Research Article

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Decoherence of steerability and coherence dynamics induced by nonlinear qubit–cavity interactions

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Abstract: In this study, we have applied the intrinsic decoherence (ID) model of the Milburn equation to investigate the temporal evolution of various quantum resources (steerability, entanglement, and coherence) in the generated dipole-coupled two-qubit states. Einstein–Podolsky–Rosen steering, EOF, and Jensen–Shannon divergence are used to analyze these quantum resources. We consider two dipole-coupled qubit interacting with two spatially separated cavities with nonlinear photonic transitions, filled with a superposition of generalized Barut–Girardello coherent fields. The exploration of steerability, entanglement, and coherence has been carried out by considering the nonlinearity of qubit–cavity interactions, the nonclassicality of the superposition of Barut–Girardello coherent fields, the dipole–dipole coupling, and the ID of qubit–cavity interactions. Our results show that the nonlinearity of qubit–cavity interactions, the initial generalized Barut–Girardello coherent states, and the qubit–cavity detuning significantly enhance the generation of quantum steerability, entanglement, and coherence. The dynamics of entanglement and steerability are consistent with the hierarchy principle. The ID effect reduces the amplitudes and frequencies of the generated quantum resources to their stationary values. It is found that the phenomena of sudden steerability arising and sudden annihilation of entanglement are influenced by the initial non-classicality, dipole coupling, and intrinsic qubit–cavity decoherence. Moreover, the strong decoherence and dipole coupling significantly enhance the steady-state values of steerability, entanglement, and coherence.

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1 Introduction

The investigation of quantum coherence and nonlocality generated through nonlinear qubit–photon interactions represents a cornerstone of quantum information science [1,2]. These studies not only enrich the foundational understanding of quantum theory but also pave the way for advancements in quantum technologies. Nonlinear qubit–photon interactions can be experimentally implemented in various systems, including trapped atomic ions [3], Rydberg atoms [4,5], and superconducting quantum circuits [6]. There are several significant nonlinearity resources for qubit–photon interactions, including the Kerr-like medium [7,8], which can be realized in superconducting circuits [9], intensity-dependent interactions [10], multi-photon qubit–field interactions observed in superconducting qubit–photon circuits [11], trapped ions [12], and multi-mode cavities [13,14]. These parametric photonic interactions have been used to realize maximally entangled photon-number states [15], generalized Bell states [16], and continuous variable entanglement generation [17]. Additionally, generalized superposed coherent states [18–20] are significant nonlinearity resources that have been widely used to achieve a variety of quantum information resources. For instance, they have enabled bidirectional quantum teleportation through multipartite Glauber coherent states [21], facilitated the experimental preparation of generalized cat states for itinerant microwave photons [22], and supported the realization of quantum information resources in single-mode quantized light fields [23]. These states have been led to enhancing the entanglement of even entangled coherent states [24], analyzing the nonextensivity of generalized dissipative SU(1,1) coherent states [25], and exploring the nonclassicality of two-coupled-qubit correlations inside nondegenerate parametric amplifier cavities [26]. Furthermore, the construction of generalized SU(1,1) coherent states and their quantum effects has been explored [27].

Generating two-qubit nonlocalities (including steerability, entanglement, and coherence) plays an important role in quantum information science. The coherence of two-qubit states arises from their quantum superposition properties [28,29]. This fundamental coherence resource can be quantified using several measures, such as relative entropy [30], l_1-norm of coherence [31], trace-distance measure [31], and a measure based on the quantum Jensen–Shannon divergence (J–S divergence) [32,33]. The J–S divergence coherence has been extensively studied in several quantum systems, including Heisenberg spin models [33,34], Ising models [35], intensity-dependent cavity–qubit, and qubit–qubit interactions [36]. Additionally, the J–S divergence quantifier has been applied to analyze coherence and entanglement dynamics in a graphene sheet of disordered electrons with quantum–memory dynamics [37]. The two-qubit entanglement has numerous applications in quantum computation [38] and communication [39]. It can be realized in various real systems [40,41]. Quantum Einstein–Podolsky–Rosen (EPR) steering [42–44], a distinct form of nonlocality intermediate between entanglement and Bell nonlocality, obeys a strict hierarchy principle [45–47]: entanglement is necessary but insufficient for two-qubit steerability. EPR steering uniquely exhibits asymmetric quantum nonlocality [48]. The hierarchy has been experimentally confirmed for the nonclassicality of single-qubit states through their potential for entanglement, steering, and Bell nonlocality [49].

After exploring quantum steerability based on the EUR steering inequality of a bipartite state [42,50], the decay of quantum steering and coherence of two-qubit X-states, due to thermal equilibrium temperature [51,52], and noisy non-Markovian environments [53], was investigated. Moreover, the decoherence of steerability in cavity–magnon and qubit–qubit states in hybrid quantum systems [54,55] in the presence of damping rates described by master equations for open system–reservoir interactions has been investigated. Furthermore, the thermal decay of steerability, due to the temperature of the bath, is evaluated using quantum steering in an anisotropic two-qubit Heisenberg model. Furthermore, a scheme has been proposed to generate entanglement and EPR steering in a hybrid optomagnonic qubit–cavity system [56], taking into account the effects of damping due to system–reservoir interactions.

