (2.8)$$

Table 4: Metrics with 10-dimensional Killing algebra

Metric no./branch no.	Metric coefficients	Additional symmetries
10d	$f = h = e^{a_1 x}$, $a_1 \neq 0$.	$V_{(4)} = \left[\frac{t^2+y^2+z^2}{2} + \frac{e^{-2a_1 x}}{2a_1^2}\right]\partial_t - \frac{t}{a_1}\partial_x + yt\partial_y + zt\partial_z$,
10	g is a constant function,	$V_{(5)} = -yt\partial_t + \frac{y}{a_1}\partial_x + \left[\frac{z^2-y^2-t^2}{2} + \frac{e^{-2a_1 x}}{2a_1^2}\right]\partial_y - yz\partial_z$,
		$V_{(6)} = zt\partial_t - \frac{z}{a_1}\partial_x + yz\partial_y + \left[\frac{t^2+z^2-y^2}{2} - \frac{e^{-2a_1 x}}{2a_1^2}\right]\partial_z$,
		$V_{(7)} = -t\partial_t + \frac{1}{a_1}\partial_x - y\partial_y - z\partial_z$
		$V_{(8)} = y\partial_t + t\partial_y$,
		$V_{(9)} = z\partial_t + t\partial_z$,
		$V_{(10)} = \partial_t$
10e	f = const.,	$V_4 = ye^{a_1 x}\partial_t - \frac{ye^{a_1 x}}{g}\partial_x + \frac{ge^{a_1 x}}{a_1}\partial_y$
16	$g = a_1 t + a_2$,	$V_5 = ye^{-a_1 x}\partial_t + \frac{ye^{-a_1 x}}{g}\partial_x + \frac{ge^{-a_1 x}}{a_1}\partial_y$
	h = const.,	$V_6 = ze^{a_1 x}\partial_t - \frac{ze^{a_1 x}}{g}\partial_x + \frac{ge^{a_1 x}}{a_1}\partial_z$
	where $a_1 \neq 0$	$V_7 = ze^{-a_1 x}\partial_t + \frac{ze^{-a_1 x}}{g}\partial_x + \frac{ge^{-a_1 x}}{a_1}\partial_z$
		$V_8 = e^{a_1 x}\partial_t - \frac{e^{a_1 x}}{g}\partial_x$
		$V_9 = e^{-a_1 x}\partial_t + \frac{e^{-a_1 x}}{g}\partial_x$
		$V_{10} = \partial_x$
10f	$f = f(t)$, $g = g(x)$,	$V_{(4)} = \frac{y}{f(t)}\partial_t + \int f(t)\mathrm{d}t\partial_y$,
17	h = const.,	$V_{(5)} = \frac{z}{f(t)}\partial_t + \int f(t)dt\partial_z$,
	where $f_{,t}(t) \neq 0$	$V_{(6)} = \frac{\int g(x)\mathrm{d}x}{f(t)}\partial_t + \frac{\int f(t)\mathrm{d}t}{g(x)}\partial_x$,
	and $g_{,x}(x) \neq 0$	$V_{(7)} = -\frac{y}{g(x)}\partial_x + \int g(x)\mathrm{d}x\partial_y$,
		$V_{(8)} = -\frac{z}{g(x)}\partial_x + \int g(x)\mathrm{d}x\partial_z$,
		$V_{(9)} = \frac{1}{f(t)}\partial_t$
		$V_{(10)} = \frac{1}{g(x)}\partial_x$
10g	$f = a_1\int g(x)\mathrm{d}x$, $g = g(x)$, h = const.,	$V_{(4)} = \frac{a_1 ye^{a_1 t}}{f(x)}\partial_t - \frac{a_1 ye^{a_1 t}}{g(x)}\partial_x + f(x)e^{a_1 t}\partial_y$,
18	where $a_1 \neq 0$ and $g_{,x}(x) \neq 0$	$V_{(5)} = -\frac{a_1 ye^{-a_1 t}}{f(x)}\partial_t - \frac{a_1 ye^{-a_1 t}}{g(x)}\partial_x + f(x)e^{-a_1 t}\partial_y$,
		$V_{(6)} = \frac{a_1 ze^{a_1 t}}{f(x)}\partial_t - \frac{a_1 ze^{a_1 t}}{g(x)}\partial_x + f(x)e^{a_1 t}\partial_z$,
		$V_{(7)} = -\frac{a_1 ze^{-a_1 t}}{f(x)}\partial_t - \frac{a_1 ze^{-a_1 t}}{g(x)}\partial_x + f(x)e^{-a_1 t}\partial_z$,
		$V_{(8)} = -\frac{e^{a_1 t}}{\int g(x)\mathrm{d}x}\partial_t + \frac{a_1 e^{a_1 t}}{g(x)}\partial_x$,
		$V_{(9)} = -\frac{e^{-a_1 t}}{\int g(x)\mathrm{d}x}\partial_t - \frac{a_1 e^{-a_1 t}}{g(x)}\partial_x$,
		$V_{(10)} = \partial_t$
10h	f = const., $g = g(x)$, h = const.,	$V_{(4)} = t\partial_y + y\partial_t$,
18	where $g_{,x}(x) \neq 0$	$V_{(5)} = t\partial_z + z\partial_t$,
		$V_{(6)} = \int g(x)\mathrm{d}x\partial_t + \frac{t}{B}\partial_x$,
		$V_{(7)} = -\frac{y}{g(x)}\partial_x + \int g(x)\mathrm{d}x\partial_y$,
		$V_{(8)} = -\frac{z}{g(x)}\partial_x + \int g(x)\mathrm{d}x\partial_z$,
		$V_{(9)} = \partial_t$
		$V_{(10)} = \frac{1}{g(x)}\partial_x$

