Research Article

Khaled Lotfy*, Ibrahim S. Elshazly, Borhen Halouani*, Anand K. Yadav, and Eslam S. Elidy

Wave propagation in nonlocal piezo-photo-hygrothermoelastic semiconductors subjected to heat and moisture flux

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Abstract: This research explores the propagation characteristics of piezo-photo-hygrothermoelastic waves in orthotropic semiconductor materials by incorporating the effect of nonlocal elasticity. The proposed model describes the complex interactions between piezoelectricity, photothermal excitation, plasma wave transport, and hygrothermal diffusion under continuous heat and moisture fluxes while accounting for the influence of spatial nonlocality on the elastic response. The nonlocal parameter is introduced into the elastic constitutive relation to capture the size-dependent and dispersive nature of elastic waves, which is essential for semiconductor devices operating at small scales. The study develops a generalized system of coupled equations involving the elastic, thermal, moisture, and plasma fields, all integrated within the framework of nonlocal piezo-photo-hygrothermoelasticity. The system is analytically solved using the normal mode technique to derive expressions for temperature distribution, moisture concentration, carrier density, electric potential, displacement fields, and stresses. Numerical simulations are carried out for the cadmium selenide (CdSe) semiconductor material. The results show that the nonlocal parameter significantly modifies the wave amplitudes, attenuation characteristics, and coupling between the involved physical fields. The proposed model offers valuable insights into the dynamic behavior of semiconductor devices subjected to coupled thermo-mechanical–electrical–moisture interactions, especially when nonlocal effects become significant due to micro- and nanoscale dimensions.

Keywords: piezo-photo-hygrothermoelasticity, semiconductor, nonlocal, plasma waves, heat and moisture diffusion, wave propagation

*** Corresponding author: Khaled Lotfy,** Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt, e-mail: khlotfy@zu.edu.eg
*** Corresponding author: Borhen Halouani,** Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh, 11451, Saudi Arabia, e-mail: halouani@ksu.edu.sa
Ibrahim S. Elshazly: Department of Basic Sciences, Common First Year, King Saud University, Riyadh, 11451, Saudi Arabia, e-mail: iali2.c@ksu.edu.sa
Anand K. Yadav: Shishu Niketan Model Senior Secondary School, Sector 22-D, Chandigarh, India, e-mail: yadavanand977@gmail.com
Eslam S. Elidy: Department of Mathematics, Faculty of Science, Zagazig University, P.O. Box 44519, Zagazig, Egypt, e-mail: eslamelidy@yahoo.com
ORCID: Khaled Lotfy 0000-0001-9383-1361

Nomenclature

T	thermal temperature
c_{ijkl}	elastic stiffness tensor
Q_{ij}^{T}	coupled thermal diffusivity
φ	electric potential
d_n^{T}	electronic deformation coefficient
$\tau_i,\ d_i$	pyroelectric constants
J	moisture concentration potential
q_{ij}^{m}	coupled moisture diffusivities
α_{ij}^{T}	coupling thermal material coefficients
D_{ij}^{T}	temperature diffusivity
κ	thermal activation coupling parameter
τ	photogenerated carrier lifetime
E_g	energy gap of the semiconductor
δ_n^{T}	difference in the deformation potential of the conduction and valence bands
κ	thermal activation coupling parameter
e_{ij}	component of strain
β_{ij}^{T}	thermal moduli tensor
D_i	electric displacement
$\eta_{ijk},\ \varepsilon_{ij}$	piezothermal moduli tensors
E_i	electric field density
C_{E}	specific heat at constant strain
k_{m}	moisture diffusion constant
ρ	density
β_{ij}^{m}	moisture moduli tensor
D_{ij}^{m}	moisture diffusivity
σ_{ij}	component of stress

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δ_{ij} Kroneckar delta
$\frac{E_g N^*}{\tau}$ photoexcitation effect

1 Introduction

The study of wave propagation in semiconductor materials subjected to coupled thermoelastic, piezoelectric, photothermal, plasma, and moisture diffusion effects has become increasingly important due to its wide applications in microelectronics, optoelectronics, MEMS/NEMS, and advanced sensing technologies. In such materials, the interaction between mechanical, thermal, electric, photothermal, and hygrothermal fields gives rise to a variety of waves whose dynamics directly affect the performance, stability, and lifespan of semiconductor-based devices. These interactions are particularly significant under harsh service conditions involving thermal shocks, moisture ingress, and high-frequency excitations. Recently, the importance of nonlocal elasticity has emerged as a critical factor when analyzing wave propagation in microscale and nanoscale systems, where classical local continuum theories fail to capture the observed dispersive and size-dependent behaviors. The inclusion of the nonlocal parameter accounts for the long-range interatomic interactions that significantly modify the elastic response, especially at a small scales. As semiconductor devices continue to miniaturize, understanding the interplay between piezoelectric, photothermal, moisture, plasma, and nonlocal elastic effects becomes essential for the accurate design and optimization of high-performance components. This coupled and nonlinear multiphysical problem forms the basis of modern research aimed at improving the functionality and reliability of next-generation semiconductor technologies. This has led to the urgent need for more accurate and comprehensive models capturing the simultaneous effects of piezoelectricity, photothermal excitation, plasma waves, and moisture transport in semiconductors. In addition to the classical couplings, the mechanical response of semiconductor materials at small scales is notably affected by nonlocal elastic effects, where the stress at a given point is influenced by strains in its neighborhood rather than just the local strain. The presence of nonlocality becomes critical when wave phenomena are studied in micro- and nanoscale semiconductor structures, as classical local theories often fail to account for observed dispersive and size-dependent behaviors. However, the majority of existing investigations have either neglected the nonlocal influence or restricted it to simple elasticity without considering its interaction with hygrothermoelastic, piezoelectric, and photothermal fields.

Previous research has provided valuable contributions to the understanding of hygrothermal and thermoelastic interactions in solids. The interaction of temperature and moisture diffusion has been explored in polymers, composite structures, and geomaterials, where the coupled diffusion of heat and moisture significantly affects stress development [1–6]. The foundation of thermoelasticity itself was laid by Biot [7], and later expanded through the development of generalized thermoelasticity theories [8,9] and their application in modeling thermal, electromagnetic, and elastic wave interactions [10–14]. Similarly, the theory of piezo-thermoelasticity, introduced by Nowacki [15,16] and further developed by Mindlin [17,18], accounts for the coupling between thermal, mechanical, and piezoelectric effects. Chandrasekharaiah [19] refined this theory by introducing finite wave propagation speeds to better describe wave behaviors in realistic materials.

The elastic response of semiconductors has also been extensively studied due to their widespread applications in electronics and optoelectronics. Early contributions from Mindlin [20] highlighted the significance of piezoelectricity in semiconductors, and subsequent studies [21–25] examined how photothermal and plasma wave effects influence the dynamic behavior of these materials. Further developments were achieved by Khamis *et al.* [22] and Lotfy [26], who examined photothermal and thermoelastic wave propagation under dual-phase-lag and memory-dependent models [27]. Hobiny and Abbas [28] further extended the framework by incorporating fractional-order thermoelastic theories in semiconductor materials. While these models provided valuable insights, they largely relied on classical elasticity, neglecting the potential influence of nonlocal elasticity on wave propagation.

Motivated by this limitation, the present work proposes a comprehensive theoretical model that incorporates nonlocal elasticity into the piezo-photo-hygrothermoelastic wave propagation framework for orthotropic semiconductor materials [29–31]. The model fully captures the simultaneous interactions between elastic, photothermal, piezoelectric, plasma, and moisture diffusion fields, while accounting for the size-dependent effects inherent in nonlocal elastic theory [32]. The inclusion of the nonlocal parameter in the constitutive relations leads to modified wave characteristics, including dispersion and attenuation patterns, which are critical for accurately predicting wave propagation in micro- and nanoscale semiconductor devices. In practical scenarios, semiconductor devices are often designed with characteristic dimensions at micro- and nanoscales, where nonlocal effects cannot be ignored [33–35]. While several models have addressed hygrothermal stresses in polymeric and composite systems, and generalized thermoelastic and piezo-thermoelastic responses in semiconductors, only a few studies have explored nonlocal elasticity in the context of coupled piezo-photo-hygrothermoelastic wave propagation [36,37]. The proposed model fills this gap by integrating nonlocal elasticity with hygrothermoelastic and photothermal effects, enabling a more accurate description of the coupled wave behavior [38].

Wave propagation in semiconductor and elastic media has been extensively investigated in recent years, particularly in the presence of complex coupled effects such as porosity, size dependence, nonlocality, and thermo-piezoelectric interactions. Studies have explored the impact of double and single porosity on SH-wave vibrations in periodic porous lattices, revealing significant alterations in wave behavior due to porosity variations [39]. Additionally, the interaction of surface waves in micropolar thermoelastic media with dual pore connectivity has been analyzed, demonstrating size-dependent effects on wave dispersion and attenuation [40]. Recent work has also addressed moisture and temperature diffusivity in orthotropic hygro-thermo-piezo-elastic materials, providing insights into coupled moisture–temperature interactions [41]. The role of nonlocality in shear wave propagation within fiber-reinforced poroelastic structures has been examined, highlighting its influence on wave speed and stability under interfacial disturbances [42]. Furthermore, vibrational analysis of size-dependent thermo-piezo-photoelectric semiconductor media has been conducted under the memory-dependent Moore–Gibson–Thompson photo-thermoelasticity framework, emphasizing the significance of memory effects in advanced semiconductor applications [43]. Additional studies have explored double poro-magneto-thermoelastic models incorporating micro-temperature and initial stress under memory-dependent heat transfer, shedding light on thermal relaxation effects in complex elastic media [44]. The propagation of Rayleigh waves in nonlocal piezo-thermoelastic materials with voids under memory-dependent heat transfer has also been examined, showcasing the intricate interplay between thermal memory and wave characteristics [45]. In a different context, flexoelectric effects and viscoelastic coatings have been shown to significantly alter surface wave dynamics in advanced geomaterial plates [46]. Moreover, fractional and memory effects on wave reflection in prestressed microstructured solids with dual porosity have been investigated, offering new insights into wave behavior in engineered materials [47]. The study of wave propagation in complex elastic media has been significantly advanced by incorporating micropolar, microstretch, and diffusion effects, leading to more refined and accurate physical models. While these studies have advanced our understanding of wave propagation in various structured media, the combined effects of nonlocality, temperature-dependent thermal conductivity, and orthotropic piezoelectric micropolar interactions in semiconductor materials remain largely unexplored [48]. The present study aims to bridge this gap by developing a comprehensive model that integrates these factors, providing a more realistic representation of wave dynamics in semiconductor devices at micro- and nanoscales according to microstretch thermoelastic diffusion [49]. Although several studies have been conducted on wave propagation in piezo-photo-thermoelastic and hygrothermoelastic semiconductor media, most of them have ignored the influence of nonlocal effects, which become critical when dealing with micro- and nanoscale semiconductor structures. The neglect of nonlocal elasticity in previous investigations leads to inaccurate predictions of stress, displacement, and temperature distributions, especially under high-frequency or small-scale conditions. Furthermore, the combined interaction of piezoelectric, photothermal, and hygrothermal fields in the presence of nonlocality has not been adequately addressed. Motivated by these shortcomings, the present work aims to develop a comprehensive theoretical model that incorporates nonlocal elasticity within the framework of photo-thermoelasticity for piezo-photo-hygrothermoelastic semiconductors. This formulation provides a more realistic understanding of the coupled wave propagation behavior, capturing the essential physics that was previously overlooked.

The primary goal of this work is to investigate how nonlocal elasticity affects the transient and steady-state responses of the semiconductor medium when exposed to heat, moisture, and photothermal excitations. Using the normal mode technique, the governing equations are analytically solved, and the obtained expressions describe the distributions of displacement, temperature, carrier density, moisture concentration, stress, and electric fields. Numerical simulations are conducted for the cadmium selenide (CdSe) semiconductor material to demonstrate the impact of the nonlocal parameter on wave profiles. The results show that the presence of nonlocal elasticity leads to substantial changes in wave amplitudes, attenuation rates, and coupling characteristics between the various fields, providing deeper insights into the behavior of semiconductor devices operating under coupled thermo-electrical–mechanical–moisture environments, particularly at small scales.

2 Basic equation

This section contains the fundamental governing equations describing the propagation of coupled piezo-photo-hygro-thermoelastic waves in an orthotropic nonlocal semiconductor with temperature-dependent thermal conductivity. In this study, we consider an orthotropic semiconductor medium subjected to simultaneous mechanical, thermal, electric, and moisture effects, where the photothermal excitation and moisture flux are applied on the surface. The material is assumed to be homogeneous, electrically and thermally conductive, and exhibits piezoelectric characteristics. The coordinate system is chosen such that the x-axis is normal to the surface of the semiconductor, while the z-axis lies along the surface, describing a semi-infinite domain ($x \geq 0$) (Figure 1). The physical quantities involved in the formulation include the displacement vector $u(\vec{r}, t)$, temperature change $T(\vec{r}, t)$, moisture concentration $m(\vec{r}, t)$, and carrier density $N(\vec{r}, t)$. These variables collectively describe the piezo-photo-hygrothermoelastic behavior of the nonlocal semiconductor, and their interactions will be mathematically modeled in the following sections [21,22].

The stress–strain–temperature–moisture relation for an orthotropic piezo-hygrothermoelastic nonlocal semiconductor is given by

$$(1 - \varepsilon^2 \nabla^2)\sigma_{ij} = \sigma'_{ij} = c_{ijkl}e_{kl} - \beta_{ij}^{\mathrm{T}} T\delta_{ij} - \delta_{ij}^{\mathrm{N}} N^* \delta_{ij} - \beta_{ij}^{\mathrm{m}} m\delta_{ij} - \eta_{ijk}E_k, \tag{1}$$

$$D_i = \eta_{ijk}e_{jk} + \varepsilon_{ij}E_j + \tau_i T + d_i m, \qquad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad E_j = -\varphi_{,j}. \tag{2}$$

The nonlocal parameter ε characterizes the influence of long-range interatomic interactions within the material, which plays a crucial role in capturing size-dependent mechanical behavior, especially when dealing with materials at the micro- and nanoscale. In the experimental findings reported by Yadav *et al.* [50–52], it has been identified that there exists a mutual interaction between the traditional Fourier heat flux and Fick moisture flux, leading to the emergence of additional cross-coupling fluxes commonly referred to as the Dufour and Soret effects. Specifically, the heat conduction process is influenced by moisture gradients, giving rise to the Dufour flux, while the moisture diffusion process is affected by temperature gradients, resulting in the Soret flux. Consequently, the expressions for both the total heat flux q_i and the total moisture flux f_i are modified to include these coupling terms, as presented by Yadav *et al.* [50–52]:

$$q_i = -D_{ij}^{\mathrm{T}} T_{,i} - q_{ij}^{\mathrm{m}} m_{,i}\,, \tag{3}$$

$$f_i = -D_{ij}^{\mathrm{m}} m_{,i} - Q_{ij}^{\mathrm{T}} T_{,i}\,. \tag{4}$$

In this formulation, q^{F} represents the conventional Fourier heat flux, which is governed by the thermal diffusivity (D^{T}) and is expressed as $q_i^{\mathrm{F}} = -D_{ij}^{\mathrm{T}} T_{,i}$. Similarly, f^{F} denotes the classical Fick moisture flux potential, which depends on the moisture diffusivity (D^{m}) and is defined as $f_i^{\mathrm{F}} = -D_{ij}^{\mathrm{m}} m_{,i}$. The term q^{D} refers to the additional heat flux generated by the Dufour effect, while f^{s} corresponds to the additional moisture flux arising from the Soret effect. These cross-effects introduce new coupling parameters, namely the cross-diffusivities q_{ij}^{d} and Q_{ij}^{T}, which describe the interaction between heat and moisture transport mechanisms. Specifically, the Dufour-related contribution is expressed as

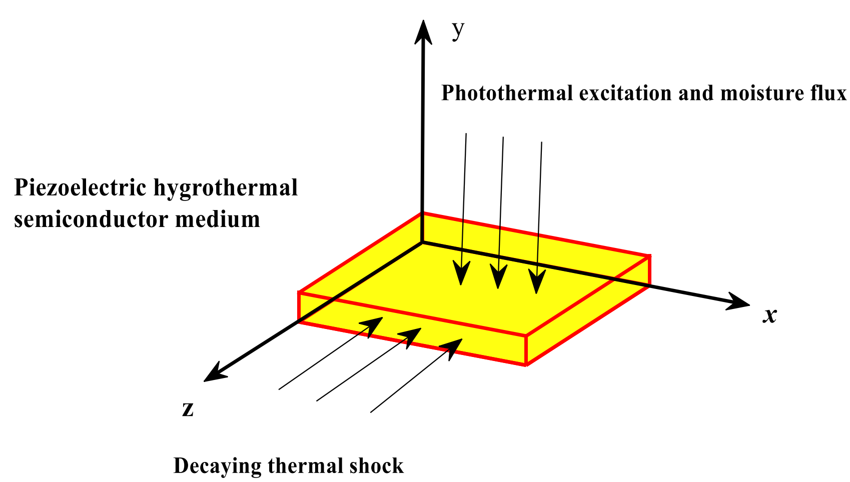

Figure 1: Geometry of the problem.

$q_i^{\mathrm{D}} = -q_{ij}^{\mathrm{d}} m_{,i}$, and the Soret-induced moisture flux is given by $f_i^{\mathrm{s}} = -Q_{ij}^{\mathrm{T}} T_{,i}$ [50].