There are several approaches to describe decoherence, one of which is ID (ID). This approach explains the degradation of quantum coherence and is governed by the Milburn equation [57]. The Milburn equation generalizes the Schrödinger equation based on the hypothesis that closed quantum systems do not exhibit unitary evolution. In this scenario, quantum coherence is automatically destroyed as the system evolves, and ID occurs without the system interacting with a reservoir and without a dissipation. The Milburn equation's ID model has been used to study the decoherence of more quantum information resources in various quantum systems [58–61]. In addition, there are other master equations that describe the interactions between open systems and reservoir, which naturally lead to irreversible effects such as dissipation (where the system's energy is not conserved) and decoherence (where the system's energy is conserved) [62,63].

Motivated by the experimental realization of nonlinear qubit–photon interactions and the significant role of generalized coherent states in achieving various quantum information resources, this work investigates the decoherence of steerability, entanglement, and coherence for the generated general two-qubit non-X-states of two dipole-coupled qubit interacting nonlinearly with nondegenerate coherent cavities filled with initial generalized GBG coherent fields. To analyze the decoherence of steerability and coherence dynamics, we use the Milburn equation's ID model.

In the following section, we introduce the ID model based on the Milburn equation and provide its analytical solution for a system described by a Hamiltonian involving two dipole-coupled qubit inside two spatially separated nondegenerate coherent cavities. Section 3 presents the measures of quantum steerability, entanglement, and coherence. The dynamics of steerability, entanglement, and coherence are discussed in Section 4. Finally, our conclusions are presented in Section 5.

2 Physical model

Here, we consider that a Hamiltonian $\hat{H}$ describes two two-level atoms (qubit), A and B, coupled via dipole–dipole interaction and situated inside two spatially separated nondegenerate cavities. Each single-mode cavity field ($r = A, B$) is governed by the raising operator $\hat{F}_r^\dagger$ and has a frequency ω_r. The qubit–cavity interactions occur through nonlinear nondegenerate two-photon transitions [64,65]. The Hamiltonian of the system is given by

$$\begin{aligned}\hat{H} = \sum_{r=A,B}\Big\{&\omega_r\left(\hat{F}_r^\dagger\hat{F}_r + \frac{1}{2}\right) + \frac{\omega}{2}[|1_r\rangle\langle 1_r| - |0_r\rangle\langle 0_r|] \\ &+ \lambda\hat{F}_A\hat{F}_B|1_r\rangle\langle 0_r| + \lambda\hat{F}_A^\dagger\hat{F}_B^\dagger|0_r\rangle\langle 1_r|\Big\} \\ &+ K|1_A0_B\rangle\langle 0_A1_B| + K|0_A1_B\rangle\langle 1_A0_B|,\end{aligned} \tag{1}$$

where ω represents the frequency of the two qubit. Each r-qubit is described by the up $|1_r\rangle$ and the down $|0_r\rangle$ states.

The parameter K represents the dipole coupling between the two qubit.

To study the decoherence of nonlinear qubit–cavity interaction dynamics, we consider the ID model of the Milburn equation [57], which depends on the Hamiltonian $\hat{H}$ of Eq. (1). With the ID coupling γ and the qubit–cavity density matrix $\hat{M}(t)$, the Milburn equation is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{M}(t) = -i[\hat{H}, \hat{M}(t)] - \frac{1}{\gamma}[\hat{H}, [\hat{H}, \hat{M}(t)]]. \tag{2}$$

To solve Eq. (2), we use the generators $\hat{G}_{\pm}$ of the SU(1,1) Lie group, taking into account the following considerations: $\omega_A = \omega_B = \frac{\omega}{2}$, $\hat{G}_- = \hat{F}_A\hat{F}_B$, $\hat{G}_+ = \hat{F}_A^{\dagger}\hat{F}_B^{\dagger}$, and $\hat{G}_0 = \frac{1}{2}(\hat{F}_A^{\dagger}\hat{F}_A + \hat{F}_B^{\dagger}\hat{F}_B + 1)$. The generator $\hat{G}_{\pm}$ satisfies

$$[\hat{G}_0, \hat{G}_{\pm}] = \pm\hat{G}_{\pm}, \quad [\hat{G}_-, \hat{G}_+] = 2\hat{G}_0. \tag{3}$$

