Research Article

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Chaotic dynamics and some solutions for the (n + 1)-dimensional modified Zakharov–Kuznetsov equation in plasma physics

https://doi.org/10.1515/phys-2025-0159
received February 26, 2025; accepted April 30, 2025

Abstract: In the ongoing work, the (n + 1)-dimensional modified Zakharov–Kuznetsov equation is discussed, which characterizes the dispersive and ion acoustic wave propagation in plasma physics. The main research content is to analyze the chaotic dynamics of the equation and provide some new traveling wave solutions. The studied equation is transformed into an ordinary differential equation by using traveling wave transformation. The bifurcation theory, Lyapunov exponent, and sensitivity of initial value condition are employed to analyze the chaotic behavior and stability of the equation. Furthermore, by utilizing the integral form of the equation and complete discrimination system for polynomial method, some new exact solutions are given, including rational, trigonometric, hyperbolic, and Jacobi elliptic function solutions. To examine the properties and shapes of the solutions, some two- and three-dimensional graphs are given with the aid of MATLAB software under appropriate parameters intuitively.

Keywords: nonlinear partial differential equation, bifurcation theory, Lyapunov exponent, sensitivity analysis, complete discrimination system

1 Introduction

Nonlinear partial differential equations (NLPDEs) have a wide range of applications in physics, engineering, and finance. For example, NLPDEs can be utilized to describe fluid mechanics [1,2], the light dispersion in the medium [3,4], heat transfer in objects [5], and the dynamics of disease propagation in human populations [6], *etc*. They are also often used for simulating nonlinear interactions between the atmosphere and the ocean [7], the process of chemical reactions and substance diffusion [8], and so on.

By analyzing the solutions of NLPDEs, we can deeply understand the dynamic behavior, stability, and evolution process of the system, and reveal the important mechanisms in the phenomena, such as limit cycles, chaotic behavior, and phase transitions. Furthermore, many practical problems can be reduced by solving the corresponding NLPDEs in order to improve industrial processes or respond to natural disasters. The study on the solutions of NLPDEs has also advanced the development of mathematical theory and provided important tools and frameworks for other scientific fields. So far, there have been many effective approaches to explore the solutions of NLPDEs, such as extended Fan sub-equation approach [9], Bäcklund transformation technique [10], Hirota's bilinear approach [11], the Sine–Gordon expansion scheme [12], the dynamics approach [13–15], $(G'/G, 1/G)$-expansion approach [16], ϕ^6 model expansion method [17], Riccati equation mapping approach [18], complete discrimination system for polynomial method [19,20], Lie symmetry method [21], and many more.

The modified Zakharov–Kuznetsov (mZK) equation describes the nonlinear wave behavior in two-dimensional plasmas, waves, and other physical phenomena. It is a generalization of the Zakharov–Kuznetsov (ZK) equation, which is an equation describing nonlinear waves of long waves and small amplitudes, commonly used in plasma physics and fluid dynamics. At present, there have been many in-depth studies on the (2+1)-, (3+1)-, and (n + 1)-dimensional forms of mZK equations. The obtained solutions provide powerful tools for the description and application of plasma physics. As regards the (2+1)-dimensional mZK equation, Zahran *et al.* [22] obtained the exact and numerical solutions by manipulating the Paul-Painleve methods and the variational iteration method to obtain some exact solutions, respectively. Naher and Abdullah [18] and Al-Amin *et al.* [23] used

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(G'/G)-expansion and enhanced auxiliary equation methods, respectively. With regard to (3+1)-dimensional mZK equation, Ali *et al.* [24,25] exploited the solutions of the (3+1)-dimensional mZK equation by Sine–Gordon expansion and $(1/G')$-expansion methods. Seadawy [26], Lu *et al.* [27], and Du *et al.* [28] obtained some exact solitary waves of the (3+1)-dimensional mZK equation by utilizing the reductive perturbation procedure, modified extended direct algebraic method, and (G'/G)-expansion method, respectively. Arshed *et al.* [29] and Arab [30] have done some research on the solutions of (3+1)-dimensional extended ZK equation by the aid of the generalized Kudryashov method, the modified Khater method, and Jacobi-elliptic functions. For the $(n+1)$-dimensional mZK equation, Ali *et al.* [31] applied modified Kudryashov methods to find the so-called bright, dark, and singular solutions. Jhangeer *et al.* [32] gave semi-dark, rational, and singular solitary wave solutions of $(n+1)$-dimensional mZK equation by using the extended direct algebraic approach. Munro and Parkes [33] studied the solitary travelling-wave solutions of the mZK equation with the aid of the method of Rowlands and Infeld.

This study discusses the $(n+1)$-dimensional mZK equation as follows:

$$H_t + aH^2H_x + (\nabla^2 H)_x = 0, \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \ldots + \frac{\partial^2}{\partial z_{n-1}^2}$ is the Laplacian of $(n-1)$-dimensional. The dynamic analysis of NLPDEs can reveal the structure of its phase space and help us understand the evolution behavior of the system under different initial conditions, including attractors, singularities, and bifurcation phenomena. The study of chaotic behavior can deepen our understanding of nonlinear system dynamics. Through Lyapunov exponent and sensitivity analysis, the stability of the periodic orbit is determined and the long-term behavior of the system is understood. Therefore, in this study, bifurcation theory, Lyapunov exponent, and sensitivity analysis are used to investigate the $(n+1)$-dimensional mZK equation first. And then, since the form of the equation conforms to certain rules, it can be solved by using complete discrimination system for polynomial method, which leads to some new traveling wave solutions. These methods mentioned above have not yet been applied to Eq. (1). The research results in this work enrich the theory of $(n+1)$-dimensional mZK equation and provide new research ideas for the study of other plasma equations as well as some high-dimensional nonlinear dynamic models.