Considering $c_{ijkl} = c_{ijkl}$, $\beta_{ij}^{\mathrm{T}} = \beta_i^{\mathrm{T}} \delta_{ij}$, $\beta_{ij}^{\mathrm{m}} = \beta_i^{\mathrm{m}} \delta_{ij}$, $\delta_{ij}^{\mathrm{N}} = \delta_i^{\mathrm{N}} \delta_{ij}$, $q_{ij}^{\mathrm{d}} = q_i^{\mathrm{d}} \delta_{ij}$, $Q_{ij}^{\mathrm{T}} = Q_i^{\mathrm{T}} \delta_{ij}$, $D_{ij}^{\mathrm{E}} = D_i^{\mathrm{E}} \delta_{ij}$ yields

$$\rho T_0 \dot{S} = -q_{i,i}, \tag{5}$$

$$\rho T_o S = T + \frac{B_i^{\mathrm{T}} T_o}{\rho C_{\mathrm{E}}} e_{kk} - T_0 \tau_i \frac{\partial \varphi}{\partial z}. \tag{6}$$

From Eqs. (5) and (6),

$$-q_{i,i} = \dot{T} + \frac{B_i^{\mathrm{T}} T_o}{\rho C_{\mathrm{E}}} \dot{e}_{kk} - T_0 \tau_k \frac{\partial \dot{\phi}}{\partial z}. \tag{7}$$

Taking into account the Thomson effect within Fourier's heat conduction law, and considering the additional influence of plasma on heat transport while disregarding the Peltier contribution, the modified heat conduction behavior for semiconductors can be formulated [51,52]:

$$q_i = -D_i^{\mathrm{T}} T_{,i} - q_i^{\mathrm{d}} m_{,i} - \int \frac{E_{\mathrm{g}}}{\tau} N^* \mathrm{d}x_i. \tag{8}$$

By differentiating Eq. (8) for x_i, the following relation is obtained [53]:

$$q_{i,i} = -D_i^{\mathrm{T}} T_{,ii} - q_i^{\mathrm{d}} m_{,ii} - \frac{E_{\mathrm{g}}}{\tau} N^*. \tag{9}$$

Substituting Eq. (9) into Eq. (10) yields the governing equation for temperature and plasma diffusion within the piezoelectric hygrothermal semiconductor medium [54]:

$$D_i^{\mathrm{T}} T_{,ii} + q_i^{\mathrm{d}} m_{,ii} + \frac{E_{\mathrm{g}}}{\tau} N^* - \dot{T} - \frac{B_i^{\mathrm{T}} T_0}{\rho C_{\mathrm{E}}} \dot{e}_{kk} + T_0 \tau_k \frac{\partial \dot{\phi}}{\partial z} = 0. \tag{10}$$

Similarly, the moisture-plasma diffusion equation governing the behavior in a piezoelectric hygrothermal semiconductor medium can be expressed as [31]

$$\begin{aligned} &D_i^{\mathrm{m}} m_{,ii} + Q_i^{\mathrm{T}} T_{,ii} + \frac{E_{\mathrm{g}}}{\tau} N^* - \dot{m} - \frac{B_i^{\mathrm{m}} m_0 D_i^{\mathrm{m}}}{k_{\mathrm{m}}} \dot{u}_{j,j} \\ &\quad + m_0 d_k \frac{\partial \dot{\phi}}{\partial z} = 0. \end{aligned} \tag{11}$$

The equation of motion for a nonlocal semiconductor medium [32,33] is

$$\rho(1 - \xi^2 \nabla^2) \frac{\partial^2 u_i(r, t)}{\partial t^2} = \sigma'_{ij,j}. \tag{12}$$

3 Problem formulation

By substituting Eqs. (1) and (2) into Eq. (12), the governing equations for the two-dimensional (2D) case are obtained under the assumption $\vec{u}(x, y, z, t) = (u, 0, w)$ that [55]

$$\begin{aligned} &c_{11} \frac{\partial^2 u}{\partial x^2} + c_{55} \frac{\partial^2 u}{\partial z^2} + (c_{13} + c_{55}) \frac{\partial^2 w}{\partial x \partial z} - \delta_1^{\mathrm{N}} \frac{\partial N^*}{\partial x} \\ &\quad - \beta_1^{\mathrm{T}} \frac{\partial T}{\partial x} - \beta_1^{\mathrm{m}} \frac{\partial m}{\partial x} + (\eta_{31} + \eta_{15}) \frac{\partial^2 \varphi}{\partial x \partial z} \\ &\quad = \rho(1 - \varepsilon^2 \nabla^2) \frac{\partial^2 u}{\partial t^2}, \end{aligned} \tag{13}$$

$$\begin{aligned} &c_{55} \frac{\partial^2 w}{\partial x^2} + c_{33} \frac{\partial^2 w}{\partial z^2} + (c_{13} + c_{55}) \frac{\partial^2 u}{\partial x \partial z} - \delta_3^{\mathrm{N}} \frac{\partial N^*}{\partial z} \\ &\quad - B_3^{\mathrm{T}} \frac{\partial T}{\partial z} - \beta_3^{\mathrm{m}} \frac{\partial m}{\partial z} + \eta_{15} \frac{\partial^2 \varphi}{\partial x^2} + \eta_{33} \frac{\partial^2 \varphi}{\partial z^2} \\ &\quad = \rho(1 - \varepsilon^2 \nabla^2) \frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{14}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$.

The corresponding 2D plasma-thermal diffusion equation governing the interaction between the carrier density and temperature fields is expressed as follows [56]:

$$\frac{\partial N^*}{\partial t} = \left(D_1^{\mathrm{E}} \frac{\partial^2 N^*}{\partial x^2} + D_3^{\mathrm{E}} \frac{\partial^2 N^*}{\partial z^2} \right) - \frac{N^*}{\tau} + \kappa T. \tag{15}$$

The 2D heat-plasma diffusion equation for the piezoelectric hygrothermal semiconductor medium, derived from Eq. (10), can be expressed as

$$\begin{aligned} &\left(D_1^{\mathrm{T}} \frac{\partial^2 T}{\partial x^2} + D_3^{\mathrm{T}} \frac{\partial^2 T}{\partial z^2} \right) + \left(q_1^{\mathrm{m}} \frac{\partial^2 m}{\partial x^2} + q_3^{\mathrm{m}} \frac{\partial^2 m}{\partial z^2} \right) \\ &\quad + \frac{E_{\mathrm{g}}}{\tau} N^* - \frac{\partial T}{\partial t} - \frac{T_0}{\rho C_{\mathrm{E}}} \left(\beta_1^T \frac{\partial^2 u}{\partial x \partial t} + \beta_3^T \frac{\partial^2 w}{\partial z \partial t} \right) \\ &\quad + T_0 \tau_3 \frac{\partial^2 \varphi}{\partial z \partial t} = 0. \end{aligned} \tag{16}$$

The moisture-plasma diffusion equation describing the coupled behavior in the piezoelectric hygrothermal semiconductor medium, as obtained from Eq. (11), takes the following form:

$$\begin{aligned} &\left(D_1^{\mathrm{m}} \frac{\partial^2 m}{\partial x^2} + D_3^{\mathrm{m}} \frac{\partial^2 m}{\partial z^2} \right) + \left(Q_1^{\mathrm{T}} \frac{\partial^2 T}{\partial x^2} + Q_3^{\mathrm{T}} \frac{\partial^2 T}{\partial z^2} \right) \\ &\quad + \frac{E_{\mathrm{g}}}{\tau} N^* - \frac{\partial m}{\partial t} - \frac{m_0}{k_{\mathrm{m}}} \left(\beta_1^{\mathrm{m}} D_1^{\mathrm{m}} \frac{\partial^2 u}{\partial x \partial t} + \beta_3^{\mathrm{m}} D_3^{\mathrm{m}} \frac{\partial^2 w}{\partial z \partial t} \right) \\ &\quad + m_0 d_3 \frac{\partial^2 \varphi}{\partial z \partial t} = 0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\eta_{15} \frac{\partial^2 w}{\partial x^2} + \eta_{33} \frac{\partial^2 w}{\partial z^2} + (\eta_{31} + \eta_{15}) \frac{\partial^2 u}{\partial x \partial z} \\ &\quad - \varepsilon_{11} \frac{\partial^2 \varphi}{\partial x^2} - \varepsilon_{33} \frac{\partial^2 \varphi}{\partial z^2} + \tau_3 \frac{\partial T}{\partial z} + d_3 \frac{\partial m}{\partial z} = 0, \end{aligned} \tag{18}$$

where $\beta_1^{\mathrm{T}} = (c_{11} + c_{13}) \alpha_{11}^{\mathrm{T}} + c_{13} \alpha_{11}^{\mathrm{T}}$, $\beta_3^{\mathrm{T}} = 2 c_{13} \alpha_{33}^{\mathrm{T}} + c_{33} \alpha_{33}^{\mathrm{T}}$, $\beta_1^{\mathrm{m}} = (c_{11} + c_{13}) \alpha_{11}^{\mathrm{m}} + c_{13} \alpha_{11}^{\mathrm{m}}$, $\beta_3^{\mathrm{T}} = 2 c_{13} \alpha_{33}^{\mathrm{m}} + c_{33} \alpha_{33}^{\mathrm{m}}$, $\delta_1^{\mathrm{N}} = (c_{11} + c_{13}) d_{11}^{\mathrm{N}} + c_{13} d_{11}^{\mathrm{N}}$, $\delta_3^{\mathrm{N}} = 2 c_{13} d_{33}^{\mathrm{N}} + c_{33} d_{33}^{\mathrm{N}}$, α_{11}^{T}, α_{33}^{T} represent the linear thermal expansion constants, α_{11}^{m}, α_{33}^{m} refer to the moisture expansion constants, and $d_{11}^{\mathrm{N}}, d_{33}^{\mathrm{N}}$ express the electronic deformation coefficient.

4 Solution to the problem

To analyze the propagation of harmonic waves in the considered medium, we focus on waves traveling, with the wave normal lying in the xz-plane. To obtain analytical solutions for Eqs. (13)–(18) governing the coupled physical fields, the normal mode method is employed by assuming the following form for the physical variables. The normal mode technique is particularly advantageous in solving multi-physics wave propagation problems, as it effectively reduces the system of partial differential equations to an algebraic system in the transformed domain. This method allows for the direct analysis of wave dispersion, attenuation, and coupling characteristics between the thermoelastic, piezoelectric, photothermal, and plasma fields, making it highly suitable for complex semiconductor models involving nonlocal effects and hygrothermal interactions [57–59]:

$$[T, N^{*}, u, w, m, \varphi](x, z, t) = [\widetilde{T}(x),\ \widetilde{N}^{*}(x),\ \widetilde{u}(x),\ \widetilde{w}(x),\ \widetilde{m}(x),\ \widetilde{\varphi}(x)]\, e^{\omega t + ibz}, \tag{19}$$

where ω expresses the angular frequency, $i = \sqrt{-1}$, b is a wave number along the z-direction, $\widetilde{T}(x)$, $\widetilde{N}^{*}(x)$, $\widetilde{u}(x)$, $\widetilde{w}(x)$, $\widetilde{m}(x)$, and ,$\widetilde{\varphi}(x)$ are the amplitude functions depending on the normal direction x. By applying the normal mode representation introduced in Eq. (19) to Eqs. (13)–(18), the system is reduced to a set of six coupled homogeneous equations governing the field variables:

$$(a_{11}D^2 + a_{12})\widetilde{u} + a_{13}D\widetilde{w} + a_{14}D\widetilde{N^{*}} + a_{15}D\widetilde{T} + a_{16}D\widetilde{m} + a_{17}D\widetilde{\varphi} = 0, \tag{20}$$

$$(a_{21}D^2 + a_{22})\widetilde{w} + a_{23}D\widetilde{u} + a_{24}\widetilde{N^{*}} + a_{25}\widetilde{T} + a_{26}\widetilde{m} + (a_{27}D^2 + a_{28})\widetilde{\varphi} = 0, \tag{21}$$

$$(a_{31}D^2 + a_{32})\widetilde{N^{*}} + a_{33}\widetilde{T} = 0, \tag{22}$$

$$(a_{41}D^2 + a_{42})\widetilde{T} + (a_{43}D^2 + a_{44})\widetilde{m} + a_{45}\widetilde{N^{*}} + a_{46}\widetilde{u} + a_{47}\widetilde{w} + a_{48}\widetilde{\varphi} = 0, \tag{23}$$

$$(a_{51}D^2 + a_{52})\widetilde{m} + (a_{53}D^2 + a_{54})\widetilde{T} + a_{55}\widetilde{N^{*}} + a_{56}D\widetilde{u} + a_{57}\widetilde{w} + a_{58}\widetilde{\varphi} = 0, \tag{24}$$

$$(a_{61}D^2 + a_{62})\widetilde{w}D + a_{63}\widetilde{u} - (a_{64}D^2 + a_{65})\widetilde{\varphi} + a_{66}\widetilde{T} + a_{67}\widetilde{m} = 0, \tag{25}$$

where $D = \frac{\mathrm{d}}{\mathrm{d}x}$,

$a_{11} = c_{11} + \rho\omega^2\varepsilon^2$, $a_{12} = -(c_{55}b^2 + \rho\omega^2(1 + \varepsilon^2 b^2))$,
$a_{13} = ib(c_{13} + c_{55})$, $a_{14} = -\delta_1^{\mathrm{N}}$, $a_{15} = -\beta_1^{\mathrm{T}}$,
$a_{16} = -\beta_1^{\mathrm{m}}$, $a_{17} = ib(\eta_{31} + \eta_{15})$, $a_{32} = \left(b^2 D_3^{\mathrm{E}} + \omega + \frac{1}{\tau}\right)$,
$a_{33} = \kappa$,
$a_{21} = c_{55} + \rho\omega^2\varepsilon^2$, $a_{22} = -(b^2 c_{33} + \rho\omega^2(1 + \varepsilon^2 b^2))$,
$a_{23} = ib(c_{13} + c_{55})$,
$a_{24} = -ib\delta_3^{\mathrm{N}}$, $a_{25} = -ibB_3^{\mathrm{T}}$, $a_{26} = -ib\beta_3^{\mathrm{m}}$, $a_{27} = \eta_{15}$,
$a_{28} = -\eta_{33}b^2$,
$a_{31} = D_1^{\mathrm{E}}$,
$a_{41} = D_1^{\mathrm{T}}$, $a_{42} = b^2 D_3^{\mathrm{T}} + \omega$, $a_{43} = q_1^{\mathrm{m}}$, $a_{44} = -b^2 q_3^{\mathrm{m}}$,
$a_{45} = \frac{E_g}{\tau}$,
$a_{46} = -\frac{T_0}{\rho C_{\mathrm{E}}}\beta_1^{\mathrm{T}}\omega$, $a_{47} = -ib\omega\frac{T_0}{\rho C_{\mathrm{E}}}\beta_3^{\mathrm{T}}$, $a_{48} = ib\omega T_0 \tau_3$,
$a_{51} = D_1^{\mathrm{m}}$, $a_{52} = -(b^2 D_3^{\mathrm{m}} + \omega)$,
$a_{53} = Q_1^{\mathrm{T}}$, $a_{54} = -b^2 Q_3^{\mathrm{T}}$, $a_{55} = a_{45}$, $a_{56} = -\frac{m_0}{k_{\mathrm{m}}}\beta_1^{\mathrm{m}} D_1^{\mathrm{m}}\omega$,
$a_{57} = -ib\omega\frac{m_0}{k_{\mathrm{m}}}\beta_3^{\mathrm{m}} D_3^{\mathrm{m}}$, $a_{58} = -ib\omega m_0 d_3$
$a_{61} = \eta_{15}$, $a_{62} = -\eta_{33}b^2$, $a_{63} = -ib(\eta_{31} + \eta_{15})$,
$a_{64} = \varepsilon_{11}$, $a_{65} = b^2\varepsilon_{33}$, $a_{66} = -ib\tau_3$, $a_{67} = -ibd_3$.

