Research Article

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Exploring the peakon solitons molecules and solitary wave structure to the nonlinear damped Kortewege–de Vries equation through efficient technique

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Abstract: This work examined solitary wave solutions to the nonlinear damped Korteweg–de Vries equation by employing the new auxiliary equation approach. The physical structure to the secured solutions visualized in dark solitons, bright solitons, periodic solitons, kink and anti-kink wave solitons, peakon bright and dark solitons, and dispersive solitary waves. The physical interpretation of constructed solutions is visually portrayed using two-dimensional, three-dimensional, and contour plots on the basis of numerical simulation, which help comprehend the physical features of nonlinear behaviour for the solitary waves. The explored solutions will be play important role in Mathematical physics, ion-acoustic waves, dust-acoustic waves, and plasma physics. This study has demonstrated that our suggested method is more beneficial, successful, strong and effective for studying analytically various nonlinear partial differential equations (NLPDEs) that arise in mathematical physics, engineering, plasma physics, and many other scientific fields.

Keywords: damped Kortewege–de Vries equation, new auxiliary equation method, solitary wave structures, Peakon solitons, exact solitons

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1 Introduction

The nonlinear evolution equations have great interest of the scientist, physicist, mathematician, and engineers in these days due to its marvellous applications in sciences and engineering. This study investigated the solitary waves one of the nonlinear models which arise in dusty plasma. It is extremely importance to research on nonlinear waves in dusty plasma due to its amazing applications in astrophysics environments including planet rings, comet tails, planet magneto-sphere, surroundings of laboratory plasma, and Earth's ionosphere [1–8]. The dust ion-acoustic waves (DIAWs) are ion-acoustic modes that are essentially modified by the presence of dust particles. It has been theoretically showed that dust plasmas containing negatively charged static dust particles reinforce DIAWs at first spell in small quantities of frequency due to charge density equilibrium and protection of storing electron depletions [9]. The DIAWs were investigated experimentally in lab plasma by the researchers [10]. In addition, the researchers investigated how the distribution of dust grains affected the DIAWs in dust plasma collisions involving dust particles with Gaussian distributions [11]. The instability of DIAWs in dust plasma collisions was investigated by the researchers [12]. In line for its understanding of lab settings, laser beam plasmas, astrophysics plasmas, and the Earth's ionosphere, the nonlinear characteristics of DIAWs in the appearance of ionized impacts, and ion neutral collisions with the negatively charged dust grain drawn within the direction of nonlinear phenomena in the research of polluted plasma [13]. A great deal of theoretical work has been done by numerous other researchers on the nonlinear behaviour of partial differential equations in various areas [14–26]. In the studies by Iqbal *et al.* [4] and Alruwaili *et al.* [27], the researchers studied the nonlinear behaviour of DIAWs in dusty plasma with two diverse temperatures trapped electrons in the event of ion dust collision and ionization effect. Nonlinear DIAWs in polluted plasma interactions with positively charged ions, negatively charged dust grains, particles that are neutral, and thermal electrons have also been investigated [28]. Researchers examined the nonlinear occurrences of DIAWs in polluted plasma with confined electrons while taking into account ions in source designs and they discovered that the magnitude of DIAWs affected signal losses in momentum ions, replication of ions sounds of dust fragments, and ions different forms of plasma [29]. Through the use of hydrodynamic analysis, researchers looked at how DIAWs were excited in plasma interactions to reflect how DIAWs were excited by the relative mobility of both electrons and ions, which generated the electrostatic field [30]. Tamang *et al.* recently examined, while accounting for the ionizing effect, ion dust, ion loss, ion neutral, and dust neutral collisions, the nonlinear behaviour of DIAWs in polluted plasma containing positive ions that are charged, negative charged particles fluid, expertise particles that are neutral, and q-nonextensive electrons. With the use of the reductive perturbation approach, they were able to construct the nonlinear damped Korteweg–de Vries and damped modified Korteweg–de Vries equations. They next found the solitary wave solitons by using the conservation law of momentum [31].

To determine the solitary wave solutions of nonlinear partial differential equations (NLPDEs), a great deal of study has been done in the previous few decades. The study of NLPDEs solitary wave solutions is crucial for improving knowledge and understanding their mechanism and applications. Therefore, several researchers and mathematicians devised numerous ways to ascertain the solitary wave solutions of NLPDEs. The extended direct algebraic method, extended mapping method, F-expansion method, Darboux transformation technique, Hirota bilinear technique, Jacobic elliptic method, Riccati equation rational expansion method, modified Sardar subequation method, Kudryashov auxiliary equation scheme, improved F-expansion method, extended simple equation method, exp $(-\Phi(\zeta))$-function method, auxiliary equation method (AEM), improved (G'/G)-expansion method, fractional unified solver method, and others are a few of the key techniques [32–54].

In this latest study, we create new form of solutions for the damped KdV equation by implementing the new AEM.

The format of this study work is as outlined follows: Section 1 discusses the introduction. We outlined the suggested technique in Section 2. In Section 3, construct new solutions to the damped KdV equation by using the new technique. We discussed the results in Section 4. Finally, Section 5 is the conclusion part of the work.

2 Summery of proposed technique

The nonlinear equation with partial derivatives are expressed as follows:

$$L(\phi_\tau, \phi_x, \phi_{xx}, \phi_{xxx}, \ldots) = 0. \tag{1}$$

The L is a polynomial function for $\phi(x,\tau)$ and its all derivatives. Consider the transformation for Eq. (1) as follows:

$$\phi(x,\tau) = u(\zeta), \quad \zeta = \Bbbk x + \wp\tau. \tag{2}$$

The nonlinear ordinary differential equation for Eq. (2) is given as follows:

$$M(\wp u', \Bbbk u', \Bbbk^2 u'', \Bbbk^3 u''', \ldots) = 0. \tag{3}$$

M is a polynomial function of $u(\zeta)$ and its all derivatives. The solution for Eq. (3) in series form is given as follows:

$$u(\zeta) = a_0 + \sum_{k=1}^{N} (a_k \varphi^k(\zeta) + b_k \varphi^{-k}(\zeta)), \tag{4}$$

where a_0, a_k, b_k $(k = 1,2,3,\ldots)$ are constants which can be calculated later. We implement the homogeneity principle technique for ascertaining the value of N in Eq. (3), and the nonlinear term and the higher order derivative in Eq. (3) are balanced. The $\varphi(\zeta)$ satisfy the following Riccati equation:

$$\left(\frac{d\varphi}{d\zeta}\right) = \varphi^2(\zeta) + f, \tag{5}$$

where f is constant. The nine exact solutions for Eq. (5) are given as follows:

When $f < 0$, the hyperbolic solutions for Eq. (5) are as follows:

$$\varphi_1(\zeta) = \frac{\sqrt{-(s^2+t^2)f} - s\sqrt{-f}\cosh(2\sqrt{-f}(\zeta+\zeta_0))}{s\sinh(2\sqrt{-f}(\zeta+\zeta_0)) + t}, \tag{6}$$

$$\varphi_2(\zeta) = \frac{-\sqrt{-(s^2+t^2)f} - s\sqrt{-f}\cosh(2\sqrt{-f}(\zeta+\zeta_0))}{s\sinh(2\sqrt{-f}(\zeta+\zeta_0)) + t}, \tag{7}$$

$$\varphi_3(\zeta) = \sqrt{-f} + \frac{-2s\sqrt{-f}}{s + \cosh(2\sqrt{-f}(\zeta+\zeta_0)) - \sinh(2\sqrt{-f}(\zeta+\zeta_0))}, \tag{8}$$

$$\varphi_4(\zeta) = -\sqrt{-f} + \frac{-2s\sqrt{-f}}{s + \cosh(2\sqrt{-f}(\zeta + \zeta_0)) - \sinh(2\sqrt{-f}(\zeta + \zeta_0))}. \quad (9)$$

When $f > 0$, the trigonometric solutions for Eq. (5) are given as follows:

$$\varphi_5(\zeta) = \frac{\sqrt{(s^2 - t^2)f} - s\sqrt{f}\cos(2\sqrt{f}(\zeta + \zeta_0))}{s\sin(2\sqrt{f}(\zeta + \zeta_0)) + t}, \quad (10)$$

$$\varphi_6(\zeta) = \frac{-\sqrt{(s^2 - t^2)f} - s\sqrt{f}\cos(2\sqrt{f}(\zeta + \zeta_0))}{s\sin(2\sqrt{f}(\zeta + \zeta_0)) + t}, \quad (11)$$

$$\varphi_7(\zeta) = i\sqrt{f} + \frac{-2si\sqrt{f}}{s + \cos(2\sqrt{f}(\zeta + \zeta_0)) - i\sin(2\sqrt{f}(\zeta + \zeta_0))}, \quad (12)$$

$$\varphi_8(\zeta) = -i\sqrt{f} + \frac{2si\sqrt{f}}{s + \cos(2\sqrt{f}(\zeta + \zeta_0)) + i\sin(2\sqrt{f}(\zeta + \zeta_0))}. \quad (13)$$

When $f = 0$, the rational solution for Eq. (5) are given as follows:

$$\varphi_9(\zeta) = -\frac{1}{\zeta + \zeta_0}, \quad (14)$$

where p, q, and r in Eqs. (6)–(14) are constants. By substituting Eq. (4) into Eq. (3) with Eq. (5), and combing the each coefficients of $\varphi^k(\zeta)$ $(k = 1, 2, 3, \ldots)$, and each coefficient make equal to zero, then we obtain set of equations. By utilizing any computational software, we can resolve these family of equations and find the values of the unknown parameters. We may ascertain the ideal solutions of Eq. (1) by substituting the values of the unknown parameters and $\varphi(\zeta)$ in Eq. (4).

