Research Article

Eugene Oks*

On solutions of the Dirac equation for 1D hydrogenic atoms or ions

https://doi.org/10.1515/phys-2025-0185
received April 25, 2025; accepted June 12, 2025

Abstract: Studies of 1D hydrogenic atoms or ions reveal interesting physics and could be relevant to the following experimental situations: (1) hydrogenic atoms in very strong magnetic fields where the electron is effectively confined in a finite-radius cylinder surrounding a field line; (2) an electron constrained to the surface of a nanotube in the field of an ion; (3) excitons in research of semiconductors, superconductivity, and polymers; (4) electron at the surface of liquid helium. In the present theoretical paper, we revisited the 1D Dirac–Coulomb problem. We employed a model where the Coulomb potential is truncated by a constant value at some relatively small but finite distance R from the origin, thus allowing for the finite size of the nucleus of charge Z. We derived analytically the wave functions in the exterior and interior regions. We showed that for nonzero energies, the wave functions in the exterior and interior regions and their derivatives cannot be matched at the boundary, so that the corresponding mathematical solutions are not physically acceptable. For zero energy, we determined the value of $R(Z)$, at which the interior and exterior wave functions and their derivatives can be matched at the boundary. We demonstrated that for the electronic hydrogenic atoms or ions, the matching value of $R(Z)$ is greater than the corresponding nuclear sizes by at least two orders of magnitude. However, for the muonic hydrogenic atoms or ions, the matching value of $R(Z)$ turned out to be approximately equal to the nuclear size $R_0(Z)$ (within the accuracy of the expressions for $R_0(Z)$) for the range of the nuclear charges $14 \le Z \le 25$ (*i.e.*, from Si to Mn), the best match of R and R_0 being for $Z = 18$ (Ar) and $Z = 19$ (K).

Keywords: 1D hydrogenic atoms/ions, Dirac equation, muonic atoms

* **Corresponding author: Eugene Oks,** Physics Department, 380 Duncan Drive, Auburn University, Auburn, AL, 36849, United States of America, e-mail: oksevgu@auburn.edu

1 Introduction

There is a very large number of studies of 1D hydrogen atoms (many of the studies are referred in this work). From the theoretical viewpoint, they are fascinating for several reasons. First, they reveal interesting physics. Second, on some occasions, they contradict each other, causing disputes (the disputes being the driving force in search for the scientific truth). For example, one disputed issue is whether it should be considered an impenetrable barrier at the origin, while another disputed issue is even whether there are bound states (specific references are given here).

Third, the total number of the relevant theoretical publications is ~10^2. The overwhelming majority of them are dealing with the Schrödinger equation: therefore, here we refer only to some works published in the last 30 years (out of about 70 years of such studies) – namely, studies by Golovinskii and Preobrazhenskii [1,2], Loudon [3], Palma and Raff [4], Nieto [5], Gordeyev and Chhajlany [6], Fischer *et al.* [7], Xianxi *et al.* [8], and references therein. Andrews [9,10] insisted that an impenetrable barrier should be considered at the origin, while some other authors disagreed. Of course, all the results of these studies were *non-relativistic* since they were obtained based on the Schrödinger equation.

There are also relevant *relativistic* theoretical studies. For such studies based on the Klein–Fock–Gordon equation, we refer to the studies by Barton [11], Spector and Lee [12], and references therein. For the relevant studies based on the Dirac equation, we refer to the studies by Audinet and Toulouse [13], Gusynin *et al.* [14], Ho and Khalilov [15], Benvegnu [16], Moss [17], Lapidus [18], and references therein.

As for the disputes in the studies based on the Dirac equation, Ho and Khalilov [15] set up an impenetrable barrier at the origin, while Audinet and Toulouse [13], Benvegnu [16], and Moss [17] did not. Last but not least, Moss [17] claimed that there are no bound states, while Ho and Khalilov [15] clearly showed the presence of the bound states.

From the experimental viewpoint, the above theoretical results could be useful for some of the following studies:

(1) Hydrogenic atoms in very strong magnetic fields where the electron is effectively confined in a finite-radius cylinder surrounding a field line (*e.g.*, book by Ruder *et al.* [19]);
(2) An electron constrained to the surface of a nanotube in the field of an ion (*e.g.*, study by Banyai *et al.* [20] and references therein);
(3) Excitons in research of semiconductors (Elliot and Loudon [21,22]), superconductivity (Ginzburg [23]), Wigner crystal (Carr [24], Wigner [25]), and polymers (Abe and Su [26], Abe [27], and Heeger *et al.* [28]);
(4) Electron at the surface of liquid helium – *e.g.*, studies by Kawakami *et al.* [29], Golovinskii and Preobrazhenskii [2], Nieto [5], Cheng *et al.* [30], Bridges and McGill [31], Grimes *et al.* [32], and references therein.

We note that among the above 1D theoretical studies, some of them incorporated the effects of the finite nuclear size and some of them did not. In the 3D situation, there are a number of studies incorporating the effects of the finite nuclear size. Examples are the studies by Shabaev [33], Franzke and Gertsmann [34], Lee and Milstein [35], Salman and Saue [36], Pachucki *et al.* [37], Oks [38–41].

Since the present work focuses on the 1D situation, the above 3D theoretical studies of the effects of the finite nuclear size are only tangentially related to our work.