By employing the elimination method to solve the system of Eqs. (20)–(25), we obtain the following results:

$$(\Delta_1 D^{12} + \Delta_2 D^{10} + \Delta_3 D^8 + \Delta_4 D^6 + \Delta_5 D^4 + \Delta_6 D^2 + \Delta_7)\ \{\widetilde{T},\ \widetilde{N}^{*},\ \widetilde{u},\ \widetilde{w},\ \widetilde{m},\ \widetilde{\varphi}\}\ (x) = 0. \tag{26}$$

The explicit expressions for the coefficients Δ_i, $i = 1, \ \ldots.. \ , \ 7$ are provided in Appendix A. The solution of Eq. (26) in factorization form is

$$(D^2 - k_1^2)(D^2 - k_2^2)(D^2 - k_3^2)(D^2 - k_4^2)(D^2 - k_5^2)(D^2 - k_6^2)\{\widetilde{T}, \widetilde{N}^{*}, \widetilde{u}, \widetilde{w}, \widetilde{m}, \widetilde{\varphi}\}(x) = 0. \tag{27}$$

where $k_i^2(i = 1 \to 7)$ are the roots of Eq. (27), chosen such that their real parts are positive to ensure the boundedness of the solution as $x \to \infty$. Owing to the linearity of the problem, the solution of Eq. (27) can be expressed as

$$T^{*}(x) = \sum_{i=1}^{6} M_i(b, \omega)\, e^{-k_i x}. \tag{28}$$

Similarly, the solutions corresponding to the other physical variables take the following form:

$$N^{*}(x) = \sum_{i=1}^{6} M_i'(b, \omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{1i} M_i(b, \omega)\, e^{-k_i x}, \tag{29}$$

$$u^*(x) = \sum_{i=1}^{6} M_i^{'}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{2i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (30)$$

$$w^*(x) = \sum_{i=1}^{6} M_i^{''}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{3i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (31)$$

$$\varphi^*(x) = \sum_{i=1}^{6} M_i^{(4)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{4i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (32)$$

$$m^*(x) = \sum_{i=1}^{6} M_i^{(5)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{5i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (33)$$

$$\sigma_{xx}^*(x) = \sum_{i=1}^{6} M_i^{(6)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{6i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (34)$$

$$\sigma_{zz}^*(x) = \sum_{i=1}^{6} M_i^{(7)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{7i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (35)$$

$$\sigma_{xz}^*(x) = \sum_{i=1}^{6} M_i^{(8)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{8i}\, M_i(b,\omega)\, e^{-k_i x}, \quad (36)$$

$$D_x^*(x) = \sum_{i=1}^{6} M_i^{(9)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{9i} M_i(b,\omega)\, e^{-k_i x}, \quad (37)$$

$$D_z^*(x) = \sum_{i=1}^{6} M_i^{(10)}(b,\omega)\, e^{-k_i x} = \sum_{i=1}^{6} h_{10i} M_i(b,\omega)\, e^{-k_i x}, \quad (38)$$

such that

$$\begin{bmatrix} h_{1i} \\ h_{2i} \\ h_{3i} \\ h_{4i} \\ h_{5i} \end{bmatrix} = \begin{bmatrix} a_{14}k_i & (a_{11}k_i^2 + a_{12}) & a_{13}k_i & a_{17}k_i & a_{16}k_i \\ a_{24} & a_{23}k_i & (a_{21}k_i^2 + a_{22}) & (a_{27}k_i^2 + a_{28}) & a_{26} \\ (a_{31}k_i^2 + a_{32}) & 0 & 0 & 0 & 0 \\ a_{45} & a_{46} & a_{47} & a_{48} & (a_{43}k_i^2 + a_{44}) \\ a_{55} & a_{56}k_i & a_{57} & a_{58} & (a_{51}k_i^2 + a_{52}) \end{bmatrix}^{-1} \begin{bmatrix} -a_{15}k_i \\ -a_{25} \\ -a_{33} \\ -(a_{41}k_i^2 + a_{42}) \\ -(a_{53}k_i^2 + a_{54}) \end{bmatrix},$$

$$\begin{aligned} h_{6i} &= c_{11}k_i h_{2i} + ic_{13} b h_{3i} - \delta_1^{\mathrm{N}} h_{1i} - \beta_1^{\mathrm{T}} - \beta_1^{\mathrm{m}} h_{5i} - ib\eta_{31} h_{4i}, \\ h_{7i} &= c_{55}(ibh_{2i} + k_i h_{3i}) + \eta_{15} k_i h_{4i}, \\ h_{8i} &= c_{13}k_i h_{2i} + ibc_{33} h_{3i} - \delta_3^{\mathrm{N}} h_{1i} - \beta_3^{\mathrm{T}} - \beta_3^{\mathrm{m}} m + ib\eta_{33} h_{4i}, \\ h_{9i} &= \eta_{15}(ibh_{2i} + k_i h_{3i}) - \varepsilon_{11} k_i h_{4i} + d_1 h_{5i}, \\ h_{10i} &= \eta_{31} k_i h_{2i} + ib\eta_{33} h_{3i} + \tau_3 - ib\varepsilon_{33} h_{4i} + d_3 h_{5i}, \end{aligned}$$

where $M_i, M_i', M_i^{''}, M_i^{(3)}, M_i^{(4)}, M_i^{(5)}, M_i^{(6)}, M_i^{(7)}, M_i^{(8)}, M_i^{(9)}$, and $M_i^{(6)}$, $i = 1, 2, 3, 4, 5$, are unknown constants dependent on the parameters b and ω, which can be determined by applying the boundary conditions.

5 Boundary conditions

The solution of the governing equations requires the specification of appropriate boundary conditions that accurately reflect the physical situation under consideration. In the present problem, the piezoelectric hygrothermal semiconductor medium is subjected to coupled mechanical, thermal, plasma, moisture, and electric fields at its surface. Therefore, boundary conditions are formulated to account for external actions such as thermal and plasma excitations, moisture variation, and mechanical stresses. These conditions play a crucial role in capturing the interaction between temperature, moisture, carrier density, electric potential, and stress fields, which are vital for understanding the behavior of the medium under practical engineering applications, including thermal management, optoelectronic devices, and moisture-sensitive semiconductor components. The boundary conditions are chosen to ensure the physical realism of the model and the proper decay of the disturbances away from the excitation region. To obtain the constants M_i, we will assume the following:

(I) Since the surface is subjected to a **decaying thermal shock**, the temperature at the boundary is not constant but follows an exponentially decaying function with time at $x = 0$:

$$T^*(0, b, \omega) = T_0 \exp(-\zeta t), \quad (39)$$

where ζ is the decay constant (controls how fast the thermal shock decays). This boundary condition describes the application of an initial thermal disturbance at the free surface that dissipates exponentially with time, modeling situations where the surface is exposed to a rapid thermal shock, such as laser pulses or transient heating, commonly encountered in semiconductor manufacturing and thermal testing. In this case, we have

$$\sum_{i=1}^{6} M_i(b,\omega) = T_0 \exp(-\zeta t). \quad (40)$$

(II) The normal component of the electric field at the free surface $x = 0$ is assumed to vanish, which is a common assumption for electrically insulated or free surfaces in piezoelectric and semiconductor media. This condition is expressed as

$$\frac{\partial \varphi^*(0, b, \omega)}{\partial x} = 0. \tag{41}$$

Therefore,

$$\sum_{i=1}^{6} \frac{h_{4i}}{k_i} M_i(b, \omega) = 0. \tag{42}$$

This boundary condition ensures that no electric flux crosses the boundary, reflecting practical situations where the surface is either free of charge carriers or insulated, which is highly relevant in microelectronic and piezoelectric device applications.

(III) At the boundary $x = 0$, the plasma diffusion function is prescribed as

$$N^*(0, b, \omega) = n_0. \tag{43}$$

Hence,

$$\sum_{i=1}^{6} h_{1i} M_i(b, \omega) = n_0, \tag{44}$$

where n_0 is a known plasma distribution or excitation function at the surface, which could result from external plasma injection, laser excitation, or boundary-controlled processes. This boundary condition plays a significant role in describing the behavior of charge carriers at the surface, which is essential for analyzing semiconductor devices, sensors, and plasma-affected systems.

(IV) Since the surface is assumed to be traction-free, the boundary conditions for mechanical stresses are given by

$$\sigma_{xx}^*(0, b, \omega) = 0. \tag{45}$$

Furthermore,

$$\sum_{i=1}^{6} h_{6i} M_i(b, \omega) = 0. \tag{46}$$

These conditions represent a mechanically free surface, common in thin films, coatings, and layered semiconductor structures, where the outer surface is not subjected to external mechanical loads.

(V) The tangent mechanical conditions at $x = 0$ are

$$\sigma_{xz}^*(0, b, \omega) = 0. \tag{47}$$

Hence,

$$\sum_{i=1}^{6} h_{8i} M_i(b, \omega) = 0. \tag{48}$$

(VI) At the boundary $x = 0$, the moisture concentration is specified by

$$m^*(0, b, \omega) = m_0. \tag{49}$$

So,

$$\sum_{i=1}^{6} h_{5i} M_i(b, \omega) = m_0, \tag{50}$$

where m_0 represents the imposed moisture concentration on the surface. This condition models the direct influence of environmental humidity or controlled hygrothermal processes on the medium. It is crucial in the analysis of hygro-thermoelastic and piezoelectric materials used in advanced sensing, coatings, and smart structures.

The simultaneous solution of the established boundary conditions yields the unknown parameters $M_i(b, \omega)$, following the methodologies outlined in previous studies [60,61].

6 Numerical results and discussions

In this section, numerical simulations are carried out based on the physical constants of the CdSe semiconductor material. The obtained computational results are valuable for illustrating the influence of the proposed model and may contribute to advancements in physics-based technologies, with potential applications across various sectors of the modern mechanical and semiconductor industries. The physical properties of CdSe used in the simulation are listed in Table 1, where all constants are expressed in SI units [36,37].

6.1 Impact of nonlocal parameters

In this section, we investigate the influence of the nonlocal effect on the behavior of the piezo-hygrothermoelastic semiconductor medium subjected to thermal, moisture, and electrical fields. The results are presented for two cases: (i) local medium and (ii) nonlocal medium. The variations of the main physical fields (temperature, moisture concentration, carrier density, displacement, and electric potential) against the dimensionless distance are plotted and analyzed, as shown in Figure 2. From subplot 1, it is evident that the temperature decays monotonically as the distance increases. However, the nonlocal medium exhibits a slower decay compared to the local medium. This indicates that the nonlocal effect enhances the thermal conduction and causes the thermal wave to penetrate deeper into the medium. Physically, this can be attributed to the fact that the nonlocal model accounts for the long-range interactions among particles, leading to an extended thermal influence beyond the immediate neighborhood. Subplot 2 shows an oscillatory behavior of moisture concentration that diminishes with distance. The amplitude and phase of oscillations are noticeably altered under the nonlocal effect, displaying a smoother attenuation compared to the local model. This is because nonlocality helps redistribute the moisture effect more gradually, reducing sharp moisture gradients, which is crucial for predicting moisture transport in micro- and nanoscale devices. Subplot 3 illustrates the distribution of electric displacement in both local and nonlocal piezo-hygrothermoelastic semiconductor media. In the local case, the electric displacement oscillates with higher peaks and attenuates quickly due to the immediate and concentrated interaction between the piezoelectric effect and the applied thermal and moisture fields. In contrast, the nonlocal case shows a smoother and more extended distribution with lower amplitude and delayed attenuation. This behavior is attributed to the long-range interactions incorporated by the nonlocal effect, which spreads the electric field response over a larger area and reduces the intensity of localized electric displacement. Such a response is beneficial in semiconductor devices where smoother and less concentrated electric displacement can improve device performance and reduce the risk of dielectric breakdown. Subplot 4 demonstrates a decreasing trend of the carrier density with distance. In the nonlocal case, carrier diffusion appears more extended, with a higher concentration maintained over larger distances compared to the local case. This is a direct consequence of nonlocality facilitating a more effective redistribution of carriers, which is important for semiconductor performance, especially under photothermal or hygrothermal loads. The dimensionless tangential (shear) stress distribution, as shown in subplot 5, exhibits a wave-like oscillatory pattern along the dimensionless distance for both the local and nonlocal semiconductor mediums. This behavior is a direct result of the combined effects of thermal, hygrothermal, and piezoelectric interactions under the applied boundary conditions and external stimuli. In the local medium, the tangential stress demonstrates higher-amplitude oscillations, which decay gradually with distance. The oscillations arise due to the coupling between mechanical deformation and the applied thermal and moisture loads. These interactions generate shear waves that propagate through the medium, leading to alternating tensile and compressive tangential stresses. The relatively sharp peaks and valleys observed in the local case are associated with the localized nature of the stress wave propagation without considering long-range atomic or microstructural interactions. In contrast, the nonlocal medium exhibits smoother, damped, and wider oscillations in the tangential stress distribution. The amplitudes of the tangential stress waves are slightly reduced, and their spatial spread is extended compared to the local case. This difference is attributed to the nonlocal effect, which accounts for the internal length scale and long-range forces in the semiconductor structure. The nonlocality acts as a stabilizing factor, reducing the sharpness of stress concentrations and distributing the tangential stress more uniformly over the medium. This is mechanically beneficial since it lowers the risk of shear-induced failures such as micro-cracking or delamination, which are critical concerns in semiconductor and MEMS structures. The displacement plots (subplot 6) show characteristic thermoelastic damped oscillations. Notably, the nonlocal model leads to more pronounced and longer-lasting oscillations compared to the local model. This behavior is physically reasonable as the nonlocal effect introduces additional stiffness to the medium due to internal interactions across finite distances, causing the mechanical waves to sustain their oscillations longer before damping out. The normal stress distribution, as shown in subplot 7, exhibits significant differences between the local and nonlocal cases. In the local medium, the normal stress displays sharp oscillations with higher amplitude and rapid decay due to the direct and localized response of the material to the coupled effects of thermal, moisture, and piezoelectric loadings.

Table 1: Physical constants of the CdSe medium

λ	6.4×10^{10},	c_{11}	74.1×10^{9}
μ	6.5×10^{10}		
ρ	5,504	c_{13}	39.3×10^{9}
T_0	298	c_{33}	83.6×10^{9}
τ	5×10^{-5}	c_{55}	13.2×10^{9}
d_n	-9×10^{-31}	q_1^{m}	2.1×10^{-7}
D_{E}	2.5×10^{-3}	q_3^{m}	2.1×10^{-7}
E_g	1.11	α_{m}	2.68×10^{-3}
α_t	4.14×10^{-6}	β_1^{T}	6.21×10^{5}
k	150	β_3^{T}	5.51×10^{5}
D_3^{T}	$\frac{k}{\rho\, C_{\mathrm{e}}}$	D_1^{T}	$\frac{k}{\rho\, C_{\mathrm{e}}}$
D_3^{m}	0.648×10^{-6}	C_{e}	260
Q_1^{T}	0.648×10^{-6}	m_0	20%
Q_3^{T}	0.648×10^{-6}	D_1^{m}	2.1×10^{-7}
k_{m}	2.2×10^{-8}		

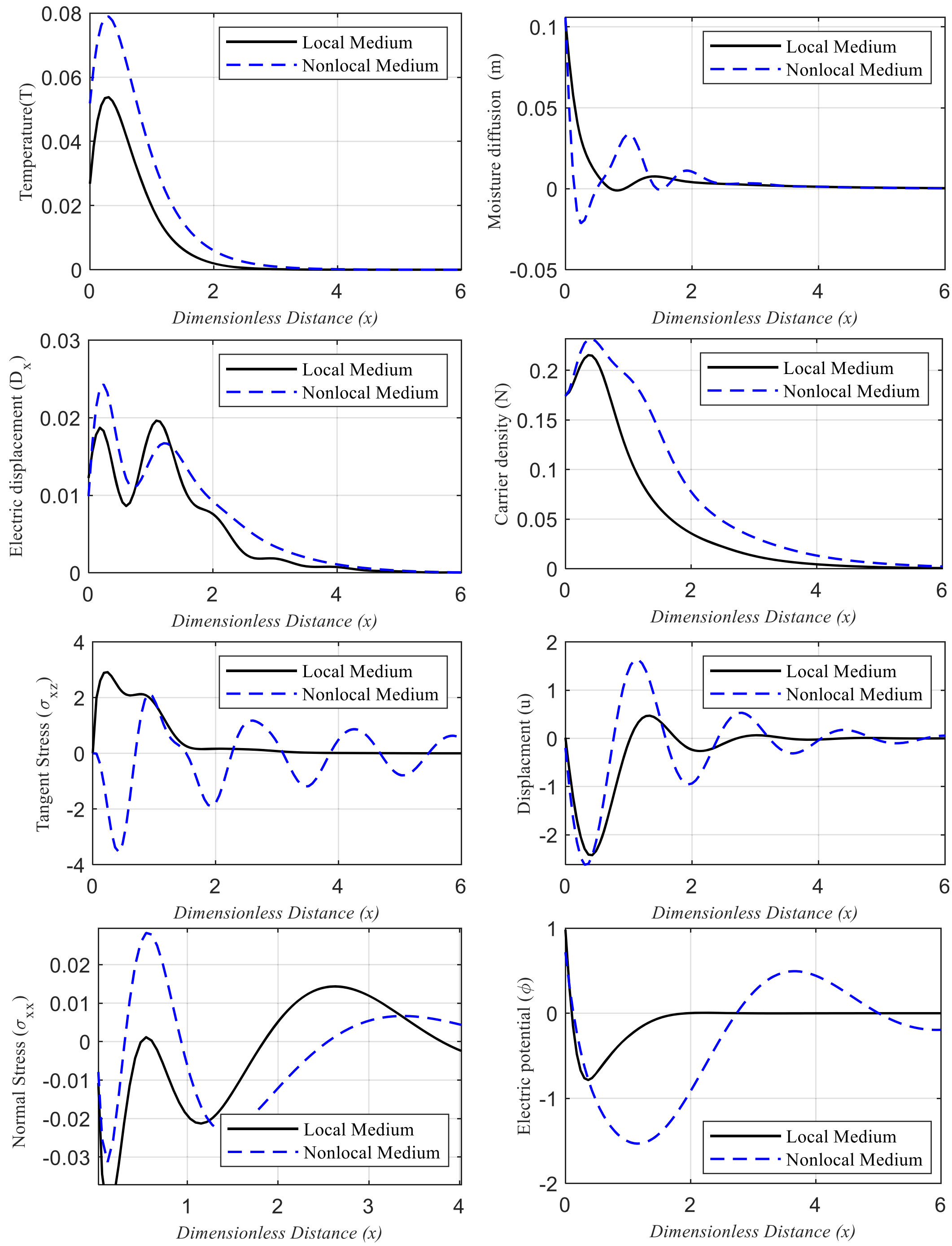

Figure 2: Variation of the dimensionless main fields *versus* the dimensionless distance x in a piezo-hygrothermoelastic semiconductor medium for both local and nonlocal cases at m_0 = 20%.

However, in the nonlocal medium, the stress distribution becomes smoother with reduced amplitude and slower decay, indicating the role of microstructural interactions in dispersing the stress over a wider region. This nonlocal behavior effectively mitigates stress concentration and delays the stress wave attenuation, which is crucial for enhancing the structural stability of the piezo-hygrothermoelastic semiconductor medium under thermo–moisture–mechanical coupling. The electric potential exhibits a similar oscillatory pattern (subplot 8). The nonlocal medium shows a shift in both amplitude and phase compared to the local medium. This shift indicates that the piezoelectric coupling is significantly influenced by the nonlocal parameter. Mechanically, the internal interactions in the nonlocal model redistribute electric charges more effectively, altering the resulting potential field distribution.

6.1.1 Mechanism analysis

From a physical standpoint, the presence of the nonlocal effect fundamentally modifies the energy transfer processes in the system. The internal length scale parameter introduced by the nonlocal theory enables the medium to account for **long-range interactions** among neighboring points, leading to the observed moderation of gradients and oscillations across all fields. This mechanism ensures smoother transitions of physical quantities and improved resistance to external disturbances, making nonlocality a valuable factor in the engineering of advanced semiconductor devices, microelectromechanical systems (MEMS), and sensors.