3 Extraction of solitary wave solutions of governing model

Here, we apply the proposed method on the damped KdV equation for the constructions of solitary wave solutions. The damped KdV equation given as follows:

$$\phi_\tau + \nabla\phi\phi_x + \eth\phi_{xxx} + \mho\phi = 0. \quad (15)$$

Eq. (15) is transformed as follows:

$$\phi(x, \tau) = u(\zeta), \quad \zeta = (\Bbbk x + \wp\tau). \quad (16)$$

By substituting Eq. (10) into Eq. (4), we obtain the following equation as follows:

$$\wp u' + \nabla\wp uu' + \eth u^3 u''' + \mho u = 0. \quad (17)$$

By implementing the homogeneity principle to Eq. (16), we achieve $N = 2$. The trial solution of Eq. (16) is given as follows:

$$u(\zeta) = a_0 + a_1\varphi(\zeta) + a_2\varphi^2(\zeta) + \frac{b_1}{\varphi(\zeta)} + \frac{b_2}{\varphi^2(\zeta)}. \quad (18)$$

By substituting Eq. (17) along with Eq. (5), in Eq. (16) and collecting the every co-efficients of $\varphi^k(\zeta)$, $(i = 1, 2, 3, \ldots)$, then every co-efficient is equal to zero and obtain a families of equations in constant parameters $a_0, a_1, a_2, b_1, b_2, \Bbbk$, and $\wp$. These parameters values in the system of equations are solved by Mathematica software to obtain the following solutions cases.

Family-I

$$a_0 = a_0,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,$$
$$b_2 = -\frac{12\eth f^2\Bbbk^2}{\nabla},\ \wp = -\nabla a_0\Bbbk - 8\eth f\Bbbk^3. \quad (19)$$

By substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1) are obtained as follows:

According to case 1 as $f < 0$ of the new AEM and using the values of the parameters arranged in family-I with Eq. (6) in Eq. (15), then we obtain the hyperbolic solutions for Eq. (9) as follows:

$$\phi_1(x, \tau) = a_0 - \frac{12\eth f^2\Bbbk^2(s\sinh(2\sqrt{-f}(\zeta + \zeta_0)) + t)^2}{\nabla(\sqrt{-f(s^2 + t^2)} - \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta + \zeta_0)))^2}, \quad (20)$$

$$\phi_2(x, \tau) = a_0 - \frac{12\eth f^2\Bbbk^2(s\sinh(2\sqrt{-f}(\zeta + \zeta_0)) + t)^2}{\nabla(\sqrt{-f(s^2 + t^2)} + \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta + \zeta_0)))^2}, \quad (21)$$

$$\phi_3(x, \tau) = a_0 + \frac{12\eth f\Bbbk^2}{\nabla\left(1 - \frac{2s}{-\sinh(2\sqrt{-f}(\zeta + \zeta_0)) + \cosh(2\sqrt{-f}(\zeta + \zeta_0)) + s}\right)^2}, \quad (22)$$

$$\phi_4(x, \tau) = a_0 + \frac{12\eth f\Bbbk^2}{\nabla\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta + \zeta_0)) + \cosh(2\sqrt{-f}(\zeta + \zeta_0)) + s} + 1\right)^2}, \quad (23)$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-I with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_5(x,\tau) = a_0 - \frac{12\eth f^2\Bbbk^2(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{\nabla(\sqrt{f(s^2-t^2)}-\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}, \tag{24}$$

$$\phi_6(x,\tau) = a_0 - \frac{12\eth f^2\Bbbk^2(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{\nabla(\sqrt{f(s^2-t^2)}+\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}, \tag{25}$$

$$\phi_7(x,\tau) = a_0 + \frac{12\eth f\Bbbk^2}{\nabla\left(1-\frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}, \tag{26}$$

$$\phi_8(x,\tau) = a_0 + \frac{12\eth f\Bbbk^2}{\nabla\left(1-\frac{2s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}. \tag{27}$$

where $\zeta = \Bbbk x + \wp\tau$.

According to case 3 $f = 0$ and using family-I with the help of Eq. (7) in Eq. (15) and obtain the rational solution for Eq. (1) as follows:

$$\phi_9(x,\tau) = a_0 + (\zeta+\zeta_0)^2, \tag{28}$$

where $\zeta = \Bbbk x + \wp\tau$.

Family-II

$$a_0 = \frac{-8\eth f\Bbbk^3-\wp}{\nabla\Bbbk},\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\quad b_2 = -\frac{12\eth f^2\Bbbk^2}{\nabla}. \tag{29}$$

By substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1) is obtained as follows:

By according to case 1 as $f < 0$ of the new AEM and using the values of the parameters arranged in family-II with Eq. (6) in Eq. (15), we obtain the hyperbolic solutions for Eq. (9) as follows:

$$\phi_{10}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{(\sqrt{-f(s^2+t^2)}-\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\wp}{\nabla\Bbbk}, \tag{30}$$

$$\phi_{11}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{(\sqrt{-f(s^2+t^2)}+\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\wp}{\nabla\Bbbk}, \tag{31}$$

$$\phi_{12}(x,\tau) = \frac{4\eth f\Bbbk^3\left(\frac{3}{\left(1-\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2}-2\right)-\wp}{\nabla\Bbbk}, \tag{32}$$

$$\phi_{13}(x,\tau) = \frac{4\eth f\Bbbk^3\left(\frac{3}{\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}+1\right)^2}-2\right)-\wp}{\nabla\Bbbk}, \tag{33}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-II with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{14}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{(\sqrt{f(s^2-t^2)}-\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\wp}{\nabla\Bbbk}, \tag{34}$$

$$\phi_{15}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{(\sqrt{f(s^2-t^2)}+\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\wp}{\nabla\Bbbk}, \tag{35}$$

$$\phi_{16}(x,\tau) = \frac{-\wp+4\eth f\Bbbk^3\left(-2+\frac{3}{\left(1-\frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}\right)}{\nabla\Bbbk}, \tag{36}$$

$$\phi_{17}(x,\tau) = \frac{-\wp+4\eth f\Bbbk^3\left(-2+\frac{3}{\left(1-\frac{2s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}\right)}{\nabla\Bbbk}, \tag{37}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f = 0$ and using family-II with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{18}(x,\tau) = (\zeta+\zeta_0)^2 - \frac{8\eth f\Bbbk^3+\wp}{\nabla\Bbbk}, \tag{38}$$

where $\zeta = \Bbbk x + \wp\tau$.

Family-III

$$a_0 = a_0,\ a_1 = 0,\ a_2 = -\frac{12\eth\Bbbk^2}{\nabla},\ b_1 = 0,\quad b_2 = 0,\ \wp = -\nabla a_0\Bbbk - 8\eth f\Bbbk^3. \tag{39}$$

By substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1) is obtained as follows:

By according to case 1 $f < 0$ and using the values of the parameters arranged in family-III with Eq. (6) in Eq. (15), we obtain the hyperbolic solutions for Eq. (9) as follows:

$$\phi_{19}(x,\tau) = a_0 - \frac{12ð\Bbbk^2(\sqrt{-f(s^2+t^2)} - \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{\nabla(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}, \quad (40)$$

$$\phi_{20}(x,\tau) = a_0 - \frac{12ð\Bbbk^2(\sqrt{-f(s^2+t^2)} + \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{\nabla(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}, \quad (41)$$

$$\phi_{21}(x,\tau) = a_0 + \frac{12ðf\left(\Bbbk - \frac{2\Bbbk s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2}{\nabla}, \quad (42)$$

$$\phi_{22}(x,\tau) = a_0 + \frac{12ðf\left(\frac{2\Bbbk s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s} + \Bbbk\right)^2}{\nabla}, \quad (43)$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-III with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{23}(x,\tau) = a_0 - \frac{12ð\Bbbk^2(\sqrt{f(s^2-t^2)} - \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{\nabla(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}, \quad (44)$$

$$\phi_{24}(x,\tau) = a_0 - \frac{12ð\Bbbk^2(\sqrt{f(s^2-t^2)} + \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{\nabla(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}, \quad (45)$$

$$\phi_{25}(x,\tau) = a_0 + \frac{12ðf\left(\Bbbk - \frac{2\Bbbk s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}{\nabla}, \quad (46)$$

$$\phi_{26}(x,\tau) = a_0 + \frac{12ðf\left(\Bbbk - \frac{2\Bbbk s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}{\nabla}, \quad (47)$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f = 0$ and using family-II with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{27}(x,\tau) = a_0 - \frac{12ð\Bbbk^2}{\nabla(\zeta+\zeta_0)^2}, \quad (48)$$

where $\zeta = \Bbbk x + \wp\tau$.

Family-IV

$$a_0 = \frac{-8ðf\Bbbk^3 - \wp}{\nabla\Bbbk}, a_1 = 0, a_2 = -\frac{12ð\Bbbk^2}{\nabla}, b_1 = 0, b_2 = 0. \quad (49)$$

Substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1), obtained as follows:

According to case 1 $f < 0$ and using the values of the parameters arranged in family-IV with Eq. (6) in Eq. (15), we obtain the hyperbolic solutions for Eq. (9) as follows:

$$\phi_{28}(x,\tau) = -\frac{8ðf\Bbbk^3 + \frac{12ð\Bbbk^3(\sqrt{-f(s^2+t^2)} - \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2} + \wp}{\nabla\Bbbk}, \quad (50)$$

$$\phi_{29}(x,\tau) = -\frac{8ðf\Bbbk^3 + \frac{12ð\Bbbk^3(\sqrt{-f(s^2+t^2)} + \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2} + \wp}{\nabla\Bbbk}, \quad (51)$$

$$\phi_{30}(x,\tau) = -\frac{8ðf\Bbbk^3 - 12ðf\Bbbk^3\left(1 - \frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2 + \wp}{\nabla\Bbbk}, \quad (52)$$

$$\phi_{31}(x,\tau) = -\frac{8ðf\Bbbk^3 - 12ðf\Bbbk^3\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s} + 1\right)^2 + \wp}{\nabla\Bbbk}, \quad (53)$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-IV with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{32}(x,\tau) = -\frac{8ðf\Bbbk^3 + \frac{12ð\Bbbk^3(\sqrt{f(s^2-t^2)} - \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2} + \wp}{\nabla\Bbbk}, \quad (54)$$

$$\phi_{33}(x,\tau) = -\frac{8ðf\Bbbk^3 + \frac{12ð\Bbbk^3(\sqrt{f(s^2-t^2)} + \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2} + \wp}{\nabla\Bbbk}, \quad (55)$$

$$\phi_{34}(x,\tau) = -\frac{8ðf\Bbbk^3 - 12ðf\Bbbk^3\left(1 - \frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2 + \wp}{\nabla\Bbbk}, \quad (56)$$

$$\phi_{35}(x,\tau) = -\frac{8ðf\Bbbk^3 - 12ðf\Bbbk^3\left(1 - \frac{2s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2 + \wp}{\nabla\Bbbk}. \quad (57)$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f = 0$ and using family-IV with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{36}(x,\tau) = -\frac{12\eth\Bbbk^2}{\nabla(\zeta+\zeta_0)^2}, \tag{58}$$

where $\zeta = \Bbbk x + \wp\tau$.