In the present theoretical study, we revisit the 1D Dirac–Coulomb problem. We start from the Dirac equation in the form utilized in the illuminating study by Ho and Khalilov [15] (hereafter, HK study). However, HK employed the model where the potential is the Coulomb one down to the origin with the impenetrable wall at the origin, the finite potential being defined only for the positive values of the coordinate. In distinction, we use a more realistic model where the Coulomb potential is truncated by a constant value at some relatively small but finite distance R from the origin (thus allowing for the nuclear size) and is defined both for the positive and negative values of the coordinate with no impenetrable wall at the origin. Besides, while HK considered only a 1D hydrogen atom, we consider a hydrogenic atom or ion of any nuclear charge Z. To the best of our knowledge, there were no studies of the solutions of the 1D Dirac equation for the potential that we use.

We obtain analytically the wave functions in the exterior and interior regions and then deduce the value of $R(Z)$, at which the interior and exterior wave functions and their derivatives can be matched at the boundary. We show that for the electronic hydrogenic atoms or ions, the matching value of $R(Z)$ exceeds the corresponding nuclear sizes by at least two orders of magnitude. However, for the muonic hydrogenic atoms or ions, the matching value of $R(Z)$ is approximately equal to the nuclear size $R_0(Z)$ (within the accuracy of the formula for $R_0(Z)$) for the range of the nuclear charges $14 \le Z \le 25$ (*i.e.*, from Si to Mn).

2 New results

2.1 General calculations

In the HK study [15], the solutions of the 1D Dirac equation have been sought on the positive half-line $x \ge 0$ for a negative fermion in the field described by the following model potential energy:

$$V_{\mathrm{HK}}(x) = -q/x, \quad (x > 0), \qquad V_{\mathrm{HK}}(0) = \infty. \tag{1}$$

In distinction, we use the potential

$$V(x) = -q/|x|, \quad (|x| \ge R), \qquad V(x) = -q/R, \quad (R \ge |x| \ge 0), \tag{2}$$

where R = const > 0.

The fermion can be either electron or muon – so we use the natural units $\hbar = c = m = 1$, where m is either the electron mass or the muon mass, respectively. For the quantity q, we take

$$q = Z\alpha, \tag{3}$$

where Z is the nuclear charge and α is the fine structure constant. (Thus, we extend the study to hydrogen-like ions, while HK considered only hydrogen atoms.) Then, the Dirac equation for the two components $\psi_1(x)$ and $\psi_2(x)$ of the wave function can be written similar to Eqs. (5) and (6) from the HK study, but only for $x \ge R$:

$$\mathrm{d}\psi_1/\mathrm{d}x + (1 - q/x)\psi_1 = E\psi_2, \tag{4}$$

$$-\mathrm{d}\psi_2/\mathrm{d}x + (1 - q/x)\psi_2 = E\psi_1, \tag{5}$$

where E is the eigenenergy. Below we attempt to find physically acceptable analytical solutions of Eqs. (4) and (5) in the range of $x \ge 0$. After finding such solution, we will properly extend it to the range of $x < 0$.

On expressing ψ_1 *via* ψ_2 from Eq. (5) and substituting in Eq. (4), we obtain

$$x^2\mathrm{d}^2\psi_2/\mathrm{d}x^2 + [(E^2-1)x^2 + 2qx - q(q+1)]\psi_2 = 0. \tag{6}$$

There is a reference book titled “Handbook of Exact Solutions for Ordinary Differential Equations” by Polyanin and Zaitsev [42], which is one of the most comprehensive handbooks on the subject. As one of the numerous examples, the authors of the handbook [42] consider the following equation (example 2.1.2.110):

$$x^2 d^2y/dx^2 + (ax^2 + bx + c)y = 0. \quad (7)$$

They note that the first step to the exact solution is the substitution $y(x) = x^\gamma u(x)$, where γ is the solution of the equation $\gamma^2 - \gamma + c = 0$, which in our case is

$$\gamma^2 - \gamma - q(q + 1) = 0. \quad (8)$$

The two solutions of Eq. (8) are

$$\gamma_+ = 1 + q, \quad \gamma_- = -q. \quad (9)$$

In the case of Eq. (6), the substitution

$$\psi_2(x) = x^\gamma u(x). \quad (10)$$

yields

$$x d^2u/dx^2 + 2\gamma du/dx + [(E^2 - 1)x + 2q]u = 0. \quad (11)$$

Here and below, we temporarily omitted the subscript of the parameter γ for brevity. The exact solutions of the equations of the type of Eq. (11) are given in the handbook [42] in example 2.1.2.103

$$u_\pm(x) = \text{const } \exp[\pm(1 - E^2)^{1/2}x]J[\gamma \pm q/(1 - E^2)^{1/2}, \; 2\gamma, \; \pm 2(1 - E^2)^{1/2}(-x)], \quad (12)$$

where $J(A, B, w)$ is either the Kummer function $\Phi(A, B, w)$ or the Tricomi function $U(A, B, w)$.