The operators $\hat{G}_{\pm}$ and $\hat{G}_0$ act on the cavity eigenstates $|n, p\rangle$ (where $n = 0, 1, 2, \ldots$) as follows:

$$\begin{aligned} &\hat{G}_-|n, p\rangle = \Lambda_{n,p}|n-1, p\rangle, \quad \hat{G}_+|n, p\rangle = \Lambda_{n+1,p}|n+1, p\rangle, \\ &\hat{G}_0|n, p\rangle = (n+k)|n, p\rangle, \quad \hat{G}^2|n, p\rangle = p(p-1)|n, p\rangle, \end{aligned} \tag{4}$$

where $\Lambda_{n,p} = \sqrt{n(n+2p-1)}$. The operator: $\hat{G}^2 = \hat{G}_0^2 - \frac{1}{2}(\hat{G}_+\hat{G}_- + \hat{G}_-\hat{G}_+) = p(p-1)\hat{I}$ is the Casimir operator with Bargmann number p. Therefore, using the SU(1,1) Lie group generators, the Hamiltonian of Eq. (1) can be rewritten as

$$\begin{aligned} \hat{H} = \omega G_o + \sum_{r=A,B} \Bigg\{ &\frac{\omega}{2}(|1_r\rangle\langle 1_r| - |0_r\rangle\langle 0_r|) \\ &+ \lambda G_-|1_r\rangle\langle 0_r| + \lambda G_+|0_r\rangle\langle 1_r| \Bigg\} \\ &+ K|1_A0_B\rangle\langle 0_A1_B| + K|0_A1_B\rangle\langle 1_A0_B|), \end{aligned} \tag{5}$$

which has the eigenstates $|V_l^{n,p}\rangle$ (where $l = 1, 2, 3, 4$) and the eigenvalues $V_l^{n,p}$, related by $\hat{H}|V_l^{n,p}\rangle = V_l^{n,p}|V_l^{n,p}\rangle$. For the space SU(1,1)-SU(2) system states: $\{|1\rangle = |1_A1_B, p, n\rangle,\ |2\rangle = |1_A0_B, p, n+1\rangle,\ |3\rangle = |0_A1_B, p, n+1\rangle,\ |4\rangle = |0_A0_B, p, n+2\rangle\}$, the eigenstates $|V_l^{n,p}\rangle$ can be rewritten as

$$|V_l^{n,p}\rangle = \sum_{m=1}^{4} Y_{lm}^{n,p}|m\rangle. \tag{6}$$

To find a particular solution for Eq. (2) with the Hamiltonian of Eq. (5), we consider that the two qubit are initially in the upper states: $|1_A1_B\rangle\langle 1_A1_B|$. While the SU(1,1)-cavity system is initially in a superposition of generalized Barut-Girardello (GBG) coherent states: $|\eta, p\rangle$ [18,19] with the average photon number $|\eta|^2$, given by

$$|\psi(0)\rangle = \frac{1}{N_\eta}[|\eta, p\rangle + k|-\eta, p\rangle] = \sum_{n=0}^{\infty} R_n|n, p\rangle, \tag{7}$$

where $N_\eta = 1 + k^2 + 2k\langle\eta, p| - \eta, p\rangle$. Here, $k = 0$ for the GBG coherent state and $k = 1$ for the GBG even coherent state. The expression for R_n is given by

$$R_n = \frac{\eta^n[1 + (-1)^n k]}{1 + k^2 + 2k\langle\eta, p| - \eta, p\rangle}\sqrt{\frac{|\eta|^{2p-1}}{n! I_{2p-1}(2|\eta|)\Gamma(2p+n)}},$$

where $I_\nu(x)$ is the modified Bessel function.

In the space of the eigenstates: $\{|V_l^{n,p}\rangle\}$, the analytical solution of Eq. (2) is given by

$$\hat{M}(t) = \sum_{m,n=0}\sum_{s=1,3,4}\Lambda_m^*\Lambda_n\left\{\sum_{\ell=1}^{4}(1 - \delta_{\ell 2})X_{mn}^{s\ell}|V_s^{m,p}\rangle\langle V_\ell^{n,p}|\right\}, \tag{8}$$

where

$$X_{mn}^{s\ell} = Y_{s1}^{m,p}Y_{\ell 1}^{n,p}e^{-i\lambda(V_s^{m,p} - V_l^{n,p})t - \frac{1}{\gamma}(V_s^{m,p} - V_l^{n,p})^2 t}.$$

To study the decoherence of the steerability and coherence dynamics of the generated two-qubit states, we need the reduced two-qubit density matrix $M^{AB}(t)$, which is obtained by tracing over the SU(1,1)-cavity system states as follows:

$$M^{AB}(t) = \mathrm{Tr}_{\mathrm{Cavity}}\{M(t)\}. \tag{9}$$

The reduced two-qubit density matrix $M^{AB}(t)$ of Eq. (9) will be used to investigate the time evolution of steerability, entanglement, and coherence using the following measures.

3 Nonlocality and coherence measures

- **EPR steering**

  Here, the maximal violation of EPR steering inequalities [42] will be used to quantify the nonlocal steerability of the two-SU(1,1)-qubit state. The EPR steering quantifier is given by

$$S(t) = \max\left\{0, \frac{\sqrt{\Delta^2 - \Delta_{\min}^2} - 1}{\sqrt{2} - 1}\right\}, \tag{10}$$

  where $\Delta = \sqrt{\vec{d}}$, with the diagonal matrix $\vec{d} = \{d_1, d_2, d_3\}$ of the correlation two-qubit matrix: $D = [d_{ij}]$, where $(i, j = x, y, z)$ [66], with $d_{ij} = \mathrm{Tr}\{M^{AB}(t)\sigma_A^i\sigma_B^j\}$ depending on the Pauli matrices $\hat{\sigma}_k^i (k = A, B)$. $\Delta_{\min} = \min\{|d_1|, |d_2|, |d_3|\}$. For a non-steerable two-qubit state, $S(t) = 0$. When $S(t) = 1$, the two-$SU(1,1)$-qubit are in a maximally steerable state.

