

Research Article

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Analyzing the role of length and radius of MWCNTs in a nanofluid flow influenced by variable thermal conductivity and viscosity considering Marangoni convection

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Abstract: Carbon nanotubes (CNTs), formed by rolling graphene sheets into cylindrical shapes, are widely used in the development of next-generation lithium-ion batteries due to their outstanding thermal conductivity and large surface area, which enhance electrode performance, enable faster charging, and improve energy capacity. Inspired by such a fascinating application of CNTs, the present investigation examines the flow behavior of an aqueous-based nanofluid containing multi-walled carbon nanotubes (MWCNTs), utilizing the well-established Yamada–Ota thermal conductivity model to assess the influence of nanoparticle radius and length. The proposed flow model novelty lies in incorporating Marangoni convection and temperature-dependent thermophysical properties. The governing partial differential equations are transformed into ordinary differential equations through appropriate similarity transformations. Numerical solutions are obtained using the boundary value problem solver bvp4c, with results illustrated through graphs and tables. The findings indicate that thermal conductivity reaches its peak at specific nanoparticle length $L = 25 \times 10^{-6}$ and radius values $R = 150 \times 10^{-9}$. Optimizing the length-to-radius ratio of MWCNTs in nanofluids can greatly improve thermal conductivity in high-performance electronic cooling systems

as extended nanotube lengths enhance heat transfer, while increased radii tend to reduce it. Additionally, Marangoni convection leads to a reduction in temperature distribution while enhancing the heat transfer rate. An increase in the variable temperature parameter results in a corresponding rise in the temperature profile. Validation of the proposed model is also included in this study.

Keywords: nanofluid flow, Yamada–Ota thermal conductivity model, energy efficiency, Marangoni convection, variable viscosity and thermal conductivity

Nomenclature

A	positive dimensional constant
C_1, C_2	constant quantities
C_p	specific heat capacity
E	variable thermal conductivity parameter
f	dimensionless velocity
$K_{NF}(T)$	variable thermal conductivity of nanofluid
L	length of MWCNTs
M_A	Marangoni parameter
m	exponent constant
Nu	Nusselt number
Pr	Prandtl number
q_c	heat flux
R	radius of MWCNTs
s_1	nanoparticles
T	temperature
T_0	ambient temperature
x, y	coordinate axis
θ_R	variable viscosity parameter
Φ	particle volume fraction
σ_0	surface tension at interface
Υ_T	temperature coefficient of surface tension
θ	dimensionless temperature
ρ	density

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$\mu_{\text{NF}}(T)$	variable viscosity of nanofluid
ψ	stream function
σ	surface tension
ζ	similarity variable
Φ_{s1}	particle volume fraction of nanoparticles

1 Introduction

Variable thermal conductivity and viscosity are crucial as they significantly influence fluid rheological properties and heat transfer performance. These characteristics are essential for designing efficient nanofluid systems used in applications like heat exchangers and renewable energy systems. Choosing variables over constant properties ensures greater accuracy for specific applications. While constant properties suffice for simple models or small temperature variations, temperature-dependent characteristics are essential for accurately modeling complex scenarios or high-temperature conditions. These variations help determine fluid flow behavior and predict actual thermal transport phenomena. Researchers increasingly emphasize the importance of temperature-dependent viscosity and thermal conductivity in advanced studies. Abu-Nada [1] conducted a groundbreaking theoretical study on nanofluid flow using an aluminum-water mixture based on the Tiwari and Das model, incorporating variable features with natural convection. Mahmoud and Megahed [2] then investigated the magnetohydrodynamic flow of a power-law non-Newtonian fluid film over an extended surface, considering unique variable characteristics. Subsequently, Abu-Nada *et al.* [3] numerically analyzed the flow of Al_2O_3 -water and CuO -water nanofluid combinations in a differentially heated enclosure with variable properties. Roslan *et al.* [4] followed with an intriguing study using the finite difference technique to evaluate heat transfer rates in a buoyancy-driven nanofluid-filled trapezoidal enclosure under variable conditions. Sebdani *et al.* [5] numerically examined mixed convective nanofluid flow in a square cavity, incorporating variable viscosity and thermal conductivity. More recently, Shamshuddin and Eid [6] applied Chebyshev spectral numerical methods to study magnetized aqueous nanofluid flow with ferromagnetic nanoparticles over a stretchable spinning disk under variable features. Additional recent works exploring variable characteristics in diverse scenarios are provided in previous literature [7–10].