Family-V

$$a_0 = \frac{-8\eth f\Bbbk^3 - \wp}{\nabla\Bbbk},\ a_1 = 0,\ a_2 = -\frac{12\eth\Bbbk^2}{\nabla},\ b_1 = 0,\ b_2 = 0. \tag{59}$$

By substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1) are obtained below as:

According to case 1 $f < 0$ and using the values of the parameters arranged in family-V with Eq. (6) in Eq. (15), we obtain the hyperbolic solutions for Eq. (9) as:

$$\begin{aligned}\phi_{37}(x,\tau) &= -\frac{12\eth\Bbbk^2\left(\sqrt{-f(s^2+t^2)} - \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0))\right)^2}{\nabla(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}\\ &+ a_0 - \frac{12\eth f^2\Bbbk^2(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{\nabla\left(\sqrt{-f(s^2+t^2)} - \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0))\right)^2},\end{aligned} \tag{60}$$

$$\begin{aligned}\phi_{38}(x,\tau) &= -\frac{12\eth\Bbbk^2\left(\sqrt{-f(s^2+t^2)} + \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0))\right)^2}{\nabla(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}\\ &+ a_0 - \frac{12\eth f^2\Bbbk^2(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{\nabla\left(\sqrt{-f(s^2+t^2)} + \sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0))\right)^2},\end{aligned} \tag{61}$$

$$\begin{aligned}\phi_{39}(x,\tau) &= \frac{12\eth f\Bbbk^2}{\nabla\left(1 - \frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2}\\ &+ a_0 + \frac{12\eth f\left(\Bbbk - \frac{2\Bbbk s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2}{\nabla},\end{aligned} \tag{62}$$

$$\begin{aligned}\phi_{40}(x,\tau) &= \frac{12\eth f\Bbbk^2}{\nabla\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s} + 1\right)^2}\\ &+ a_0 + \frac{12\eth f\left(\frac{2\Bbbk s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s} + \Bbbk\right)^2}{\nabla},\end{aligned} \tag{63}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-V with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\begin{aligned}\phi_{41}(x,\tau) &= -\frac{12\eth\Bbbk^2\left(\sqrt{f(s^2-t^2)} - \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0))\right)^2}{\nabla(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}\\ &+ a_0 - \frac{12\eth f^2\Bbbk^2(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{\nabla\left(\sqrt{f(s^2-t^2)} - \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0))\right)^2},\end{aligned} \tag{64}$$

$$\begin{aligned}\phi_{42}(x,\tau) &= -\frac{12\eth\Bbbk^2\left(\sqrt{f(s^2-t^2)} + \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0))\right)^2}{\nabla(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}\\ &+ a_0 - \frac{12\eth f^2\Bbbk^2(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{\nabla\left(\sqrt{f(s^2-t^2)} + \sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0))\right)^2},\end{aligned} \tag{65}$$

$$\begin{aligned}\phi_{43}(x,\tau) &= \frac{12\eth f\Bbbk^2}{\nabla\left(1 - \frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}\\ &+ a_0 + \frac{12\eth f\left(\Bbbk - \frac{2\Bbbk s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}{\nabla},\end{aligned} \tag{66}$$

$$\begin{aligned}\phi_{44}(x,\tau) &= \frac{12\eth f\Bbbk^2}{\nabla\left(1 - \frac{2s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}\\ &+ a_0 + \frac{12\eth f\left(\Bbbk - \frac{2\Bbbk s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}{\nabla},\end{aligned} \tag{67}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f = 0$ and using family-V with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{45}(x,\tau) = a_0 - \frac{12\eth\Bbbk^2}{\nabla(\zeta+\zeta_0)^2}, \tag{68}$$

where $\zeta = \Bbbk x + \wp\tau$.

Family-VI

$$\begin{aligned}&a_0 = \frac{-8\eth f\Bbbk^3 - \wp}{\nabla\Bbbk},\ a_1 = 0,\ a_2 = -\frac{12\eth\Bbbk^2}{\nabla},\\ &b_1 = 0,\ b_2 = -\frac{12\eth f^2\Bbbk^2}{\nabla}.\end{aligned} \tag{69}$$

By substituting Eq. (10) into Eq. (9), solitary wave solutions for Eq. (1), obtained as follows:

According to case 1 $f < 0$ and using the values of the parameters arranged in family-VI with Eq. (6) in Eq. (15), we obtain the hyperbolic solutions for Eq. (9) as follows:

$$\begin{aligned}\phi_{46}(x,\tau) = &-\frac{12\nabla\eth f^2\Bbbk^4(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{(\sqrt{-f(s^2+t^2)}-\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}-8\nabla\eth f\Bbbk^4\\&-\frac{12\nabla\eth\Bbbk^4(\sqrt{-f(s^2+t^2)}-\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}-\nabla\Bbbk\wp,\end{aligned} \tag{70}$$

$$\begin{aligned}\phi_{47}(x,\tau) = &-8\nabla\eth f\Bbbk^4-\frac{12\nabla\eth\Bbbk^4(\sqrt{-f(s^2+t^2)}+\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2}{(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}-\nabla\Bbbk\wp\\&-\frac{12\nabla\eth f^2\Bbbk^4(s\sinh(2\sqrt{-f}(\zeta+\zeta_0))+t)^2}{(\sqrt{-f(s^2+t^2)}+\sqrt{-f}s\cosh(2\sqrt{-f}(\zeta+\zeta_0)))^2},\end{aligned} \tag{71}$$

$$\begin{aligned}\phi_{48}(x,\tau) = &-8\nabla\eth f\Bbbk^4-\nabla\Bbbk\wp+\frac{12\nabla\eth f\Bbbk^4}{\left(1-\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2}\\&+12\eth f\Bbbk^3\left(1-\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}\right)^2,\end{aligned} \tag{72}$$

$$\begin{aligned}\phi_{49}(x,\tau) = &-8\nabla\eth f\Bbbk^4+\frac{12\nabla\eth f\Bbbk^4}{\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}+1\right)^2}-\nabla\Bbbk\wp\\&+12\nabla\eth f\Bbbk^4\left(\frac{2s}{-\sinh(2\sqrt{-f}(\zeta+\zeta_0))+\cosh(2\sqrt{-f}(\zeta+\zeta_0))+s}+1\right)^2,\end{aligned} \tag{73}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f > 0$ and using family-VI with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

$$\phi_{50}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{(\sqrt{f(s^2-t^2)}-\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\frac{12\eth\Bbbk^3(\sqrt{f(s^2-t^2)}-\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}+\wp}{\nabla\Bbbk}, \tag{74}$$

$$\phi_{51}(x,\tau) = -\frac{\frac{12\eth f^2\Bbbk^3(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}{(\sqrt{f(s^2-t^2)}+\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}+8\eth f\Bbbk^3+\frac{12\eth\Bbbk^3(\sqrt{f(s^2-t^2)}+\sqrt{f}s\cos(2\sqrt{f}(\zeta+\zeta_0)))^2}{(s\sin(2\sqrt{f}(\zeta+\zeta_0))+t)^2}+\wp}{\nabla\Bbbk}, \tag{75}$$

$$\begin{aligned}\phi_{52}(x,\tau) = &-8\nabla\eth f\Bbbk^4+12\nabla\eth f\Bbbk^4\left(1-\frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2\\&+\frac{12\nabla\eth f\Bbbk^4}{\left(1-\frac{2s}{-i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}-\wp\nabla\Bbbk\end{aligned} \tag{76}$$

$$\begin{aligned}\phi_{53}(x,\tau) = &-8\nabla\eth f\Bbbk^4+\frac{12\nabla\eth f\Bbbk^4}{\left(1-\frac{2p}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2}-\nabla\Bbbk\wp\\&+12\nabla\eth f\Bbbk^4\left(1-\frac{2s}{i\sin(2\sqrt{f}(\zeta+\zeta_0))+\cos(2\sqrt{f}(\zeta+\zeta_0))+s}\right)^2,\end{aligned} \tag{77}$$

where $\zeta = \Bbbk x + \wp\tau$.

We consider case 2 $f = 0$ and using family-VI with the help of Eq. (7) in Eq. (15) and obtain the trigonometric solutions of Eq. (1) as follows:

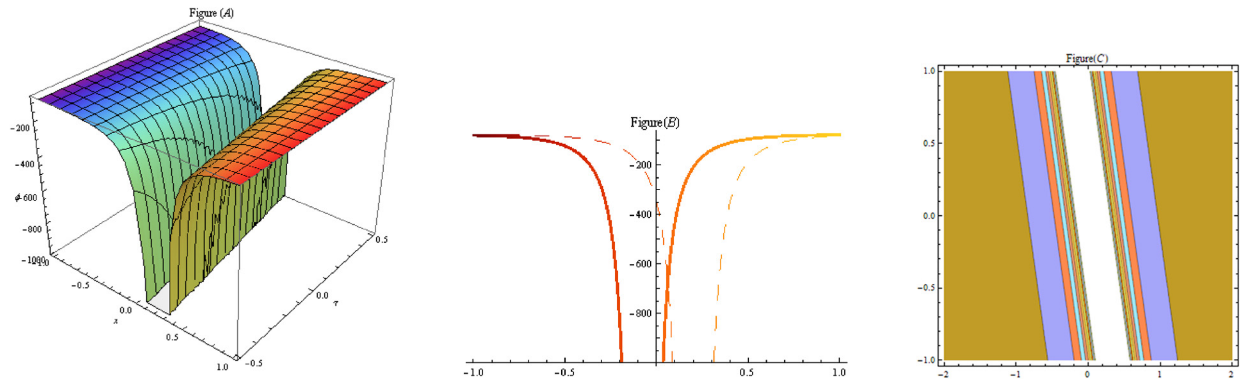

Figure 1: The dark soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f=-2$, $s=0.5$, $t=0.5$, $\zeta_0=0.2$, $\wp=0.5$, $\Bbbk=1.8$, $\nabla=1$, $\eth=1$, $a_0=1$.