First, we explore the case where there is the Kummer function in Eq. (12) and the minus sign in the argument of the exponential. The intermediate result is

$$\psi_2(x) = \text{const } x^\gamma \exp[-(1 - E^2)^{1/2}x]\Phi[\gamma - q/(1 - E^2)^{1/2}, \; 2\gamma, \; 2(1 - E^2)^{1/2}x]. \quad (13)$$

The final step in the fulfillment of the boundary condition $\psi_2(\infty) = 0$ is to ensure that the Kummer function in Eq. (13) reduces to a polynomial of the finite degree. This is achieved by the requirement that

$$\gamma - q/(1 - E^2)^{1/2} = -k, \quad k = 0, 1, 2, \ldots \quad (14)$$

On solving Eq. (14) for the energy E, we obtain (for the positive set of E)

$$E_n = \{1 - q^2/(\gamma + k)^2\}^{1/2}, \quad k = 0, 1, 2, \ldots \quad (15)$$

For $\gamma = \gamma_+ = 1 + q$, Eq. (15) for E_n coincides with the corresponding expression from the HK study (*i.e.*, with their Eq. (8)), which in our notations (*i.e.*, with $m = 1$) is

$$E_{n,\text{ HK}} = [1 - q^2/(q + n)^2]^{1/2}, \quad n = k + 1, \quad (16)$$

where the subscript HK refers to the result from the HK study.

Now, we go back to the general form of $\psi_2(x)$ from Eq. (12). It turned out that the choice of the Tricomi function coupled with the choice of the minus sign in the argument of the exponential, as well as the choice of the Kummer function coupled with the choice of the plus sign in the argument of the exponential, yielded the results for $\psi_2(x)$ – for the energies E_n from Eq. (15) – that are identical to the above case (presented in detail) where the choice was of the Kummer function coupled with the choice of the minus sign in the argument of the exponential. As for the last remaining choice of the Tricomi function coupled with the choice of the plus sign in the argument of the exponential, it yielded a (complex-valued) function that does not satisfy the condition $\psi_2(\infty) = 0$ for any energy.

Next we will solve the Dirac equation in the range $R \geq x \geq 0$, where the potential energy is $V(x) = -q/R = \text{const}$. In this range, the Dirac equation has the form

$$d\psi_1/dx + (1 - q/R)\psi_1 = E\psi_2, \quad (17)$$

$$-d\psi_2/dx + (1 - q/R)\psi_2 = E\psi_1. \quad (18)$$

At $x = 0$, the solutions of these equations should be limited, but do not have to vanish – because we did not require $V(0) = \infty$. Below we show that all solutions are limited at $x = 0$. So, the crucial test of whether the solutions are physically acceptable will be to try to match them with the corresponding exterior solutions and their logarithmic derivatives at $x = R$. As x approaches R from the outside, for the exterior functions we obtain (Eq. (13))

$$\psi_{2,\text{ext}} \approx R^\gamma, \quad (1/\psi_{2,\text{ext}})d\psi_{2,\text{ext}}/dx \approx \gamma/R. \quad (19)$$

If $E \neq 0$, then (Eq. (5))

$$\psi_{1,\text{ext}} \approx (-\gamma/E)R^{\gamma-1}, \quad (1/\psi_{1,\text{ext}})d\psi_{1,\text{ext}}/dx \approx (\gamma - 1)/R. \quad (20)$$

For the interior solutions, in the case of $E \neq 0$, from Eq. (18), we obtain

$$\psi_1 = (1/E)[-d\psi_2/dx + (1 - q/R)\psi_2], \quad (21)$$

and on substituting Eq. (21) in Eq. (17), we obtain

$$d^2\psi_2/dx^2 + [E^2 - (1 - q/R)^2]\psi_2 = 0. \quad (22)$$

Below we explore three subcases of the case $E \neq 0$.

2.2 Subcase of $E^2 < (1 - q/R)^2$

If $E^2 < (1 - q/R)^2$, then the solutions of the Eq. (20) are

$$\psi_{2,\text{int}}(x) = \exp\{\pm[(1 - q/R)^2 - E^2]^{1/2}x\}. \quad (23)$$

Then, for the logarithmic derivative at $x = R$, we obtain

$$(1/\psi_{2,\mathrm{int}})\mathrm{d}\psi_{2,\mathrm{int}}/\mathrm{d}x = \pm[(1-q/R)^2 - E^2]^{1/2}. \tag{24}$$

By equating the logarithmic derivatives of $\psi_2(x)$ on both sides of $x = R$, we obtain

$$\pm[(1-q/R)^2 - E^2]^{1/2} = \gamma_\pm/R. \tag{25}$$

On substituting $\psi_2(x)$ from Eq. (23) in Eq. (22), we obtain

$$\psi_{1,\mathrm{int}}(x) = (1/E)\{1 - q/R \pm (-1)[(1-q/R)^2 - E^2]^{1/2}\} \exp\{\pm[(1-q/R)^2 - E^2]^{1/2}x\}. \tag{26}$$

Then, for the logarithmic derivative at $x = R$, we obtain

$$(1/\psi_{1,\mathrm{int}})\mathrm{d}\psi_{1,\mathrm{int}}/\mathrm{d}x = \pm[(1-q/R)^2 - E^2]^{1/2}. \tag{27}$$

By equating the logarithmic derivatives of $\psi_1(x)$ on both sides of $x = R$, we get

$$\pm[(1-q/R)^2 - E^2]^{1/2} = (\gamma_\pm - 1)/R. \tag{28}$$

Obviously, Eqs. (25) and (28) are incompatible with each other. Thus, in the subcase of $E^2 < (1 - q/R)^2]$, there are no physically acceptable solutions.