The Marangoni effect signifies mass transfer driven by a surface tension gradient at the interface between two liquids. When this gradient arises from temperature

differences, it is identified as thermo-capillary convection. The phenomenon was first described in 1855 by physicist James Thomson, who observed it in the “tears of wine.” In this process, regions with higher surface tension pull more strongly on surrounding particles, causing the liquid to flow from areas of high to low surface tension. This gradient can result from temperature or concentration variations. The Marangoni effect has applications in crystal growth, metallurgy, soap film stabilization, Benard cells, and integrated circuit manufacturing, where it aids in drying silicon wafers. Golia and Viviani [11] introduced the concept of the Marangoni boundary layer in non-isobaric conditions, which was later explored by Montgomery and Wang [12] for a flat plate geometry. Arifin *et al.* [13] extended this work by studying nanoliquid flow influenced by Marangoni convection over a static flat plate using the Tiwari and Das model. Their findings showed that nanoliquids with titanium dioxide exhibited lower thermal diffusivity compared to those with copper nanoparticles. Hayat *et al.* [14] analytically investigated Carbon nanotube (CNT)-based nanofluid flows under Marangoni convection and thermal radiative heat flux, concluding that increasing suction reduced velocity distribution for both CNTs and nanofluids. Qayyum [15] examined the Marangoni convection flow of hybrid nanoliquids in two base fluids under gravity, finding that velocity increased with the Marangoni number, while temperature decreased. Abdullah *et al.* [16] analyzed the electrically conducting flow of hybrid nanoliquids along a porous extending plate under Marangoni convection, observing significant temperature increases with higher magnetic parameters. Recently, Anusha *et al.* [17] analyzed heat and mass transfer in Marangoni convective nanofluid flow through porous media, reporting increased axial and transverse velocities with rising Marangoni numbers. Additional recent works on Marangoni convection are cited in [18–20].

A nanofluid is a combination of a conventional liquid (e.g., water or oil) and suspended nanoparticles (such as oxides, carbides, or CNTs) with sizes below 100 nm. Recognized as next-generation fluids, nanofluids exhibit superior heat transfer properties, making them invaluable for various engineering and industrial applications, including electronics, food processing, medical technologies, and energy production. Recognizing the inadequate heat transfer capabilities of conventional liquids in high-tech processes, Maxwell [21] proposed the idea of mixing solid particles (particles with sizes ranging from millimeters and micrometers) with traditional liquids in 1881, to improve heat transfer capabilities. However, this concept initially faced criticism due to drawbacks such as clogging, sedimentation, and corrosion. In 1995, Choi and Eastman [22] pioneered the idea of nanofluids, updating

Maxwell's theory after nearly a century. Their idea involved adding nano-sized metallic nanoparticles to conventional liquids, rather than using millimeter or micrometer-sized particles. CNTs are considered the most effective nanoparticles. Made from graphene sheet structures, CNTs are grouped as either single-walled (SWCNTs) or multi-walled (MWCNTs). SWCNTs consist of a single layer with a diameter varying from 0.5 to 1.5 nm, while MWCNTs contain interconnected tubes with progressively larger diameters. CNTs exhibit thermal, mechanical, and physico-chemical properties that are six times superior, making them crucial in chemical production, optics, microelectronic cooling, and material science. Researchers have shown keen interest in exploring nanofluid flows containing CNTs over varied geometries. Yan *et al.* [23] experimentally studied the trend of CNTs-based hybrid non-Newtonian nanofluid flow at different temperatures. They determined that viscosity enhances for greater estimations of particle volume fraction. Harish Babu *et al.* [24] analyzed the inclined magnetic flux on the hybrid nanoliquid along an exponentially stretching sheet. They came to the conclusion that compared to a fundamental liquid, the temperature distribution of a hybrid nanoliquid including CNTs boosts more significantly. Next Sneha *et al.* [25] assessed the role of Marangoni convection and magnetic flux over the SWCNTs-based nanofluid flow in a porous media. It was shown that the rate of heat transfer was enhanced by Marangoni convection. El Ghil and Driouch [26] examined the non-orthogonal stagnation flow of CNTs-nano liquid along an extending surface. Their numerical results showed that SWCNTs have a more substantial heat transmission rate compared with MWCNTs. Dero *et al.* [27] obtained multiple solutions of CNTs-based nanofluid flow on a permeable shrinking sheet with viscous dissipation effects. They discovered that the velocity distribution for the nanoparticles volume fraction for MWCNTs-kerosene oil is larger than that of SWCNTs-kerosene oil nanoliquid. More work on the CNTs-based nanofluid along different geometries is available in previous studies [28–31].

The reviewed literature revealed extensive studies on nanofluid flows, but only few of them have focused on Marangoni convection in such systems. Notably, none of the existing works, including those cited, provide a numerical solution for aqueous-based nanofluid flow with MWCNTs under Marangoni convection while incorporating variable thermal conductivity and viscosity. The proposed model's uniqueness lies in considering the influence of MWCNTs length and radius, analyzed using the Yamada–Ota (Y–O) thermal conductivity model. Numerical solutions are obtained through the bvp4c technique, with results presented through illustrations and tables. Table 1 highlights the novelty of the proposed model by highlighting gap analysis.

2 Mathematical formulation

The proposed mathematical model is developed based on the following assumptions:

- We consider an incompressible, aqueous-based nanofluid flow containing MWCNTs, following the Y–O thermal conductivity model.
- The Y–O thermal conductivity model is engaged to gauge the radius and length of nanoparticles in a nanofluid flow.
- The assumed nanofluid combination is in thermal balance and there is no slip between customary liquid and the assumed nanoparticles.
- Temperature-dependent thermophysical characteristics are also considered to enhance the novelty of the model.
- The flow is subjected to Marangoni convection.
- A cartesian coordinate system (x, y) , where x is determined along the interface s and y is normal to it. The whole scenario is shown in Figure 1.
- Moreover, T_o is the temperature considered far off from the surface and σ is the surface tension, with expression

$$\sigma = \sigma_o + \left(\frac{Y_T}{2} \right) (T - T_o), \text{ where } Y_T \text{ is the temperature coefficient of surface tension [14].}$$

Table 1: Gap analysis

Published works	Marangoni convection	Y–O thermal conductivity model	Aqueous-based nanofluid	Variable viscosity and thermal conductivity
[2]	×	×	×	✓
[3]	×	×	✓	✓
[14]	✓	×	✓	×
[16]	✓	×	✓	×
Present	✓	✓	✓	✓

✓ (Effect present), × (Effect absent).