The present work is divided into four different sections. Section 2 analyzes equilibrium point and chaotic behavior of Eq. (1). Section 3 provides some hyperbolic, trigonometric, and Jacobi elliptic solutions of Eq. (1) with the aid of complete discrimination system for polynomial method, and gives some two- and three-dimensional graphs to visualize parts of these solutions. Section 4 summarizes the whole study.

2 Dynamical behavior analysis of the $(n+1)$-dimensional mZK equation

Applying the wave transformation as follows:

$$H(t, x, z_1, \ldots, z_{n-1}) = \Phi(\mu), \quad \mu = -t + bx + \sum_{k=1}^{n-1} c_k z_k, \tag{2}$$

where $t > 0$ and $b, c_k, k = 1, \ldots, n-1$ are arbitrary parameters. Thereby, we have

$$\begin{cases} H_t = -\Phi', \\ H_x = b\Phi', \\ (\nabla^2 H)_x = \left(b\sum_{k=1}^{n-1} c_k^2\right)\Phi'''. \end{cases} \tag{3}$$

Substituting Eq. (3) in Eq. (1), the nonlinear ordinary differential equation is obtained as follows:

$$-\Phi' + ab\Phi^2\Phi' + \left(b\sum_{k=1}^{n-1} c_k^2\right)\Phi''' = 0. \tag{4}$$

Denote $\tau = \sum_{k=1}^{n-1} c_k^2$. Obviously $\tau \geq 0$. Integrating Eq. (4) once by taking zero as the integration constant, the nonlinear ordinary differential equation with second-order derivative are presented as

$$-\Phi + \frac{ab}{3}\Phi^3 + b\tau\Phi'' = 0. \tag{5}$$

2.1 Bifurcation theory

Assume that $b \neq 0$ and $\tau > 0$. Eq. (5) can be transformed into the following planar dynamical system denoted by $\Upsilon = \Phi'$:

$$\begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}\mu} = \Upsilon, \\ \dfrac{\mathrm{d}\Upsilon}{\mathrm{d}\mu} = -\dfrac{a}{3\tau}\Phi^3 + \dfrac{1}{b\tau}\Phi, \end{cases} \tag{6}$$

with the Jacobian matrix

$$M = \begin{bmatrix} 0 & 1 \\ -\dfrac{a}{\tau}\Phi^2 + \dfrac{1}{b\tau} & 0 \end{bmatrix}, \tag{7}$$

and the Jacobian determinant

$$J(\Phi) = \frac{a}{\tau}\Phi^2 - \frac{1}{b\tau}. \tag{8}$$

We can see that the trace of M is equal to zero for arbitrary Φ, that is Trace(M) ≡ 0. Let

$$Y(\Phi) = -\frac{a}{3\tau}\Phi^3 + \frac{1}{b\tau}\Phi.$$

The three roots of $Y(\Phi)$ might be

$$\Phi_1 = 0, \Phi_2 = \sqrt{\frac{3}{ab}}, \Phi_3 = -\sqrt{\frac{3}{ab}}. \tag{9}$$

Furthermore,

$$J(\Phi_1) = -\frac{1}{b\tau}, \quad J(\Phi_i) = \frac{2}{b\tau}, \quad i = 2, 3.$$

In the next work, the bifurcation theory of planar dynamical system [34] will be used to analyze the three equilibrium points of Eq. (6) and provide phase diagrams with the aid of MATLAB software (version R2023b).

(I) Consider that $a < 0$ and $b > 0$ (resp. $a > 0$ and $b < 0$). Then, $Y(\Phi) = 0$ has only one real root Φ_1. So Φ_1 is a saddle (resp. center) point since $J(\Phi_1) < 0$ (resp. $J(\Phi_1) > 0$) (Figure 1(a); (resp. Figure 1(b)).

(II) Consider that $a > 0$ and $b > 0$. Then, $Y(\Phi) = 0$ has three different real roots Φ_i, $i = 1, 2, 3$. We have $J(\Phi_1) < 0$ and $J(\Phi_{2,3}) > 0$. Thereby, there will be a saddle point at Φ_1, and two center points at $\Phi_{2,3}$ respectively (Figure 1(c)).

(III) Consider that $a < 0$ and $b < 0$. Then, $Y(\Phi) = 0$ also has three different real roots. Thus, there will be a center point at Φ_1 and two saddle points at $\Phi_{2,3}$ due to $J(\Phi_1) > 0$ and $J(\Phi_{2,3}) < 0$, respectively. (Figure 1(d)).

(IV) Consider that $a = 0$. Then, $Y(\Phi) = 0$ has only one real root Φ_1. Then, if $b < 0$ (resp. $b > 0$), then there will be a center (resp. saddle) point at Φ_1 (Figure 1(e); resp. Figure 1(f)).

2.2 Chaotic behavior under periodic external disturbance

Periodic disturbances exist in many real systems, such as physical field, ecosystem, *etc*. The addition of periodic external perturbations can help to understand the response ability and stability boundary of nonlinear differential systems in the face of uncertainty in practice, and improve the accuracy and practicability of the model. In this subsection, the cosine function with amplitude C_0 and frequency w is added to Eq. (6) as a periodic external disturbance. Denote $\Gamma = w\mu$. The $(n + 1)$-dimensional mZK system under the periodic external disturbance $C_0 \cos w\mu$ is presented as follows:

$$\begin{cases} \dfrac{\mathrm{d}\Phi}{\mathrm{d}\mu} = \Upsilon, \\ \dfrac{\mathrm{d}\Upsilon}{\mathrm{d}\mu} = -\dfrac{a}{3\tau}\Phi^3 + \dfrac{1}{b\tau}\Phi + C_0\cos\Gamma, \\ \dfrac{\mathrm{d}\Gamma}{\mathrm{d}\mu} = w. \end{cases} \tag{10}$$

It can be seen from the simulation experiment that the chaotic behavior of Eq. (10) appears under the conditions $a = 8$, $b = 4$, $\tau = 7$, $C_0 = 3$, $w = 3$ with the aid of MATLAB software (version R2023b, Figure 2). The initial state is taken as $(\Phi_0, \Upsilon_0) = (0,0)$. From both the two-dimensional graph of time series about Φ (Figure 2(a)) or Υ (Figure 2(b)) and the trajectory graphs (Figure 2(c) and (d)), it can be observed that the system shows a highly nonlinear periodic wave. When other initial values are taken, this conclusion can still be reached.