(*Continued*)

Table 4: *Continued*

Metric no./branch no.	Metric coefficients	Additional symmetries
10i 23	$f = f(t)$, $g =$ const. $h =$ const., where $f_{,t}(t) \neq 0$	$V_{(4)} = \frac{y}{f(t)}\partial_t + \int f(t)\mathrm{d}t\partial_y$, $V_{(5)} = \frac{z}{f(t)}\partial_t + \int f(t)\mathrm{d}t\partial_z$, $V_{(6)} = \frac{x}{f(t)}\partial_t + \int f(t)\mathrm{d}t\partial_x$, $V_{(7)} = x\partial_y - y\partial_x$, $V_{(8)} = x\partial_z - z\partial_x$, $V_{(9)} = \frac{1}{f(t)}\partial_t$ $V_{(10)} = \partial_x$
10j 24	$f = a_1x + a_2$, $g =$ const., $h =$ const., where $a_1 \neq 0$	$V_{(4)} = \frac{ye^{a_1t}}{f}\partial_t - ye^{a_1t}\partial_x + \frac{fe^{a_1t}}{a_1}\partial_y$ $V_{(5)} = -\frac{ye^{-a_1t}}{f}\partial_t - ye^{-a_1t}\partial_x + \frac{fe^{-a_1t}}{a_1}\partial_y$ $V_{(6)} = \frac{ze^{a_1t}}{f}\partial_t - ze^{a_1t}\partial_x + \frac{fe^{a_1t}}{a_1}\partial_z$ $V_{(7)} = -\frac{ze^{-a_1t}}{f}\partial_t - ze^{-a_1t}\partial_x + \frac{fe^{-a_1t}}{a_1}\partial_z$ $V_{(8)} = -\frac{e^{a_1t}}{f}\partial_t + e^{a_1t}\partial_x$ $V_{(9)} = \frac{e^{-a_1t}}{f}\partial_t + e^{-a_1t}\partial_x$ $V_{(10)} = \partial_t$

$$f^2\eta^0_{,z} - h^2\eta^3_{,t} = 0, \tag{2.9}$$

$$g^2\eta^1_{,y} + h^2\eta^2_{,x} = 0, \tag{2.10}$$

$$g^2\eta^1_{,z} + h^2\eta^3_{,x} = 0, \tag{2.11}$$

$$\eta^2_{,z} + \eta^3_{,y} = 0. \tag{2.12}$$

The aforementioned symmetry equations must be solved in order to find the exact form of the KVF η. The metric functions $f(t,x)$ $g(t,x)$, and $h(t,x)$ must be subject to specific constraints for these equations to be solved generally. The Rif algorithm is used to examine these equations, which offers various metrics admitting different dimensional Killing algebras. Figure 1 displays the generated Rif tree. The Rif tree's nodes, also known as pivots, are derived as given in (2.13).

The dependent and independent variables ordering is crucial while employing the Rif algorithm, since it has a visible impact on how complicated the resulting Rif tree is. There is no universal rule for selecting an optimal variable order to simplify the Rif tree. However, through trial and error, we found that ordering the dependent variables as $\eta^0 > \eta^1 > \eta^2 > \eta^3$ and the independent variables as $t > x > y > z$ yields the most simplified Rif tree in our case.