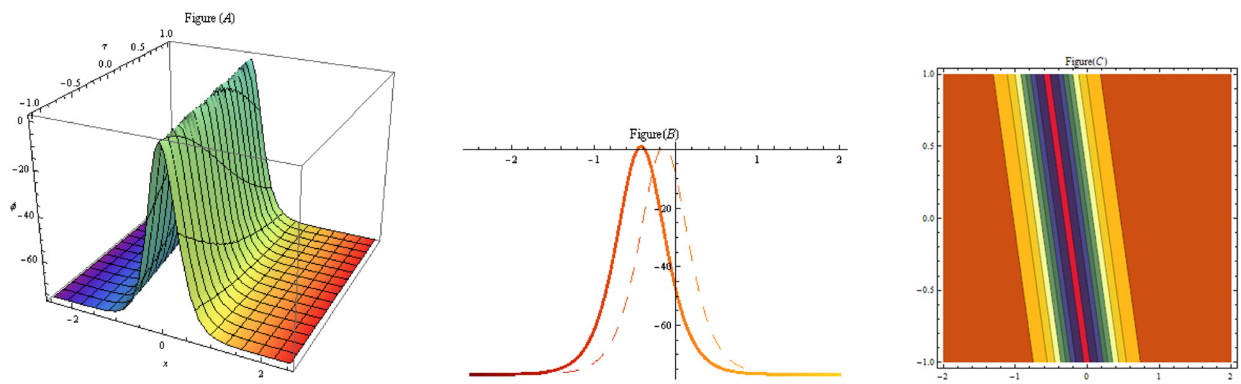

Figure 2: The peakon bright soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f=-2$, $s=0.5$, $t=0.5$, $\zeta_0=0.2$, $\wp=0.5$, $\Bbbk=1.8$, $\nabla=1$, $\eth=1$, $a_0=1$.

$$\phi_{54}(x,\tau)=-\frac{12\eth\Bbbk^2}{\nabla(\zeta+\zeta_0)^2}, \tag{78}$$

where $\zeta=\Bbbk x+\wp\tau$.

4 Discussion

In this section, we compare and discuss the findings we obtained with those already discovered in the literature for

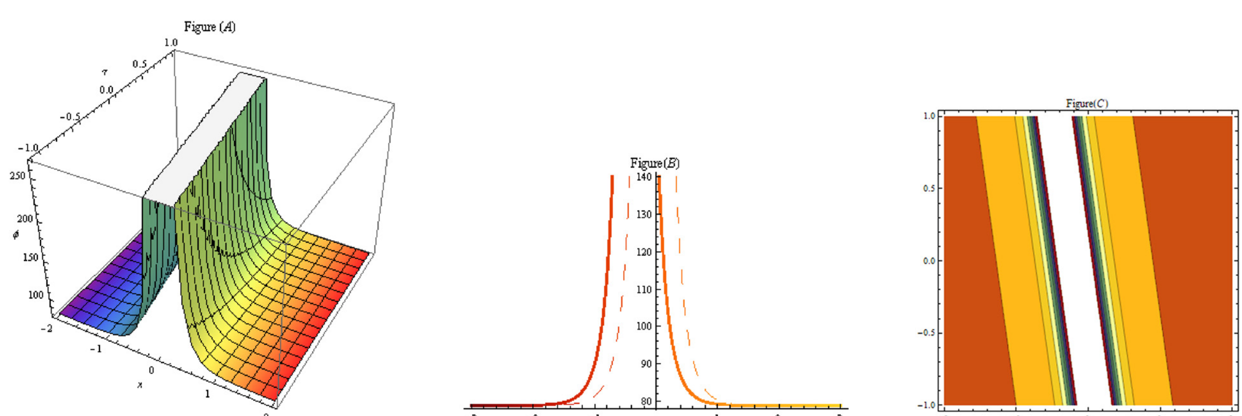

Figure 3: The bright soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f=-2$, $s=0.5$, $t=0.5$, $\zeta_0=0.2$, $\wp=0.5$, $\Bbbk=1.8$, $\nabla=-1$, $\eth=1$, $a_0=1$.

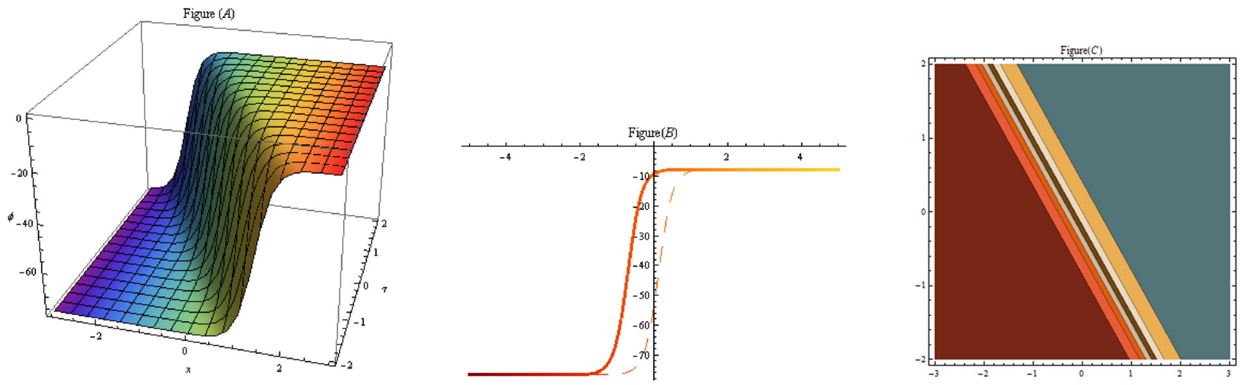

Figure 4: The anti-kink wave soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 1.5$, $\Bbbk = 1.8$, $\nabla = 1$, $\eth = 1$, $a_0 = 1$.

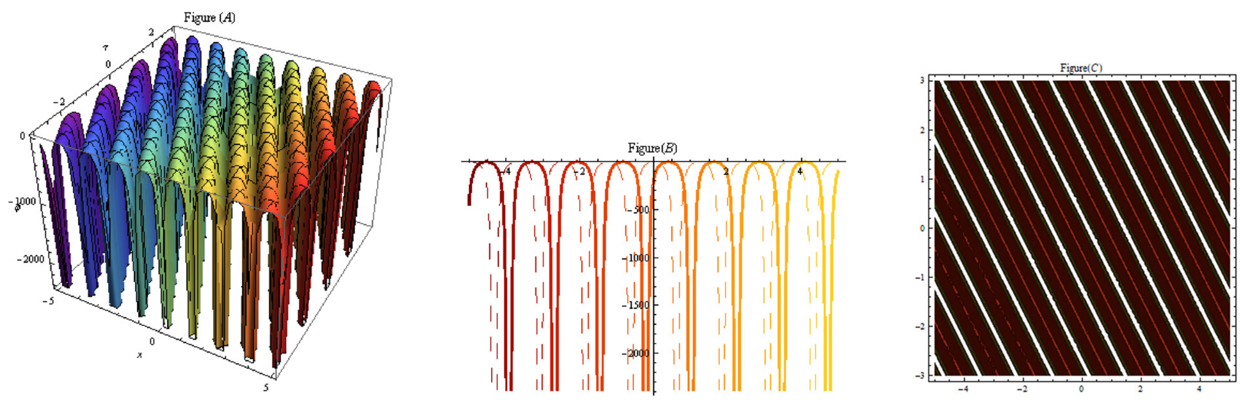

Figure 5: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 1.5$, $\Bbbk = 1.8$, $\nabla = 1$, $\eth = 1$, $a_0 = 1$.

which they used different methodologies. We have created novel and more general solitary wave solutions for the studied problem in the current study. The generic solution of Eq. (4) with a range of two constant parameters that can be interpreted in many ways is the key contribution in this work. By suggesting the symbolic calculation, the values of the parameters a_k and b_k are merged. There are several solutions to Eq. (5) that correspond to trigonometric,

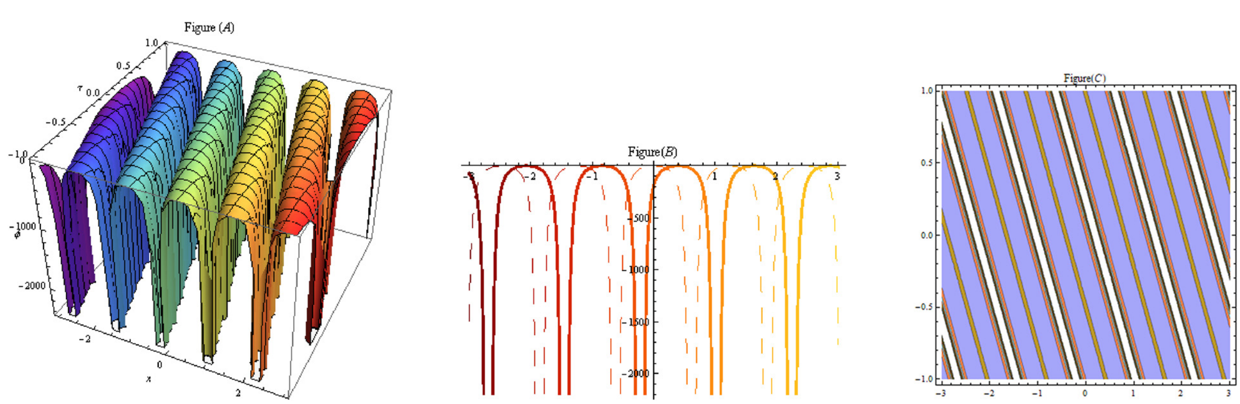

Figure 6: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 1.5$, $\Bbbk = 1.8$, $\nabla = 1$, $\eth = 1$, $a_0 = 1$.