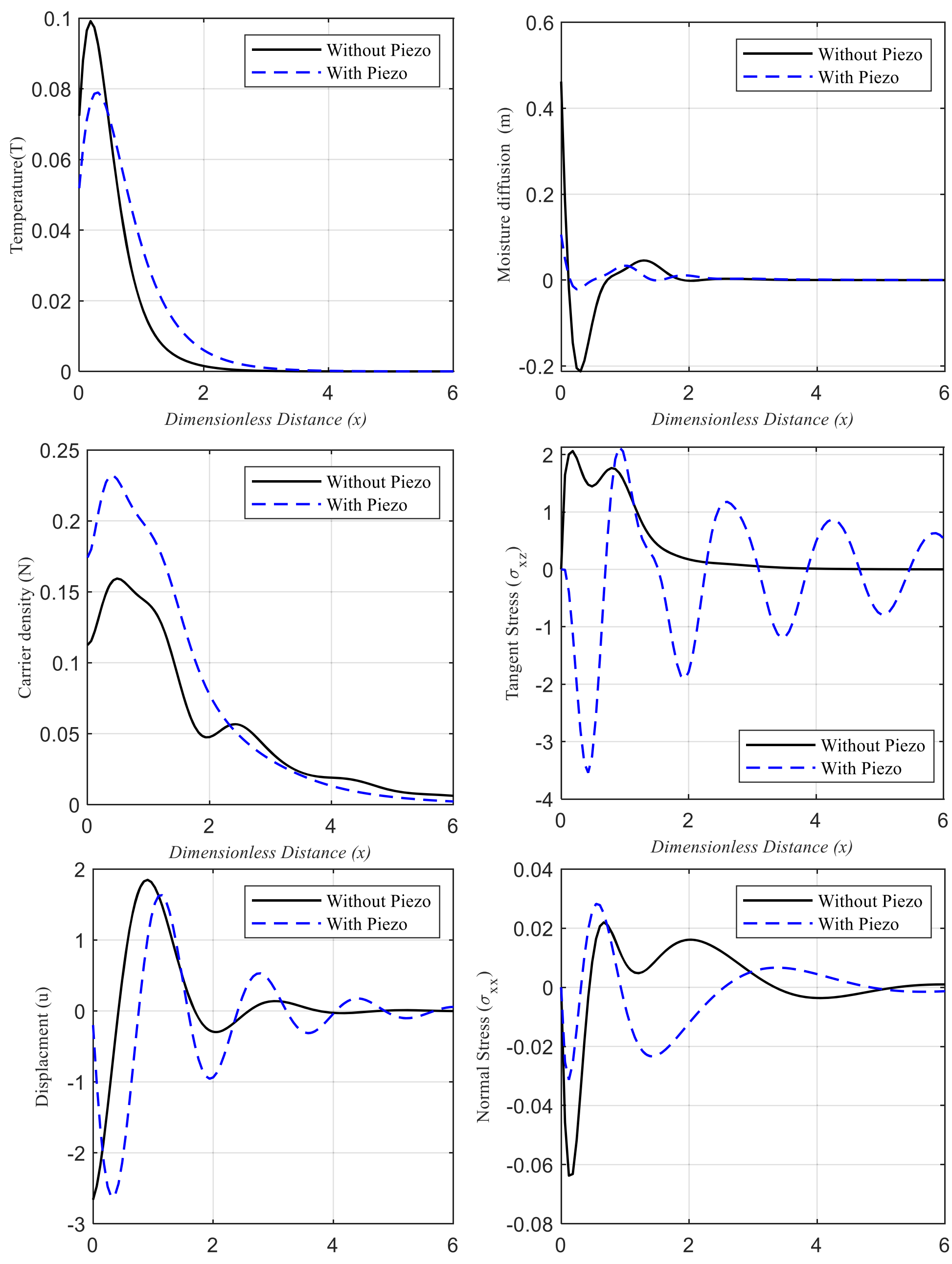

Figure 3: Distributions of the dimensionless physical fields *versus* the dimensionless distance in a piezo-hygrothermoelastic nonlocal semiconductor medium, illustrating the comparison between two cases: With piezoelectric effect and without piezoelectric effect at m_0 = 20%.

6.1.2 Physical interpretation

The nonlocal behavior helps the material dissipate stress more efficiently, which is beneficial for preventing failure due to normal stress concentration, such as **microcracking**, **delamination**, or **debonding**. In semiconductor and MEMS, where miniaturization makes nonlocality more pronounced, this effect is crucial for ensuring mechanical integrity under thermal and moisture-induced loads.

6.2 Piezoelectric effect

Figure 3 displays the plotted fields of (1) temperature, (2) moisture concentration, (3) carrier density, (4) tangent stress, (5) displacement, and (6) normal stress. The results demonstrate the significant influence of the piezoelectric effect on enhancing or altering the amplitude, phase, and attenuation behavior of the thermal, mechanical, electric, and diffusion responses in the medium. The nonlocal nature of the semiconductor further modifies the wave characteristics, making the interactions more complex and physically meaningful. The temperature distribution decreases progressively with the increase in the dimensionless distance for both cases (without and with the piezoelectric effect). However, the temperature field in the presence of the piezoelectric effect shows slightly lower values and a more gradual decay compared to the case without piezoelectricity. This behavior is mainly due to the energy conversion mechanism induced by the piezoelectric coupling, where part of the thermal energy is converted into electrical and mechanical energies. Consequently, the heat conduction process is altered, leading to a reduced thermal diffusion rate and a smoother temperature profile. The piezoelectric effect also interacts with the hygrothermal and mechanical fields, modifying the thermal stress and influencing the heat propagation characteristics in the semiconductor medium. The observed attenuation indicates the combined influence of nonlocality, moisture, and piezoelectric coupling on heat transfer within the material. The moisture concentration decreases smoothly with the dimensionless distance in both cases (with and without the piezoelectric effect). However, the presence of the piezoelectric effect slightly reduces the moisture concentration throughout the medium. This is attributed to the coupling between the electrical and mechanical fields caused by the piezoelectric effect, which influences the diffusion of moisture. The piezoelectric effect alters stress distribution, leading to a redistribution of the moisture concentration, resulting in slightly lower values compared to the case without the piezoelectric effect. The carrier density exhibits a decaying behavior with increasing distance, similar to the moisture concentration. It is observed that the presence of the piezoelectric effect leads to a noticeable reduction in the carrier density. The physical mechanism behind this is that the additional electric field generated by the piezoelectric effect alters the electrochemical potential, which in turn affects the carrier distribution. This coupling causes carriers to redistribute differently, reducing their concentration compared to the purely hygrothermoelastic case without piezoelectricity. The tangent stress distribution shows significant oscillations near the boundary, and these oscillations are more pronounced when the piezoelectric effect is considered. The piezoelectric coupling intensifies the mechanical wave interactions due to the induced electric field, which amplifies the stress fluctuations. Physically, this is due to the energy exchange between the mechanical and electric fields, which causes more prominent stress variations and higher peaks and valleys compared to the nonpiezoelectric case. The displacement distribution exhibits an oscillatory behavior that gradually attenuates as the dimensionless distance increases. It is evident that the displacement amplitude is higher in the case with the piezoelectric effect compared to the case without it. This enhancement can be physically explained by the fact that the piezoelectric coupling generates additional internal stresses and electric fields, which in turn reinforce the mechanical response of the medium. The interaction between the hygrothermal field, moisture diffusion, and the induced electric field leads to stronger deformation when piezoelectricity is present. The nonlocal effect also contributes to the persistence of oscillations over longer distances due to the memory effect and microstructural interactions in the semiconductor. The oscillatory nature of the displacement reflects the wave-like propagation of the coupled thermoelastic and piezoelectric fields, which are further modulated by moisture content and nonlocality, making the displacement more pronounced and spread over a larger region when the piezoelectric effect is active. The normal stress distribution shows clear oscillations that decay with increasing dimensionless distance. The presence of the piezoelectric effect significantly amplifies the magnitude of the normal stress compared to the case without the piezoelectric contribution. This behavior is attributed to the coupling between the electric and mechanical fields through the piezoelectric effect, which induces additional internal forces resisting deformation. As the piezoelectric coupling generates extra electric displacements under mechanical loading, it modifies the stress state and results in more intense stress oscillations. The nonlocality also plays a role by allowing the stress waves to be influenced by the microstructure of the semiconductor, leading to smoother decay and an extended influence range. Physically, this indicates that the piezoelectric mechanism not only strengthens the normal stress response but also delays the attenuation of stress waves as they propagate through the medium, resulting in larger and longer-lasting oscillations compared to the nonpiezoelectric case.

6.3 Three-dimensional (3D) representation

Figure 4 presents 3D visualizations of the key physical fields, namely temperature, moisture diffusion, electric displacement, carrier density, tangential stress, displacement, normal stress, and electric potential, within the piezo-hygrothermoelastic nonlocal semiconductor medium. The temperature and normal stress fields exhibit relatively smooth wave patterns, with noticeable attenuation along the z direction due to thermal and mechanical dissipation. The moisture diffusion and carrier density fields show oscillatory and rapidly decaying behavior along both the x and z directions, indicating a strong coupling between moisture and carrier dynamics. The electric displacement and electric potential plots highlight the localized nature of the piezoelectric effect, contributing to sharp gradients near the excitation region and influencing the wave propagation. Tangential stress and displacement fields exhibit complex wavefronts with noticeable reflections and interference patterns, reflecting the combined influence of mechanical, thermal, and piezoelectric interactions. Overall, the 3D plots reveal the synergistic behavior between the involved multiphysics fields and demonstrate how nonlocality and piezoelectricity alter the spatial evolution of each field.

However, the present study still has some limitations. The model is based on linear material behavior and considers an idealized 2D geometry, which does not fully capture 3D or nonlinear effects that may occur in real semiconductor structures. In addition, the material was assumed to be homogeneous and isotropic, although actual semiconductors often exhibit anisotropy and graded properties. Moreover, phenomena such as temperature-dependent thermal conductivity, plastic deformation, and damage mechanisms were not considered. Future studies may address these aspects by extending the model to a more general 3D formulation, incorporating nonlinear and anisotropic effects, and including additional physical mechanisms to provide a more realistic and comprehensive analysis.

6.4 Comparison between the present study and existing literature

To highlight the novelty and significance of the present work, a comprehensive comparison with recent related studies is summarized in Table 2. Previous investigations mainly focused on simplified models considering partial coupling between physical fields such as photo-thermoelastic or magneto-thermoelastic effects, while often neglecting essential factors like nonlocal elasticity, hygrothermal interaction, or temperature-dependent thermal conductivity. Some works addressed nonlocal effects but were limited to one-dimensional geometries or lacked the inclusion of piezoelectric and photo-induced effects, which are critical in modern semiconductor applications. In contrast, the current study develops an advanced and more realistic 2D model that integrates piezoelectricity, photothermal interaction, hygrothermal effects, nonlocal elasticity, and temperature-dependent thermal conductivity. This combination provides a comprehensive framework suitable for analyzing wave propagation phenomena in semiconductor devices at micro- and nanoscales.

To validate the accuracy of the present analytical model, a comparative analysis is conducted by juxtaposing the obtained numerical results with those from the existing literature. Figures 2 and 3 illustrate the variation of key physical parameters, such as displacement, temperature distribution, and stress components, in comparison with results reported by Lotfy *et al.* [62]. The trends observed in our findings exhibit strong agreement with previously established results, demonstrating the reliability of the proposed model. Additionally, minor deviations can be attributed to differences in boundary conditions, material properties, and the incorporation of nonlocal and temperature-dependent effects in the present study. This comparison substantiates the correctness of our approach and reinforces the significance of the newly introduced physical effects in semiconductor materials.

The graphical results presented in Figures 2–4 illustrate the behavior of key physical parameters, including displacement, temperature distribution, and stress variations, under the influence of different governing factors. It is observed that the temperature profile exhibits a nonmonotonic variation, initially increasing due to rapid thermal diffusion and subsequently stabilizing as heat conduction reaches equilibrium. This trend is primarily driven by the temperature-dependent thermal conductivity, which influences heat transfer rates across the semiconductor medium. Similarly, the displacement field shows oscillatory behavior, particularly in the presence of nonlocal effects, which account for size-dependent material responses. The amplitude of oscillations increases in regions of higher thermal gradient, highlighting the coupling between thermal and mechanical fields. Furthermore, stress distributions reveal significant variations in regions with strong electromagnetic interactions, indicating the interplay between piezoelectric, thermal, and elastic effects. The observed differences between the present results and classical models emphasize the impact of nonlocality, fractional-order derivatives, and semiconductor microstructural characteristics on wave propagation.

These graphical trends align well with theoretical expectations and previous studies, validating the effectiveness of the proposed model in capturing complex thermoelastic interactions in hydro-poroelastic semiconductor media.

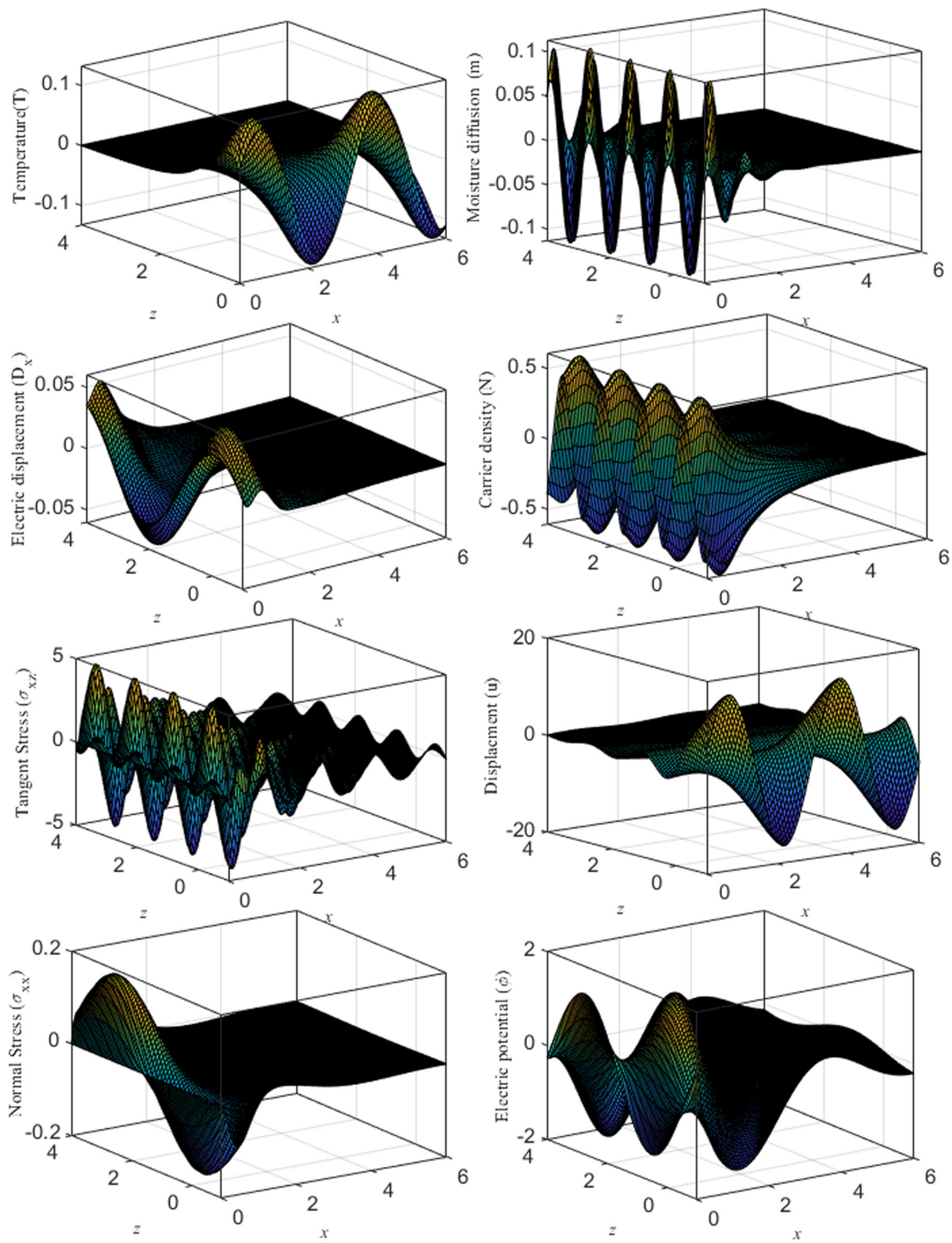

Figure 4: 3D representations of the main physical fields within the piezo-hygrothermoelastic nonlocal semiconductor medium, illustrating the variations of main fields as functions of the dimensionless coordinates *x* and *z*.

Table 2: Comparison between the present study and related previous works

Study	Model	Coupled fields	Nonlocal effect	Geometry	Solution technique	Explanation
Lotfy *et al.* (2023)	Photo-thermoelastic semiconductor	Photo + thermoelastic	Not considered	2D	Normal mode	Investigated photo-thermoelastic waves but neglected nonlocality, hygrothermal, and piezoelectric effects. Limited for modern semiconductors
Zenkour and Abbas (2022)	Magneto-thermoelastic semiconductor	Magnetic + thermoelastic	Not considered	2D	Laplace transform	Focused on magneto-thermoelasticity but ignored nonlocality and photo/piezoeffects, limiting applicability to advanced semiconductor devices
Shah *et al.* (2021)	Thermoelastic micro/nanorods	Thermoelastic only	Considered	1D	Normal mode	Included nonlocality, but limited to 1D geometry without considering optical, piezoelectric, or hygrothermal interactions
Present Study	Piezo-photo-hygrothermoelastic semiconductor	Piezoelectric + photo + hygrothermal + thermoelastic	Fully considered	2D	Normal mode	Novel combination of nonlocality, piezoelectricity, photo-thermoelasticity, hygrothermal effect, and temperature-dependent conductivity, making it suitable for modern optoelectronic and semiconductor systems

7 Conclusion and applications

In this study, the coupled interactions of thermal, moisture, electrical, and mechanical fields in a piezo-hygrothermoelastic nonlocal semiconductor medium were investigated. The presented results, both in 2D and 3D representations, show that the piezoelectric effect significantly influences the dynamic behavior of all the studied fields. The inclusion of piezoelectricity leads to noticeable alterations in the amplitudes, oscillations, and decay patterns of temperature, moisture diffusion, carrier density, electric displacement, electric potential, displacement, and stress distributions. Notably, the nonlocal effect introduces spatial smoothing and modifies the wave propagation characteristics, while moisture diffusion and carrier density show strong localized responses near the excitation region. The mechanical fields (displacement, tangential, and normal stresses) experience considerable amplification and spatial redistribution due to the piezoelectric coupling. These observations confirm that piezoelectricity, moisture, and temperature changes play a dominant role in controlling the wave behavior and field distributions in semiconductor media.