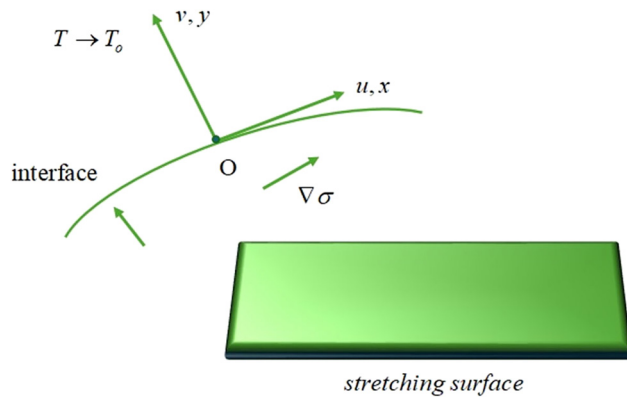


Figure 1: Geometry of the flow model.

The model equations [13] following the above assumptions are

$$u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0, \quad (1)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \left(\frac{1}{\rho_{NF}} \right) \left[\left(\frac{\partial \mu_{NF}(T)}{\partial y} \right) \left(\frac{\partial u}{\partial y} \right) + \mu_{NF}(T) \left(\frac{\partial^2 u}{\partial y^2} \right) \right], \quad (2)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \left(\frac{1}{(\rho C_p)_{NF}} \right) \left[\left(\frac{\partial K_{NF}(T)}{\partial y} \right) \left(\frac{\partial T}{\partial y} \right) + K_{NF}(T) \left(\frac{\partial^2 T}{\partial y^2} \right) \right], \quad (3)$$

where $\mu_{NF}(T)$ is the variable viscosity given as [32]

$$\mu_{NF}(T) = \mu_{NF} \left(\frac{\theta_R}{\theta_R - \theta} \right), \quad (4)$$

$K_{NF}(T)$ is the variable thermal conductivity and θ_R is the variable viscosity parameter, given by [32]

$$K_{NF}(T) = K_{NF}(1 + E\theta), \quad (5)$$

where E is the variable thermal conductivity parameter. The appropriate boundary constraints [3] are

$$\begin{aligned} \mu_{NF}(T) \left(\frac{\partial u}{\partial y} \right) &= - \left(\frac{\partial \sigma}{\partial T} \right) \left(\frac{\partial T}{\partial x} \right), \quad v = 0, \\ T &= T_w = T_o + Ax^{m+1}, \quad \text{at } y = 0, \\ u &\rightarrow 0, \quad T \rightarrow T_o, \quad \text{as } y \rightarrow \infty. \end{aligned} \quad (6)$$

where m is the exponent constant. Similarity transformations are appended as follows:

$$\begin{aligned} \zeta &= C_2 x^{\left(\frac{m-1}{3} \right)} y, \quad \psi = C_1 x^{\left(\frac{2+m}{3} \right)} f(\zeta), \\ u &= C_1 C_2 x^{\left(\frac{2+m}{3} \right)} f'(\zeta), \quad T = T_o + Ax^{m+1} \theta(\zeta), \\ v &= - \left[C_1 \left(\frac{2+m}{3} \right) x^{\left(\frac{m-1}{3} \right)} f(\zeta) + C_1 x^{\left(\frac{m-1}{3} \right)} \left(\frac{m-1}{3} \right) \zeta f'(\zeta) \right]. \end{aligned} \quad (7)$$

By substituting Eqs. (1)–(6) in Eq. (7), we get the subsequent system

$$\begin{aligned} \left(\frac{\mu_{NF}}{\rho_{NF}} \right) \left(\frac{\theta_R}{\theta_R - \theta} \right) f''' + \left(\frac{\theta_R}{[\theta_R - \theta]^2} \right) \theta' f'' + \left(\frac{2+m}{3} \right) f f'' \\ - \left(\frac{2m+1}{3} \right) f'^2 = 0, \end{aligned} \quad (8)$$

$$\begin{aligned} \left(\frac{K_{NF}}{(\rho C_p)_{NF}} \right) (E\theta'^2 + (1 + E\theta)\theta'') - \text{Pr} \left[(m+1)f'\theta \right. \\ \left. - \left(\frac{2+m}{3} \right) f\theta' \right] = 0, \end{aligned} \quad (9)$$

The constraints on the surface are outlined below

$$\begin{aligned} f(0) = 0, \quad f''(0) = - \left(\frac{\mu_{NF}}{\mu_F} \right) \left(\frac{\theta_R - \theta}{\theta_R} \right) M_A, \quad \theta(0) = 1, \\ f'(\infty) \rightarrow 0, \quad \theta(\infty) \rightarrow 0. \end{aligned} \quad (10)$$

The above equations include the following dimensionless parameters:

$$M_A = \frac{AY_T(m+1)}{\mu_F C_1 C_2^2}, \quad \text{Pr} = \frac{\mu_F (C_p)_F}{k_F}, \quad (11)$$

where M_A is the Marangoni parameter and Pr is the Prandtl number. The base liquid and MWCNTs thermophysical attributes are shown in Table 2.