- **Entanglement of formation (EOF)**

Here, we use the EOF [67] to explore the generated entanglement in the two-SU(1,1)-qubit state. EOF is a monotonically increasing quantifier that correlates with concurrence [68]. The EOF has the following formula:

$$E(t) = \sum_{n=0,1} (n - W)\log_2(n - (-1)^{n+1}W),$$
$$W = \frac{1}{2}[1 + \sqrt{1 - [2\max\{0, Z\}]^2}], \quad (11)$$

where $Z = \sqrt{e_1} - \sqrt{e_2} - \sqrt{e_3} - \sqrt{e_4}$, and $e_i (i = 1 - 4)$ are the eigenvalues of the matrix: $M = M^{AB}(t)(\hat{\sigma}_A^y \otimes \hat{\sigma}_B^y) M^{*AB}(t)(\hat{\sigma}_A^y \otimes \hat{\sigma}_B^y)$, listed in decreasing order.

- **J–S coherence**

To measure the generated coherence in the two-$SU(2)$-qubit system $M^{AB}(t)$ of Eq. (9), the square root of the J–S divergence [32,33] will be used as a significant quantifier for two-qubit coherence, which is defined as

$$J(t) = \sqrt{S[M_T^{AB}] - \frac{1}{2}S\{[M^{AB}(t)] - S[M_d^{AB}]\}}, \quad (12)$$

where $S[\cdot]$ is the von Neumann entropy, $M_T^{AB} = [M^{AB}(t) + M_d^{AB}(t)]/2$, and M_d^{AB} is the diagonal two-qubit matrix $M^{AB}(t)$ of Eq. (9).

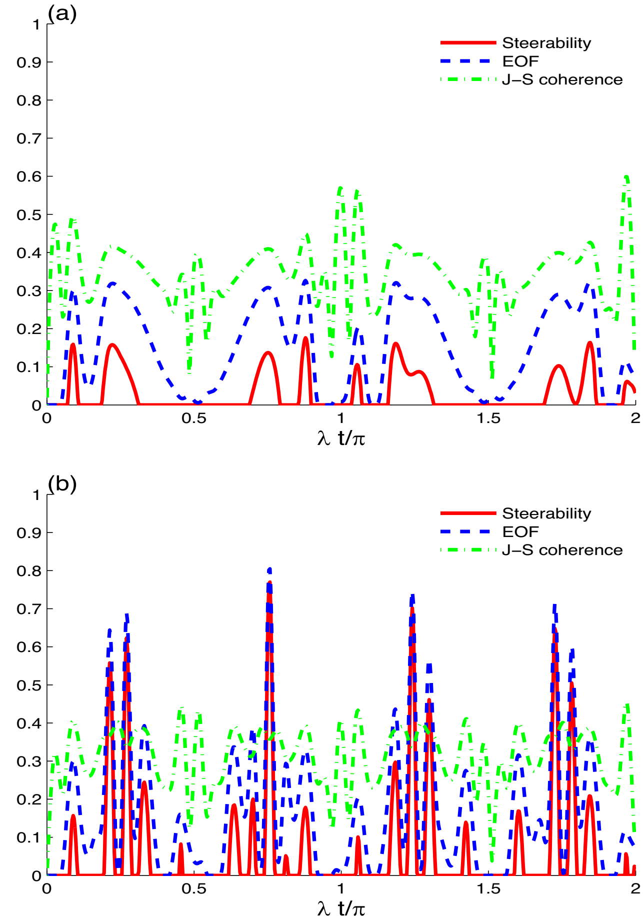

Figure 1: Time evolution of steerability, EOF, and J-S coherence when the two qubit initially enter a GBG coherent cavity in (a) and a GBG even coherent cavity in (b) with $p = \frac{1}{4}$, $\eta = 7$, $K = 0.0$, and $\gamma = 10^5$.

4 Steerability, entanglement, and coherence dynamics

The generation of the steerability, EOF, and J–S coherence will be explored under the influence of the following physical parameters: (1) the nonlinear interactions between the two dipole-coupled qubit and the two nondegenerate coherent cavities. (2) The non-classicality of the superposition of GBG coherent states. (3) The dipole coupling of the two- SU(2)-qubit, represented by the term: $K|1_A 0_B\rangle\langle 0_A 1_B| + K|0_A 1_B\rangle\langle 1_A 0_B|$. (4) The ID coupling induced by the SU(1,1)-SU(2) interactions.