2.3 Lyapunov exponent

In addition to the chaotic trajectories shown above, Lyapunov exponent is also an index that accurately reflects the chaotic degree of Eq. (10). If a system has a positive Lyapunov index, this indicates that a tiny change in initial conditions can lead to a significant difference in the system trajectory, *i.e.*, the system appears to be unstable in this direction. If the system has a negative Lyapunov exponent, this indicates that the system tends to be stable in that direction. Thereby, an appropriate set of parameters is taken to compute the Lyapunov exponents of Eq. (10) and the corresponding figure is provided by using MATLAB software (version R2023b). Let $a = 48$, $b = \frac{1}{26}$, $\tau = 2$, and $w = 1.9$. As the parameter C_0 increases, the three exponents of Eq. (10) change, and one of the indices is always positive (Figure 3(a)).

Taking $C_0 = 4$, the three Lyapunov exponents ξ_1, ξ_2, and ξ_3 approach 0.34, 0.014, and, −0.35, respectively (Figure 3(b)). It can be seen that Eq. (10) is very sensitive to initial conditions and has chaotic behavior since the first Lyapunov exponent ξ_1 is positive, which leads to an increase in the complexity of the system.

2.4 Sensitivity analysis

In dynamic systems, sensitivity analysis of initial conditions can help us understand the dependence of the system on the initial conditions, which helps to improve the prediction ability, especially in long-term prediction or complex systems. In practical applications, initial conditions

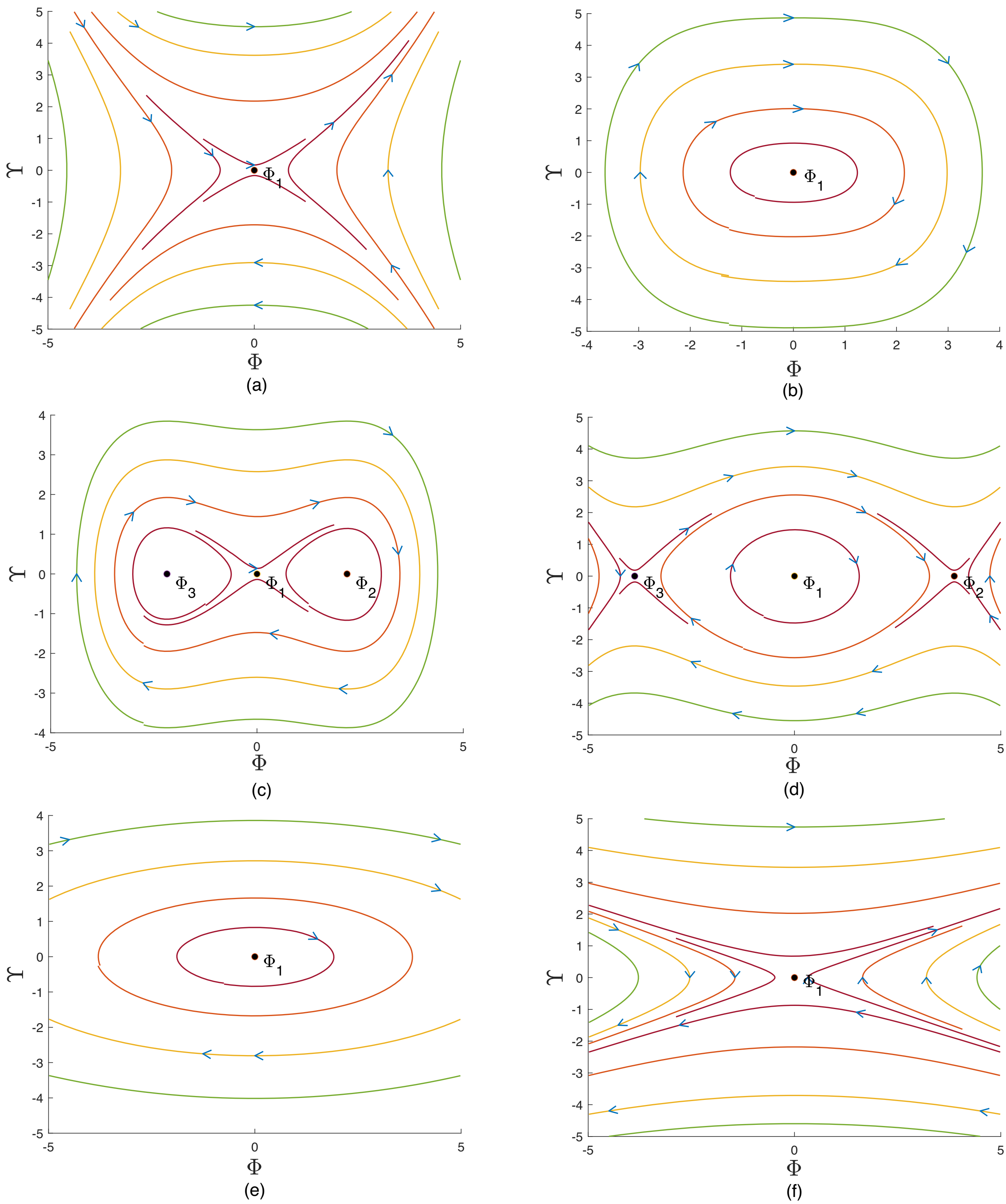

Figure 1: Phase portraits of Eq. (6). (a) $a < 0, b > 0$, (b) $a > 0, b < 0$, (c) $a > 0, b > 0$, (d) $a < 0, b < 0$, (e) $a = 0, b < 0$, and (f) $a = 0, b > 0$.