2.1 Thermophysical models for nanofluid

The subsequent models are considered to reflect the thermophysical characteristics [34,35]

Dynamic viscosity

$$\mu_{NF} = \mu_F / (1 - \Phi_{s1})^{2.5}.$$

Density

$$\frac{\rho_{NF}}{\rho_F} = (1 - \Phi_{s1}) + \Phi_{s1} \frac{\rho_{s1}}{\rho_F}.$$

Specific heat capacity

$$\frac{(\rho C_p)_{NF}}{(\rho C_p)_F} = (1 - \Phi_{s1}) + \Phi_{s1} \frac{(\rho C_p)_{s1}}{(\rho C_p)_F}.$$

Table 2: Thermophysical values for base fluid and nanoparticles [33]

Base fluid/nanoparticles	k	ρ	C_p
H ₂ O	0.613	997.1	4,179
MWCNTs	3,000	1,600	796

Thermal conductivity

$$\frac{k_{NF}}{k_F} = \frac{1 + \frac{k_F L}{k_{s1} R} \Phi_{s1}^{0.2} + \left(1 - \frac{k_F}{k_{s1}}\right) \Phi_{s1} \frac{L}{R} \Phi_{s1}^{0.2} + 2\Phi_{s1} \left(\frac{k_{s1}}{k_{s1} - k_F}\right) \ln \left(\frac{k_{s1} + k_F}{2k_{s1}}\right)}{1 - \Phi_{s1} + 2\Phi_{s1} \left(\frac{k_F}{k_{s1} - k_F}\right) \ln \left(\frac{k_{s1} + k_F}{2k_F}\right)}, \quad (12)$$

where L represents the length, R is the radius of CNTs, (k_F) is the thermal conductivity of fluid, (k_{s1}) is the thermal conductivity of CNTs, and (Φ_{s1}) is the particle volume fraction of CNTs.

2.2 Engineering quantity

The rate of heat flux in non-dimensional notation is given as

$$Nu = \frac{x(q_c)}{K_F(T_w - T_o)},$$

where

$$q_c = -K_{NF}(T) \frac{\partial T}{\partial y}. \quad (13)$$

The Nusselt number in dimensionless form is

$$Nu = -\left[\frac{k_{NF}}{k_F}(1 + E\theta)\right] C_2 X^{\left(\frac{m+2}{3}\right)} \theta'(\zeta). \quad (14)$$

2.3 Numerical scheme

We employed the collocation technique `bvp4c`, an efficient computational method (a finite difference mechanism) for solving sets of coupled differential equations. Using this technique, which is elaborated below, the order of the equations governing the flow is reduced to one. A technique to transform the higher-order system to one is appended as under

$$y_1 \rightarrow f, \quad y_2 \rightarrow f', \quad y_3 \rightarrow f'', \quad yy_1 \rightarrow f''', \\ y_4 \rightarrow \theta, \quad y_5 \rightarrow \theta', \quad yy_2 \rightarrow \theta'',$$

$$yy_1 = \left[\frac{1}{\left(\frac{\mu_{NF}}{\rho_{NF}} \left(\frac{\theta_R}{\theta_R - y_4} \right) \right)} \right] \left[- \left(\frac{\theta_R}{[\theta_R - y_4]^2} \right) y_5 y_3 - \left(\frac{2+m}{3} \right) y_1 y_3 + \left(\frac{2m+1}{3} \right) (y_2)^2 \right],$$

$$yy_2 = \left[\frac{1}{\left(\frac{K_{NF}}{(\rho C_p)_{NF}} (E y_5^2 + (1 + E y_4)) \right)} \right] \text{Pr} \left[(m+1) y_2 y_4 - \left(\frac{2+m}{3} \right) y_1 y_5 \right],$$

$$y_1(0), \quad y_3(0) + \left(\frac{\mu_{NF}}{\mu_F} \right) \left(\frac{\theta_R - y_4}{\theta_R} \right) M_A, \quad y_4(0) - 1, \quad (15) \\ y_2(\infty) \rightarrow 0, \quad y_4(\infty) \rightarrow 0.$$