The obtained results are highly relevant to the design and optimization of advanced semiconductor devices such as photoacoustic sensors, MEMS/NEMS, piezoelectric transducers, and smart materials subjected to thermal, mechanical, and electrical excitations. Moreover, the insights from this work can be applied to improve semiconductor-based devices operating in harsh environments where moisture, temperature, and electric effects are present simultaneously, such as aerospace, biomedical devices, and microelectronics industries. Future extensions of this study may include the consideration of fractional-order models, dynamic boundary conditions, and more complex geometries to further enhance the applicability to modern semiconductor technologies.

Specifically, the current model can be further advanced by considering the following. The extension to 3D formulations captures more realistic geometrical effects. The inclusion of material anisotropies and functionally graded properties, which are common in advanced semiconductor devices. Incorporation of nonlinear thermoelasticity and large deformation effects to handle high-intensity or shock-type loading. Studying time-fractional derivatives or memory-dependent heat conduction models to account for relaxation phenomena and non-Fourier heat conduction. Considering temperature-dependent physical parameters such as thermal conductivity, elasticity moduli, and piezoelectric coefficients for more accurate temperature-field predictions. Investigation of dynamic boundary conditions, surface effects, and microstructural interactions in nanostructured semiconductors.

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Appendix A

The main coefficients of Eq. (26) are as follows:

$$\Delta_1 = -a_{11}a_{31}(a_{41}a_{51} - a_{43}a_{53})(a_{27}a_{61} - a_{21}a_{64}),$$

$$\begin{aligned}
\Delta_2 = {} & a_{17}a_{31}(a_{41}a_{51} - a_{43}a_{53})(a_{23}a_{61} - a_{21}a_{63}) \\
& - a_{12}a_{31}(a_{41}a_{51} - a_{43}a_{53})(a_{27}a_{61} - a_{21}a_{64}) \\
& - a_{16}a_{31}(a_{46}a_{53} - a_{41}a_{56})(a_{27}a_{61} - a_{21}a_{64}) \\
& + a_{15}a_{31}(a_{46}a_{51} - a_{43}a_{56})(a_{27}a_{61} - a_{21}a_{64}) \\
& + a_{13}a_{31}(a_{41}a_{51} - a_{43}a_{53})(a_{27}a_{63} - a_{23}a_{64}) \\
& + a_{11}(a_{28}a_{31}(-a_{41}a_{51} + a_{43}a_{53})a_{61} \\
& + a_{27}(a_{32}(-a_{41}a_{51} + a_{43}a_{53})a_{61} \\
& + a_{31}((-a_{42}a_{51} - a_{41}a_{52} + a_{44}a_{53} + a_{43}a_{54})a_{61} \\
& + (-a_{41}a_{51} + a_{43}a_{53})a_{62})) + (a_{22}a_{31}(a_{41}a_{51} - a_{43}a_{53}) \\
& + a_{21}(a_{32}(a_{41}a_{51} - a_{43}a_{53}) \\
& + a_{31}(a_{42}a_{51} + a_{41}a_{52} - a_{44}a_{53} - a_{43}a_{54})))a_{64} \\
& + a_{21}a_{31}(a_{41}a_{51} - a_{43}a_{53})a_{65}),
\end{aligned}$$

$$\begin{aligned}
\Delta_3 = {} & -a_{14}a_{33}(a_{46}a_{51} - a_{43}a_{56})(a_{27}a_{61} - a_{21}a_{64}) \\
& + a_{12}(a_{28}a_{31}(-a_{41}a_{51} + a_{43}a_{53})a_{61} \\
& + a_{27}(a_{32}(-a_{41}a_{51} + a_{43}a_{53})a_{61} \\
& + a_{31}((-a_{42}a_{51} - a_{41}a_{52} + a_{44}a_{53} + a_{43}a_{54})a_{61} \\
& + (-a_{41}a_{51} + a_{43}a_{53})a_{62})) \\
& + (a_{22}a_{31}(a_{41}a_{51} - a_{43}a_{53}) \\
& + a_{21}(a_{32}(a_{41}a_{51} - a_{43}a_{53}) \\
& + a_{31}(a_{42}a_{51} + a_{41}a_{52} - a_{44}a_{53} - a_{43}a_{54}))a_{64} \\
& + a_{21}a_{31}(a_{41}a_{51} - a_{43}a_{53})a_{65}) \\
& + a_{16}(a_{28}a_{31}(-a_{46}a_{53} + a_{41}a_{56})a_{61} \\
& + a_{27}(a_{32}(-a_{46}a_{53} + a_{41}a_{56})a_{61} + a_{31}(a_{56}(a_{42}a_{61} + a_{41}a_{62}) \\
& - a_{46}(a_{54}a_{61} + a_{53}a_{62}) + (a_{47}a_{53} - a_{41}a_{57})a_{63})) \\
& + a_{22}a_{31}(a_{46}a_{53} - a_{41}a_{56})a_{64} \\
& + a_{23}a_{31}(a_{48}a_{53}a_{61} - a_{47}a_{53}a_{64} + a_{41}(-a_{58}a_{61} + a_{57}a_{64})) \\
& + a_{21}(a_{32}(a_{46}a_{53} - a_{41}a_{56})a_{64} + a_{31} \\
& + a_{31}((-a_{48}a_{53} + a_{41}a_{58})a_{63} \\
& + (a_{46}a_{54} - a_{42}a_{56})a_{64} + (a_{46}a_{53} - a_{41}a_{56})a_{65})) \\
& + a_{15}(a_{28}a_{31}(a_{46}a_{51} - a_{43}a_{56})a_{61} + a_{27}(a_{32}(a_{46}a_{51} - a_{43}a_{56})a_{61} \\
& + a_{31}(-a_{56}(a_{44}a_{61} + a_{43}a_{62}) + a_{46}(a_{52}a_{61} + a_{51}a_{62}) \\
& + (-a_{47}a_{51} + a_{43}a_{57})a_{63})) + a_{22}a_{31}(-a_{46}a_{51} + a_{43}a_{56})a_{64} \\
& + a_{23}a_{31}(-a_{48}a_{51}a_{61} + a_{47}a_{51}a_{64} + a_{43}(a_{58}a_{61} - a_{57}a_{64})) \\
& + a_{21}(a_{32}(-a_{46}a_{51} + a_{43}a_{56})a_{64} + a_{31}((a_{48}a_{51} - a_{43}a_{58})a_{63} \\
& + (-a_{46}a_{52} + a_{44}a_{56})a_{64} + (-a_{46}a_{51} + a_{43}a_{56})a_{65}))) \\
& + a_{13}(a_{28}a_{31}(a_{41}a_{51} - a_{43}a_{53})a_{63} \\
& + (a_{23}(a_{32}(-a_{41}a_{51} + a_{43}a_{53}) \\
& + a_{31}(-a_{42}a_{51} - a_{41}a_{52} + a_{44}a_{53} + a_{43}a_{54})) \\
& + a_{31}(a_{26}(-a_{46}a_{53} + a_{41}a_{56}) + a_{25}(a_{46}a_{51} - a_{43}a_{56})))a_{64} \\
& + a_{23}a_{31}(-a_{41}a_{51} + a_{43}a_{53})a_{65} + a_{27}(a_{32}(a_{41}a_{51} - a_{43}a_{53})a_{63} \\
& + a_{31}(a_{42}a_{51}a_{63} + a_{41}a_{52}a_{63} - a_{44}a_{53}a_{63} - a_{43}a_{54}a_{63} \\
& - a_{46}a_{51}a_{66} \\
& + a_{43}a_{56}a_{66} + (a_{46}a_{53} - a_{41}a_{56})a_{67}))) \\
& + a_{17}(a_{26}a_{31}(a_{46}a_{53} - a_{41}a_{56})a_{61} + a_{25}a_{31}(-a_{46}a_{51} + a_{43}a_{56})a_{61} \\
& + a_{23}(a_{32}(a_{41}a_{51} - a_{43}a_{53})a_{61} \\
& + a_{31}((a_{42}a_{51} + a_{41}a_{52} - a_{44}a_{53} - a_{43}a_{54})a_{61} \\
& + (a_{41}a_{51} - a_{43}a_{53})a_{62})) \\
& + a_{22}a_{31}(-a_{41}a_{51} + a_{43}a_{53})a_{63} \\
& + a_{21}(a_{32}(-a_{41}a_{51} + a_{43}a_{53})a_{63} + a_{44}a_{53}a_{63} \\
& + a_{43}(a_{54}a_{63} - a_{56}a_{66}) \\
& + a_{46}(a_{51}a_{66} - a_{53}a_{67}) + a_{41}(-a_{52}a_{63} + a_{56}a_{67})))) \\
& + a_{11}(a_{28}(a_{32}(-a_{41}a_{51} + a_{43}a_{53})a_{61} \\
& + a_{31}((-a_{42}a_{51} - a_{41}a_{52} + a_{44}a_{53} + a_{43}a_{54})a_{61} \\
& + (-a_{41}a_{51} + a_{43}a_{53})a_{62})) \\
& + a_{26}a_{31}(-a_{48}a_{53}a_{61} + a_{47}a_{53}a_{64} + a_{41}(a_{58}a_{61} - a_{57}a_{64})) \\
& + a_{25}a_{31}(a_{48}a_{51}a_{61} - a_{47}a_{51}a_{64} + a_{43}(-a_{58}a_{61} + a_{57}a_{64})) \\
& + a_{22}(a_{32}(a_{41}a_{51} - a_{43}a_{53})a_{64} \\
& + a_{31}((a_{42}a_{51} + a_{41}a_{52} - a_{44}a_{53} - a_{43}a_{54})a_{64} \\
& + (a_{41}a_{51} - a_{43}a_{53})a_{65})) \\
& + a_{27}(a_{33}a_{45}(-a_{43} + a_{51})a_{61} \\
& + a_{32}((-a_{42}a_{51} - a_{41}a_{52} + a_{44}a_{53} + a_{43}a_{54})a_{61} \\
& + (-a_{41}a_{51} + a_{43}a_{53})a_{62}) \\
& + a_{31}(-a_{42}(a_{52}a_{61} + a_{51}a_{62}) + a_{44}(a_{54}a_{61} + a_{53}a_{62}) \\
& + a_{43}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{51}a_{66} - a_{53}a_{67}) \\
& + a_{41}(-a_{52}a_{62} + a_{57}a_{67}))) + a_{21}(a_{33}a_{45}(a_{43} - a_{51})a_{64} \\
& + a_{32}((a_{42}a_{51} + a_{41}a_{52} - a_{44}a_{53} - a_{43}a_{54})a_{64} \\
& + (a_{41}a_{51} - a_{43}a_{53})a_{65}) \\
& + a_{31}(a_{42}(a_{52}a_{64} + a_{51}a_{65}) \\
& - a_{44}(a_{54}a_{64} + a_{53}a_{65}) + a_{43}(-a_{54}a_{65} + a_{58}a_{66}) \\
& + a_{48}(-a_{51}a_{66} + a_{53}a_{67}) + a_{41}(a_{52}a_{65} - a_{58}a_{67}))))
\end{aligned}$$

$$\begin{aligned}
\Delta_4 = {} & a_{14}a_{33}(a_{28}(-a_{46}a_{51} + a_{43}a_{56})a_{61} \\
& + a_{27}(a_{56}(a_{44}a_{61} + a_{43}a_{62}) \\
& - a_{46}(a_{52}a_{61} + a_{51}a_{62}) + (a_{47}a_{51} - a_{43}a_{57})a_{63}) \\
& + a_{22}(a_{46}a_{51} - a_{43}a_{56})a_{64} \\
& + a_{23}(a_{48}a_{51}a_{61} - a_{47}a_{51}a_{64} + a_{43}(-a_{58}a_{61} + a_{57}a_{64})) \\
& + a_{21}((-a_{48}a_{51} + a_{43}a_{58})a_{63} + a_{46}a_{52} - a_{44}a_{56})a_{64} \\
& + (a_{46}a_{51} - a_{43}a_{56})a_{65})) + a_{16}(a_{28}(a_{32}(-a_{46}a_{53} + a_{41}a_{56})a_{61} \\
& + a_{31}(a_{56}(a_{42}a_{61} + a_{41}a_{62}) - a_{46}(a_{54}a_{61} + a_{53}a_{62}) \\
& + (a_{47}a_{53} - a_{41}a_{57})a_{63}))
\end{aligned}$$

$$
\begin{aligned}
&+a_{25}a_{31}(-a_{48}a_{56}a_{61}+a_{47}a_{56}a_{64}+a_{46}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{22}(a_{32}(a_{46}a_{53}-a_{41}a_{56})a_{64}\\
&+a_{31}((-a_{48}a_{53}+a_{41}a_{58})a_{63}\\
&+(a_{46}a_{54}-a_{42}a_{56})a_{64}+(a_{46}a_{53}-a_{41}a_{56})a_{65}))\\
&+a_{23}(a_{32}(a_{48}a_{53}a_{61}-a_{47}a_{53}a_{64}+a_{41}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{31}(a_{48}(a_{54}a_{61}+a_{53}a_{62})+a_{42}(-a_{58}a_{61}+a_{57}a_{64})\\
&-a_{47}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{41}(-a_{58}a_{62}+a_{57}a_{65})))\\
&+a_{27}(a_{33}a_{45}(a_{46}-a_{56})a_{61}\\
&+a_{32}(a_{56}(a_{42}a_{61}+a_{41}a_{62})-a_{46}(a_{54}a_{61}+a_{53}a_{62})\\
&+(a_{47}a_{53}-a_{41}a_{57})a_{63})\\
&+a_{31}(a_{42}(a_{56}a_{62}-a_{57}a_{63})+a_{47}(a_{54}a_{63}-a_{56}a_{66})\\
&+a_{46}(-a_{54}a_{62}+a_{57}a_{66})))+a_{21}(a_{33}a_{45}(-a_{46}+a_{56})a_{64}\\
&+a_{32}((-a_{48}a_{53}+a_{41}a_{58})a_{63}+(a_{46}a_{54}-a_{42}a_{56})a_{64}\\
&+(a_{46}a_{53}-a_{41}a_{56})a_{65})\\
&+a_{31}(a_{42}(a_{58}a_{63}-a_{56}a_{65})+a_{48}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(a_{54}a_{65}-a_{58}a_{66})))\\
&+a_{17}(a_{24}a_{33}(a_{46}a_{51}-a_{43}a_{56})a_{61}\\
&+a_{26}(a_{32}(a_{46}a_{53}-a_{41}a_{56})a_{61}\\
&+a_{31}(-a_{56}(a_{42}a_{61}+a_{41}a_{62})\\
&+a_{46}(a_{54}a_{61}+a_{53}a_{62})+(-a_{47}a_{53}+a_{41}a_{57})a_{63}))\\
&+a_{25}(a_{32}(-a_{46}a_{51}+a_{43}a_{56})a_{61}\\
&+a_{31}(a_{56}(a_{44}a_{61}+a_{43}a_{62})\\
&-a_{46}(a_{52}a_{61}+a_{51}a_{62})+(a_{47}a_{51}-a_{43}a_{57})a_{63}))\\
&+a_{22}(a_{32}(-a_{41}a_{51}+a_{43}a_{53})a_{63}\\
&+a_{31}(-a_{42}a_{51}a_{63}+a_{44}a_{53}a_{63}\\
&+a_{43}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{63}+a_{56}a_{67})))\\
&+a_{21}(a_{33}a_{45}(-a_{43}+a_{51})a_{63}\\
&+a_{32}(-a_{42}a_{51}a_{63}+a_{44}a_{53}a_{63}\\
&+a_{43}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{63}+a_{56}a_{67}))\\
&+a_{31}(a_{44}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{63}+a_{56}a_{67})))\\
&+a_{23}(a_{33}a_{45}(a_{43}-a_{51})a_{61}\\
&+a_{32}((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{61}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{62})\\
&+a_{31}(a_{42}(a_{52}a_{61}+a_{51}a_{62})-a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(-a_{54}a_{62}+a_{57}a_{66})+a_{47}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{62}-a_{57}a_{67}))))+a_{12}(a_{28}(a_{32}(-a_{41}a_{51}+a_{43}a_{53})a_{61}\\
&+a_{31}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62}))\\
&+a_{26}a_{31}(-a_{48}a_{53}a_{61}+a_{47}a_{53}a_{64}+a_{41}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{25}a_{31}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{22}(a_{32}(a_{41}a_{51}-a_{43}a_{53})a_{64}\\
&+a_{31}(a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{65}))\\
&+a_{27}(a_{33}a_{45}(-a_{43}+a_{51})a_{61}
\end{aligned}
$$