often contain certain uncertainties. Sensitivity analysis also can quantify the effect of these uncertainties on the behavior of the system. This subsection gives visual graphics for the sensitivity of both Eq. (6) (having no periodic external disturbance) and Eq. (10) (adding external periodic disturbances) by using MATLAB software (version R2023b). Let $a = 6.6, b = 1, \tau = 2, C_0 = 3, w = 1.9$. Considering two sets of the initial values: $(\Phi_0, \Upsilon_0) = (0, 0.001)$ (represented by the green curve) and $(0, 0.002)$ (represented by the blue curve); $(\Phi_0, \Upsilon_0) = (0.001, 0)$ (represented by the green curve) and $(0.002, 0)$ (represented by the blue curve). The time series of Eqs (6) and (10) are shown in Figures 4 and 5, respectively. These figures show that regardless of the existence of periodic external disturbance in the system, when the initial position changes slightly, the system will evolve to different paths.

3 Traveling wave solutions of the $(n + 1)$-dimensional mZK equation

In this section, complete discrimination system for polynomial method is utilized to solve Eq. (1). Then, a variety of new traveling wave solutions including isolated wave and Jacobi elliptic periodic solutions are given and graphically displayed.

3.1 Solving the traveling wave solutions of Eq. (1)

By integrating Eq. (5) again, we obtain

$$-\Phi^2 + \frac{ab}{6}\Phi^4 + b\tau(\Phi')^2 = h_0, \tag{11}$$

where h_0 is an integral constant. Denote

$$\vartheta_2 = -\frac{a}{6\tau}, \vartheta_1 = \frac{1}{b\tau}, \vartheta_0 = \frac{h_0}{b\tau}. \tag{12}$$

Then, Eq. (11) can be simplified as

$$(\Phi')^2 = \vartheta_2\Phi^4 + \vartheta_1\Phi^2 + \vartheta_0. \tag{13}$$

Furthermore, assume that

$$\begin{aligned} &\Phi = \pm\sqrt{(4\vartheta_2)^{-\frac{1}{3}}\phi}\,, p_1 = 4\vartheta_1(4\vartheta_2)^{-\frac{2}{3}}, \\ &p_0 = 4\vartheta_0(4\vartheta_2)^{-\frac{1}{3}}, \upsilon = (4\vartheta_2)^{\frac{1}{3}}\mu. \end{aligned} \tag{14}$$

So, Eq. (13) has the following form:

$$(\phi_\upsilon)^2 = \upsilon(\upsilon^2 + p_1\upsilon + p_0), \tag{15}$$

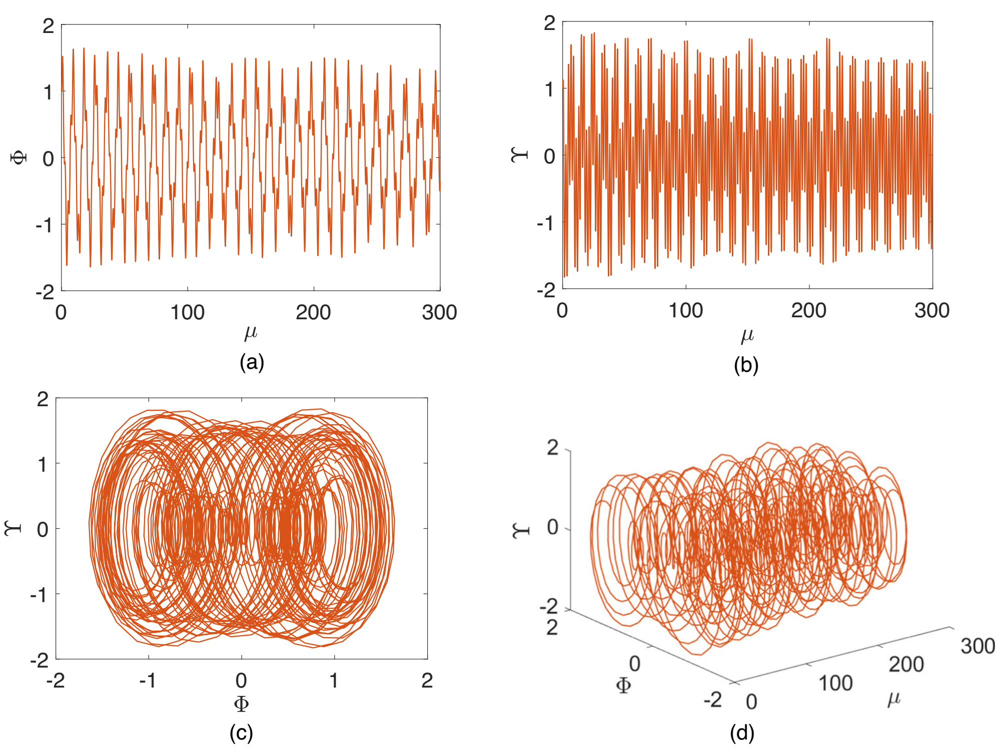

Figure 2: Phase portraits of Eq. (10) with $a = 8, b = 4, \tau = 7, C_0 = 3, w = 3$. (a) Time series about Φ, (b) time series about Υ, (c) 2D phase portrait, and (d) 3D phase portrait.