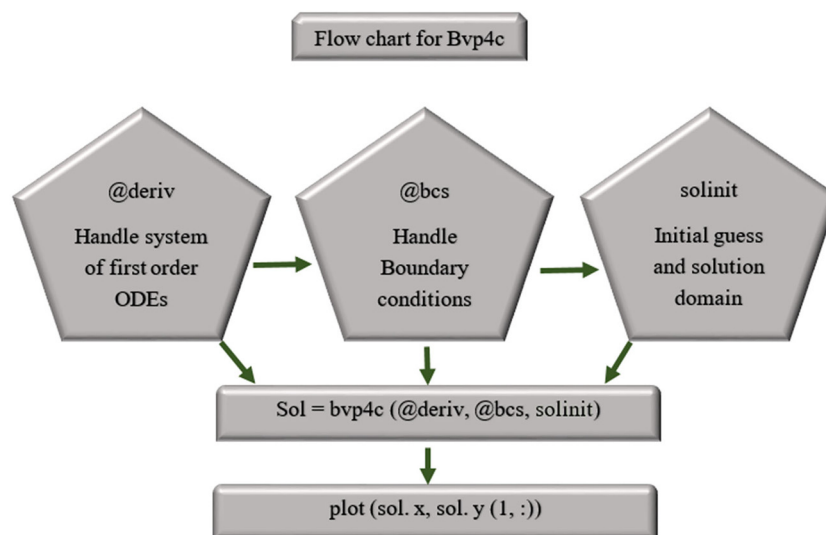


Figure 2: Flow chart of bvp4c numerical scheme.

A step size of $\Delta\eta = 0.01$ with $\eta_\infty = 14$ is considered for all profiles for convergence purposes. Another important ingredient of this numerical scheme is the consideration of the initial guess that must satisfy the assumed boundary conditions. A complete flow diagram of the numerical technique (bvp4c) is given in Figure 2.

This technique possesses the following advantages over the contemporary analytical and numerical schemes:

- The bvp4c automatically adapts the mesh size for enhanced accuracy. By doing so we can achieve accuracy without user interference.
- In comparison to the traditional techniques, bvp4c efficiently handles the discontinuities.
- Contrary to analytical techniques, it handles nonlinear boundary value problems effectively.
- The bvp4c is simpler and straightforward to implement with less effort.
- In comparison to finite difference, the bvp4c method engages the collocation method leading to higher accuracy.

Table 3 portrays the numerical results of heat transmission rate for different estimations of variable thermal conductivity parameter (E), Marangoni parameter (M_A), and exponent constant (m). It is noted that the heat transmission rate improves for mounting estimations of all three parameters. This is because the improved values of variable thermal conductivity parameter result in enhanced diffusion that eventually triggers the heat transfer rate. Similarly, stronger thermocapillary convection is observed for increased Marangoni parameter that improves the fluid movement at the surface. Thus, enhanced heat transfer efficacy at the fluid interface is seen. Likewise, a large value of exponent triggers the thermal conductivity in high temperatures that eventually leads to an efficient heat transfer rate.

Table 3: Numerical estimates of Nusselt number Nu against varied parameters

(E)	(M_A)	(m)	(Nu)
0.1	0.10	0.1	0.84463
0.2			0.90408
0.3			0.95993
0.4			1.01320
	0.12	0.2	0.89717
	0.13		0.92157
	0.14		0.94454
			0.86797
			0.89031
			0.91179

Table 4: Numerical estimates of thermal conductivity against higher values of particle volume fraction

Φ_{NF}	k_{NF}
0.01	1.344517154332972
0.02	1.830524151848501
0.03	2.498053161467530
0.04	3.440034976804251
0.05	4.841142856090153
0.06	7.111676268043935

The variation in thermal conductivity with particle volume fractions ranging from 1 to 6% is shown in Table 4, with the MCNTs length held constant. As the particle volume fraction of MCNTs increases at a fixed length, thermal conductivity is observed to rise. This finding aligns well with the experimental results stated by Assael *et al.* [35].

3 Graphical results analysis

This section aims to inspect the consequences of emerging factors on various flow and thermal distributions.

From Figure 3, the impact of length (L) of MWCNTs on the thermal conductivity can be observed. It is seen that for raising the length of MWCNTs, the thermal conductivity of nanofluid flow is significantly enhanced. This enhancement in the thermal conductivity may be owing to the high intrinsic thermal conductivity of MWCNTs, their aspect ratio, and uniform dispersion of MWCNTs in water, and volume fraction.

The role of the radius (R) of MWCNTs over the thermal conductivity can be visualized in Figure 4. It is inferred that for improving the radius of MWCNTs, the thermal

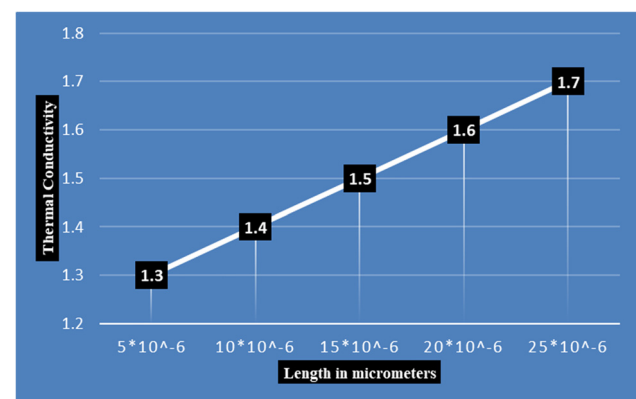


Figure 3: Upshot of length of MWCNTs on the thermal conductivity K_{NF} .