The results in Figure 1 demonstrate that the generation of steerability, EOF, and J–S coherence in the two-SU(2) qubit, due to nonlinear qubit–cavity interactions, significantly depends on the non-classicality of the superposition of the GBG coherent states. In Figure 1(a), the time evolution of steerability, EOF, and coherence (measured by the square root of the J–S divergence) are shown for the initial GBG coherent cavity with $\eta = 7$, without considering the effects of SU(1,1)-SU(2) ID and dipole two-SU(2)-qubit interactions. We observe that EPR steering and EOF arise suddenly from zero, causing the generated two-SU(2)-qubit states having a partial steerability and entanglement. Meanwhile, the J–S coherence and EOF display distinct quasi-periodic dynamics with a period of π. As seen from the EOF curve, the sudden birth and death phenomena [69] occur for the two-SU(2)-qubit entanglement at the ends of the disentanglement intervals. Recently, these sudden birth and death entanglement phenomena have been investigated in various real two-qubit systems, such as disordered quantum systems assisted by auxiliary qubit [70], two atoms interacting with a coherent cavity [71], and magnons with an atomic ensemble linked by an optical cavity [72]. Additionally, in this figure, the generated two-SU(2)-qubit entanglement nonlocality develops more rapidly than the EPR steering. After a specific time, known as the “steerability birth time (SBT),” the EPR steering suddenly increases to its partial steerability. The steerability of the two-SU(2)-qubit exhibits a sudden death for a particular time interval. At the ends of these sudden birth–death intervals in the EPR steering dynamics, the steerability of the two-qubit experiences sudden arising and sudden annihilation phenomena, which have been experimentally confirmed [73]. From the irregular oscillatory dynamics of the steerability and EOF, we find that the two SU(2)-qubit do not return to their initial pure state during the disappearance intervals of the EPR steering nonlocality. This indicates that the generated two-SU(2)-qubit states possess EOF during these intervals, i.e., there is an entangled two-SU(2)-qubit state during the unsteerability intervals. The results confirm that the two-qubit steerability and EOF dynamics agree with the hierarchy principle [45–47]. The dash-dotted curve indicates that the nonlinear interactions of the two qubit with GBG coherent states have a high capacity to store the J–S coherence and the EOF in the two-SU(2)-qubit state. The amount of stored coherence, measured by the J–S divergence square root, is greater than that of the nonlocality related to the EOF and EPR steering. The generated two-SU(2)-qubit states exhibit the J–S coherence during the disappearance intervals of steerability and EOF.

In Figure 1(b), the time evolution of steerability, EOF, and J–S coherence of Figure 1(a) are shown but with the initial GBG even coherent cavity with $\eta = 7$. We observe that the non-classicality of GBG even coherent states generates more steerability, EOF, and J–S divergence coherence. When two nondegenerate cavities with nonlinear 2-photonic transitions are filled with GBG even coherent fields, the high non-classicality enhances the generated amount of steerability and EOF. In this scenario, the amplitudes and frequencies of the irregular oscillatory dynamics of the two-SU(2)-qubit' steerability and EOF increase. However, for the J–S divergence square root, its amplitudes decrease while its frequencies increase. Moreover, the intervals of the sudden birth and death phenomena for the steerability decrease, while those for the EOF increase.

Figures 2 and 3 demonstrate that the generation of the two-SU(2)-qubit' steerability, EOF, and J–S divergence coherence – resulting from nonlinear interactions between the two dipole-coupled qubit and the two nondegenerate coherent cavities – depends on the dipole two-SU(2)-qubit interaction coupling $K = 20\lambda$. The dipole two-SU(2)-qubit interaction coupling influences the term $K|1_A 0_B\rangle\langle 0_A 1_B| + K|0_A 1_B\rangle\langle 1_A 0_B|$ in the Hamiltonian of Eq. (1). This term involves the two-SU(2)-qubit operators, which naturally transform the pure state $|1_A 0_B\rangle$ (lacking two-SU(2)-qubit nonlocality and coherence) into an entangled state $(\alpha_1|1_A 0_B\rangle + \alpha_2|0_A 1_B\rangle)$, where $|\alpha_1|^2 + |\alpha_2|^2 = 1$. This entangled state is close to exhibiting two-SU(2)-qubit nonlocality and coherence. Additionally, Figures 2 and 3 confirm the transformation of quantum information resources to two-