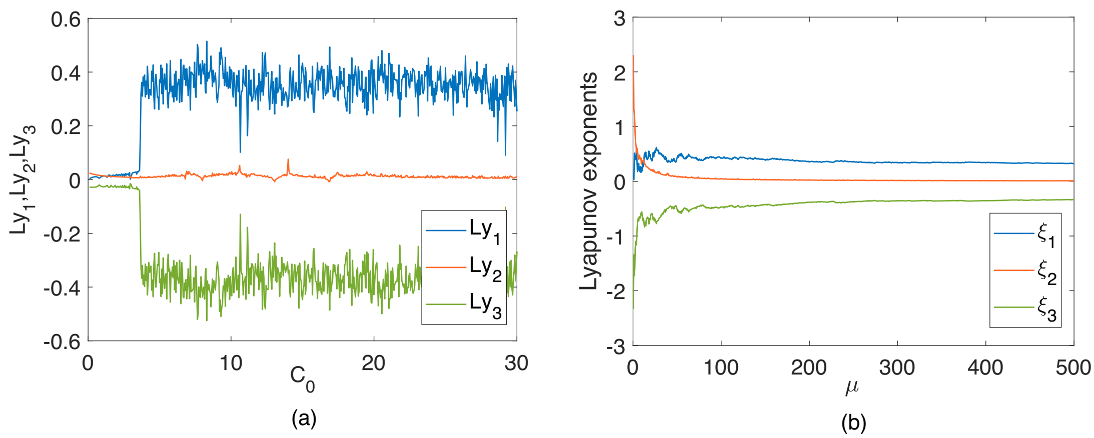

Figure 3: Dynamics of Lyapunov exponents with $a = 48$, $b = \frac{1}{26}$, $\tau = 2$, $w = 1.9$. (a) Lyapunov exponents spectrum depending C_0 and (b) time series of Lyapunov exponents with $C_0 = 4$.

which has the integral form

$$\pm(\upsilon - \upsilon_0) = \int \frac{1}{\sqrt{\phi(\phi^2 + p_1\phi + p_0)}} \mathrm{d}\phi, \tag{16}$$

where υ_0 is the integration constant. Note that, Φ and ϕ are the functions with the variables μ and υ, respectively.

Let

$$F(\phi) = \phi^2 + p_1\phi + p_0.$$

The discriminant of the quadratic polynomial $F(\phi) = 0$ is $\Delta = p_1^2 - 4p_0$.

Case 1: $\Delta = 0$ and $\phi = (4\vartheta_2)^{\frac{1}{3}}\Phi^2 > 0$, *i.e.* $\frac{a}{6\tau} < 0$.

When $p_1 = 4\vartheta_1(4\vartheta_2)^{-\frac{2}{3}} < 0$, *i.e.*, $b\tau < 0$, the solutions of Eq. (16) are

$$\phi_1(\upsilon) = -\frac{p_1}{2}\tanh^2\left(\frac{1}{2}\sqrt{-\frac{p_1}{2}}(\upsilon - \upsilon_0)\right), \tag{17}$$

$$\phi_2(\upsilon) = -\frac{p_1}{2}\coth^2\left(\frac{1}{2}\sqrt{-\frac{p_1}{2}}(\upsilon - \upsilon_0)\right). \tag{18}$$

Substitute Eqs (2), (12), and (14) in Eqs (17) and (18), the solutions of Eq. (1) are obtained as follows:

$$\begin{aligned} &H_1(t, x, z_1, \ldots, z_{n-1}) \\ &= \pm\sqrt{\frac{3}{ab}}\tanh\left[\frac{1}{2}\sqrt{-\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t + bx\right.\right.\right. \\ &\quad \left.\left.\left. + \sum_{k=1}^{n-1} c_k z_k\right) - \upsilon_0\right)\right], \end{aligned} \tag{19}$$

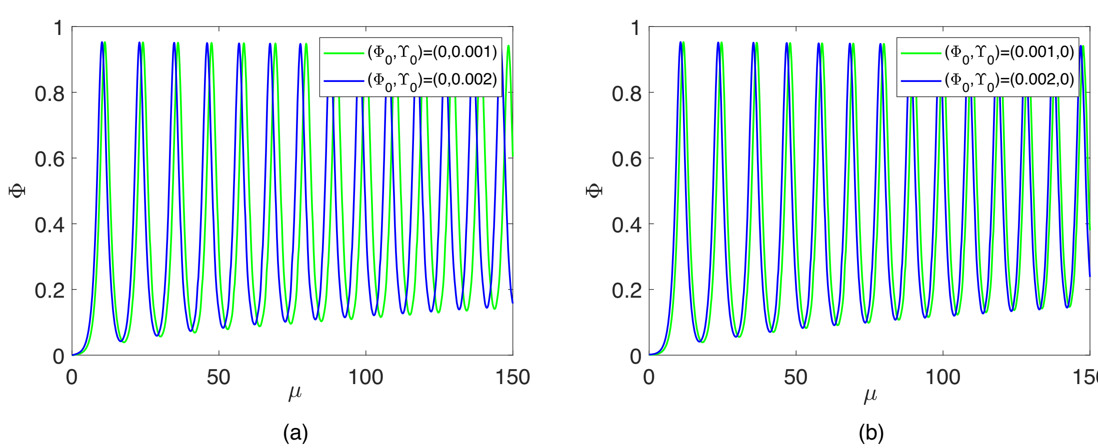

Figure 4: Sensitivity analysis of Eq. (6) with $a = 6.6$, $b = 1$, $\tau = 2$ (having no periodic external disturbance). (a) Time series with $\Phi_0 = 0$ under different Υ_0 and (b) time series with $\Upsilon_0 = 0$ under different Φ_0.