$$
\begin{aligned}
&+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62})\\
&+a_{31}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62}+a_{57}a_{67})))+a_{21}(a_{33}a_{45}(a_{43}-a_{51})a_{64}\\
&+a_{32}((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+a_{41}a_{51}-a_{43}a_{53})a_{65})+a_{31}(a_{42}(a_{52}a_{64}+a_{51}a_{65})\\
&-a_{44}(a_{54}a_{64}+a_{53}a_{65})+a_{43}(-a_{54}a_{65}+a_{58}a_{66})\\
&+a_{48}(-a_{51}a_{66}+a_{53}a_{67})+a_{41}(a_{52}a_{65}-a_{58}a_{67}))))\\
&+a_{11}(-a_{24}a_{33}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{26}(a_{32}(-a_{48}a_{53}a_{61}+a_{47}a_{53}a_{64}+a_{41}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{31}(-a_{48}(a_{54}a_{61}+a_{53}a_{62})+a_{42}(a_{58}a_{61}-a_{57}a_{64})\\
&+a_{47}(a_{54}a_{64}+a_{53}a_{65})+a_{41}(a_{58}a_{62}-a_{57}a_{65}))\\
&+a_{25}(a_{32}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{31}(a_{48}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(-a_{58}a_{61}+a_{57}a_{64})\\
&-a_{47}(a_{52}a_{64}+a_{51}a_{65})+a_{43}(-a_{58}a_{62}+a_{57}a_{65})))\\
&+a_{28}(a_{33}a_{45}(-a_{43}+a_{51})a_{61}+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}\\
&+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62})\\
&+a_{31}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{27}(a_{33}a_{45}((-a_{44}+a_{52})a_{61}+(-a_{43}+a_{51})a_{62})\\
&+a_{32}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})\\
&+a_{47}(a_{51}a_{66}-a_{53}a_{67})+a_{41}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{31}(a_{44}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{22}(a_{33}a_{45}(a_{43}-a_{51})a_{64}+a_{32}\\
&((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{65})\\
&+a_{31}(a_{42}(a_{52}a_{64}+a_{51}a_{65})-a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65}-a_{58}a_{67})))+a_{21}(a_{33}a_{45}((a_{44}-a_{52})a_{64}\\
&+(a_{43}-a_{51})a_{65})\\
&+a_{32}(a_{42}(a_{52}a_{64}+a_{51}a_{65})-a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{31}(a_{44}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{65}-a_{58}a_{67})))\\
&+a_{13}(a_{24}a_{33}(-a_{46}a_{51}+a_{43}a_{56})a_{64}\\
&+a_{26}(a_{32}(-a_{46}a_{53}+a_{41}a_{56})a_{64}\\
&+(a_{46}a_{52}-a_{44}a_{56})a_{64}+(a_{46}a_{51}-a_{43}a_{56})a_{65}))\\
&+a_{28}(a_{32}(a_{41}a_{51}-a_{43}a_{53})a_{63}\\
&+a_{31}(a_{42}a_{51}a_{63}-a_{44}a_{53}a_{63}+a_{43}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(-a_{51}a_{66}+a_{53}a_{67})+a_{41}(a_{52}a_{63}-a_{56}a_{67})))\\
&+a_{27}(a_{33}a_{45}(a_{43}-a_{51})a_{63}+a_{32}(a_{42}a_{51}a_{63}-a_{44}a_{53}a_{63}
\end{aligned}
$$

$$
\begin{aligned}
&+a_{43}(-a_{54}a_{63}+a_{56}a_{66})+a_{46}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{63}-a_{56}a_{67}))\\
&+a_{31}(a_{44}(-a_{54}a_{63}+a_{56}a_{66})+a_{46}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{63}-a_{56}a_{67}))\\
&+a_{23}(a_{33}a_{45}(-a_{43}+a_{51})a_{64}\\
&+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{64}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{65})\\
&+a_{31}(-a_{42}(a_{52}a_{64}+a_{51}a_{65})+a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(a_{54}a_{65}-a_{58}a_{66})+a_{48}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{65}+a_{58}a_{67})))+a_{15}(a_{28}(a_{32}(a_{46}a_{51}-a_{43}a_{56})a_{61}\\
&+a_{31}(-a_{56}(a_{44}a_{61}+a_{43}a_{62})\\
&+a_{46}(a_{52}a_{61}+a_{51}a_{62})+(-a_{47}a_{51}+a_{43}a_{57})a_{63}))\\
&+a_{26}a_{31}(a_{48}a_{56}a_{61}-a_{47}a_{56}a_{64}+a_{46}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{22}(a_{32}(-a_{46}a_{51}+a_{43}a_{56})a_{64}\\
&+a_{31}((a_{48}a_{51}-a_{43}a_{58})a_{63}\\
&+(-a_{46}a_{52}+a_{44}a_{56})a_{64}+(-a_{46}a_{51}+a_{43}a_{56})a_{65}))\\
&+a_{23}(a_{32}(-a_{48}a_{51}a_{61}+a_{47}a_{51}a_{64}+a_{43}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{31}(-a_{48}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{58}a_{61}-a_{57}a_{64})\\
&+a_{47}(a_{52}a_{64}+a_{51}a_{65})+a_{43}(a_{58}a_{62}-a_{57}a_{65})))\\
&+a_{27}(a_{32}(-a_{56}(a_{44}a_{61}+a_{43}a_{62})+a_{46}(a_{52}a_{61}+a_{51}a_{62})\\
&+(-a_{47}a_{51}+a_{43}a_{57})a_{63}\\
&+a_{31}(a_{44}(-a_{56}a_{62}+a_{57}a_{63})+a_{47}(-a_{52}a_{63}+a_{56}a_{67})\\
&+a_{46}(a_{52}a_{62}-a_{57}a_{67}))\\
&+a_{21}(a_{32}(a_{48}a_{51}a_{63}+a_{44}a_{56}a_{64}-a_{46}(a_{52}a_{64}+a_{51}a_{65})\\
&+a_{43}(-a_{58}a_{63}+a_{56}a_{65}))\\
&+a_{31}(a_{44}(-a_{58}a_{63}+a_{56}a_{65})\\
&+a_{48}(a_{52}a_{63}-a_{56}a_{67})+a_{46}(-a_{52}a_{65}+a_{58}a_{67})))
\end{aligned}
$$

$$
\begin{aligned}
\Delta_4 =\ & a_{14}a_{33}(a_{28}(-a_{46}a_{51}+a_{43}a_{56})a_{61}\\
&+a_{27}(a_{56}(a_{44}a_{61}+a_{43}a_{62})\\
&-a_{46}(a_{52}a_{61}+a_{51}a_{62})+(a_{47}a_{51}-a_{43}a_{57})a_{63})\\
&+a_{22}(a_{46}a_{51}-a_{43}a_{56})a_{64}\\
&+a_{23}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{21}((-a_{48}a_{51}+a_{43}a_{58})a_{63}+a_{46}a_{52}-a_{44}a_{56})a_{64}\\
&+(a_{46}a_{51}-a_{43}a_{56})a_{65}))+a_{16}(a_{28}(a_{32}(-a_{46}a_{53}+a_{41}a_{56})a_{61}\\
&+a_{31}(a_{56}(a_{42}a_{61}+a_{41}a_{62})-a_{46}(a_{54}a_{61}+a_{53}a_{62})\\
&+(a_{47}a_{53}-a_{41}a_{57})a_{63}))\\
&+a_{25}a_{31}(-a_{48}a_{56}a_{61}+a_{47}a_{56}a_{64}+a_{46}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{22}(a_{32}(a_{46}a_{53}-a_{41}a_{56})a_{64}\\
&+a_{31}((-a_{48}a_{53}+a_{41}a_{58})a_{63}\\
&+(a_{46}a_{54}-a_{42}a_{56})a_{64}+(a_{46}a_{53}-a_{41}a_{56})a_{65}))\\
&+a_{23}(a_{32}(a_{48}a_{53}a_{61}-a_{47}a_{53}a_{64}+a_{41}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{31}(a_{48}(a_{54}a_{61}+a_{53}a_{62})+a_{42}(-a_{58}a_{61}+a_{57}a_{64})\\
&-a_{47}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{41}(-a_{58}a_{62}+a_{57}a_{65})))\\
&+a_{27}(a_{33}a_{45}(a_{46}-a_{56})a_{61}\\
&+a_{32}(a_{56}(a_{42}a_{61}+a_{41}a_{62})-a_{46}(a_{54}a_{61}+a_{53}a_{62})\\
&+(a_{47}a_{53}-a_{41}a_{57})a_{63})\\
&+a_{31}(a_{42}(a_{56}a_{62}-a_{57}a_{63})+a_{47}(a_{54}a_{63}-a_{56}a_{66})\\
&+a_{46}(-a_{54}a_{62}+a_{57}a_{66})))+a_{21}(a_{33}a_{45}(-a_{46}+a_{56})a_{64}\\
&+a_{32}((-a_{48}a_{53}+a_{41}a_{58})a_{63}+(a_{46}a_{54}-a_{42}a_{56})a_{64}\\
&+(a_{46}a_{53}-a_{41}a_{56})a_{65})\\
&+a_{31}(a_{42}(a_{58}a_{63}-a_{56}a_{65})+a_{48}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(a_{54}a_{65}-a_{58}a_{66})))\\
&+a_{17}(a_{24}a_{33}(a_{46}a_{51}-a_{43}a_{56})a_{61}\\
&+a_{26}(a_{32}(a_{46}a_{53}-a_{41}a_{56})a_{61}\\
&+a_{31}(-a_{56}(a_{42}a_{61}+a_{41}a_{62})\\
&+a_{46}(a_{54}a_{61}+a_{53}a_{62})+(-a_{47}a_{53}+a_{41}a_{57})a_{63}))\\
&+a_{25}(a_{32}(-a_{46}a_{51}+a_{43}a_{56})a_{61}\\
&+a_{31}(a_{56}(a_{44}a_{61}+a_{43}a_{62})\\
&-a_{46}(a_{52}a_{61}+a_{51}a_{62})+(a_{47}a_{51}-a_{43}a_{57})a_{63}))\\
&+a_{22}(a_{32}(-a_{41}a_{51}+a_{43}a_{53})a_{63}\\
&+a_{31}(-a_{42}a_{51}a_{63}+a_{44}a_{53}a_{63}\\
&+a_{43}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{63}+a_{56}a_{67})))\\
&+a_{21}(a_{33}a_{45}(-a_{43}+a_{51})a_{63}\\
&+a_{32}(-a_{42}a_{51}a_{63}+a_{44}a_{53}a_{63}\\
&+a_{43}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{63}+a_{56}a_{67}))\\
&+a_{31}(a_{44}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{63}+a_{56}a_{67})))\\
&+a_{23}(a_{33}a_{45}(a_{43}-a_{51})a_{61}\\
&+a_{32}((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{61}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{62})\\
&+a_{31}(a_{42}(a_{52}a_{61}+a_{51}a_{62})-a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(-a_{54}a_{62}+a_{57}a_{66})+a_{47}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{62}-a_{57}a_{67}))))+a_{12}(a_{28}(a_{32}(-a_{41}a_{51}+a_{43}a_{53})a_{61}\\
&+a_{31}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62}))\\
&+a_{26}a_{31}(-a_{48}a_{53}a_{61}+a_{47}a_{53}a_{64}+a_{41}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{25}a_{31}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{22}(a_{32}(a_{41}a_{51}-a_{43}a_{53})a_{64}\\
&+a_{31}(a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{65}))\\
&+a_{27}(a_{33}a_{45}(-a_{43}+a_{51})a_{61}\\
&+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62})\\
&+a_{31}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62}+a_{57}a_{67})))+a_{21}(a_{33}a_{45}(a_{43}-a_{51})a_{64}\\
&+a_{32}((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+a_{41}a_{51}-a_{43}a_{53})a_{65})+a_{31}(a_{42}(a_{52}a_{64}+a_{51}a_{65})\\
&-a_{44}(a_{54}a_{64}+a_{53}a_{65})+a_{43}(-a_{54}a_{65}+a_{58}a_{66})\\
&+a_{48}(-a_{51}a_{66}+a_{53}a_{67})+a_{41}(a_{52}a_{65}-a_{58}a_{67}))))
\end{aligned}
$$

$$\begin{aligned}
&+a_{11}(-a_{24}a_{33}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{26}(a_{32}(-a_{48}a_{53}a_{61}+a_{47}a_{53}a_{64}+a_{41}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{31}(-a_{48}(a_{54}a_{61}+a_{53}a_{62})+a_{42}(a_{58}a_{61}-a_{57}a_{64})\\
&+a_{47}(a_{54}a_{64}+a_{53}a_{65})+a_{41}(a_{58}a_{62}-a_{57}a_{65}))\\
&+a_{25}(a_{32}(a_{48}a_{51}a_{61}-a_{47}a_{51}a_{64}+a_{43}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{31}(a_{48}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(-a_{58}a_{61}+a_{57}a_{64})\\
&-a_{47}(a_{52}a_{64}+a_{51}a_{65})+a_{43}(-a_{58}a_{62}+a_{57}a_{65})))\\
&+a_{28}(a_{33}a_{45}(-a_{43}+a_{51})a_{61}\\
&+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{61}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{62})\\
&+a_{31}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{27}(a_{33}a_{45}((-a_{44}+a_{52})a_{61}+(-a_{43}+a_{51})a_{62})\\
&+a_{32}(-a_{42}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})\\
&+a_{47}(a_{51}a_{66}-a_{53}a_{67})+a_{41}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{31}(a_{44}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{22}(a_{33}a_{45}(a_{43}-a_{51})a_{64}+a_{32}\\
&((a_{42}a_{51}+a_{41}a_{52}-a_{44}a_{53}-a_{43}a_{54})a_{64}\\
&+(a_{41}a_{51}-a_{43}a_{53})a_{65})\\
&+a_{31}(a_{42}(a_{52}a_{64}+a_{51}a_{65})-a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65}-a_{58}a_{67})))+a_{21}(a_{33}a_{45}((a_{44}-a_{52})a_{64}\\
&+(a_{43}-a_{51})a_{65})\\
&+a_{32}(a_{42}(a_{52}a_{64}+a_{51}a_{65})-a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{31}(a_{44}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{65}-a_{58}a_{67})))\\
&+a_{13}(a_{24}a_{33}(-a_{46}a_{51}+a_{43}a_{56})a_{64}\\
&+a_{26}(a_{32}(-a_{46}a_{53}+a_{41}a_{56})a_{64}\\
&+(a_{46}a_{52}-a_{44}a_{56})a_{64}+(a_{46}a_{51}-a_{43}a_{56})a_{65}))\\
&+a_{28}(a_{32}(a_{41}a_{51}-a_{43}a_{53})a_{63}\\
&+a_{31}(a_{42}a_{51}a_{63}-a_{44}a_{53}a_{63}+a_{43}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(-a_{51}a_{66}+a_{53}a_{67})+a_{41}(a_{52}a_{63}-a_{56}a_{67})))\\
&+a_{27}(a_{33}a_{45}(a_{43}-a_{51})a_{63}+a_{32}(a_{42}a_{51}a_{63}-a_{44}a_{53}a_{63}\\
&+a_{43}(-a_{54}a_{63}+a_{56}a_{66})+a_{46}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{63}-a_{56}a_{67}))\\
&+a_{31}(a_{44}(-a_{54}a_{63}+a_{56}a_{66})+a_{46}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{63}-a_{56}a_{67}))\\
&+a_{23}(a_{33}a_{45}(-a_{43}+a_{51})a_{64}\\
&+a_{32}((-a_{42}a_{51}-a_{41}a_{52}+a_{44}a_{53}+a_{43}a_{54})a_{64}\\
&+(-a_{41}a_{51}+a_{43}a_{53})a_{65})\\
&+a_{31}(-a_{42}(a_{52}a_{64}+a_{51}a_{65})+a_{44}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{43}(a_{54}a_{65}-a_{58}a_{66})+a_{48}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{65}+a_{58}a_{67})))+a_{15}(a_{28}(a_{32}(a_{46}a_{51}-a_{43}a_{56})a_{61}\\
&+a_{31}(-a_{56}(a_{44}a_{61}+a_{43}a_{62})\\
&+a_{46}(a_{52}a_{61}+a_{51}a_{62})+(-a_{47}a_{51}+a_{43}a_{57})a_{63}))\\
&+a_{26}a_{31}(a_{48}a_{56}a_{61}-a_{47}a_{56}a_{64}+a_{46}(-a_{58}a_{61}+a_{57}a_{64}))\\
&+a_{22}(a_{32}(-a_{46}a_{51}+a_{43}a_{56})a_{64}\\
&+a_{31}((a_{48}a_{51}-a_{43}a_{58})a_{63}\\
&+(-a_{46}a_{52}+a_{44}a_{56})a_{64}+(-a_{46}a_{51}+a_{43}a_{56})a_{65}))\\
&+a_{23}(a_{32}(-a_{48}a_{51}a_{61}+a_{47}a_{51}a_{64}+a_{43}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{31}(-a_{48}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(a_{58}a_{61}-a_{57}a_{64})\\
&+a_{47}(a_{52}a_{64}+a_{51}a_{65})+a_{43}(a_{58}a_{62}-a_{57}a_{65})))\\
&+a_{27}(a_{32}(-a_{56}(a_{44}a_{61}+a_{43}a_{62})+a_{46}(a_{52}a_{61}+a_{51}a_{62})\\
&+(-a_{47}a_{51}+a_{43}a_{57})a_{63}\\
&+a_{31}(a_{44}(-a_{56}a_{62}+a_{57}a_{63})+a_{47}(-a_{52}a_{63}+a_{56}a_{67})\\
&+a_{46}(a_{52}a_{62}-a_{57}a_{67}))\\
&+a_{21}(a_{32}(a_{48}a_{51}a_{63}+a_{44}a_{56}a_{64}-a_{46}(a_{52}a_{64}+a_{51}a_{65})\\
&+a_{43}(-a_{58}a_{63}+a_{56}a_{65}))\\
&+a_{31}(a_{44}(-a_{58}a_{63}+a_{56}a_{65})\\
&+a_{48}(a_{52}a_{63}-a_{56}a_{67})+a_{46}(-a_{52}a_{65}+a_{58}a_{67})))
\end{aligned}$$