$$\begin{aligned}
p1 &= h_{,t}\\
p2 &= h_{,t}g_{,x} - h_{,x}g_{,t}\\
p3 &= h_{,t}f_{,x} - h_{,x}f_{,t}\\
p4 &= h_{,t}h_{,tx} - h_{,tt}h_{,x}\\
p5 &= (h_{,t})^2h_{,xx} - 2h_{,tx}h_{,x} + h_{,tt}(h_{,x})^2\\
p6 &= (h_{,t})^2h_{,ttx} - 2h_{,t}h_{,tt}h_{,tx} - h_{,t}h_{,x}h_{,ttt} + 2(h_{,tt})^2h_{,x}\\
p7 &= (h_{,t})^2h_{,xx} - h_{,tt}(h_{,x})^2\\
p8 &= h_{,x}\\
p9 &= g_{,t}\\
p10 &= f_{,t}\\
p11 &= g_{,t}f_{,x} - g_{,x}f_{,t}\\
p12 &= g_{,t}g_{,tx} - g_{,x}g_{,tt}\\
p13 &= -(g_{,t})^2g_{,xx} + 2g_{,t}g_{,x}g_{,tx} - (g_{,x})^2g_{,tt}\\
p14 &= g_{,ttt}g_{,t}g_{,x} - (g_{,t})^2g_{,ttx} + 2g_{,t}g_{,tt}g_{,tx} - 2g_{,x}(g_{,tt})^2\\
p15 &= (g_{,t})^2g_{,xx} - (g_{,x})^2g_{,tt}\\
p16 &= g_{x}\\
p17 &= f_{,t}f_{,tx} - f_{,x}f_{,tt}\\
p18 &= (f_{,t})^2f_{,xx} - 2f_{,t}f_{,x}f_{,tx} + (f_{,x})^2f_{,tt}\\
p19 &= (f_{,t})^2f_{,ttx} - f_{,t}f_{,x}f_{,ttt} - 2f_{,t}f_{,tx}f_{,tt} + 2f_{,x}(f_{,tt})^2\\
p20 &= (f_{,t})^2f_{,xx} - (f_{,x})^2f_{,tt}\\
p21 &= f_{,x}.
\end{aligned} \tag{2.13}$$

Table 5: Energy conditions

Metric no.	Physical terms	Energy conditions
4b, 4j	$\rho = \frac{a_1^2}{(a_1t+a_2)^2}$ $p_{\parallel} = -\rho$ $p_{\perp} = 0$	All energy conditions are identically satisfied
4c	$\rho = \frac{(a_1+a_3)^2}{(a_1t+a_2)^2}$ $p_{\parallel} = -\frac{(a_1+a_3)(a_1+3a_3)}{(a_1t+a_2)^2}$ $p_{\perp} = -\frac{a_3(a_1+a_3)}{(a_1t+a_2)^2}$	WEC, NEC and SEC are satisfied if $a_3(a_1+a_3) < 0$ and $a_1(a_1+a_3) > 0$ DEC is satisfied if $a_3(a_1+a_3) < 0$, $a_1(a_1+a_3) > 0$ and $(a_1+a_3)(a_1+2a_3) > 0$
4d	$\rho = \frac{a_2^2-a_1^2}{[a_1x+a_2\int f(t)dt]^2}$ $p_{\parallel} = -\rho$ $p_{\perp} = 0$	All energy conditions are satisfied if $a_2^2 - a_1^2 > 0$
4f	$\rho = \frac{4[2a_1^2(t^2-x^2)+4a_1(a_2x+a_3t)+2a_1a_4-a_2^2+a_3^2]}{[a_1(t^2-x^2)+2(a_2x+a_3t+a_4)]^2}$ $p_{\parallel} = -\rho$ $p_{\perp} = -\frac{4a_1}{a_1(t^2-x^2)+2(a_2x+a_3t+a_4)}$	WEC is satisfied if $2a_1^2(t^2 - x^2) + 4a_1(a_2x + a_3t)$ $+2a_1a_4 - a_2^2 + a_3^2 \geq 0$ and $a_1^2(t^2-x^2) + 2a_1(a_2x+a_3t) - a_2^2 + a_3^2 \geq 0$ NEC is satisfied if $a_1^2(t^2-x^2) + 2a_1(a_2x+a_3t) - a_2^2 + a_3^2 \geq 0$ SEC is satisfied if $a_1^2(t^2-x^2) + 2a_1(a_2x+a_3t) - a_2^2 + a_3^2 \geq 0$ and $\frac{8a_1}{a_1(t^2-x^2)+2(a_2x+a_3t+a_4)} < 0$ DEC is satisfied if $2a_1^2(t^2 - x^2) + 4a_1(a_2x + a_3t)$ $+2a_1a_4 - a_2^2 + a_3^2 \geq 0$ and $8a_1^2(t^2-x^2) + 24a_1(a_2x+a_3t)$ $+16a_1a_4 - 4a_2^2 + 4a_3^2 \geq 0$