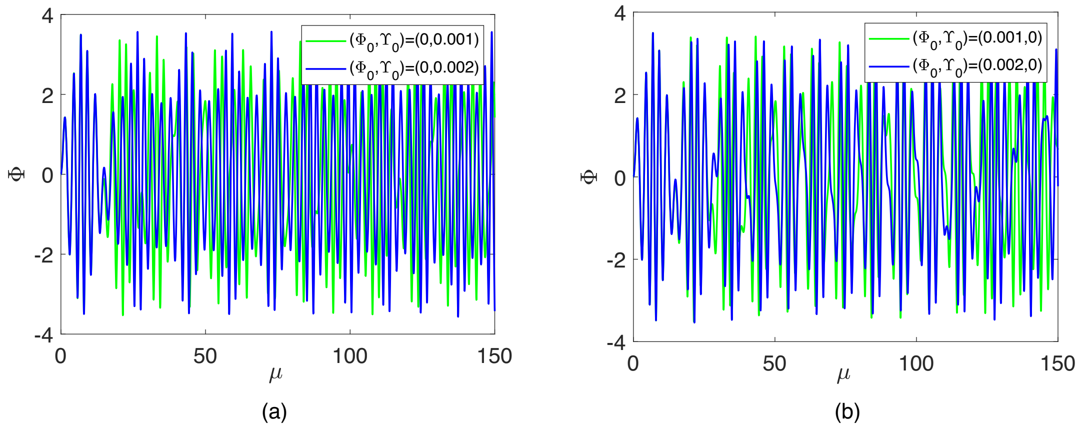

Figure 5: Sensitivity analysis of Eq. (10) with $a = 6.6$, $b = 1$, $\tau = 2$, $C_0 = 3$, $w = 1.9$ (adding periodic external disturbances). (a) Time series with $\Phi_0 = 0$ under different Υ_0 and (b) time series with $\Upsilon_0 = 0$ under different Φ_0.

$$H_2(t,x,z_1,\ldots,z_{n-1}) = \pm\sqrt{\frac{3}{ab}}\coth\left[\frac{1}{2}\sqrt{-\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-v_0\right)\right]. \tag{20}$$

When $p_1 > 0$, *i.e.*, $b\tau > 0$, the solution of Eq. (16) is

$$\phi_3 = \frac{p_1}{2}\tan^2\left(\frac{1}{2}\sqrt{\frac{p_1}{2}}(v-v_0)\right).$$

Thereby, the solution of Eq. (1) is

$$H_3(t,x,z_1,\ldots,z_{n-1}) = \pm\sqrt{-\frac{3}{ab}}\tan\left[\frac{1}{2}\sqrt{\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-v_0\right)\right]. \tag{21}$$

When $p_1 = 0$, *i.e.*, $b\tau = 0$, the solution of Eq. (16) is

$$\phi_4 = \frac{4}{(v-v_0)^2}.$$

So the solution of Eq. (1) is

$$H_4(t,x,z_1,\ldots,z_{n-1}) = \pm\frac{2}{\left(-\frac{2a}{3\tau}\right)^{\frac{1}{2}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\left(-\frac{2a}{3\tau}\right)^{\frac{1}{6}}v_0}. \tag{22}$$

Case 2: $\Delta > 0$, $p_0 = 0$, and $\phi > -p_1$. When $p_1 > 0$, *i.e.*, $b\tau > 0$, the solutions of Eq. (16) are

$$\phi_5 = p_1\left(\frac{1}{2}\tanh^2\left(\frac{1}{2}\sqrt{\frac{p_1}{2}}(v-v_0)\right)-1\right),$$

$$\phi_6 = p_1\left(\frac{1}{2}\coth^2\left(\frac{1}{2}\sqrt{\frac{p_1}{2}}(v-v_0)\right)-1\right).$$

We can obtain the solutions of Eq. (1) as follows:

$$H_5(t,x,z_1,\ldots,z_{n-1}) = \pm\sqrt{-\frac{6}{ab}}\left\{\frac{1}{2}\tanh^2\left[\frac{1}{2}\sqrt{\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx\,,+\sum_{k=1}^{n-1}c_kz_k\right)-v_0\right)\right]-1\right\}^{\frac{1}{2}} \tag{23}$$

$$H_6(t,x,z_1,\ldots,z_{n-1}) = \pm\sqrt{-\frac{6}{ab}}\left\{\frac{1}{2}\coth^2\left[\frac{1}{2}\sqrt{\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx\,.+\sum_{k=1}^{n-1}c_kz_k\right)-v_0\right)\right]-1\right\}^{\frac{1}{2}} \tag{24}$$

When $p_1 < 0$, *i.e.*, $b\tau < 0$, there is

$$\phi_7 = -p_1\left(\frac{1}{2}\tan^2\left(\frac{1}{2}\sqrt{-\frac{p_1}{2}}(v-v_0)\right)+1\right).$$

The solutions of Eq. (1) is

$$H_7(t,x,z_1,\ldots,z_{n-1}) = \pm\sqrt{\frac{6}{ab}}\left\{\frac{1}{2}\tan^2\left[\frac{1}{2}\sqrt{-\frac{2}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}}}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right)\right]-1\right\}^{\frac{1}{2}} \tag{25}$$

Case 3: $\Delta > 0$ and $p_0 \neq 0$. Then, $\phi F(\phi) = (\phi - g_1)(\phi - g_2)(\phi - g_3) = 0$, where $g_1 < g_2 < g_3$. It is known that one of g_i, $i = 1, 2, 3$ must be zero. When $g_1 < \phi < g_2$, there is

$$\phi_8 = g_1 + (g_2 - g_1)\,\mathrm{sn}^2\left[\frac{\sqrt{g_3 - g_1}}{2}(\upsilon - \upsilon_0), w_1\right], \quad \text{when } g_1 < \phi < g_2,$$

$$\phi_9 = \frac{(-g_2)\,\mathrm{sn}^2\left[\frac{\sqrt{g_3-g_1}}{2}(\upsilon-\upsilon_0), w_1\right] + g_3}{\mathrm{cn}^2\left[\frac{\sqrt{g_3-g_1}}{2}(\upsilon-\upsilon_0), w_1\right]}, \quad \text{when } \phi > g_3,$$

where $w_1^2 = \frac{g_2 - g_1}{g_3 - g_1}$, sn and cn are the Jacobi elliptic sine and cosine functions respectively.