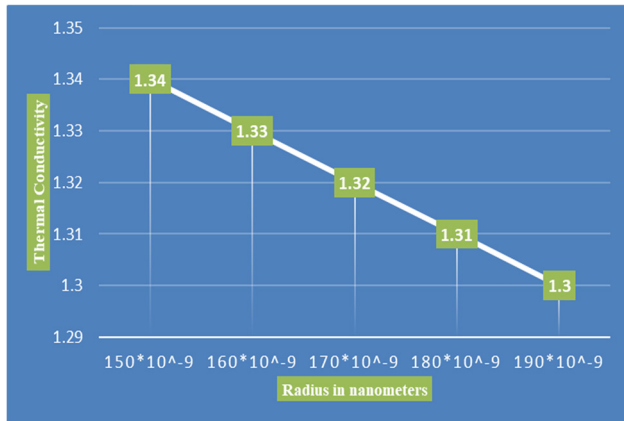


Figure 4: Upshot of radius of MWCNTs on the thermal conductivity K_{NF} .

conductivity drops. Here the factors affecting thermal conductivity may include increased inter-layer scattering, thermal boundary resistance, and contact resistance in composites.

The role of the Marangoni parameter (M_A) on the velocity distribution $f'(\zeta)$ can be seen in Figure 5. It is seen that velocity distribution escalates for mounting values of (M_A). Marangoni convection causes the fluid to flow at higher velocities in areas with a noticeable surface tension noted. When there are significant temperature differences close to borders or interfaces, this impact may be very noticeable.

The trend of velocity distribution $f'(\zeta)$ against the particle volume fraction (Φ), is shown in Figure 6. It is seen that raising counts of particle volume fraction (Φ) upsurges the velocity distribution. Physically, strengthening the particle volume fraction causes a rapid collision of nanoparticles, which improves the velocity distribution.

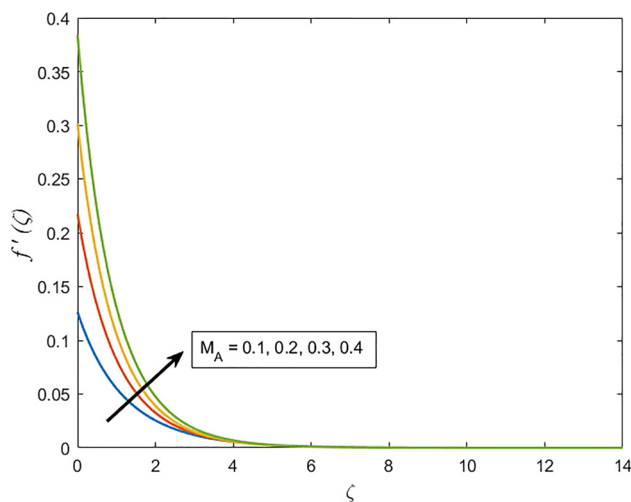


Figure 5: Upshot of Marangoni parameter (M_A) on $f'(\zeta)$.

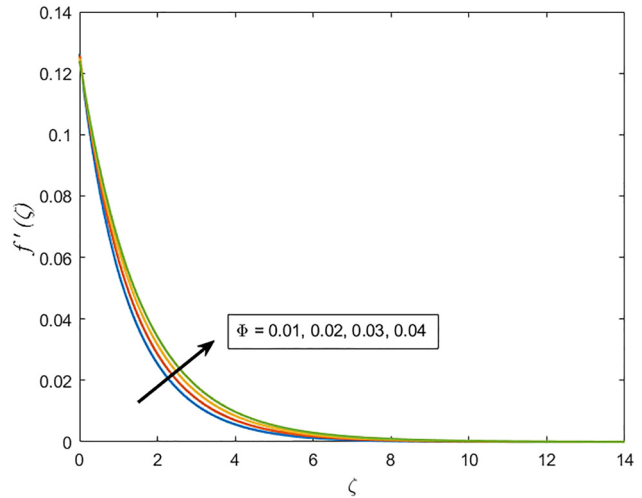


Figure 6: Upshot of particle volume fraction (Φ) on $f'(\zeta)$.

Figure 7 reveals the impact of the variable viscosity parameter (θ_R) on the velocity distribution $f'(\zeta)$. It is seen that velocity distribution escalates by greater values of (θ_R). This is due to the fact that rising (θ_R) implies zero shear rate viscosity, which lowers viscous drag force and speeds up flow.

The influence of the exponent constant (m) on the velocity distribution $f'(\zeta)$ is displayed in Figure 8. It is seen that velocity distribution drops significantly for greater counts of (m). When m rises, the temperature variation throughout the surface reduces, resulting in a lesser velocity distribution.