2.3 Subcase of $E^2 > (1 - q/R)^2$

Now we proceed to the subcase where $E^2 > (1 - q/R)^2$. We denote

$$p = [E^2 - (1-q/R)^2]^{1/2}. \tag{29}$$

Then, from Eq. (22) follows that either

$$\psi_{2,\mathrm{int}}(x) = \sin(px), \tag{30}$$

or

$$\psi_{2,\mathrm{int}}(x) = \cos(px). \tag{31}$$

For the sub-subcase where $\psi_{2,\mathrm{int}}(x) = \sin(px)$, by equating the corresponding logarithmic derivatives at $x = R$, we obtain

$$p/\tan(pR) = \gamma/R, \tag{32}$$

so that

$$\tan(pR) = pR/\gamma. \tag{33}$$

Here we temporarily omitted the subscript ± at γ for brevity. For $\psi_{1,\mathrm{int}}(x)$ in this sub-subcase we obtain (Eq. (21))

$$\psi_{1,\mathrm{int}}(x) = (1/E)[(1-q/R)\sin(px) - p\cos(px)]. \tag{34}$$

The logarithmic derivative calculated at $x = R$ is

$$(1/\psi_{1,\mathrm{int}})\mathrm{d}\psi_{1,\mathrm{int}}/\mathrm{d}x = p[p\tan(pR) + 1 - q/R] /[(1-q/R)\tan(pR) - p]. \tag{35}$$

By equating it to the corresponding logarithmic derivative in the exterior, we obtain (Eq. (20))

$$p[p\tan(pR) + 1 - q/R]/[(1-q/R)\tan(pR) - p] = (\gamma - 1)/R. \tag{36}$$

On substituting Eq. (33) in Eq. (36), after some elementary calculations, we obtain the following equation with respect to unknown R:

$$R^2 + R/p^2(R) + \gamma(\gamma - 1)/p^2(R) = 0. \tag{37}$$

In view of Eq. (9), for $\gamma_+ = 1 + q$, the quantity $\gamma(\gamma - 1) = q(1 + q)$; for $\gamma_- = -q$, the result for the quantity $\gamma(\gamma - 1)$ is the same: $\gamma(\gamma - 1) = q(1 + q)$. So, Eq. (37) becomes

$$R^2 + R/p^2(R) + q(1+q)/p^2(R) = 0. \tag{38}$$

Since $R > 0$, all three terms in the left side of Eq. (38) are positive, so that Eq. (38) does not have solutions for $R > 0$. Consequently, the sub-subcase where $\psi_{2,\mathrm{int}} = \sin(px)$, does not lead to a physically acceptable solution.

Now, we analyze the sub-subcase where $\psi_{2,\mathrm{int}}(x) = \cos(px)$. By equating the corresponding logarithmic derivatives at $x = R$, we obtain

$$\tan(pR) = -\gamma/(pR). \tag{39}$$

For $\psi_{1,\mathrm{int}}(x)$ in this subcase, we obtain

$$\Psi_{1,\mathrm{int}}(x) = (1/E)[p\sin(px) + (1-q/R)\cos(px)]. \tag{40}$$

The logarithmic derivative calculated at $x = R$ is

$$(1/\psi_{1,\mathrm{int}})\mathrm{d}\psi_{1,\mathrm{int}}/\mathrm{d}x = p[p - (1-q/R)\tan(pR)] /[p\tan(pR) + 1 - q/R]. \tag{41}$$

By equating it to the corresponding logarithmic derivative in the exterior, we obtain

$$p[p - (1-q/R)\tan(pR)]/[p\tan(pR) + 1 - q/R] = (\gamma - 1)/R. \tag{42}$$

On substituting Eq. (39) in Eq. (42), after denoting $y = 1/R$ and performing some elementary calculations, we obtain the following equation with respect to unknown y:

$$[\gamma(\gamma - 1) - q]y^2 + y + p^2 = 0. \tag{43}$$

We found above that both for γ_+ and for γ_-, we have $\gamma(\gamma - 1) = q(1 + q)$, then Eq. (43) becomes

$$q^2y^2 + y + p^2 = 0. \tag{44}$$

Since $R > 0$, so that $y > 0$, then all three terms in the left side of Eq. (44) are positive, so that Eq. (38) does not have solutions for $R > 0$. Consequently, the sub-subcase where $\psi_{2,\mathrm{int}} = \cos(px)$, does not lead to a physically acceptable solution either.

2.4 Subcase of $E^2 = (1 - q/R)^2$

Now, we proceed to the subcase where $E^2 = (1 - q/R)^2$]. Then, the solutions of Eq. (22) are

$$\psi_{2,\mathrm{int}}(x) = x^{\beta}, \quad \beta = 0 \text{ or } \beta = 1. \tag{45}$$

If $\beta = 0$, so that $\psi_{2,\mathrm{int}}(x) = \mathrm{const}$ and $\mathrm{d}\psi_{2,\mathrm{int}}(x)/\mathrm{d}x = 0$, then the logarithmic derivative $(1/\psi_{2,\mathrm{int}})\mathrm{d}\psi_{2,\mathrm{int}}/\mathrm{d}x = 0$: it does not match the corresponding logarithmic derivative of the exterior function $\psi_{2,\mathrm{ext}}(x)$ from Eq. (19), which is equal to γ/R.