With the aid of the root formulas, we can exploit the two different roots of $F(\phi) = 0$ with $r_1 < r_2$ as follows:

$$\begin{aligned} r_1 &= \frac{1}{2}\left(-\frac{4}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}} - \sqrt{\frac{16}{b^2\tau^2}\left(\frac{3\tau}{2a}\right)^{\frac{4}{3}} - 4p_0}\right), \\ r_2 &= \frac{1}{2}\left(-\frac{4}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}} + \sqrt{\frac{16}{b^2\tau^2}\left(\frac{3\tau}{2a}\right)^{\frac{4}{3}} - 4p_0}\right). \end{aligned} \tag{26}$$

(I) When $r_1 < 0 < r_2$ and $r_1 < \phi < 0$, the solution of Eq. (1) is

$$H_8(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}}\left\{r_1 + (-r_1)\,\mathrm{sn}^2\left[\frac{\sqrt{r_2-r_1}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}} \times\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]\right\}^{\frac{1}{2}}, \tag{27}$$

where $w_1^2 = \frac{-r_1}{r_2 - r_1}$.

(II) When $r_1 < 0 < r_2$ and $\phi > r_2$, the solution of Eq. (1) is

$$H_9(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}} \times \frac{r_2^{\frac{1}{2}}}{\mathrm{cn}\left[\frac{\sqrt{r_2-r_1}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]}, \tag{28}$$

where $w_1^2 = \frac{-r_1}{r_2 - r_1}$.

(III) When $r_1 < r_2 < 0$ and $r_1 < \phi < r_2$, the solution of Eq. (1) is

$$H_{10}(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}}\left\{r_1 + (r_2 - r_1)\,\mathrm{sn}^2\left[\frac{\sqrt{-r_1}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}} \times\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]\right\}^{\frac{1}{2}}, \tag{29}$$

where $w_1^2 = \frac{r_2 - r_1}{-r_1}$.

(IV) When $r_1 < r_2 < 0$ and $\phi > r_2$, the solution of Eq. (1) is

$$H_{11}(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}} \times \frac{\sqrt{-r_2}\,\mathrm{sn}\left[\frac{\sqrt{-r_1}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]}{\mathrm{cn}\left[\frac{\sqrt{-r_1}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]}, \tag{30}$$

where $w_1^2 = \frac{r_2 - r_1}{-r_1}$.

(V) When $0 < r_1 < r_2$ and $0 < \phi < r_1$, the solution of Eq. (1) is

$$H_{12}(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}} r_1^{\frac{1}{2}}\,\mathrm{sn}\left[\frac{\sqrt{r_2}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}} \times\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right], \tag{31}$$

where $w_1^2 = \frac{r_1}{r_2}$.

(VI) When $0 < r_1 < r_2$ and $\phi > r_2$, the solution of Eq. (1) is

$$H_{13}(t,x,z_1,\ldots,z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}} \times \frac{\left\{(-r_1)\,\mathrm{sn}^2\left[\frac{\sqrt{r_2}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right] + r_2\right\}^{\frac{1}{2}}}{\mathrm{cn}\left[\frac{\sqrt{r_2}}{2}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t+bx+\sum_{k=1}^{n-1}c_kz_k\right)-\upsilon_0\right), w_1\right]}, \tag{32}$$

where $w_1^2 = \frac{r_1}{r_2}$.

Case 4: $\Delta < 0$ and $\phi > 0$. The solution of Eq. (16) is

$$\phi_{10} = \frac{2\sqrt{p_0}}{1 + \mathrm{cn}[\sqrt[4]{p_0}(\upsilon - \upsilon_0), w_2]} - \sqrt{p_0},$$

where $w_2^2 = \frac{1}{2}\left(1 - \frac{p_1}{2\sqrt{p_0}}\right)$. The solution of Eq. (1) is

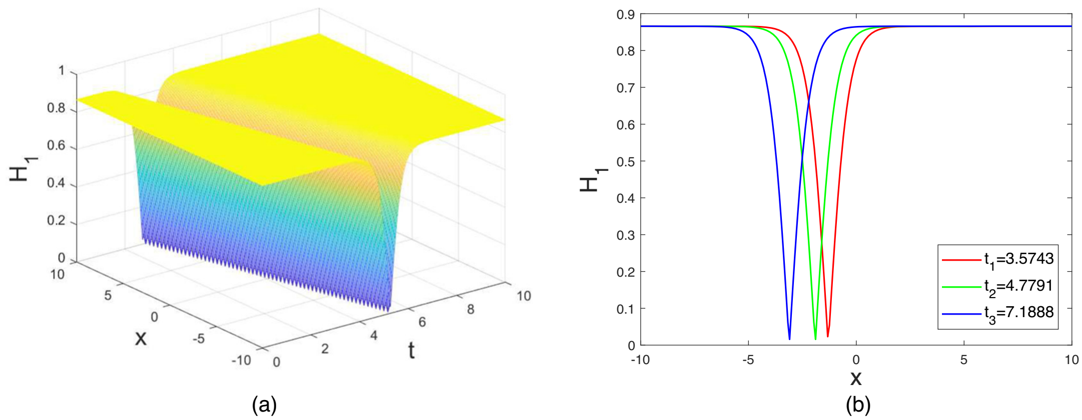

Figure 6: The figures of the solution H_1 with $a = -2$, $b = -2$, $p_0 = 0$, $\upsilon_0 = 5$, $\tau = 6$, $z_\tau = 4$. (a) 3D plot and (b) 2D plot.

$$H_{14}(t, x, z_1, \ldots, z_{n-1}) = \pm\left(-\frac{3\tau}{2a}\right)^{\frac{1}{6}} \times \left\{\frac{2\sqrt{p_0}}{1 + \mathrm{cn}\left[\sqrt[4]{p_0}\left(\left(-\frac{2a}{3\tau}\right)^{\frac{1}{3}}\left(-t + bx + \sum_{k=1}^{n-1} c_k z_k\right) - \upsilon_0\right), w_2\right]} - \sqrt{p_0}\right\}^{\frac{1}{2}}, \tag{33}$$

where $w_2^2 = \frac{1}{2} - \frac{1}{b\tau}\left(\frac{3\tau}{2a}\right)^{\frac{2}{3}} p_0^{-\frac{1}{2}}$.