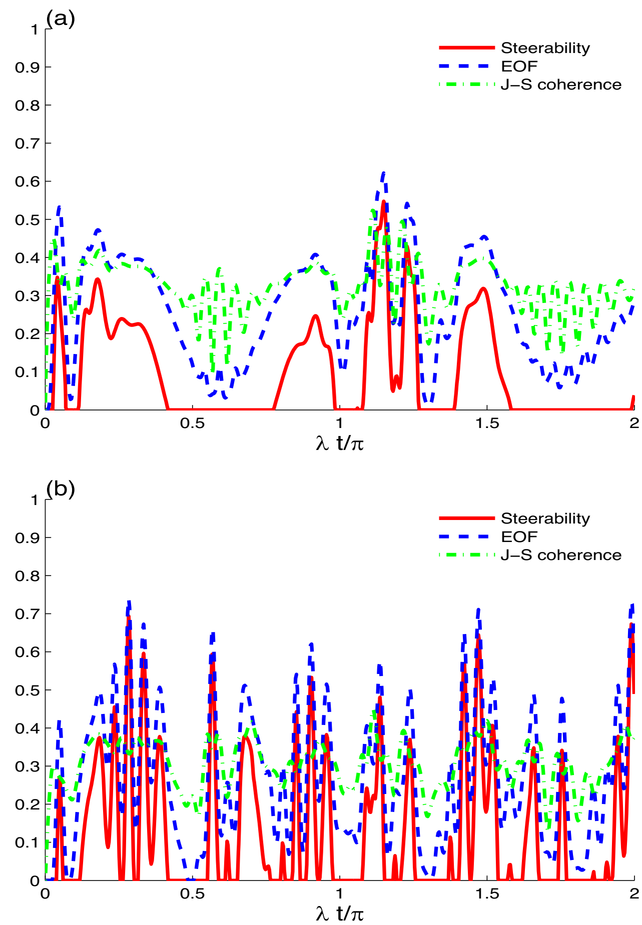

Figure 2: Time evolution of steerability, EOF, and J–S coherence of Figure 1 but for $K = 20\lambda$.

SU(2)-qubit. Figure 2(a) illustrates that the nonlinear interactions enhance the amplitudes and frequencies of the irregular oscillatory dynamics of the steerability, J–S divergence coherence, and EOF. The intervals of the sudden birth and death phenomena for the EOF have completely disappeared. The time intervals of the generated non-steerable states, which exhibit sudden arising and sudden annihilation phenomena at their ends, are clearly visible in the two-SU(2)-qubit' steerability. The generated two-SU(2)-qubit states exhibit J–S coherence and entanglement during the disappearance intervals of steerability. Figure 2(b) illustrates the impact of the high non-classicality of GBG even coherent states on the generation of steerability, EOF, and J–S divergence square root coherence. The non-classicality of GBG even coherent states enhances the amplitudes and frequencies of steerability and EOF. However, for J–S divergence square root coherence, the amplitudes decrease. Additionally, the sudden birth and death phenomena of steerability are reduced.

Figure 2(b) illustrates the impact of the high non-classicality of the initial GBG even coherent state on the

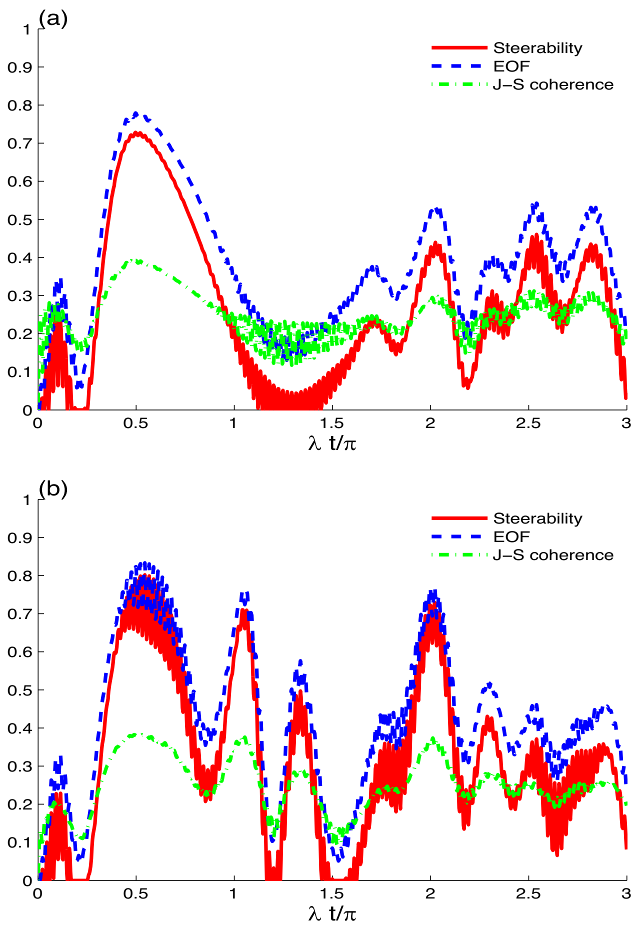

Figure 3: Time evolution of steerability, EOF, and J–S coherence of Figure 1 but for strong detuning $K = 80\lambda$.

generation of steerability, entanglement, and J–S coherence. The non-classicality of GBG even coherent states enhances the amplitudes and frequencies of steerability and entanglement. However, for J–S divergence square root coherence, the amplitudes are decreased. Additionally, the sudden birth and death phenomena of the steerability are reduced.

Figure 3 shows the oscillatory dynamics of the two-SU(2)-qubit quantifiers of Figure 1, but with a strong detuning of $K = 80\lambda$. From Figure 3(a), we observe that the generated steerability, EOF, and J–S coherence are more significant compared to the case of $K = 20\lambda$. This enhancement is due to the strong detuning of the two-SU(2) qubit. The strong detuning increases the amplitudes and frequencies of the two-SU(2)-qubit quantifiers. However, the sudden birth and death phenomena and the time disappearance intervals of the two-SU(2) qubit' steerability are reduced. Meanwhile, the sudden arising and annihilation phenomena of the generated entanglement have completely disappeared. Figure 3(b) shows the effect of the non-classicality of GBG even coherent states on the generation of steerability, EOF, and J–S coherence. The generated nonlocality and J–S coherence can be enhanced by increasing the detuning of the two-SU(2) qubit and the initial non-classicality. The irregular oscillatory dynamics of the two-SU(2)-qubit quantifiers exhibit more secondary oscillations.