The Rif tree uses the symbols "=" and "<>" to indicate whether the associated p_i is zero or nonzero. In branch 1, for example, we have $p_1 = h_{,t} \neq 0$ and $p_2 = h_{,t}g_{,x} - h_{,x}g_{,t} \neq 0$. In branch 2, we have $p_1 = h_{,t} \neq 0$, $p_3 = h_{,t}f_{,x} - h_{,x}f_{,t} \neq 0$ and $p_2 = h_{,t}g_{,x} - h_{,x}g_{,t} = 0$. Similar restrictions are placed on the metric coefficients by each branch of the Rif tree. It can be seen that each branch of the Rif tree restricts the metric coefficients in a different way. These restrictions are then used to solve Eqs. (2.3)–(2.12), that explicitly determines the components of KVFs and the values of the metric coefficients. In such a way, one obtains a complete classification of the spacetime under consideration via KVFs. In the present case the solution of Eqs. (2.3)–(2.12), under the restrictions of all branches of the Rif tree, yields various plane symmetric metrics admitting 3-, 4-, 5-, 6-, 7-, and 10-dimensional Killing algebras. The branches labeled by 4 and 6 yield only three KVFs, as given in the set M_3. The metrics of the remaining branches produce extra KVFs in addition to those given in the set M_3. Tables 1–4 provide a summary of these branches' outcomes, where the first two columns of each table show the metric numbering, the metric coefficients and the corresponding branch of the Rif tree that produces the metric. The last column of each table contains extra symmetries admitted by the associated metric.

Each of the metrics 4a-4s admits four-dimensional Killing algebra. The dimension of Killing algebra for the metrics 6a-6f is 6, while its dimension for the metrics 7a-7d is 7. Finally, the metrics 10a-10j admit 10 KVFs. Of the derived metrics, those labeled by 6b, 6c and 10c are Friedmann metrics, while the metrics 4g-4j and 7b represent locally rotationally symmetric Bianchi type I models. All these metrics are same as given in ref. [25], where Bianchi type I spacetime was classified via its KVFs. Similarly, the metrics 4o-4s, 6d-6f, 7d, 10d, 10g, 10h, and 10j are static plane symmetric metrics. All the remaining derived metrics are nonstatic plane symmetric metrics. Though the