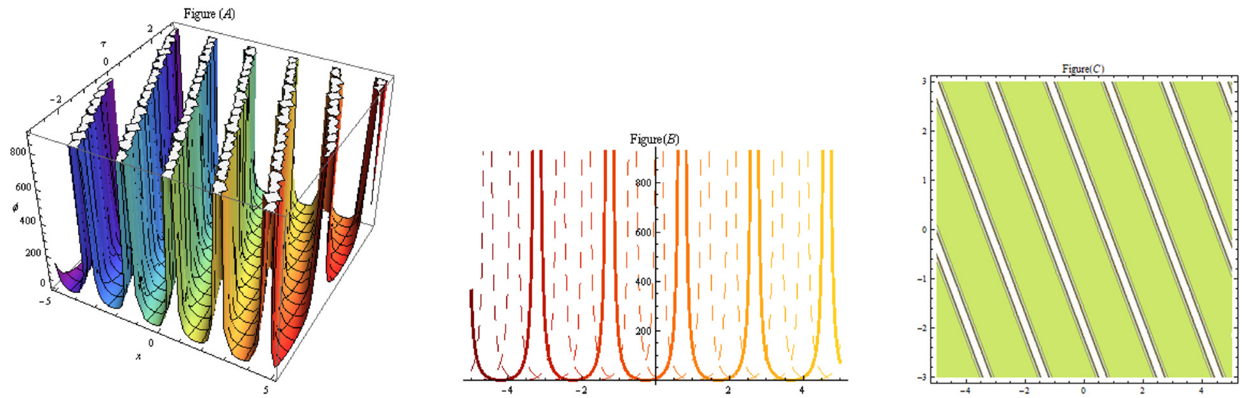

Figure 7: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) $f = 4, s = 1, t = 1, \zeta_0 = 1, \wp = 0.5, \Bbbk = 0.8, \nabla = -1, \eth = 1, a_0 = 1$.

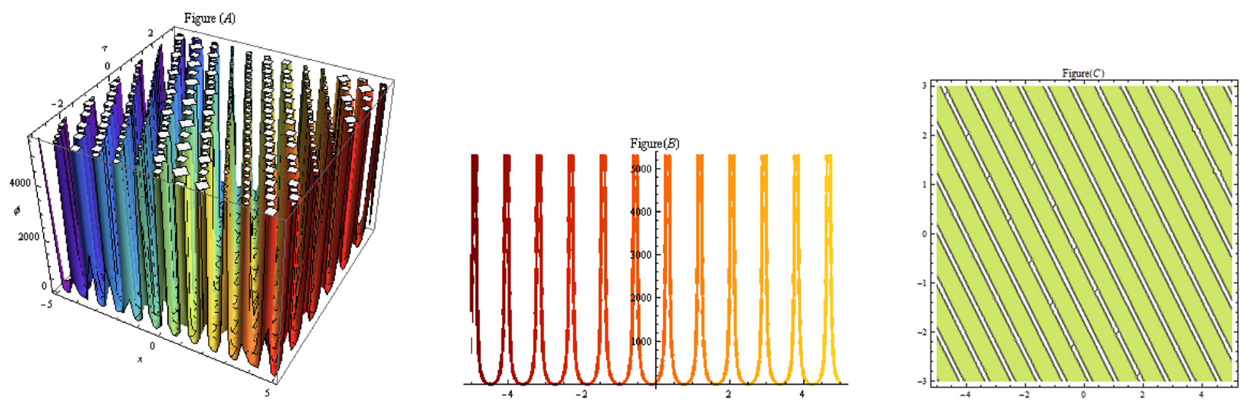

Figure 8: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4, s = 1, t = 1, \zeta_0 = 1, \wp = 1.5, \Bbbk = 1.8, \nabla = -1, \eth = 1, a_0 = 1$.

hyperbolic, and rational functions. Our novel results which provide solitary wave solution is achieved with the aid of our potent and efficient approach. We now compared the outcomes we got with those of other methods.

By implementing the extended and modified mathematical methods, the researchers discovered that the findings were in the nature of solitary waves, and the motion of the ascertain results was quasi–periodic and chaotic [1,2]. However, our ascertain findings have multiple scientific interpretations, which include dark solitons, bright solitons, periodic solitons, kink and anti-kink solitons, peakon bright and dark solitons in (Figures 1–26).

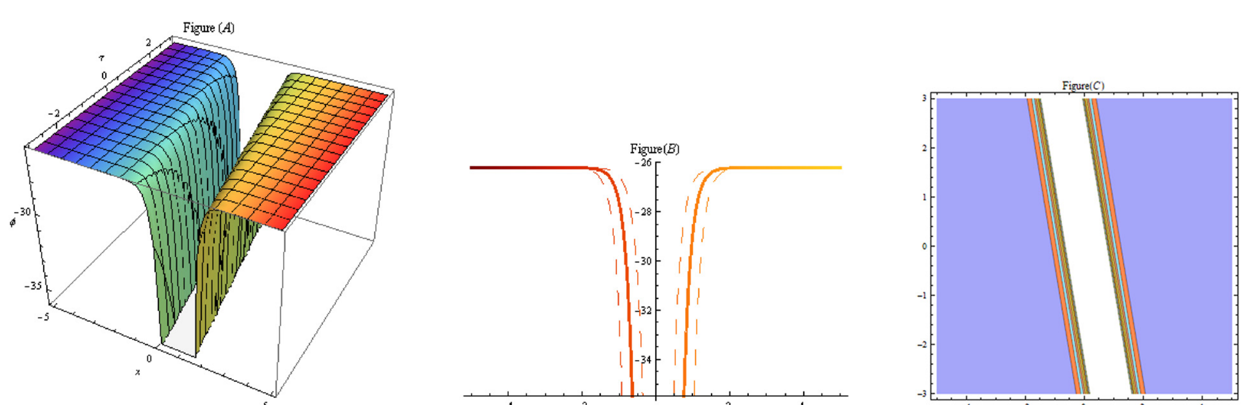

Figure 9: The dark soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2, s = 0.5, t = 0.5, \zeta_0 = 0.2, \wp = 0.5, \Bbbk = 1.8, \nabla = 1, \eth = 1$.

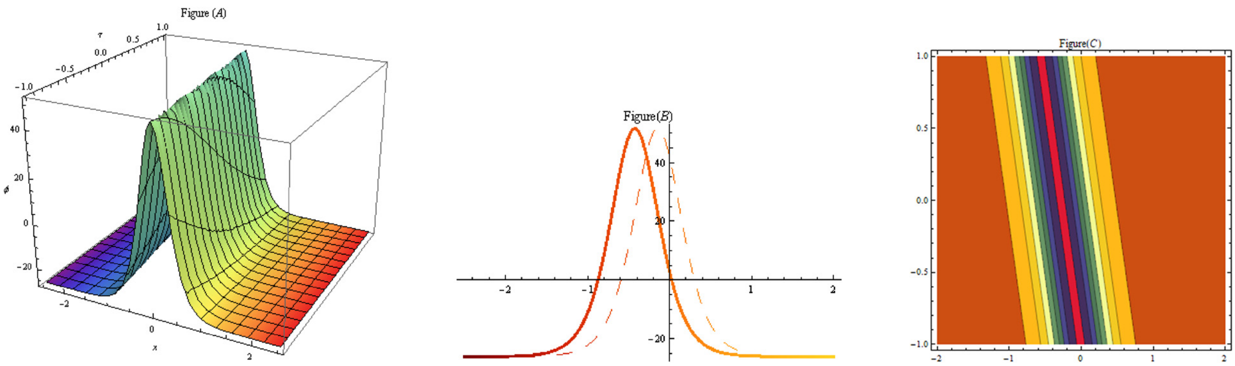

Figure 10: The peakon bright soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $\eth = 1$.

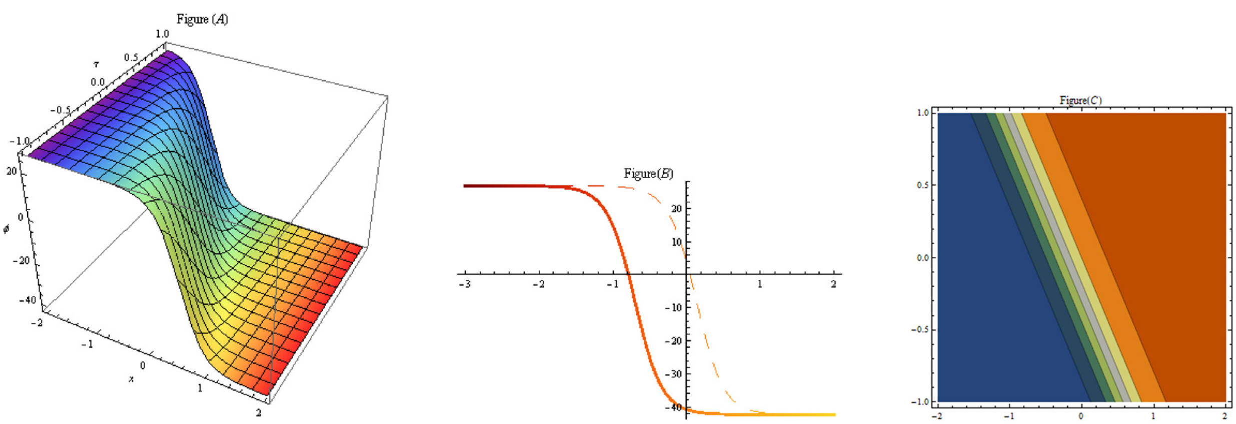

Figure 11: The kink wave soliton solution represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 1.5$, $\Bbbk = 1.8$, $\nabla = -1$, $\eth = 1$.