Figures 4 and 5 illustrate the time evolution of steerability, EOF, and J–S coherence of Figure 1, taking into account a weak SU(1,1)-SU(2) ID coupling of $\gamma = 10^3/3$. After accounting for decoherence, Figure 4(a) demonstrates that the amplitudes and frequencies of the generated two-SU(2)-qubit' quantum information resources have decayed. The coherence and entanglement evolve into stationary sine curves, with maxima occurring at times $\lambda t = \frac{1}{4}(2n+1)\pi$ $(n = 0, 1, 2, \dots)$ and minima at times $\lambda t = \frac{n}{2}\pi$ $(n = 1, 2, \dots)$. Meanwhile, the generated two-SU(2)-qubit' steerability reduces to a stationary zero value. The generated two-SU(2)-qubit states exhibit stationary J–S divergence coherence

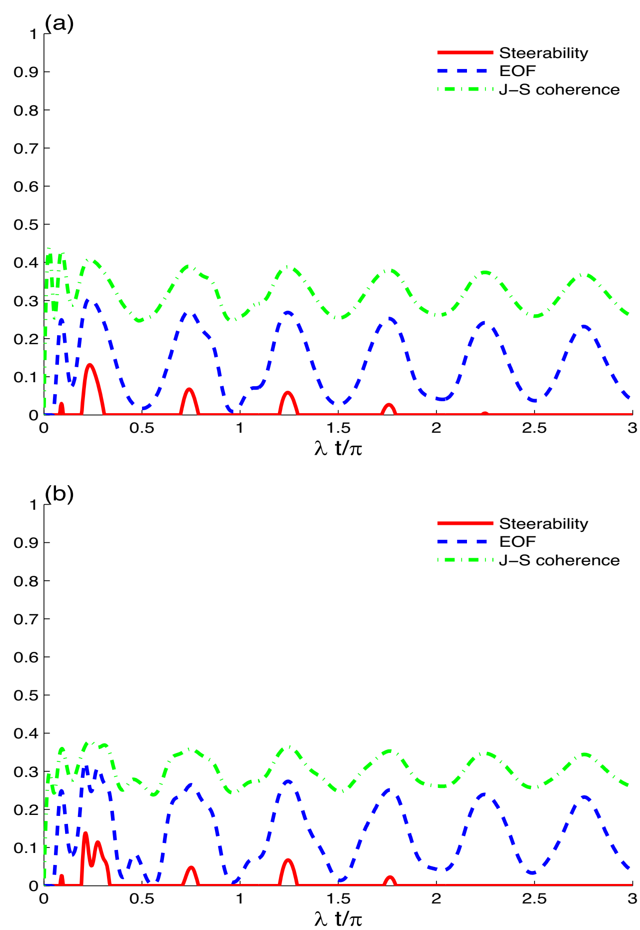

Figure 4: Time evolution of steerability, EOF, and J–S coherence of Figure 1 but when taking into consideration the weak the SU(1,1)–SU(2) ID coupling $\gamma = 10^3/3$.