Table 6: Energy conditions

Metric no.	Physical terms	Energy conditions
4g	$\rho = \frac{(a_1-2a_3)(3a_1-2a_3)}{a_1^2t^2}$ $p_{\|} = -\frac{(a_1-2a_3)(a_1-6a_3)}{a_1^2t^2}$ $p_{\perp} = -\frac{(a_1-2a_3)^2}{a_1^2t^2}$	WEC is satisfied if $(a_1 - 2a_3)(3a_1 - 2a_3) > 0$ $(a_1 - 2a_3)(a_1 + 2a_3) > 0$ and $a_1(a_1 - 2a_3) > 0$ NEC is satisfied if $(a_1 - 2a_3)(a_1 + 2a_3) > 0$ and $a_1(a_1 - 2a_3) > 0$ SEC is satisfied if $(a_1 - 2a_3)(a_1 + 2a_3) > 0$ $a_1(a_1 - 2a_3) > 0$ and $a_3(a_1 - 2a_3) > 0$ DEC is satisfied if $(a_1 - 2a_3)(3a_1 - 2a_3) > 0$ and $a_1(a_1 - 2a_3) > 0$
4h	$\rho = \frac{3a_1^2+4a_4^2-4a_1a_3-8a_1a_4+8a_3a_4}{(a_1t+2a_2)^2}$ $p_{\|} = -\frac{a_1^2+12a_4^2-8a_1a_4}{(a_1t+2a_2)^2}$ $p_{\perp} = -\frac{a_1^2+4a_3^2+4a_4^2-4a_1a_3-4a_1a_4+4a_3a_4}{(a_1t+2a_2)^2}$	WEC is satisfied if $3a_1^2 + 4a_4^2 - 4a_1a_3 - 8a_1a_4 + 8a_3a_4 \geq 0,$ $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$ and $2a_1^2 - 4a_1a_4 - 4a_3^2 + 4a_3a_4 \geq 0$ NEC is satisfied if $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$ and $2a_1^2 - 4a_1a_4 - 4a_3^2 + 4a_3a_4 \geq 0$ SEC is satisfied if $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$ $2a_1^2 - 4a_1a_4 - 4a_3^2 + 4a_3a_4 \geq 0$ and $4a_1a_3 + 8a_1a_4 - 8a_3^2 - 16a_4^2 \geq 0$ DEC is satisfied if $3a_1^2 + 4a_4^2 - 4a_1a_3 - 8a_1a_4 + 8a_3a_4 \geq 0,$ $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0,$ $4a_1^2 + 16a_4 2 - 4a_1a_3 - 16a_1a_4 + 8a_3a_4 \geq 0$ $2a_1^2 - 4a_1a_4 - 4a_3^2 + 4a_3a_4 \geq 0$ and $4a_1^2 - 8a_1a_3 - 12a_1a_4 + 12a_3a_4 + 8a_4^2 + 4a_3^2 \geq 0$
4i	$\rho = \frac{(a_1-2a_3)^2}{(a_1t+2a_2)^2}$ $p_{\|} = -\frac{(a_1-2a_3)(a_1-6a_3)}{(a_1t+2a_2)^2}$ $p_{\perp} = \frac{2a_3(a_1-2a_3)}{(a_1t+2a_2)^2}$	WEC, NEC and SEC are satisfied if $a_3(a_1 - 2a_3) > 0$ and $a_1(a_1 - 2a_3) > 0$ DEC is satisfied if $a_3(a_1 - 2a_3) > 0$, $a_1(a_1 - 2a_3) > 0$ $(a_1 - 2a_3)(a_1 - 4a_3) > 0$
4j	$\rho = \frac{a_1^2}{(a_1t+a_2)^2}$ $p_{\|} = -\rho$ $p_{\perp} = 0$	All energy conditions are identically satisfied
4k	$\rho = \frac{a_2^2-a_1^2}{(a_1x+a_2t)^2}$ $p_{\|} = -\rho$ $p_{\perp} = 0$	All energy conditions are satisfied if $a_2^2 - a_1^2 > 0$
4m, 4p	$\rho = -\frac{(a_1-2a_3)(a_1-6a_3)}{(a_1x+2a_2)^2}$ $p_{\|} = \frac{(a_1-2a_3)^2}{(a_1x+2a_2)^2}$ $p_{\perp} = -\frac{2a_3(a_1-2a_3)}{(a_1x+2a_2)^2}$	WEC is satisfied if $(a_1 - 2a_3)(a_1 - 6a_3) > 0$, $a_3(a_1 - 2a_3) > 0$ $(a_1 - 2a_3)(4a_3 - a_1) > 0$ NEC and SEC are satisfied if $a_3(a_1 - 2a_3) > 0$ and $(a_1 - 2a_3)(4a_3 - a_1) > 0$ DEC is satisfied if $a_3(a_1 - 2a_3) > 0$ $(a_1 - 2a_3)(4a_3 - a_1) > 0$ and $(a_1 - 2a_3)(8a_3 - a_1) > 0$

(*Continued*)

Table 6: *Continued*

Metric no.	Physical terms	Energy conditions
4n, 4q	$\rho = -\frac{a_1^2}{(a_1x + a_2)^2}$ $p_{\parallel} = -\rho$ $p_{\perp} = 0$	An un-physical model with $\rho < 0$ and satisfying none of the energy conditions

KVFs of plane symmetric spacetime were explored in an earlier study [12], these metrics were not listed there. This shows the significance of Rif tree approach for achieving a complete classification of the spacetimes.

It is remarkable that the number of KVFs admitted by a spacetime reflects the degree of its symmetry. A higher number of KVFs corresponds to a larger isometry group, which constrains the curvature tensors and is helpful in reducing the complexity of the EFEs. A four-dimensional Lorentzian manifold can admit a maximum of 10 independent KVFs. This maximum number of KVFs is admitted by maximally symmetric spacetimes such as Minkowski, de Sitter, and anti-de Sitter spacetimes. Physically, the presence of higher number of KVFs implies more conservation laws via Noether's theorem, which are essential in understanding the motion of particles and the behavior of fields. Hence, the existence of an extended symmetry algebra, such as a 10-dimensional Killing algebra in a nonstatic spacetime indicates that the spacetime is of constant curvature and plays a role in fundamental models in general relativity.

3 Solutions of EFEs

For nonstatic plane symmetric spacetime, we have constructed various Lorentzian metrics with different dimensional Killing algebras by solving the Killing symmetry equations. Of these metrics, those satisfying EFEs and having the energy-momentum tensor associated with some known matter provide the exact solutions to the EFEs. The corresponding energy-momentum tensor T_{ab} for each of these metrics can be found using the EFEs. Moreover, T_{ab} can be used to assess the physical realism of the obtained metrics and to check various energy conditions satisfied by these metrics. In this section, we follow this procedure to check which of the obtained metrics are physically realistic solutions of EFEs. The metric (2.1) has four diagonal and one off-diagonal nonvanishing components of T_{ab}, given by:

$$\begin{aligned}
T_{00} &= -\frac{1}{g^3h^2}[2gf^2hh_{,xx} - g^3(h_{,t})^2 - 2g^2hg_{,t}h_{,t} \\
&\quad + f^2g(h_{,x})^2 - 2f^2hg_{,x}h_{,x}], \\
T_{01} &= \frac{1}{fgh}[2fgh_{,tx} - gf_{,x}h_{,t} - fg_{,t}h_{,x}], \\
T_{11} &= -\frac{1}{f^3h^2}[2fg^2hh_{,tt} + fg^2(h_{,t})^2 - 2g^2hf_{,t}h_{,t} \\
&\quad - f^3(h_{,x})^2 - 2f^2hf_{,x}h_{,x}], \\
T_{22} = T_{33} &= \frac{h}{f^3g^3}[f^3gh_{,xx} - fg^3h_{,tt} - fg^2hg_{,tt} + f^2ghf_{,xx} \\
&\quad + g^3f_{,t}h_{,t} - fg^2g_{,t}h_{,t} + g^2hf_{,t}g_{,t} + f^2gf_{,x}h_{,x} \\
&\quad - f^3g_{,x}h_{,x} - f^2hf_{,x}g_{,x}].
\end{aligned} \tag{3.1}$$

In the aforementioned expressions, $f_{,t}$, $g_{,t}$, $h_{,t}$, $f_{,x}$, $g_{,x}$, and $h_{,x}$ signify the first order partial derivatives of the metric functions $f(t,x)$, $g(t,x)$ and $h(t,x)$ with respect to t and x, respectively, while the terms like $h_{,xx}$, $f_{,xx}$, $g_{,tt}$, $h_{,tt}$, and $h_{,tx}$ define the second-order partial derivatives. To find the above components of T_{ab}, first, we have found the Christoffel symbols for the metric (2.1). These symbols are used to find the components of Riemann curvature tensor, which are then contracted to find the components of the Ricci tensor. The Ricci tensor components are contracted with the inverse of the metric tensor to obtain the Ricci scalar. Finally, all these expressions are used in the EFEs, Eqs. (1.1), with $k = 1$ to obtain the desired components of T_{ab}, given in (3.1).

The structure of T_{ab} varies for several known sources of matter. Assuming that the matter source for the metric (2.1) is an anisotropic fluid, the components of T_{ab} are obtained as $T_{00} = \rho f^2$, $T_{11} = p_{\parallel}g^2$, $T_{22} = T_{33} = p_{\perp}h^2$, and $T_{01} = 0$, where ρ is the density, while $p_{\parallel}$ and $p_{\perp}$ are pressures in two directions. A perfect fluid is obtained when $p_{\parallel} = p_{\perp} = p$. Thus, among the classified metrics, those for which $T_{01} = 0$ indicate anisotropic or perfect fluids. It is simple to calculate ρ, $p_{\parallel}$, and $p_{\perp}$ for such metrics as follows:

$$\rho = \frac{T_{00}}{f^2}, p_{\parallel} = \frac{T_{11}}{g^2}, p_{\perp} = \frac{T_{22}}{h^2}. \tag{3.2}$$