$$\begin{aligned}
\Delta_5 = {}& a_{16}(a_{33}(a_{24}a_{48}a_{56}a_{61}-a_{24}a_{47}a_{56}a_{64}\\
&+a_{46}(-a_{24}a_{58}a_{61}+a_{24}a_{57}a_{64}))\\
&+a_{25}(a_{32}(-a_{48}a_{56}a_{61}+a_{47}a_{56}a_{64}+a_{46}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{31}(a_{48}(-a_{56}a_{62}+a_{57}a_{63})+a_{47}(-a_{58}a_{63}+a_{56}a_{65})\\
&+a_{46}(a_{58}a_{62}-a_{57}a_{65}))\\
&+a_{28}(a_{33}a_{45}(a_{46}a_{61}-a_{56}a_{61})\\
&+a_{32}(a_{42}a_{56}a_{61}+a_{46}(-a_{54}a_{61}-a_{53}a_{62})\\
&+a_{47}a_{53}a_{63}+a_{41}(a_{56}a_{62}-a_{57}a_{63}))\\
&+a_{31}(a_{42}(a_{56}a_{62}-a_{57}a_{63})\\
&+a_{47}(a_{54}a_{63}-a_{56}a_{66})+a_{46}(-a_{54}a_{62}+a_{57}a_{66}))\\
&+a_{32}(-a_{48}a_{53}a_{63}-a_{42}a_{56}a_{64}+a_{46}(a_{54}a_{64}+a_{53}a_{65})\\
&+a_{41}(a_{58}a_{63}-a_{56}a_{65}))\\
&+a_{31}(a_{42}(a_{58}a_{63}-a_{56}a_{65})+a_{48}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(a_{54}a_{65}-a_{58}a_{66})))\\
&+a_{21}(a_{33}a_{45}(a_{48}a_{63}-a_{58}a_{63}-a_{46}a_{65}+a_{56}a_{65})\\
&+a_{32}(a_{42}(a_{58}a_{63}-a_{56}a_{65})+a_{48}(-a_{54}a_{63}+a_{56}a_{66})\\
&+a_{46}(a_{54}a_{65}-a_{58}a_{66}))\\
&+a_{23}(a_{33}a_{45}(-a_{48}a_{61}+a_{58}a_{61}+a_{47}a_{64}-a_{57}a_{64})\\
&+a_{32}(a_{48}(a_{54}a_{61}+a_{53}a_{62})+a_{42}(-a_{58}a_{61}+a_{57}a_{64})\\
&+a_{47}(-a_{54}a_{64}-a_{53}a_{65})+a_{41}(-a_{58}a_{62}+a_{57}a_{65}))\\
&+a_{31}(a_{42}(-a_{58}a_{62}+a_{57}a_{65})+a_{48}(a_{54}a_{62}-a_{57}a_{66})\\
&+a_{47}(-a_{54}a_{65}+a_{58}a_{66})))\\
&+a_{14}(a_{28}a_{33}(a_{44}a_{56}a_{61}+a_{46}(-a_{52}a_{61}-a_{51}a_{62})\\
&+a_{47}a_{51}a_{63}+a_{43}(a_{56}a_{62}-a_{57}a_{63}))\\
&+a_{26}a_{33}(-a_{48}a_{56}a_{61}+a_{47}a_{56}a_{64}+a_{46}(a_{58}a_{61}-a_{57}a_{64}))\\
&+a_{22}a_{33}(-a_{48}a_{51}a_{63}-a_{44}a_{56}a_{64}\\
&+a_{46}(a_{52}a_{64}+a_{51}a_{65})+a_{43}(a_{58}a_{63}-a_{56}a_{65}))\\
&+a_{23}a_{33}(a_{48}(a_{52}a_{61}+a_{51}a_{62})+a_{44}(-a_{58}a_{61}+a_{57}a_{64})\\
&+a_{47}(-a_{52}a_{64}-a_{51}a_{65})+a_{43}(-a_{58}a_{62}+a_{57}a_{65}))
\end{aligned}$$

$$
\begin{aligned}
&+a_{27}a_{33}(a_{44}(a_{56}a_{62} - a_{57}a_{63}) + a_{47}(a_{52}a_{63} - a_{56}a_{67})\\
&+ a_{46}(-a_{52}a_{62} + a_{57}a_{67}))\\
&+ a_{21}a_{33}(a_{44}(a_{58}a_{63} - a_{56}a_{65}) + a_{48}(-a_{52}a_{63} + a_{56}a_{67})\\
&+ a_{46}(a_{52}a_{65} - a_{58}a_{67}))\\
&+a_{17}(a_{33}(-a_{24}a_{44}a_{56}a_{61} + a_{46}(a_{24}a_{52}a_{61} + a_{24}a_{51}a_{62})\\
&- a_{24}a_{47}a_{51}a_{63} + a_{43}(-a_{24}a_{56}a_{62} + a_{24}a_{57}a_{63}))\\
&+ a_{26}(a_{33}a_{45}(-a_{46}a_{61} + a_{56}a_{61}) + a_{32}(-a_{42}a_{56}a_{61}\\
&+ a_{46}(a_{54}a_{61} + a_{53}a_{62}) - a_{47}a_{53}a_{63} + a_{41}(-a_{56}a_{62} + a_{57}a_{63}))\\
&+a_{31}(a_{42}(-a_{56}a_{62} + a_{57}a_{63}) + a_{47}(-a_{54}a_{63} + a_{56}a_{66})\\
&+ a_{46}(a_{54}a_{62} - a_{57}a_{66}))\\
&+ a_{22}(a_{33}(-a_{43}a_{45}a_{63} + a_{45}a_{51}a_{63})\\
&+a_{32}(-a_{42}a_{51}a_{63} + a_{44}a_{53}a_{63}\\
&+ a_{43}(a_{54}a_{63} - a_{56}a_{66}) + a_{46}(a_{51}a_{66} - a_{53}a_{67})\\
&+ a_{41}(-a_{52}a_{63} + a_{56}a_{67}))\\
&+ a_{31}(a_{44}(a_{54}a_{63} - a_{56}a_{66}) + a_{46}(a_{52}a_{66} - a_{54}a_{67})\\
&+ a_{42}(-a_{52}a_{63} + a_{56}a_{67}))\\
&+a_{21}(a_{33}(-a_{44}a_{45}a_{63} + a_{45}(a_{52}a_{63} + a_{46}a_{67} - a_{56}a_{67}))\\
&+ a_{32}(a_{44}(a_{54}a_{63} - a_{56}a_{66}) + a_{46}(a_{52}a_{66} - a_{54}a_{67})\\
&+ a_{42}(-a_{52}a_{63} + a_{56}a_{67}))\\
&+ a_{23}(a_{33}(a_{44}a_{45}a_{61} + a_{43}a_{45}a_{62} + a_{45}(-a_{52}a_{61} - a_{51}a_{62}))\\
&+a_{32}(a_{42}(a_{52}a_{61} + a_{51}a_{62}) + a_{44}(-a_{54}a_{61} - a_{53}a_{62})\\
&+ a_{43}(-a_{54}a_{62} + a_{57}a_{66}) + a_{47}(-a_{51}a_{66} + a_{53}a_{67})\\
&+ a_{41}(a_{52}a_{62} - a_{57}a_{67}))\\
&+ a_{31}(a_{44}(-a_{54}a_{62} + a_{57}a_{66}) + a_{47}(-a_{52}a_{66} + a_{54}a_{67})\\
&+ a_{42}(a_{52}a_{62} - a_{57}a_{67}))\\
&+ a_{25}(a_{32}(a_{44}a_{56}a_{61} + a_{46}(-a_{52}a_{61} - a_{51}a_{62})\\
&+ a_{47}a_{51}a_{63} + a_{43}(a_{56}a_{62} - a_{57}a_{63}))\\
&+a_{31}(a_{44}(a_{56}a_{62} - a_{57}a_{63})\\
&+ a_{47}(a_{52}a_{63} - a_{56}a_{67}) + a_{46}(-a_{52}a_{62} + a_{57}a_{67})))\\
&+ a_{12}(a_{33}(-a_{24}a_{48}a_{51}a_{61} + a_{24}a_{47}a_{51}a_{64}\\
&+ a_{43}(a_{24}a_{58}a_{61} - a_{24}a_{57}a_{64}))\\
&+ a_{26}(a_{32}(-a_{48}a_{53}a_{61} + a_{47}a_{53}a_{64} + a_{41}(a_{58}a_{61} - a_{57}a_{64}))\\
&+a_{31}(a_{48}(-a_{54}a_{61} - a_{53}a_{62}) + a_{42}(a_{58}a_{61} - a_{57}a_{64})\\
&+ a_{47}(a_{54}a_{64} + a_{53}a_{65}) + a_{41}(a_{58}a_{62} - a_{57}a_{65}))\\
&+ a_{25}(a_{32}(a_{48}a_{51}a_{61} - a_{47}a_{51}a_{64}\\
&+ a_{43}(-a_{58}a_{61} + a_{57}a_{64}))\\
&+a_{31}(a_{48}(a_{52}a_{61} + a_{51}a_{62})\\
&+ a_{44}(-a_{58}a_{61} + a_{57}a_{64}) + a_{47}(-a_{52}a_{64} - a_{51}a_{65})\\
&+ a_{43}(-a_{58}a_{62} + a_{57}a_{65}))\\
&+ a_{28}(a_{33}(-a_{43}a_{45}a_{61} + a_{45}a_{51}a_{61})\\
&+a_{32}(-a_{42}a_{51}a_{61} + a_{44}a_{53}a_{61}\\
&+ a_{41}(-a_{52}a_{61} - a_{51}a_{62}) + a_{43}(a_{54}a_{61} + a_{53}a_{62}))\\
&+ a_{31}(a_{42}(-a_{52}a_{61} - a_{51}a_{62}) + a_{44}(a_{54}a_{61} + a_{53}a_{62})\\
&+ a_{43}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{51}a_{66} - a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62} + a_{57}a_{67}))\\
&+ a_{27}(a_{33}(-a_{44}a_{45}a_{61} - a_{43}a_{45}a_{62} + a_{45}(a_{52}a_{61} + a_{51}a_{62}))\\
&+ a_{32}(a_{42}(-a_{52}a_{61} - a_{51}a_{62}) + a_{44}(a_{54}a_{61} + a_{53}a_{62})\\
&+ a_{43}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{51}a_{66} - a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62} + a_{57}a_{67}))\\
&+ a_{31}(a_{44}(a_{54}a_{62} - a_{57}a_{66})\\
&+ a_{47}(a_{52}a_{66} - a_{54}a_{67}) + a_{42}(-a_{52}a_{62} + a_{57}a_{67}))\\
&+ a_{22}(a_{33}(a_{43}a_{45}a_{64} - a_{45}a_{51}a_{64})\\
&+a_{32}(a_{42}a_{51}a_{64} - a_{44}a_{53}a_{64}\\
&+ a_{41}(a_{52}a_{64} + a_{51}a_{65}) + a_{43}(-a_{54}a_{64} - a_{53}a_{65}))\\
&+ a_{31}(a_{42}(a_{52}a_{64} + a_{51}a_{65}) + a_{44}(-a_{54}a_{64} - a_{53}a_{65})\\
&+ a_{43}(-a_{54}a_{65} + a_{58}a_{66}) + a_{48}(-a_{51}a_{66} + a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65} - a_{58}a_{67}))) + a_{21}(a_{33}\\
&(a_{44}a_{45}a_{64} + a_{43}a_{45}a_{65} + a_{45}(-a_{52}a_{64} - a_{51}a_{65}))\\
&+ a_{32}(a_{42}(a_{52}a_{64} + a_{51}a_{65}) + a_{44}(-a_{54}a_{64} - a_{53}a_{65})\\
&+ a_{43}(-a_{54}a_{65} + a_{58}a_{66}) + a_{48}(-a_{51}a_{66} + a_{53}a_{67})\\
&+ a_{41}(a_{52}a_{65} - a_{58}a_{67})\\
&+a_{31}(a_{44}(-a_{54}a_{65} + a_{58}a_{66}) + a_{48}(-a_{52}a_{66} + a_{54}a_{67})\\
&+ a_{42}(a_{52}a_{65} - a_{58}a_{67})))\\
&+ a_{13}(a_{33}(a_{24}a_{48}a_{51}a_{63} + a_{24}a_{44}a_{56}a_{64}\\
&+ a_{46}(-a_{54}a_{64} - a_{53}a_{65}) + a_{41}(-a_{58}a_{63} + a_{56}a_{65}))\\
&+a_{31}(a_{42}(-a_{58}a_{63} + a_{56}a_{65}) + a_{48}(a_{54}a_{63} - a_{56}a_{66})\\
&+ a_{46}(-a_{54}a_{65} + a_{58}a_{66})\\
&+ a_{28}(a_{33}(a_{43}a_{45}a_{63} - a_{45}a_{51}a_{63})\\
&+ a_{32}(a_{42}a_{51}a_{63} - a_{44}a_{53}a_{63}\\
&+ a_{43}(-a_{54}a_{63} + a_{56}a_{66}) + a_{46}(-a_{51}a_{66} + a_{53}a_{67})\\
&+a_{41}(a_{52}a_{63} - a_{56}a_{67})\\
&+ a_{31}(a_{44}(-a_{54}a_{63} + a_{56}a_{66}) + a_{46}(-a_{52}a_{66} + a_{54}a_{67})\\
&+ a_{42}(a_{52}a_{63} - a_{56}a_{67}))\\
&+ a_{27}(a_{32}(a_{44}(-a_{54}a_{63} + a_{56}a_{66})\\
&+a_{46}(-a_{52}a_{66} + a_{54}a_{67}) + a_{42}(a_{52}a_{63} - a_{56}a_{67}))\\
&+ a_{33}(a_{44}a_{45}a_{63} + a_{45}(-a_{52}a_{63} - a_{46}a_{67} + a_{56}a_{67}))\\
&+ a_{25}(a_{32}(-a_{48}a_{51}a_{63} - a_{44}a_{56}a_{64} + a_{46}(a_{52}a_{64} + a_{51}a_{65})\\
&+ a_{43}(a_{58}a_{63} - a_{56}a_{65}))\\
&+a_{31}(a_{44}(a_{58}a_{63} - a_{56}a_{65}) + a_{48}(-a_{52}a_{63} + a_{56}a_{67})\\
&+ a_{46}(a_{52}a_{65} - a_{58}a_{67}))\\
&+ a_{23}(a_{33}(-a_{44}a_{45}a_{64} - a_{43}a_{45}a_{65} + a_{45}(a_{52}a_{64} + a_{51}a_{65}))\\
&+ a_{32}(a_{42}(-a_{52}a_{64} - a_{51}a_{65}) + a_{44}(a_{54}a_{64} + a_{53}a_{65})\\
&+a_{43}(a_{54}a_{65} - a_{58}a_{66}) + a_{48}(a_{51}a_{66} - a_{53}a_{67})\\
&+ a_{41}(-a_{52}a_{65} + a_{58}a_{67}))\\
&+ a_{31}(a_{44}(a_{54}a_{65} - a_{58}a_{66}) + a_{48}(a_{52}a_{66} - a_{54}a_{67})\\
&+ a_{42}(-a_{52}a_{65} + a_{58}a_{67}))\\
&+a_{15}(a_{26}(a_{32}(a_{48}a_{56}a_{61} - a_{47}a_{56}a_{64}\\
&+ a_{46}(-a_{58}a_{61} + a_{57}a_{64})\\
&+ a_{31}(a_{48}(a_{56}a_{62} - a_{57}a_{63}) + a_{47}(a_{58}a_{63} - a_{56}a_{65})\\
&+ a_{46}(-a_{58}a_{62} + a_{57}a_{65})) + a_{27}a_{32}(a_{44}(-a_{56}a_{62} + a_{57}a_{63})\\
&+a_{47}(-a_{52}a_{63} + a_{56}a_{67}) + a_{46}(a_{52}a_{62} - a_{57}a_{67}))\\
&+a_{21}a_{32}(a_{44}(-a_{58}a_{63} + a_{56}a_{65}) + a_{48}(a_{52}a_{63} - a_{56}a_{67})\\
&+ a_{46}(-a_{52}a_{65} + a_{58}a_{67})\\
&+ a_{28}(a_{32}(-a_{44}a_{56}a_{61} + a_{46}(a_{52}a_{61} + a_{51}a_{62}) - a_{47}a_{51}a_{63}\\
&+ a_{43}(-a_{56}a_{62} + a_{57}a_{63}))
\end{aligned}
$$