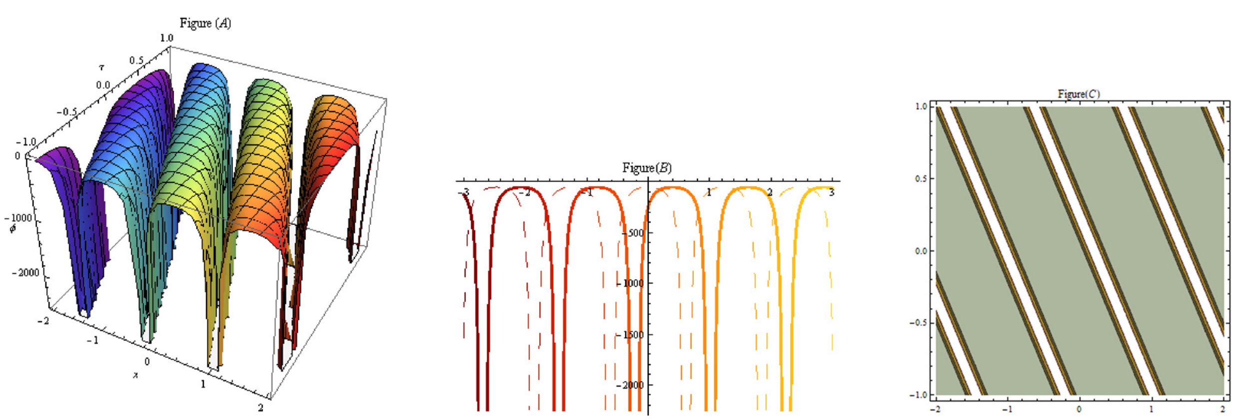

Figure 12: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 1.5$, $\Bbbk = 1.8$, $\nabla = 1$, $\eth = 1$.

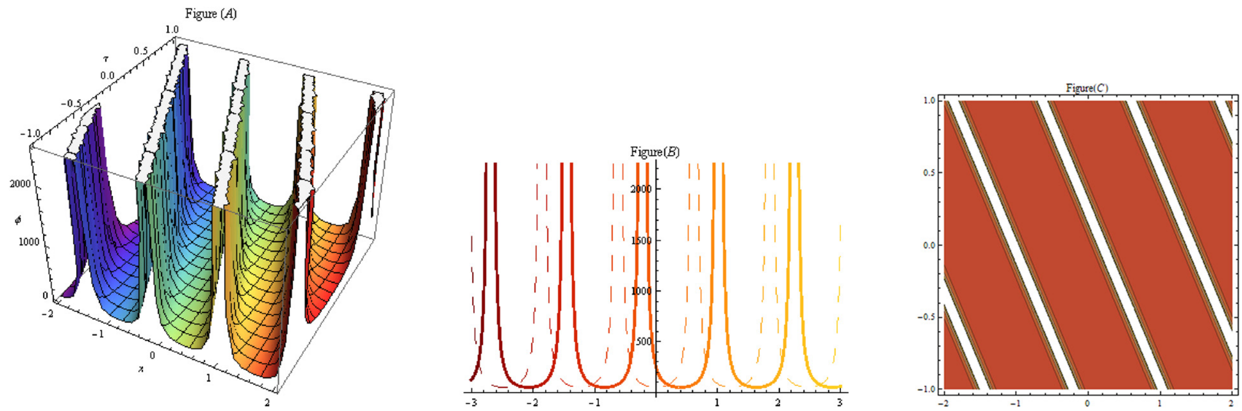

Figure 13: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2, s = 0.5, t = 0.5, \zeta_0 = 0.2, \wp = 1.5, \Bbbk = 1.8, \nabla = -1, \eth = 1$.

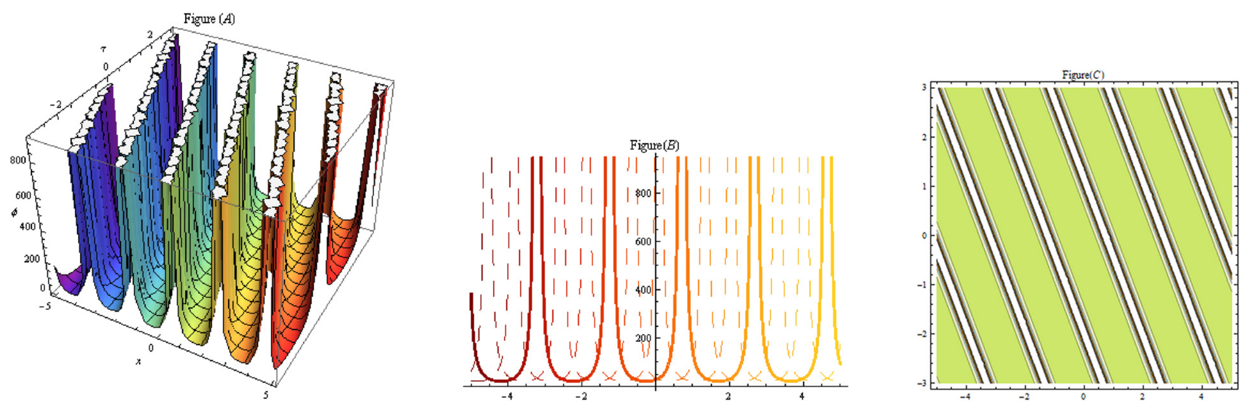

Figure 14: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4, s = 1, t = 1, \zeta_0 = 1, \wp = 0.5, \Bbbk = 0.8, \nabla = -1, \eth = 1$.

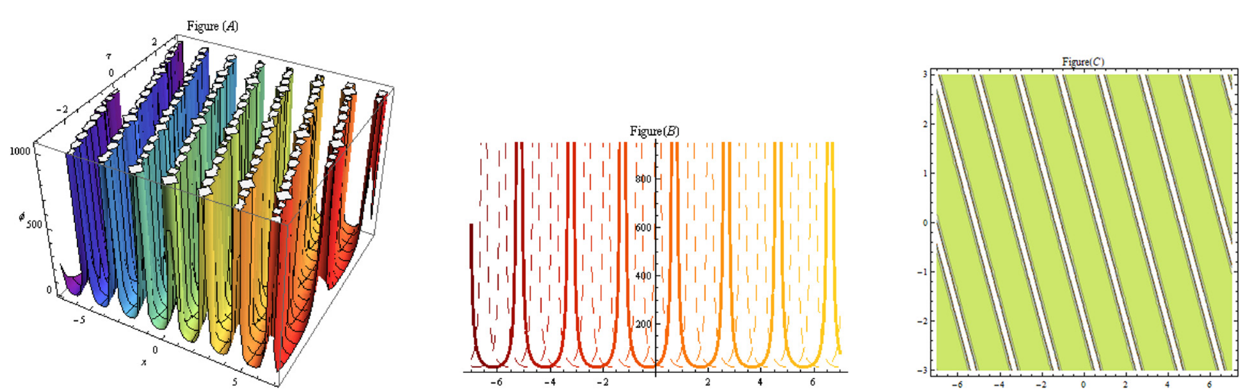

Figure 15: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4, s = 1, t = 1, \zeta_0 = 1, \wp = 0.5, \Bbbk = 0.8, \nabla = -1, \eth = 1$.

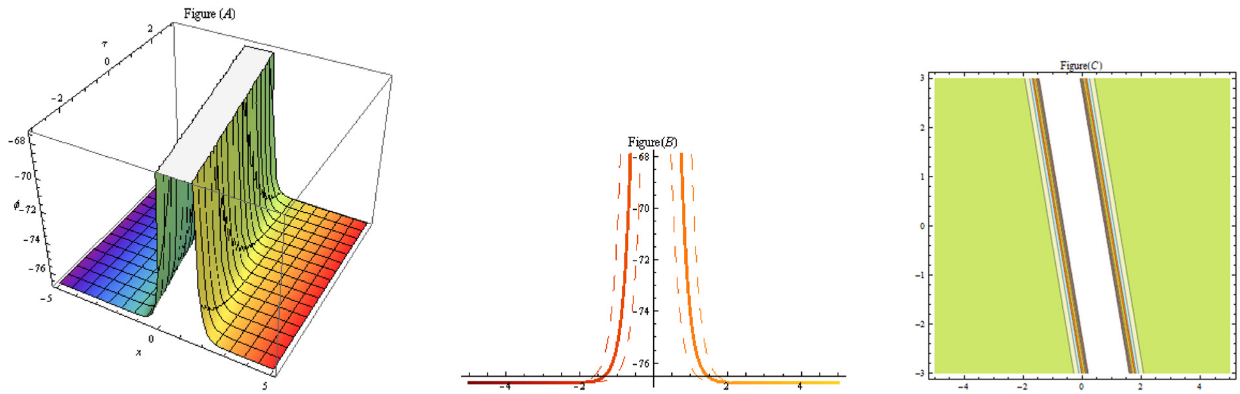

Figure 16: The bright soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = 1$, $a_0 = 1$.

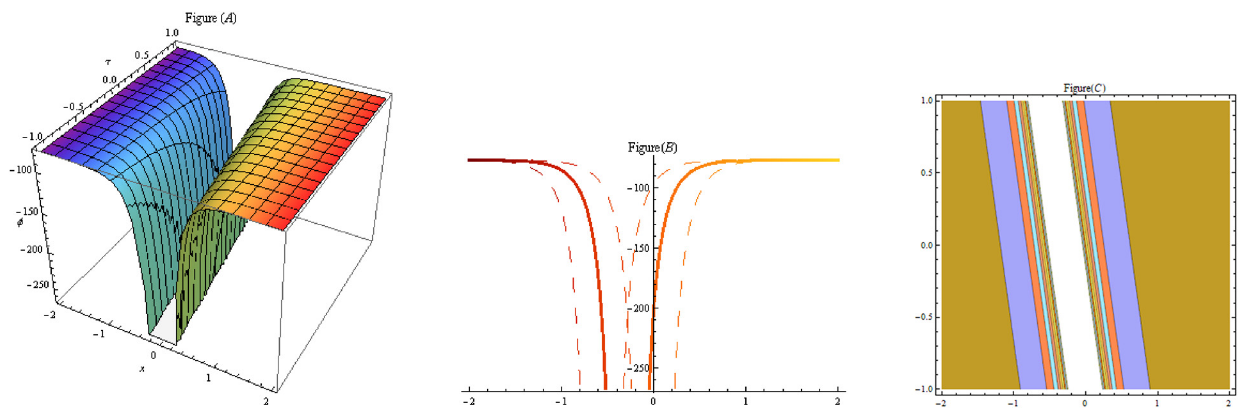

Figure 17: The dark soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = 1$, $a_0 = 1$.