Table 7: Energy conditions

Metric no.	Physical terms	Energy conditions
4o	$\rho = \frac{8a_1a_4 - a_1^2 - 12a_4^2}{(a_1x + 2a_2)^2}$	WEC iis satisfied if $8a_1a_4 - a_1^2 - 12a_4^2 \geq 0$,
		$2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$
		and $4a_1a_4 - 8a_4^2 - 4a_1a_3 + 4a_3^2 + 4a_3a_4 \geq 0$
	$p_\parallel = \frac{3a_1^2 + 4a_4^2 - 4a_1a_3 - 8a_1a_4 + 8a_3a_4}{(a_1x + 2a_2)^2}$	NEC is satisfied if $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$
		and $4a_1a_4 - 8a_4^2 - 4a_1a_3 + 4a_3^2 + 4a_3a_4 \geq 0$
	$p_\perp = \frac{a_1^2 + 4a_3^2 + 4a_4^2 - 4a_1a_3 - 4a_1a_4 + 4a_3a_4}{(a_1x + 2a_2)^2}$	SEC is satisfied if $2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$,
		$4a_1a_4 - 8a_4^2 - 4a_1a_3 + 4a_3^2 + 4a_3a_4 \geq 0$
		and $3a_1^2 - 8a_1a_3 - 4a_4^2 + 12a_3a_4 - 4a_1a_4 + 4a_3^2 \geq 0$
		DEC is satisfied if $8a_1a_4 - a_1^2 - 12a_4^2 \geq 0$,
		$2a_1^2 - 8a_4^2 - 4a_1a_3 + 8a_3a_4 \geq 0$
		$4a_1a_4 - 8a_4^2 - 4a_1a_3 + 4a_3^2 + 4a_3a_4 \geq 0$
		$16a_1a_4 - 4a_1^2 - 12a_4^2 + 4a_1a_3 - 8a_3a_4 \geq 0$
		and $12a_1a_4 - 2a_1^2 - 16a_4^2 + 4a_1a_3 - 4a_3^2 - 4a_3a_4 \geq 0$
4r	$\rho = 0$	All energy conditions are satisfied if $a_3(2a_3 - a_1) > 0$
	$p_\parallel = 0$	
	$p_\perp = \frac{2a_3(2a_3 - a_1)}{[a_1\int g(x)\mathrm{d}x + 2a_2]^2}$	
4s	$\rho = 0$	All energy conditions are satisfied if $a_3(2a_3 - a_1) > 0$
	$p_\parallel = 0$	
	$p_\perp = \frac{2a_3(2a_3 - a_1)}{[a_1x + 2a_2]^2}$	
6a	$\rho = \frac{3(a_1^2 - a_3^2)}{(a_1t + a_2)^2}$	All energy conditions are satisfied if $a_1^2 - a_3^2 > 0$
	$p_\parallel = p_\perp = -\frac{\rho}{3}$	
6b	$\rho = \frac{3(a_1 - 2a_3)^2}{(a_1t + 2a_2)^2}$	WEC and NEC satisfied if $a_1(a_1 - 2a_3) > 0$
	$p_\parallel = p_\perp = -\frac{(a_1 - 2a_3)(a_1 - 6a_3)}{(a_1t + 2a_2)^2}$	SEC is satisfied if $a_1(a_1 - 2a_3) > 0$
		and $a_3(a_1 - 2a_3) > 0$
		DEC is satisfied if $a_1(a_1 - 2a_3) > 0$
		and $(a_1 - 3a_3)(a_1 - 2a_3) > 0$
6c	$\rho = \frac{3}{t^2}$	All energy conditions are identically satisfied
	$p_\parallel = p_\perp = -\frac{1}{t^2}$	
6d	$\rho = -\frac{(a_1 - 2a_3)(a_1 - 6a_3)}{(a_1x + 2a_2)^2}$	WEC is satisfied if $(a_1 - 2a_3)(a_1 - 6a_3) < 0$
		$a_1(a_1 - 2a_3) > 0$
	$p_\parallel = \frac{3(a_1 - 2a_3)^2}{(a_1x + 2a_2)^2}$	NEC is satisfied if $a_1(a_1 - 2a_3) > 0$
	$p_\perp = -\rho$	SEC is satisfied if $a_1(a_1 - 2a_3) > 0$
		and $(a_1 - 2a_3)(a_1 - 3a_3) > 0$
		DEC is satisfied if $(a_1 - 2a_3)(a_1 - 6a_3) < 0$
		and $(a_1 - 2a_3)(3a_3 - a_1) > 0$
6e	$\rho = -\frac{a_1^2}{(a_1x + a_2)^2}$	WEC and DEC are failed, while NEC and
	$p_\parallel = \frac{3a_1^2}{(a_1x + a_2)^2}$	SEC are satisfied
	$p_\perp = -\rho$	
6f	$\rho = p_\parallel = 0$	DEC is failed, while
		WEC, NEC and SEC are satisfied if

(*Continued*)

Table 7: *Continued*

Metric no.	Physical terms	Energy conditions
	$p_\perp = \frac{a_2^2[a_1^4e^{8a_2x} + 4a_1^3a_3e^{6a_2x} + 6a_1^2a_3^2e^{4a_2x} + 4a_1a_3^2e^{2a_2x} + a_3^2]}{(a_1e^{2a_2x} + a_3)^4}$	$a_1^4e^{8a_2x} + 4a_1^3a_3e^{6a_2x} + 6a_1^2a_3^2e^{4a_2x}$ $+4a_1a_3^2e^{2a_2x} + a_3^2 > 0$
10c	$\rho = 3$ $p_{\|\|} = p_\perp = -3$	WEC, NEC and DEC are satisfied, while SEC is failed.
10d	$\rho = -3a_1^2$ $p_{\|\|} = p_\perp = 3a_1^2$	NEC and SEC are satisfied, while WEC and DEC are failed.