Figure 9 is portrayed to analyze the consequences of the Marangoni parameter (M_A) on the temperature distribution $\theta(\zeta)$. It is seen that temperature distribution dwindled for the greater counts of (M_A). The higher estimates of (M_A) enhance the surface tension resulting in

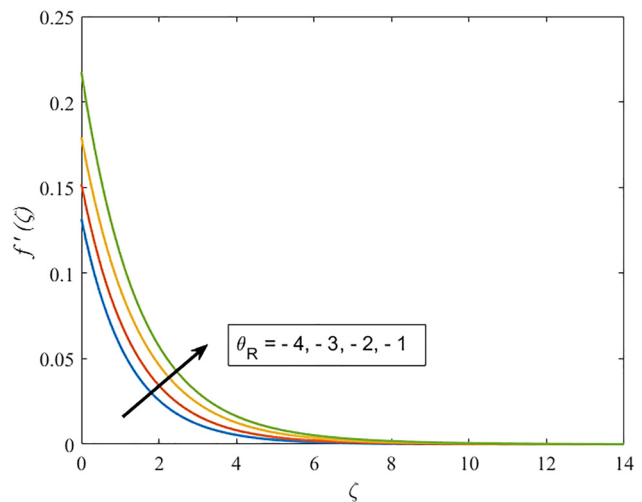


Figure 7: Upshot of variable viscosity parameter (θ_R) on $f'(\zeta)$.

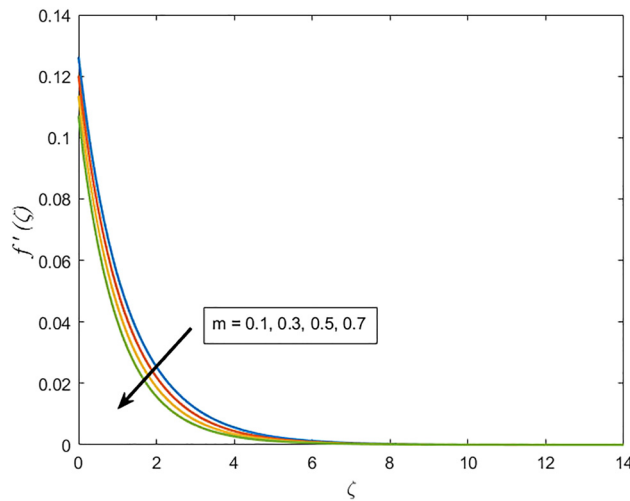


Figure 8: Upshot of exponent constant (m) on $f'(\zeta)$.

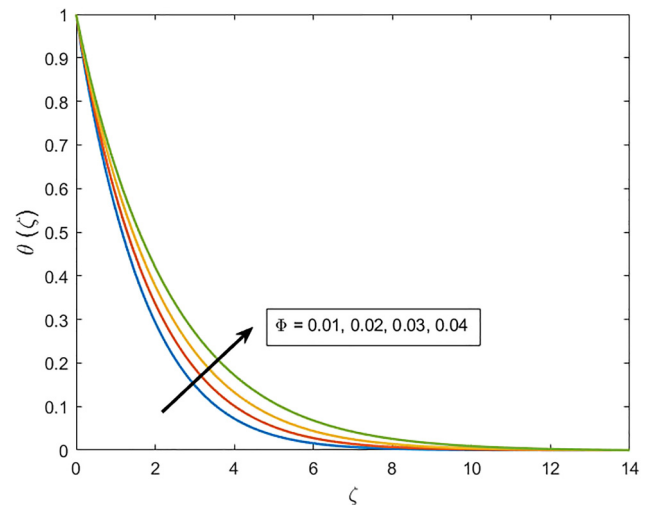


Figure 10: Upshot of particle volume fraction (Φ) on $\theta(\zeta)$.

improved heat transfer, thus reducing the temperature gradient with the assumed fluid flow.

The role of particle volume fraction (Φ) on the temperature distribution is displayed in Figure 10. It is revealed that the temperature distribution of nanofluid containing MWCNTs nanoparticles escalates for mounting (Φ). Adding more MWCNTs nanoparticles to the fluid enhances heat transfer capabilities owing to the high thermal conductivity of MWCNTs. This results in an improved heat transfer rate.

Finally, the role of variable thermal conductivity (E) on the temperature distribution can be analyzed from Figure 11. It is revealed that temperature distribution escalates for greater counts of (E). This is because the temperature is increased as a consequence of the quicker heat transfer process from sheet to fluid.

To validate the outcomes of the proposed mode, a comparison with Montgomery and Wang [12] when $m = 0$ is made. An excellent correlation is achieved (Figure 12).

4 Conclusion

This study examined the thermal properties of aqueous nanofluid flow containing immersed MWCNTs influenced by Marangoni convection. Temperature-dependent models for viscosity and thermal conductivity were utilized instead of constant values. The Y–O model was applied to analyze the impact of radius and length on nanofluid thermal

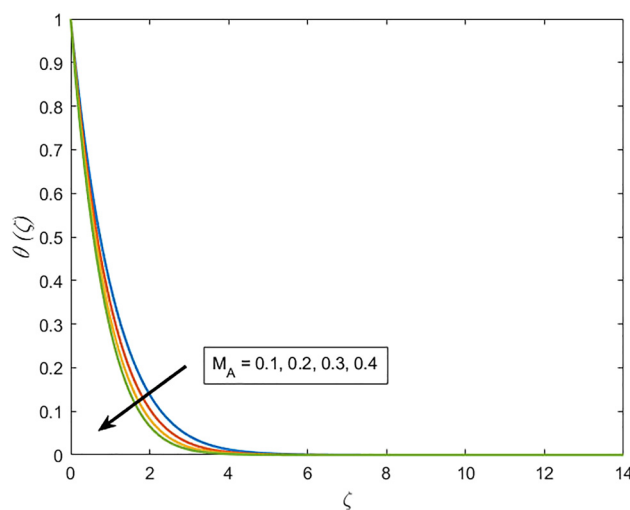


Figure 9: Upshot of Marangoni parameter (M_A) on $\theta(\zeta)$.