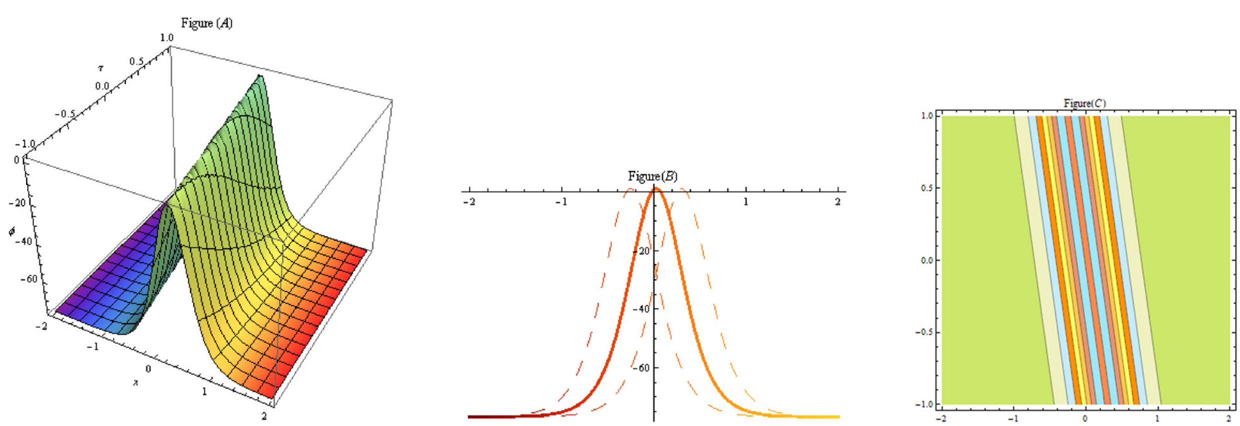

Figure 18: The peakon bright soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = 1$, $a_0 = 1$.

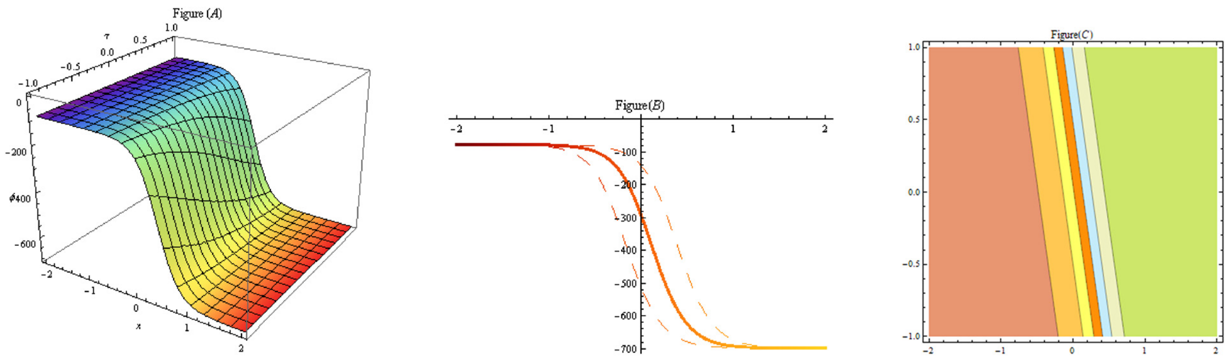

Figure 19: The kink wave soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = 1$, $a_0 = 1$.

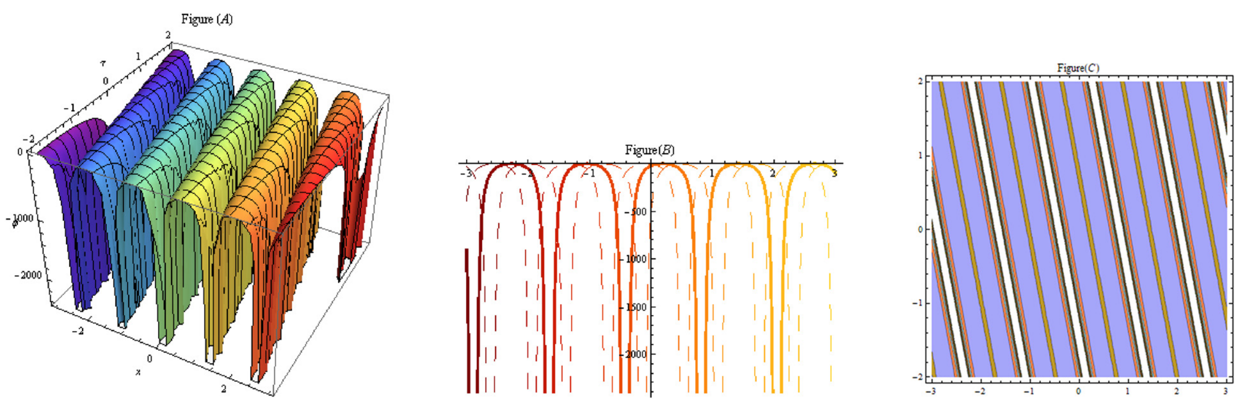

Figure 20: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = 1$, $a_0 = 1$.

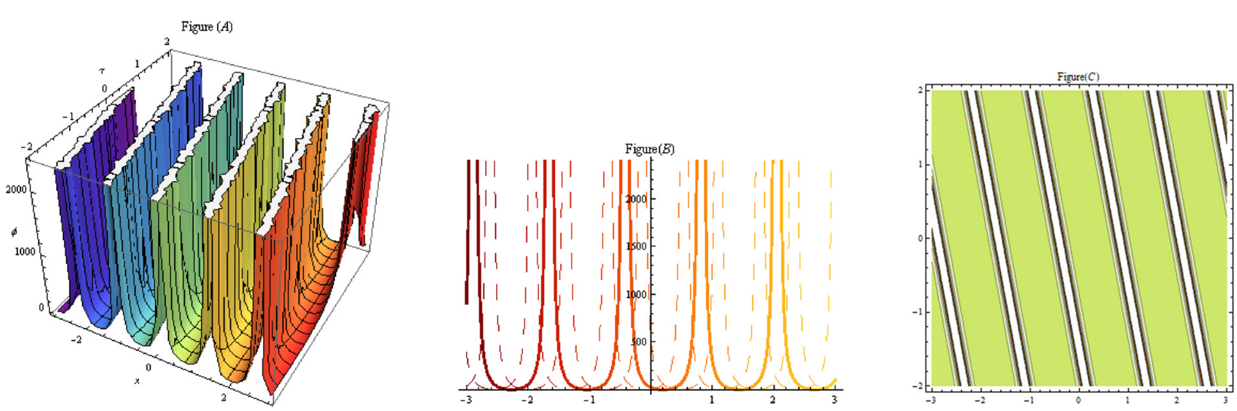

Figure 21: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 0.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = -1$, $a_0 = 1$.

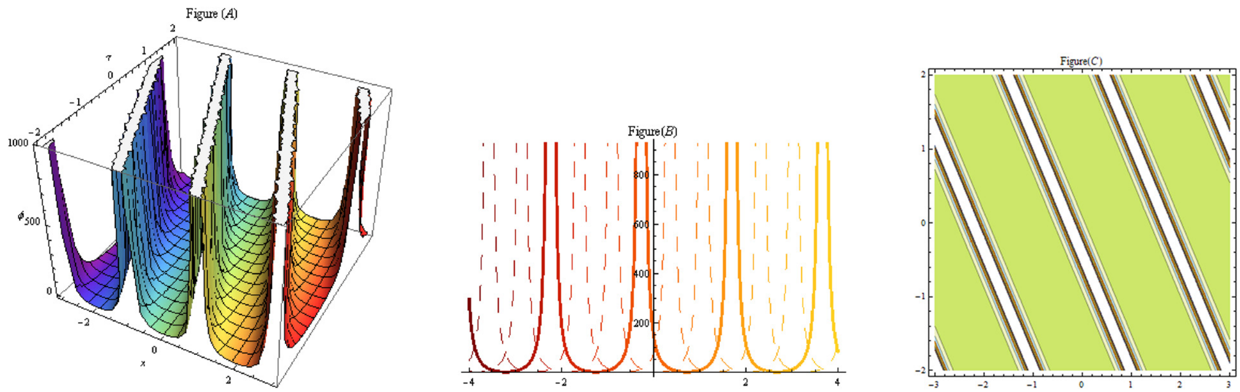

Figure 22: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4$, $s = 1$, $t = 1$, $\zeta_0 = 1$, $\wp = 0.5$, $\Bbbk = 0.8$, $\nabla = -1$, $ð = 1$, $a_0 = 1$.

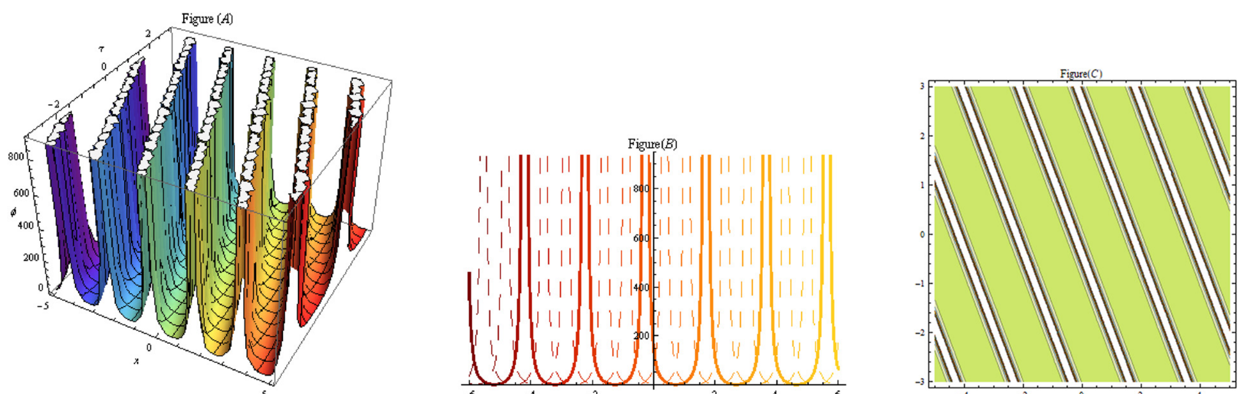

Figure 23: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4$, $s = 1$, $t = 1$, $\zeta_0 = 1$, $\wp = 0.5$, $\Bbbk = 0.8$, $\nabla = -1$, $ð = 1$, $a_0 = 1$.