If $\beta = 1$, so that $\psi_{2,\mathrm{int}}(x) = x$ and $\mathrm{d}\psi_{2,\mathrm{int}}(x)/\mathrm{d}x = 1$, so that the logarithmic derivative of the function $\psi_{2,\mathrm{int}}(x)$ calculated at $x = R$ is $1/R$, it still does not match the corresponding logarithmic derivative of the exterior function $\psi_{2,\mathrm{ext}}(x)$ from Eq. (19), which is equal to γ/R.

Alternatively, we can express ψ_2 *via* ψ_1 from Eq. (17), substitute in Eq. (18), and obtain (for $E^2 = (1 - q/R)^2$]): $\mathrm{d}^2\psi_{1,\mathrm{int}}/\mathrm{d}x^2 = 0$. Then,

$$\psi_{1,\mathrm{int}}(x) = x^{\beta}, \quad \beta = 0 \text{ or } \beta = 1. \tag{46}$$

If $\beta = 0$, then $(1/\psi_{1,\mathrm{int}})\mathrm{d}\psi_{1,\mathrm{int}}/\mathrm{d}x = 0$, so that it does not match the corresponding logarithmic derivative of the exterior function $\psi_{1,\mathrm{ext}}(x)$ from Eq. (19), which is equal to $(\gamma - 1)/R$.

If $\beta = 1$, so that $\psi_{1,\mathrm{int}}(x) = x$ and $\mathrm{d}\psi_{1,\mathrm{int}}(x)/\mathrm{d}x = 1$, so that the logarithmic derivative of the function $\psi_{1,\mathrm{int}}(x)$ calculated at $x = R$ is $1/R$, it still does not match the corresponding logarithmic derivative of the exterior function $\psi_{1,\mathrm{ext}}(x)$ from Eq. (20), which is equal to $(\gamma - 1)/R$.

Thus, there are no physically acceptable solutions for $E \neq 0$.

2.5 Subcase of $E = 0$

Finally, we study the case of $E = 0$. In this case, the Dirac equation simplifies to

$$\mathrm{d}\psi_1/\mathrm{d}x + [1 + V(x)]\psi_1 = 0, \tag{47}$$

$$-\mathrm{d}\psi_2/\mathrm{d}x + [1 + V(x)]\psi_2 = 0. \tag{48}$$

In HK study, the case of $E = 0$ was studied in the simpler model where the potential energy was $V_{\mathrm{HK}}(x) = -q/x$, $(x > 0)$, $V_{\mathrm{HK}}(0) = \infty$. Within this simpler model, HK obtained the following result for the components of the Dirac bispinor:

$$\psi_2(x) = 0, \quad \psi_1(x) = x^q \exp(-x). \tag{49}$$

After we normalized that solution – what was not done by HK – we found that the normalization constant in front of x^q in Eq. (29) should be $2^q/[q\Gamma(2q)]^{1/2}$, where $\Gamma(2q)$ is the gamma function.

For our more complicated model, where the potential energy is given by Eq. (2), the solution given by Eq. (49) is valid only in the exterior, *i.e.*, at $x \geq R$. In the interior region $(|x| \leq R)$, where $V(x) = -q/R$, from Eqs. (47) and (48), we find

$$\psi_{2,\mathrm{int}}(x) = 0, \quad \psi_{1,\mathrm{int}}(x) = C \exp[(q/R - 1)|x|], \tag{50}$$

where $C = \mathrm{const}$ will be determined below.

We emphasize that in the interior region $(|x| \leq R)$, there is no antisymmetric solution, but only the symmetric one. Therefore, the solution must be symmetric in the entire range of $-\infty < x < \infty$. In particular, the exterior solution for $\psi_{1,\mathrm{ext}}(x)$ must be $|x|^q \exp(-|x|)$.

Obviously, $\psi_2(x)$, being identically zero both in the interior and exterior regions, satisfies the matching condition at $x = R$.

So, we focus on $\psi_1(x)$. In the exterior region, where $\psi_1(x)$ has the form given in Eq. (49) with the substitution of x by $|x|$, its logarithmic derivative, calculated at $x = R$, is

$$(1/\psi_{1,\mathrm{ext}})\mathrm{d}\psi_{1,\mathrm{ext}}/\mathrm{d}x = q/R - 1. \tag{51}$$

The logarithmic derivative of $\psi_{1,\mathrm{int}}(x)$ given in Eq. (50), being calculated at $x = R$, yields

$$(1/\psi_{1,\mathrm{int}})\mathrm{d}\psi_{1,\mathrm{int}}/\mathrm{d}x = q/R - 1. \tag{52}$$

Thus, the logarithmic derivatives of $\psi_{1,\mathrm{int}}(x)$ and $\psi_{1,\mathrm{ext}}(x)$ match at $x = R$ regardless of the value of R.

Now, let us require the functions $\psi_{1,\mathrm{int}}(x)$ and $\psi_{1,\mathrm{ext}}(x)$ to match at $x = R$:

$$C \exp[q - R] = R^q \exp(-R). \tag{53}$$

Thus, we find the value of the constant C

$$C = R^q \exp(-q). \tag{54}$$

The normalization integral N is as follows:

$$N = 2\int_0^R \mathrm{d}x \; R^{2q} \exp(-2q - 2 + 2q/R) + 2\int_R^{\infty} \mathrm{d}x \; x^{2q} \exp(-2x). \tag{55}$$

On calculating the integral, we obtain

$$N = R^{2q}[\exp(-2R) - \exp(-2q)]/(q/R-1) + 2^{-2q}\Gamma(1 + 2q, \; 2R), \tag{56}$$

where $\Gamma(1 + 2q, 2R)$ is the incomplete gamma function.