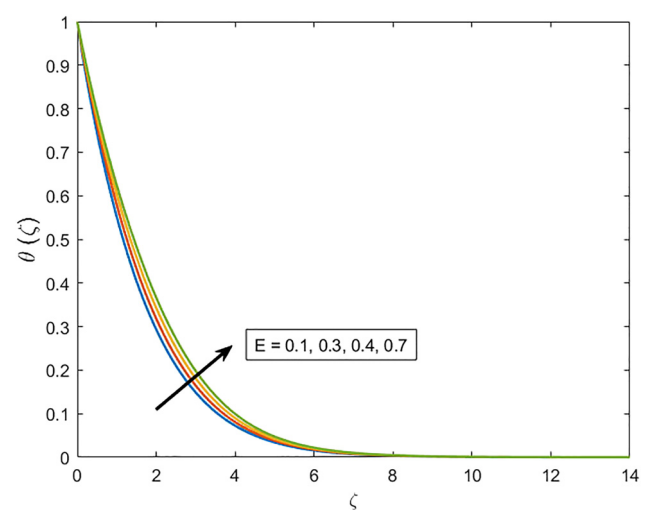


Figure 11: Upshot of variable thermal conductivity (E) on $\theta(\zeta)$.

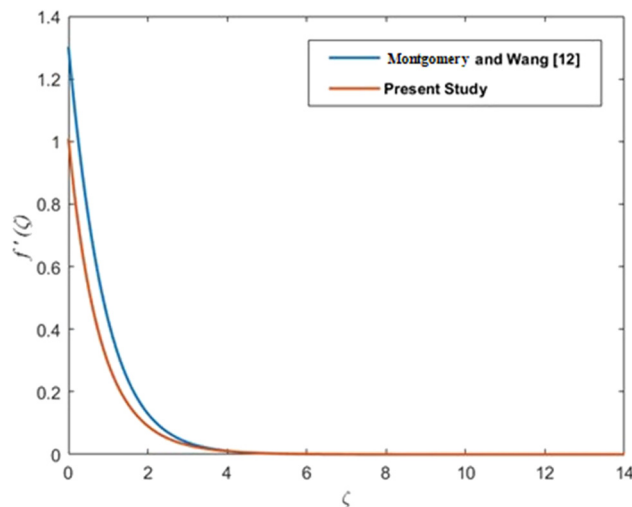


Figure 12: Comparison with published work [12] when $m = 0$.

conductivity. Numerical results were obtained using the bvp4c method. Key findings are summarized below:

- Increasing the length of MWCNTs improves the nanofluid's thermal conductivity, whereas increasing the radius reduces it.
- The thermal conductivity is improved in order to increase the particle volume fraction while maintaining the fixed length of MWCNTs.
- The Marangoni parameter boosts the velocity distribution while diminishing the temperature distribution.
- Velocity distribution escalates by raising the viscosity parameter and particle volume fraction.
- It is seen that velocity distribution drops significantly for greater counts of exponent constant.
- By raising the variable temperature parameter, the temperature distribution escalates.
- Marangoni convection drops the temperature distribution but enhances the heat transmission rate.

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Appendix

$$\frac{\partial u}{\partial x} = C_1 C_2 \left(\frac{2m+1}{3} \right) x^{\frac{2m-2}{3}} f' + C_1 C_2 \zeta \left(\frac{m-1}{3} \right) x^{\frac{2m-2}{3}} f'',$$

$$\frac{\partial v}{\partial y} = -C_1 C_2 \left(\frac{2m+1}{3} \right) x^{\frac{2m-2}{3}} f' - C_1 C_2 \zeta \left(\frac{m-1}{3} \right) x^{\frac{2m-2}{3}} f'',$$

$$\frac{\partial u}{\partial y} = C_1 C_2^2 x^m f'',$$

$$\frac{\partial^2 u}{\partial y^2} = C_1 C_2^3 x^{\frac{4m-1}{3}} f''',$$

$$\frac{\partial T}{\partial x} = A(m+1)x^m \theta + A x^m \zeta \theta',$$

$$\frac{\partial T}{\partial y} = A C_2 x^{\frac{4m+2}{3}} \theta',$$

$$\frac{\partial^2 T}{\partial y^2} = A C_2^2 x^{\frac{3m-1}{3}} \theta'',$$

$$\frac{\partial \mu_{\text{NF}}(T)}{\partial y} = \mu_{\text{NF}} \left(\frac{\theta_R}{(\theta_R - \theta)^2} \right) C_2 x^{\frac{m-1}{3}} \theta',$$

$$\frac{\partial K_{\text{NF}}(T)}{\partial y} = K_{\text{NF}} E C_2 x^{\frac{m-1}{3}} \theta'. \quad (\text{A1})$$

Using the above values of partial derivative in model equations, the set of reduced differential equations will be obtained.