$$+a_{31}(a_{44}(-a_{56}a_{62} + a_{57}a_{63}) + a_{47}(-a_{52}a_{63} + a_{56}a_{67})$$
$$+ a_{46}(a_{52}a_{62} - a_{57}a_{67})) + a_{23}(a_{32}(a_{48}(-a_{52}a_{61} - a_{51}a_{62})$$
$$+ a_{44}(a_{58}a_{61} - a_{57}a_{64}) + a_{47}(a_{52}a_{64} + a_{51}a_{65})$$
$$+ a_{43}(a_{58}a_{62} - a_{57}a_{65}))$$
$$+a_{31}(a_{44}(a_{58}a_{62} - a_{57}a_{65}) + a_{48}(-a_{52}a_{62} + a_{57}a_{67})$$
$$+ a_{47}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+ a_{22}(a_{32}(a_{48}a_{51}a_{63} + a_{44}a_{56}a_{64} + a_{46}(-a_{52}a_{64} - a_{51}a_{65})$$
$$+ a_{43}(-a_{58}a_{63} + a_{56}a_{65}))$$
$$+ a_{31}(a_{44}(-a_{58}a_{63} + a_{56}a_{65}) + a_{48}(a_{52}a_{63} - a_{56}a_{67})$$
$$+a_{46}(-a_{52}a_{65} + a_{58}a_{67})))$$
$$+ a_{11}(a_{33}(a_{48}(-a_{24}a_{52}a_{61} - a_{24}a_{51}a_{62})$$
$$+ a_{44}(a_{24}a_{58}a_{61} - a_{24}a_{57}a_{64}) + a_{47}(a_{24}a_{52}a_{64} + a_{24}a_{51}a_{65})$$
$$+ a_{43}(a_{24}a_{58}a_{62} - a_{24}a_{57}a_{65}))$$
$$+ a_{26}(a_{33}a_{45}(a_{48}a_{61} - a_{58}a_{61} - a_{47}a_{64} + a_{57}a_{64})$$
$$+a_{32}(a_{48}(-a_{54}a_{61} - a_{53}a_{62}) + a_{42}(a_{58}a_{61} - a_{57}a_{64})$$
$$+ a_{47}(a_{54}a_{64} + a_{53}a_{65}) + a_{41}(a_{58}a_{62} - a_{57}a_{65})$$
$$+ a_{31}(a_{42}(a_{58}a_{62} - a_{57}a_{65}) + a_{48}(-a_{54}a_{62} + a_{57}a_{66})$$
$$+ a_{47}(a_{54}a_{65} - a_{58}a_{66})))+ a_{28}(a_{33}$$
$$(-a_{44}a_{45}a_{61} - a_{43}a_{45}a_{62} + a_{45}(a_{52}a_{61} + a_{51}a_{62}))$$
$$+ a_{32}(a_{42}(-a_{52}a_{61} - a_{51}a_{62}) + a_{44}(a_{54}a_{61} + a_{53}a_{62})$$
$$+ a_{43}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{51}a_{66} - a_{53}a_{67})$$
$$+ a_{41}(-a_{52}a_{62} + a_{57}a_{67})$$
$$+a_{31}(a_{44}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{52}a_{66} - a_{54}a_{67})$$
$$+ a_{42}(-a_{52}a_{62} + a_{57}a_{67}))$$
$$+ a_{27}(a_{33}(-a_{44}a_{45}a_{62} + a_{45}(a_{52}a_{62} + a_{47}a_{67} - a_{57}a_{67}))$$
$$+ a_{32}(a_{44}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(a_{52}a_{66} - a_{54}a_{67})$$
$$+ a_{42}(-a_{52}a_{62} + a_{57}a_{67}))$$
$$+a_{22}(a_{33}(a_{44}a_{45}a_{64} + a_{43}a_{45}a_{65} + a_{45}(-a_{52}a_{64} - a_{51}a_{65}))$$
$$+ a_{32}(a_{42}(a_{52}a_{64} + a_{51}a_{65}) + a_{44}(-a_{54}a_{64} - a_{53}a_{65})$$
$$+ a_{43}(-a_{54}a_{65} + a_{58}a_{66})$$
$$+ a_{48}(-a_{51}a_{66} + a_{53}a_{67}) + a_{41}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+a_{31}(a_{44}(-a_{54}a_{65} + a_{58}a_{66}) + a_{48}(-a_{52}a_{66} + a_{54}a_{67})$$
$$+ a_{42}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+ a_{25}(a_{32}(a_{48}(a_{52}a_{61} + a_{51}a_{62}) + a_{44}(-a_{58}a_{61} + a_{57}a_{64})$$
$$+ a_{47}(-a_{52}a_{64} - a_{51}a_{65}) + a_{43}(-a_{58}a_{62} + a_{57}a_{65}))$$
$$+a_{31}(a_{44}(-a_{58}a_{62} + a_{57}a_{65}) + a_{48}(a_{52}a_{62} - a_{57}a_{67})$$
$$+ a_{47}(-a_{52}a_{65} + a_{58}a_{67})) + a_{21}(a_{32}(a_{44}(-a_{54}a_{65} + a_{58}a_{66})$$
$$+ a_{48}(-a_{52}a_{66} + a_{54}a_{67}) + a_{42}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+ a_{33}(a_{44}a_{45}a_{65} + a_{45}(-a_{52}a_{65} - a_{48}a_{67} + a_{58}a_{67})))),$$

$$\Delta_6 = a_{16}(a_{25}a_{32}(a_{48}(-a_{56}a_{62} + a_{57}a_{63})$$
$$+ a_{47}(-a_{58}a_{63} + a_{56}a_{65}) + a_{46}(a_{58}a_{62} - a_{57}a_{65}))$$
$$+ a_{33}(a_{48}(a_{24}a_{56}a_{62} - a_{24}a_{57}a_{63}) + a_{47}(a_{24}a_{58}a_{63} - a_{24}a_{56}a_{65})$$
$$+ a_{46}(-a_{24}a_{58}a_{62} + a_{24}a_{57}a_{65}))$$
$$+a_{28}(a_{33}a_{45}(a_{46}a_{62} - a_{56}a_{62} - a_{47}a_{63} + a_{57}a_{63})$$
$$+ a_{32}(a_{42}(a_{56}a_{62} - a_{57}a_{63})$$
$$+ a_{47}(a_{54}a_{63} - a_{56}a_{66}) + a_{46}(-a_{54}a_{62} + a_{57}a_{66})))$$
$$+a_{22}(a_{33}a_{45}(a_{48}a_{63} - a_{58}a_{63} - a_{46}a_{65} + a_{56}a_{65})$$
$$+ a_{32}(a_{42}(a_{58}a_{63} - a_{56}a_{65}) + a_{48}(-a_{54}a_{63} + a_{56}a_{66})$$
$$+ a_{46}(a_{54}a_{65} - a_{58}a_{66})))$$
$$+ a_{23}(a_{33}a_{45}(-a_{48}a_{62} + a_{58}a_{62} + a_{47}a_{65} - a_{57}a_{65})$$
$$+a_{32}(a_{42}(-a_{58}a_{62} + a_{57}a_{65})$$
$$+ a_{48}(a_{54}a_{62} - a_{57}a_{66}) + a_{47}(-a_{54}a_{65} + a_{58}a_{66})))$$
$$+ a_{15}(a_{26}a_{32}(a_{48}(a_{56}a_{62} - a_{57}a_{63}) + a_{47}(a_{58}a_{63} - a_{56}a_{65})$$
$$+ a_{46}(-a_{58}a_{62} + a_{57}a_{65}))$$
$$+a_{28}a_{32}(a_{44}(-a_{56}a_{62} + a_{57}a_{63}) + a_{47}(-a_{52}a_{63} + a_{56}a_{67})$$
$$+ a_{46}(a_{52}a_{62} - a_{57}a_{67}))$$
$$+ a_{23}a_{32}(a_{44}(a_{58}a_{62} - a_{57}a_{65}) + a_{48}(-a_{52}a_{62} + a_{57}a_{67})$$
$$+a_{47}(a_{52}a_{65} - a_{58}a_{67})) + a_{22}a_{32}$$
$$(a_{44}(-a_{58}a_{63} + a_{56}a_{65}) + a_{48}(a_{52}a_{63} - a_{56}a_{67})$$
$$+ a_{46}(-a_{52}a_{65} + a_{58}a_{67}))$$
$$+ a_{14}(a_{26}a_{33}(a_{48}(-a_{56}a_{62} + a_{57}a_{63})$$
$$+ a_{47}(-a_{58}a_{63} + a_{56}a_{65}) + a_{46}(a_{58}a_{62} - a_{57}a_{65}))$$
$$+a_{28}a_{33}(a_{44}(a_{56}a_{62} - a_{57}a_{63}) + a_{47}(a_{52}a_{63} - a_{56}a_{67})$$
$$+ a_{46}(-a_{52}a_{62} + a_{57}a_{67})) + a_{22}a_{33}(a_{44}(a_{58}a_{63} - a_{56}a_{65})$$
$$+ a_{48}(-a_{52}a_{63} + a_{56}a_{67}) + a_{46}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+ a_{23}a_{33}(a_{44}(-a_{58}a_{62} + a_{57}a_{65}) + a_{48}(a_{52}a_{62} - a_{57}a_{67})$$
$$+a_{47}(-a_{52}a_{65} + a_{58}a_{67}))$$
$$+ a_{17}(a_{26}(a_{33}a_{45}(-a_{46}a_{62} + a_{56}a_{62} + a_{47}a_{63} - a_{57}a_{63})$$
$$+ a_{32}(a_{42}(-a_{56}a_{62} + a_{57}a_{63}) + a_{47}(-a_{54}a_{63} + a_{56}a_{66})$$
$$+ a_{46}(a_{54}a_{62} - a_{57}a_{66}))$$
$$+a_{25}a_{32}(a_{44}(a_{56}a_{62} - a_{57}a_{63}) + a_{47}(a_{52}a_{63} - a_{56}a_{67})$$
$$+ a_{46}(-a_{52}a_{62} + a_{57}a_{67}))$$
$$+ a_{33}(a_{44}(-a_{24}a_{56}a_{62} + a_{24}a_{57}a_{63})$$
$$+ a_{47}(-a_{24}a_{52}a_{63} + a_{24}a_{56}a_{67}) + a_{46}(a_{24}a_{52}a_{62} - a_{24}a_{57}a_{67}))$$
$$+a_{22}(a_{33}(-a_{44}a_{45}a_{63} + a_{45}(a_{52}a_{63} + a_{46}a_{67} - a_{56}a_{67}))$$
$$+ a_{32}(a_{44}(a_{54}a_{63} - a_{56}a_{66}) + a_{46}(a_{52}a_{66} - a_{54}a_{67})$$
$$+ a_{42}(-a_{52}a_{63} + a_{56}a_{67}))$$
$$+a_{23}(a_{32}(a_{44}(-a_{54}a_{62} + a_{57}a_{66}) + a_{47}(-a_{52}a_{66} + a_{54}a_{67})$$
$$+ a_{42}(a_{52}a_{62} - a_{57}a_{67}))$$
$$+ a_{33}(a_{44}a_{45}a_{62} + a_{45}(-a_{52}a_{62} - a_{47}a_{67} + a_{57}a_{67})))$$
$$+ a_{13}(a_{26}(a_{33}a_{45}(-a_{48}a_{63} + a_{58}a_{63} + a_{46}a_{65} - a_{56}a_{65})$$
$$(+a_{32}(a_{42}(-a_{58}a_{63} + a_{56}a_{65}) + a_{48}(a_{54}a_{63} - a_{56}a_{66})$$
$$+ a_{46}(-a_{54}a_{65} + a_{58}a_{66}))$$
$$+ a_{25}a_{32}(a_{44}(a_{58}a_{63} - a_{56}a_{65}) + a_{48}(-a_{52}a_{63} + a_{56}a_{67})$$
$$+ a_{46}(a_{52}a_{65} - a_{58}a_{67}))$$
$$+a_{33}(a_{44}(-a_{24}a_{58}a_{63} + a_{24}a_{56}a_{65})$$
$$+ a_{48}(a_{24}a_{52}a_{63} - a_{24}a_{56}a_{67})$$
$$+ a_{46}(-a_{24}a_{52}a_{65} + a_{24}a_{58}a_{67}))$$
$$+ a_{28}(a_{32}(a_{44}(-a_{54}a_{63} + a_{56}a_{66})$$
$$+ a_{46}(-a_{52}a_{66} + a_{54}a_{67}) + a_{42}(a_{52}a_{63} - a_{56}a_{67}))$$
$$+ a_{33}(a_{44}a_{45}a_{63} + a_{45}(-a_{52}a_{63} - a_{46}a_{67} + a_{56}a_{67})))$$

$$
\begin{aligned}
&+a_{23}(a_{33}(-a_{44}a_{45}a_{65}+a_{45}(a_{52}a_{65}+a_{48}a_{67}-a_{58}a_{67}))\\
&+a_{32}(a_{44}(a_{54}a_{65}-a_{58}a_{66})+a_{48}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{65}+a_{58}a_{67}))\\
&+a_{12}(a_{33}(a_{48}(-a_{24}a_{52}a_{61}-a_{24}a_{51}a_{62})\\
&+a_{44}(a_{24}a_{58}a_{61}-a_{24}a_{57}a_{64})\\
&+a_{47}(a_{24}a_{52}a_{64}+a_{24}a_{51}a_{65})+a_{43}(a_{24}a_{58}a_{62}-a_{24}a_{57}a_{65}))\\
&+a_{26}(a_{33}a_{45}(a_{48}a_{61}-a_{58}a_{61}-a_{47}a_{64}+a_{57}a_{64})\\
&+a_{32}(a_{48}(-a_{54}a_{61}-a_{53}a_{62})+a_{42}(a_{58}a_{61}-a_{57}a_{64})\\
&+a_{47}(a_{54}a_{64}+a_{53}a_{65})+a_{41}(a_{58}a_{62}-a_{57}a_{65}))\\
&+a_{31}(a_{42}(a_{58}a_{62}-a_{57}a_{65})+a_{48}(-a_{54}a_{62}+a_{57}a_{66})\\
&+a_{47}(a_{54}a_{65}-a_{58}a_{66}))\\
&+a_{28}(a_{33}(-a_{44}a_{45}a_{61}-a_{43}a_{45}a_{62}+a_{45}(a_{52}a_{61}+a_{51}a_{62}))\\
&+a_{32}(a_{42}(-a_{52}a_{61}-a_{51}a_{62})+a_{44}(a_{54}a_{61}+a_{53}a_{62})\\
&+a_{43}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{51}a_{66}-a_{53}a_{67})\\
&+a_{41}(-a_{52}a_{62}+a_{57}a_{67})\\
&+a_{31}(a_{44}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{27}(a_{33}(-a_{44}a_{45}a_{62}+a_{45}(a_{52}a_{62}+a_{47}a_{67}-a_{57}a_{67}))\\
&+a_{32}(a_{44}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{62}+a_{57}a_{67}))\\
&+a_{22}(a_{33}(a_{44}a_{45}a_{64}+a_{43}a_{45}a_{65}+a_{45}(-a_{52}a_{64}-a_{51}a_{65}))\\
&+a_{32}(a_{42}(a_{52}a_{64}+a_{51}a_{65})+a_{44}(-a_{54}a_{64}-a_{53}a_{65})\\
&+a_{43}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{51}a_{66}+a_{53}a_{67})\\
&+a_{41}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{31}(a_{44}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{25}(a_{32}(a_{48}(a_{52}a_{61}+a_{51}a_{62})\\
&+a_{44}(-a_{58}a_{61}+a_{57}a_{64})+a_{47}(-a_{52}a_{64}-a_{51}a_{65})\\
&+a_{43}(-a_{58}a_{62}+a_{57}a_{65}))+a_{31}(a_{44}(-a_{58}a_{62}+a_{57}a_{65})\\
&+a_{48}(a_{52}a_{62}-a_{57}a_{67})+a_{47}(-a_{52}a_{65}+a_{58}a_{67})))\\
&+a_{21}(a_{32}(a_{44}(-a_{54}a_{65}+a_{58}a_{66})+a_{48}(-a_{52}a_{66}+a_{54}a_{67})\\
&+a_{42}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{33}(a_{44}a_{45}a_{65}+a_{45}(-a_{52}a_{65}-a_{48}a_{67}+a_{58}a_{67}))\\
&+a_{11}(a_{26}(a_{33}a_{45}(a_{48}a_{62}-a_{58}a_{62}-a_{47}a_{65}+a_{57}a_{65})\\
&+a_{32}(a_{42}(a_{58}a_{62}-a_{57}a_{65})+a_{48}(-a_{54}a_{62}+a_{57}a_{66})\\
&+a_{47}(a_{54}a_{65}-a_{58}a_{66}))\\
&+a_{25}a_{32}(a_{44}(-a_{58}a_{62}+a_{57}a_{65})+a_{48}(a_{52}a_{62}-a_{57}a_{67})\\
&+a_{47}(-a_{52}a_{65}+a_{58}a_{67}))+a_{33}(a_{44}(a_{24}a_{58}a_{62}-a_{24}a_{57}a_{65})\\
&+a_{48}(-a_{24}a_{52}a_{62}+a_{24}a_{57}a_{67})+a_{47}(a_{24}a_{52}a_{65}-a_{24}a_{58}a_{67}))\\
&+a_{28}(a_{33}(-a_{44}a_{45}a_{62}+a_{45}(a_{52}a_{62}+a_{47}a_{67}-a_{57}a_{67}))\\
&+a_{32}(a_{44}(a_{54}a_{62}-a_{57}a_{66})+a_{47}(a_{52}a_{66}-a_{54}a_{67})\\
&+a_{42}(-a_{52}a_{62}+a_{57}a_{67})))+a_{22}(a_{32}(a_{44}(-a_{54}a_{65}+a_{58}a_{66})\\
&+a_{48}(-a_{52}a_{66}+a_{54}a_{67})+a_{42}(a_{52}a_{65}-a_{58}a_{67}))\\
&+a_{33}(a_{44}a_{45}a_{65}+a_{45}(-a_{52}a_{65}-a_{48}a_{67}+a_{58}a_{67}))),
\end{aligned}
$$

$$
\begin{aligned}
\Delta_7 = {}& a_{12}(a_{28}(a_{33}a_{45}(-a_{44}+a_{52})\\
&+a_{32}(-a_{42}a_{52}+a_{44}a_{54}))a_{62}\\
&+((a_{25}a_{32}-a_{24}a_{33})(a_{48}a_{52}-a_{44}a_{58})\\
&+a_{26}(a_{33}a_{45}(a_{48}-a_{58})+a_{32}(-a_{48}a_{54}+a_{42}a_{58})))a_{62}\\
&+(a_{22}(a_{33}a_{45}(a_{44}-a_{52})+a_{32}(a_{42}a_{52}-a_{44}a_{54}))\\
&-a_{25}a_{32}-a_{24}a_{33})(a_{47}a_{52}-a_{44}a_{57})\\
&+a_{26}(a_{33}a_{45}(-a_{47}+a_{57})\\
&+a_{32}(a_{47}a_{54}-a_{42}a_{57}))a_{65}).
\end{aligned}
$$