After equating

$$N = 1, \tag{57}$$

we obtain the following relation between the value of boundary R (for which the interior and exterior wave functions and their derivatives can be matched at $x = R$) and q

$$\frac{R^{2q}[\exp(-2R) - \exp(-2q))}{q/R - 1} + 2^{-2q}\Gamma(1 + 2q, \quad 2R) = 1. \tag{58}$$

Figure 1 presents the dependence of the boundary R (in natural units) on the nuclear charge Z (by using $q = Z\alpha$).

Figure 2 presents the dependence of R (in fm) on the nuclear charge Z (by using $q = Z\alpha$) – specifically for the situation where the negative fermion is electron. We remind that for electrons, the natural unit of length is 386.1 fm.

Figure 3 presents the dependence of R (in fm) on the nuclear charge Z (by using $q = Z\alpha$) – specifically for the situation where the negative fermion is muon. We remind that for muons, the natural unit of length is 1.867 fm.

From Figures 1–3 (that were created using the software Mathematica) and the corresponding calculations, we see the following.

A. As Z varies from 1 to 137, *i.e.*, by 2 orders of magnitude, the value of the boundary R decreases only by about 12%.
B. For the situation where the negative fermion is electron, the value of the boundary R exceeds the corresponding nuclear radii by at least 2 orders of magnitude. So, this situation seems to have only academic interest.

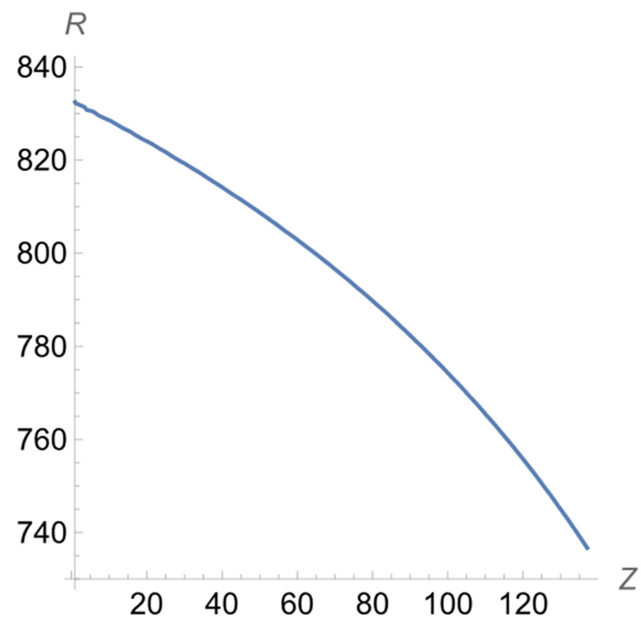

Figure 2: Dependence of R (in fm) on the nuclear charge Z (by using $q = Z\alpha$) – specifically for the situation where the negative fermion is electron.

C. For the situation where the negative fermion is muon, the value of the boundary R can be of the same order of magnitude as the corresponding nuclear radii. Thus, this situation seems to be more reasonable.

Below we provide more details related to the above item *C*. The dependence of the nuclear size R_0 on the nuclear charge is

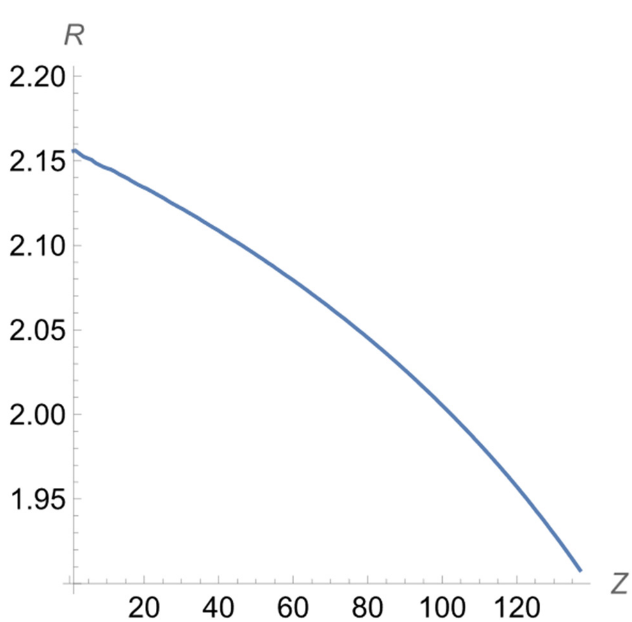

Figure 1: Dependence of the boundary R (in natural units) on the nuclear charge Z (by using $q = Z\alpha$).