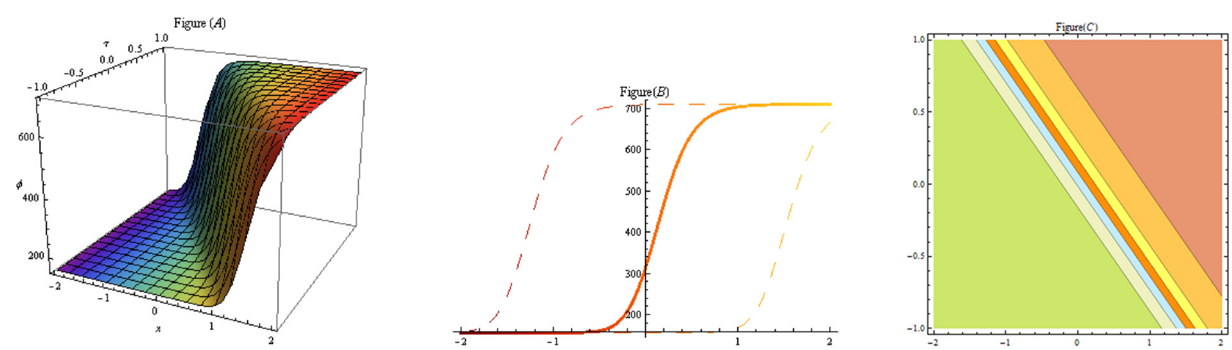

Figure 24: The anti-kink wave soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = -2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 2.5$, $\Bbbk = 1.8$, $\nabla = 1$, $ð = -1$, $a_0 = 1$.

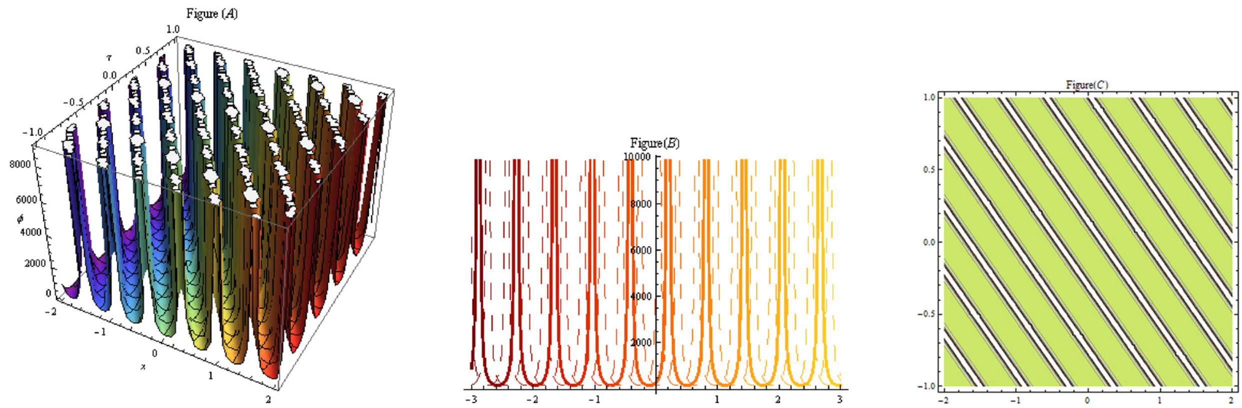

Figure 25: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 2.5$, $\Bbbk = 1.8$, $\nabla = 1$, ð $= -1$, $a_0 = 1$.

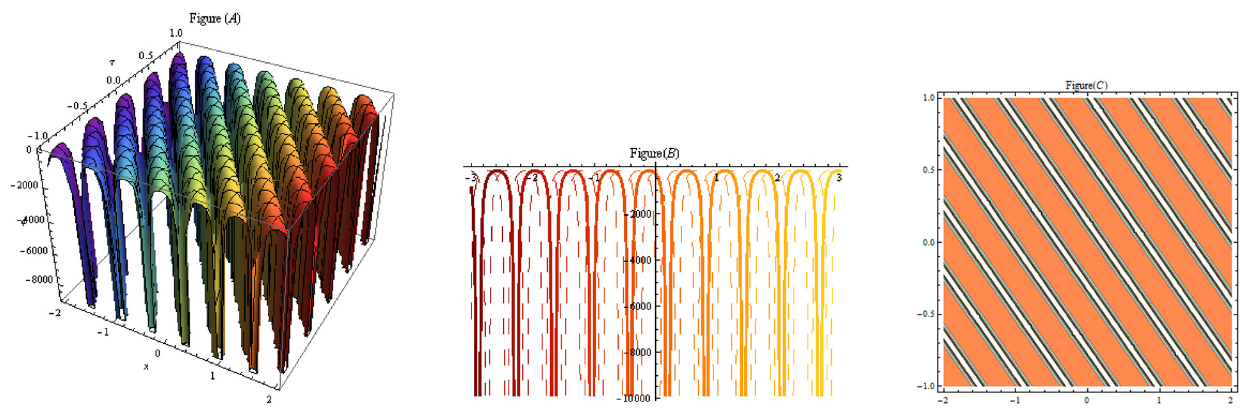

Figure 26: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 2$, $s = 0.5$, $t = 0.5$, $\zeta_0 = 0.2$, $\wp = 2.5$, $\Bbbk = 1.8$, $\nabla = 1$, ð $= 1$, $a_0 = 1$.

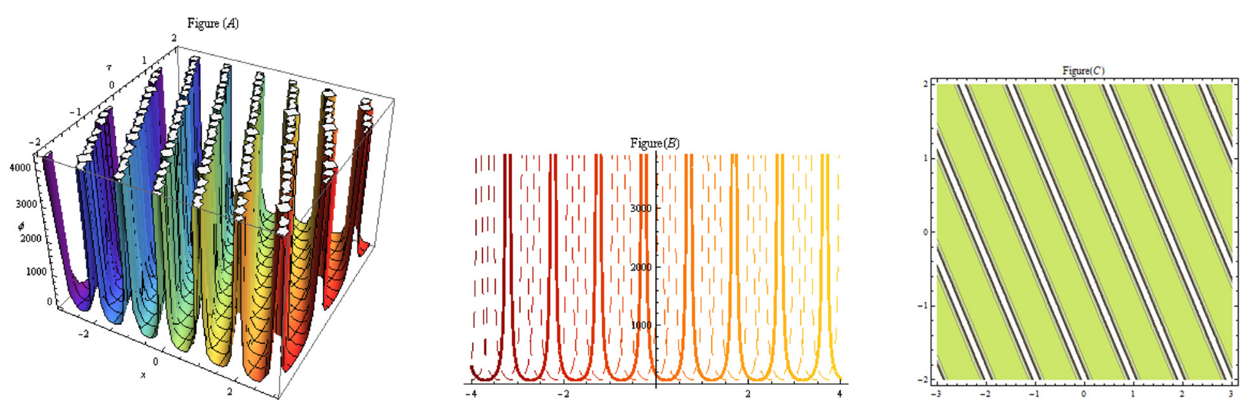

Figure 27: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4$, $s = 1$, $t = 1$, $\zeta_0 = 1$, $\wp = 0.5$, $\Bbbk = 0.8$, $\nabla = -1$, ð $= 1$, $a_0 = 1$.

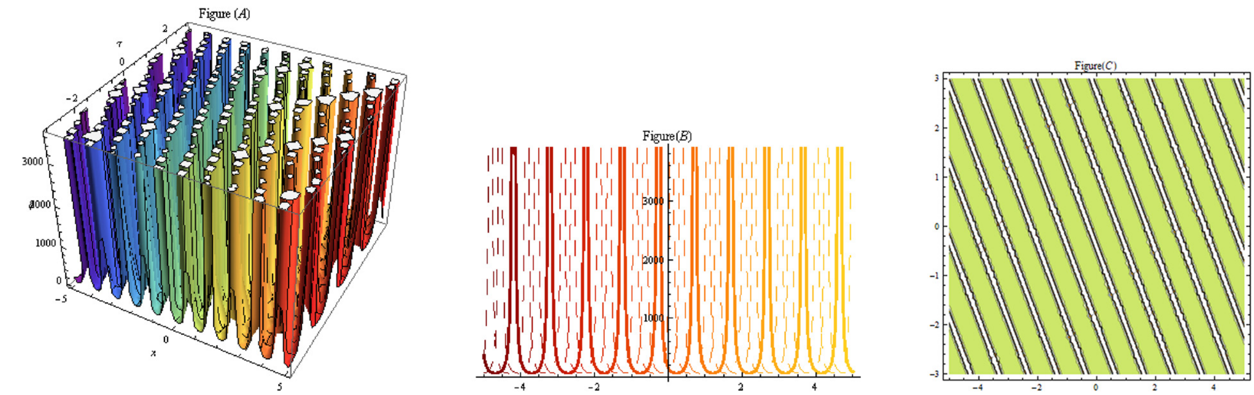

Figure 28: The periodic soliton represent in 3D, 2D, and contour shape for Eq. (11) with $f = 4$, $s = 1$, $t = 1$, $\zeta_0 = 1$, $\wp = 0.5$, $\Bbbk = 0.8$, $\nabla = -1$, $ð = 1$, $a_0 = 1$.

On the basis of the aforementioned information, discussion, and comparison, we can say that the results we acquired are novel, distinct, and broader than anything discovered in the prior literature. All of these computations demonstrate the strength, effectiveness, and productivity of our mathematical techniques in solving different nonlinear equations (Figures 27 and 28).

5 Conclusion

In the recent research, we efficiently applied the new auxiliary method to discover a new solitary waves solution to the nonlinear damped KdV equation. The results of the investigations have demonstrated the greater analytical power, efficacy, simplicity, and productivity of this new method for studying other nonlinear complex physical models of partial differential equations found in mathematical physics, fluid mechanics, hydrodynamics, chemistry, engineering, and others. By using Mathematica (9.4), we are able to acquire the novel solutions in the forms of dark solitons, bright solitons, periodic solitons, kink and anti-kink wave solitons, peakon bright and dark solitons as well as depict their physical structure through two-dimensional, three-dimensional, and contour plots graphics. The results acquired have practical implementations in the fields of quantum plasma, solitons, adiabatic parameter dynamics, fluid dynamics, biological difficulties, industrial, and several other areas.

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