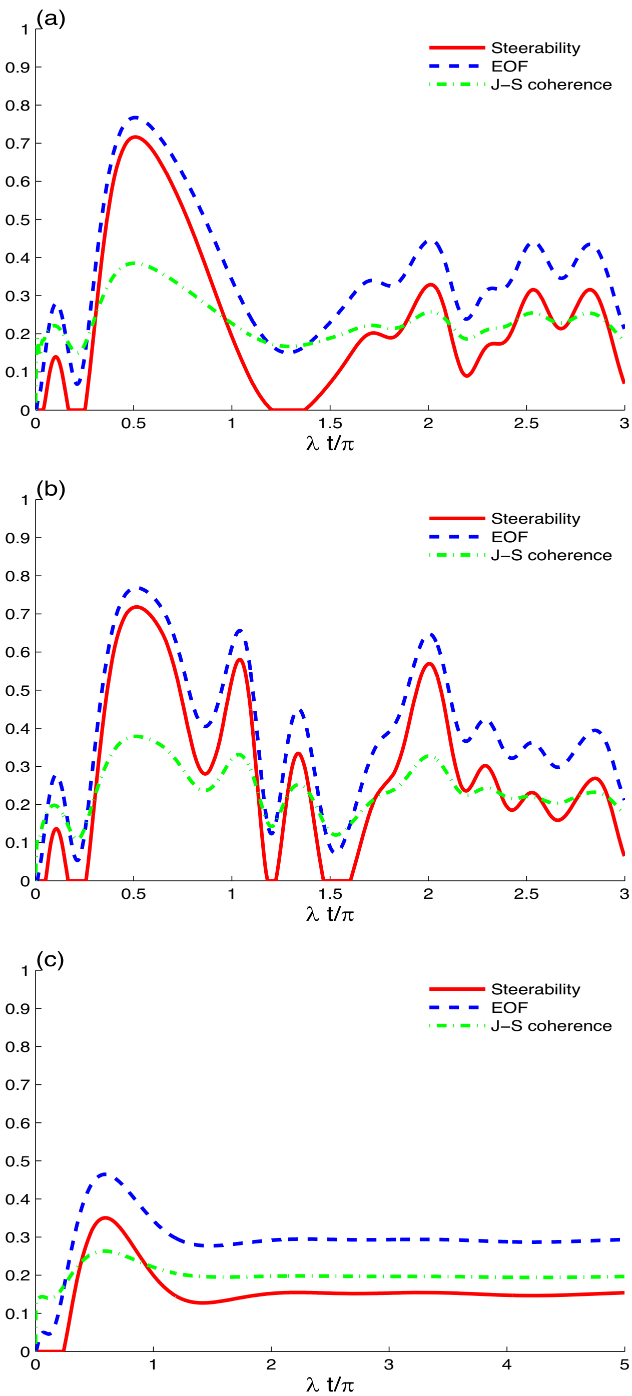

Figure 5: Time evolution of steerability, EOF, and J–S coherence of 3 but for the weak SU(1,1)–SU(2) ID coupling $\gamma = 10^3/3$ in (a,b). In Figure 3(c) is plotted for the strong SU(1,1)–SU(2) ID coupling $\gamma = 5$.

and stationary entanglement during the disappearance intervals of steerability. These disappearance intervals increase with the SU(1,1)–SU(2) ID coupling. The robustness against ID of the J–S coherence, steerability, and entanglement depends on the non-classicality of the initial GBG coherent states, as shown in Figure 4(b).

Figure 5(a) and (b) shows the time evolution of quantum information resources of Figure 3, but with a weak SU(1,1)–SU(2) ID coupling $\gamma = 10^3/3$.

As time increases, the amplitudes and densities of the irregular oscillations associated with steerability, J–S divergence coherence, and EOF are reduced. The secondary oscillations completely disappear. Figure 5(b) demonstrates the robustness against intrinsic SU(1,1)-SU (2) decoherence of the generated nonlocality and coherence dynamics when the two nonlinear nondegenerate cavities are filled with GBG even coherent fields. We find that the appearance of the sudden arising and sudden annihilation phenomena of the generated two-SU(2)-qubit' steerability depends on the non-classicality of the two nonlinear nondegenerate cavities. The J–S divergence coherence generated with the initial GBG coherent fields is more robust than that generated with the initial GBG even coherent fields. We find that with a strong ID coupling $\gamma = 5$ and two-SU(2)-qubit detuning $K = 80\lambda$, the initial two-SU(2)-qubit state, which initially has no nonlocality or coherence, evolves into a steady state with partial stationary steerability, entanglement, and coherence. Their stationary values agree with the hierarchy principle [45–47]. This means that the strong SU(1,1)–SU(2) ID coupling does not completely destroy the generated two-SU(2)-qubit' nonlocality and coherence. But it generates stationary correlated two-SU(2)-qubit states [74,75]. We find that the partial stationary steerability, entanglement, and coherence depend on the non-classicality of the superposition of the GBG coherent states, the dipole two-SU(2)-qubit interaction coupling, and the intrinsic SU(1,1)–SU(2) interaction decoherence coupling.

5 Conclusion

In this study, we use the ID model of the Milburn equation to study the dynamics of two dipole-coupled qubit embedded in spatially separated nondegenerate coherent cavities with nonlinear photonic transitions. Each qubit is initially prepared in its excited state, while the cavities start with a superposition of generalized Barut–Girardello coherent fields. The decoherence of steerability, entanglement, and coherence dynamics of the generated two dipole-coupled qubit states is analyzed using EPR steering, EOF, and the square root of J–S divergence. The generation of steerability, entanglement and coherence in the two qubit has been investigated under the following effects: nonlinear qubit–photon interactions, the non-classicality of the superposition of the GBG coherent fields, the dipole interaction coupling between the two coupled qubit, and the intrinsic qubit–cavity interaction decoherence. Our results confirm that the dynamics of steerability and entanglement formation follow the hierarchy principle. However, nonlocality related to entanglement formation appears during intervals of unsteerability. Moreover, high non-classicality of initial GBG coherent fields enhances EPR steering and entanglement, while decreasing amplitudes and increasing frequencies of J–S coherence. Strong detuning of the two SU(2) qubit increases both amplitudes and frequencies of steerability, J–S coherence, and entanglement. With weak decoherence, the amplitudes and frequencies of steerability, J–S divergence coherence, and entanglement reduce, resulting in time-independent stationary behaviors for coherence and entanglement. Finally, it is observed that sudden arising and annihilation phenomena, occurring with steerability and entanglement formation, are significantly influenced by the initial non-classicality of the cavities, dipole coupling between qubit, and ID.

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