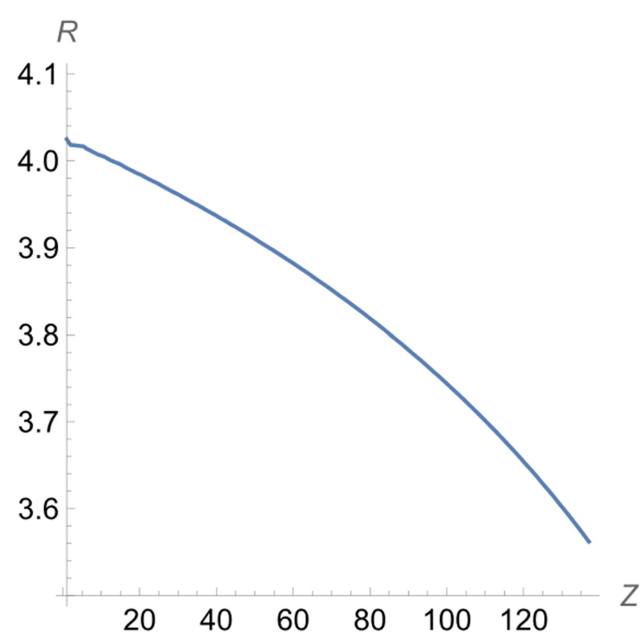

Figure 3: Dependence of R (in fm) on the nuclear charge Z (by using $q = Z\alpha$) – specifically for the situation where the negative fermion is muon.

$$R_0 = 1.2(\lambda Z)^{1/3}\ \text{fm}, \quad 2 \le \lambda \le 2.5. \tag{59}$$

For the muonic case, the boundary R is approximately equal to the nuclear size R_0 within the accuracy of 10% (the uncertainty of R_0 from Eq. (59) being about 10%) for

$$14 \le Z \le 25, \tag{60}$$

that is, from Si to Mn. The best match of R and R_0 occurs for $Z = 18$ (Ar) and $Z = 19$ (K).

3 Conclusion

We revisited the 1D Dirac–Coulomb problem. Instead of the HK model [15] where the potential was the Coulomb one down to the origin with the impenetrable wall at the origin and where the finite potential was defined only for the positive values of the coordinate, we employed a more realistic model where the Coulomb potential is truncated by a constant value at some relatively small but finite distance R from the origin (thus allowing for the nuclear size) and is defined both for the positive and negative values of the coordinate with no impenetrable wall at the origin. Besides, while HK considered only a 1D hydrogen atom, we considered a hydrogenic atom or ion of any nuclear charge Z.

For the case of the non-zero energy E, we derived analytically the wave functions in the exterior and interior regions for all possible subcases and showed that the exterior and interior wavefunctions cannot be matched (together with their derivatives) at the boundary. In other words, we demonstrated that there are no physically acceptable solutions for any non-zero energy.

For the case of zero energy, we also derived analytically the wave functions in the exterior and interior regions. We determined the value of $R(Z)$, at which the interior and exterior wave functions can be matched at the boundary (*i.e.*, ensuring the continuity of the wave function and of its derivative at the boundary). We demonstrated that for the electronic hydrogenic atoms or ions, the matching value of $R(Z)$ is greater than the corresponding nuclear sizes by at least two orders of magnitude. However, for the muonic hydrogenic atoms or ions, the matching value of $R(Z)$ turned out to be approximately equal to the nuclear size $R_0(Z)$ (within the accuracy of the expressions for $R_0(Z)$) for the range of the nuclear charges $14 \le Z \le 25$ (*i.e.*, from Si to Mn), the best match of R and R_0 being for $Z = 18$ (Ar) and $Z = 19$ (K).

We note that the presence of the zero-energy state causes the vacuum fermion number to be nonzero and to be fractional, as mentioned by HK. This happens also when the conjugation-symmetric Dirac Hamiltonian possesses an energy spectrum that is symmetric about zero, and a soliton is present, so that there is also an unpaired zero-energy state, which is self-conjugate (Jackiw [43]).

We also note that Barton [11] found a zero-energy solution while considering a truncated Coulomb potential in frames of the Klein–Gordon equation for a spinless charged particle. So, it seems that the relativistic effect of the presence of the zero-energy state is unrelated to the spin.

Funding information: The author states no funding involved.

Author contribution: The author has accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The author states no conflict of interest.

Data availability statement: All data generated or analyzed during this study are included in this published article.

References

[1] Golovinskii PA, Preobrazhenskii MA. Quasi-classical description of the Stark effect for an electron in an image-potential state. Opt Spectrosc. 2017;122:120.
[2] Golovinskii PA, Preobrazhenskii MA. Stimulated radiative transitions of an electron in an image potential. Opt Spectrosc. 2015;122:191.
[3] Loudon R One-dimensional hydrogen atom. Proc R Soc. 2016;472: 20150534.
[4] Palma G, Raff U. The one-dimensional hydrogen atom revisited. Canad J Phys. 2006;84:787.
[5] Nieto MM. Electrons above a helium surface and one-dimensional Rydberg atom. Phys Rev A. 2005;61:034901.
[6] Gordeyev AN, Chhajlany SC. One-dimensional hydrogen atom: a singular potential in quantum mechanics. J Phys A: Math Gen. 1997;30:6893.
[7] Fischer W, Leschke H, Muller P. Comment on 'On the Coulomb potential in one dimension' by P Kurasov. J Phys A: Math Gen. 1997;30:5579.
[8] Xianxi D, Dai J, Dai J. Orthogonality criteria for singular states and the nonexistence of stationary states with even parity for the one-dimensional hydrogen atom. Phys Rev A. 1997;55:2617.
[9] Andrews M. The one-dimensional hydrogen atom. Am J Phys. 1988;56:776.
[10] Andrews M. Singular potentials in one dimension. Am J Phys. 1976;44:1064.
[11] Barton GJ. On the 1D Coulomb Klein–Gordon equation. Phys A: Math Theor. 2007;40:1011.