Of all the obtained metrics in our classification, for only three metrics (labeled by $4a$, $4e$, and $4l$), we have $T_{01} \neq 0$, indicating that these models do not describe perfect or anisotropic fluids. We exclude these three metrics from our discussion. For all other classified metrics, we have $T_{01} = 0$, giving anisotropic or perfect fluid solutions to EFEs. For all these models, the components of T_{ab} can be found using Eq. (3.1), and these components can be used in Eq. (3.2) to find the physical terms ρ, $p_{||}$, and $p_\perp$. Consequently, these terms can be used to check different energy conditions, including weak energy condition (WEC, $\rho \geq 0, \rho + p_{||} \geq 0, \rho + p_\perp \geq 0$), null energy condition (NEC, $\rho + p_{||} \geq 0, \rho + p_\perp \geq 0$), strong energy condition (SEC, $\rho + p_{||} \geq 0, \rho + p_\perp \geq 0, \rho + p_{||} + 2p_\perp \geq 0$), and dominant energy condition (DEC, $\rho \geq 0, \rho \geq |p_{||}|, \rho \geq |p_\perp|$) [30].

For some of the obtained metrics, labeled by 10a, 10b, and 10e-10j the components of T_{ab} vanish, giving vacuum solutions. Consequently, the terms ρ, $p_{||}$, and $p_\perp$ vanish for all these metrics. All the energy conditions are identically satisfied for these metrics. On the other hand, the metrics labeled by 7a–7d do not satisfy any of these energy conditions. For the metrics 7a and 7b, we have $\rho = a_1^2, p_{||} = -3a_1^2$, and $p_\perp = -a_1^2$. Though none of the energy conditions is satisfied by these terms, the positive value of energy density indicates that it is a physically realistic model. Similarly, for metrics 7c and 7d, we have $\rho = -3a_1^2, p_{||} = a_1^2$, and $p_\perp = a_1^2$. These terms do not satisfy any energy condition and the negative value of ρ shows that it is not a physically realistic model. In Tables 5–7, we present the physical terms ρ, $p_{||}$ and $p_\perp$ for all the remaining metrics and give the details of energy conditions satisfied by these metrics. For metrics 6a, 6b, 6c, 10c, and 10d, we have $p_{||} = p_\perp$, giving perfect fluids, while all the remaining metrics denote anisotropic fluids.

4 Physical interpretation and connection to known models

While the work presented in this article focuses the classification of nonstatic plane symmetric spacetime using KVFs, many of the derived metrics during the classification have clear relevance in cosmological and astrophysical contexts. For example, consider the metrics labeled by 4j, 4n, and 4q, where the metric coefficient $h(t)$ or $h(x)$ is linear. These metrics exhibit directional anisotropy and are closely related to the Kasner-type or Bianchi type I cosmological models, which are commonly used to describe anisotropic expansion in the early universe. On the other hand, the metrics 4d and 4k involve mixed dependencies in both space and time and can be interpreted as models with shear or inhomogeneous anisotropic expansion, making them applicable in anisotropic gravitational collapse or structure formation scenarios. Next, in the metric 10c, all scale factors evolve exponentially and it resembles the vacuum inflationary models, like de Sitter spacetime. The metric 10a admitting ten KVFs gives a vacuum solution with plane symmetry and resembles the pp-wave or plane gravitational wave model that is widely used in theoretical studies of exact wave propagation in general relativity.

The energy-momentum tensor components for the derived metrics further support their physical relevance. For example, the metric 6c describes a perfect fluid with time-dependent energy density $\rho = 3/t^2$, typical of radiation-dominated cosmological eras. Hence, these solutions offer not only mathematical classifications but also idealized models for gravitational waves, anisotropic cosmological evolution, and collapsing systems.

5 Conclusion

We have presented a novel approach, the Rif tree method, for the classification of KVFs in nonstatic plane symmetric spacetime. By systematically applying this method, several distinct spacetime metrics with varying dimensional Killing algebras were derived. These metrics offer deeper insights into spacetime symmetries and provide valuable tools for simplifying EFEs in the context of general relativity.

Our analysis has shown that nonstatic plane symmetric spacetime can admit additional KVFs beyond the minimal set, depending on the imposed conditions on the metric coefficients. This expansion of symmetries opens up new avenues for exploring exact solutions to EFEs, with potential applications in cosmological models and the study of gravitational fields.

Moreover, the physical implications of the obtained metrics were explored by deriving the corresponding energy-momentum tensors and evaluating the energy conditions. Several metrics were found to describe physically realistic models, including anisotropic and perfect fluid solutions, while few were shown to be unphysical due to negative energy densities.

Overall, this work extends the understanding of EFEs in nonstatic spacetimes and highlights the usefulness of the Rif tree approach in identifying and classifying spacetime symmetries. Future research may focus on applying this method to other classes of spacetimes and investigating the physical significance of the derived metrics in more specific gravitational and cosmological scenarios.

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