[12] Spector HN, Lee J. Relativistic one-dimensional hydrogen atom. Am J Phys. 1985;53:248.
[13] Audinet T, Toulouse J. Effective quantum electrodynamics: One-dimensional model of the relativistic hydrogen-like atom. J Chem Phys. 2023;158:244108.
[14] Gusynin VP, Sobol OO, Zolotaryuk AV, Zolotaryuk Y. Bound states of a one-dimensional Dirac equation with multiple delta-potentials. Low Temp Phys. 2022;48:1022.
[15] Ho C-L, Khalilov VR. Fractional fermion number in a (1 + 1)-dimensional Dirac equation with a scalar Coulomb field. Phys Rev D. 2000;63:027701.
[16] Benvegnu S. Relativistic point interaction with Coulomb potential in one dimension. J Math Phys. 1997;38:556.
[17] Moss RE. The hydrogen atom in one dimension. Am J Phys. 1987;55:397.
[18] Lapidus IR. Relativistic one-dimensional hydrogen atom. Am J Phys. 1983;51:1036.
[19] Ruder H, Wunner G, Herold H, Geyer F. Atoms in strong magnetic fields. Berlin: Springer; 1994.
[20] Banyai L, Galbraith I, Ell C, Haug H. Excitons and biexcitons in semiconductor quantum wires. Phys Rev B. 1987;36:6099.
[21] Elliot RJ, Loudon R. Theory of the absorption edge in semiconductors in a high magnetic field. J Phys Chem Solids. 1960;15:196.
[22] Elliot RL, Loudon R. Theory of fine structure on the absorption edge in semiconductors. J Phys Chem Solids. 1959;8:382.
[23] Ginzburg VL. Once again about high-temperature superconductivity. Contemp Phys. 1992;33:15.
[24] Carr Jr WJ. Energy, specific heat, and magnetic properties of the low-density electron gas. Phys Rev. 1961;122:1437.
[25] Wigner EP. Effects of the electron interaction on the energy levels of electrons in metals. Trans Faradey Soc. 1938;34:678.
[26] Abe S, Su WP. Excitons and charge transfer states in one-dimensional semiconductors. Mol Cryst Liq Cryst. 1991;194:357.
[27] Abe S. Exciton versus interband absorption in Peierls insulators. J Phys Soc Jap. 1989;58:62.
[28] Heeger AJ, Kivelson S, Schrieffer JR, Su WP. Solitons in conducting polymers. Rev Mod Phys. 1988;60:781.
[29] Kawakami E, Elarabi A, Konstantinov D. Relaxation of the excited Rydberg states of surface electrons on liquid helium. Phys Rev Lett. 2021;126:106802.
[30] Cheng E, Cole MW, Cohen MH. Binding of electrons to the surface of liquid helium. Phys Rev B. 1994;50:1136.
[31] Bridges F, McGill JF. Mobility of electrons on the surface of liquid helium. Phys Rev B. 1977;15:1324.
[32] Grimes CC, Brown TR, Burns ML, Zipfel CL. Spectroscopy of electrons in image-potential-induced surface states outside liquid helium. Phys Rev B. 1976;13:140.
[33] Shabaev VM. Finite nuclear size corrections to the energy levels of the multicharged ions. J Phys B: Mol Opt Phys. 1993;26:1103.
[34] Franzke KL, Gertsmann U. Finite nuclear size effect on the relativistic hyperfine splittings of 2s and 2p excited states of hydrogen-like atoms. Foundations. 2024;4:513.
[35] Lee RN, Milstein AI. Finite nuclear size and vacuum polarization in heavy atoms. Phys Lett A. 1994;189:72.
[36] Salman M, Saue T. Calculating the many-potential vacuum polarization density of the Dirac equation in the finite-basis approximation. Phys Rev A. 2023;108:012808.
[37] Pachucki K, Patcos V, Yerokhin VA. Recoil corrections with finite nuclear size in hydrogenic systems. Phys Rev A. 2025;111:032820.
[38] Oks E. High-energy tail of the linear momentum distribution in the ground state of hydrogen atoms or hydrogen-like ions. J Phys B: Mol Opt Phys. 2001;34:2235.
[39] Oks E. Alternative kind of hydrogen atoms as a possible explanation for the latest puzzling observation of the 21 cm radio line from the early Universe. Res Astron Astrophys. 2020;20:109.
[40] Oks E. Symmetry of the non-analytic solution of the Dirac equation inside the proton of hydrogen atoms. Symmetry. 2025;17:517.
[41] Oks E. Resolution of the neutron lifetime puzzle and the conceptual design of its experimental confirmation. Nucl Phys B. 2025;1014:116879.
[42] Polyanin AD, Zaitsev VF. Handbook of exact solutions for ordinary differential equations. Boca Raton, FL: CRS Press; 1995.
[43] Jackiw R. Fractional charge and zero modes for planar systems in a magnetic field. Phys Rev D. 1984;29:2375.