3.2 Graphics for the travelling wave solutions

Here five typical traveling wave solutions of Eq. (1) are exhibited in the form of two- and three-dimensional graphs with the aid of MATLAB software (version R2023b). According to the solution conditions and adopting appropriate parameters, the structures of the hyperbolic tangent, hyperbolic cotangent, tangent, and Jacobi elliptic function solutions are shown in turn. To make parameter setting easier, we denote $z_\tau = \sum_{k=1}^{n-1} c_k z_k$. Note that, $\tau \geq 0$ since $\tau = \sum_{k=1}^{n-1} c_k^2$. It is required that $b\tau < 0$ for both H_1 and H_2, while $b\tau > 0$ for H_3. Moreover, the values of the parameters should ensure that $r_1 < r_2 < 0$ and $\phi > r_2$ for H_{11}, while $0 < r_1 < r_2$ and $0 < \phi < r_1$ for H_{12}. Figure 6 shows the structure of the solution H_1 with $a = -2, b = -2, p_0 = 0, \upsilon_0 = 5, \tau = 6$, and $z_\tau = 4$; Figure 7 exhibits the structure of the solution H_2 with $a = -2, b = -2, p_0 = 0, \upsilon_0 = 5, \tau = 7$, and $z_\tau = 4$; Figure 8 illustrates the structure of the solution H_3 with $a = -2, b = 2, p_0 = 0, \upsilon_0 = 3, \tau = 4, z_\tau = 8$. As can be seen from Figures 6–8, all the solutions H_1, H_2, and H_3 are isolated waves. Figure 9 denotes that solution H_{11} with $a = -2$, $b = 2, p_0 = 1, \upsilon_0 = 1, \tau = 0.3, z_\tau = 2$ is a quasi-periodic wave; Figure 10 appears that H_{12} with $a = -2, b = 2, p_0 = 1, \upsilon_0 = 1$, $\tau = 0.3, z_\tau = 2$ is a periodic wave.

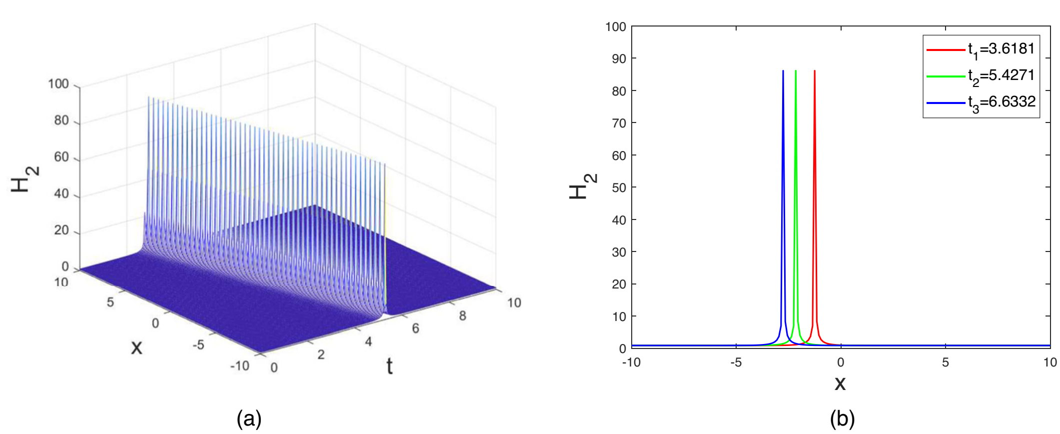

Figure 7: The figures of the solution H_2 with $a = -2$, $b = -2$, $p_0 = 0$, $\upsilon_0 = 5$, $\tau = 7$, $z_\tau = 4$. (a) 3D plot and (b) 2D plot.

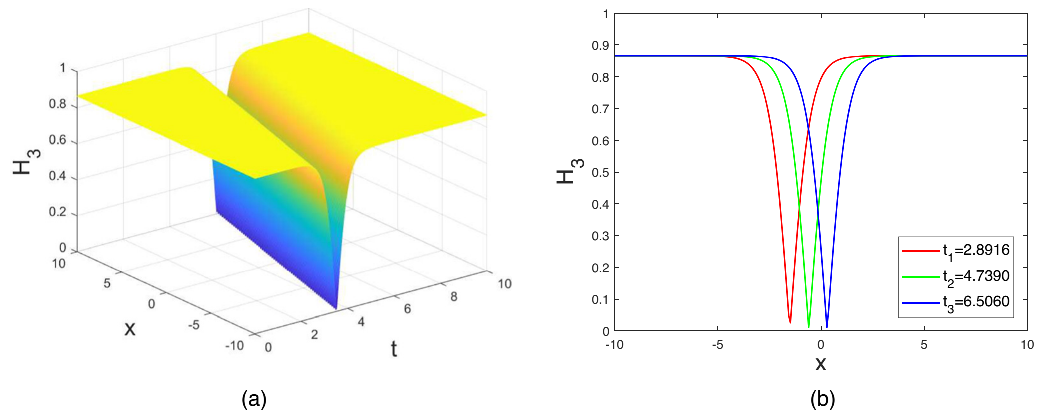

Figure 8: The figures of the solution H_3 with $a = -2$, $b = 2$, $p_0 = 0$, $\upsilon_0 = 3$, $\tau = 4$, $z_\tau = 8$. (a) 3D plot and (b) 2D plot.

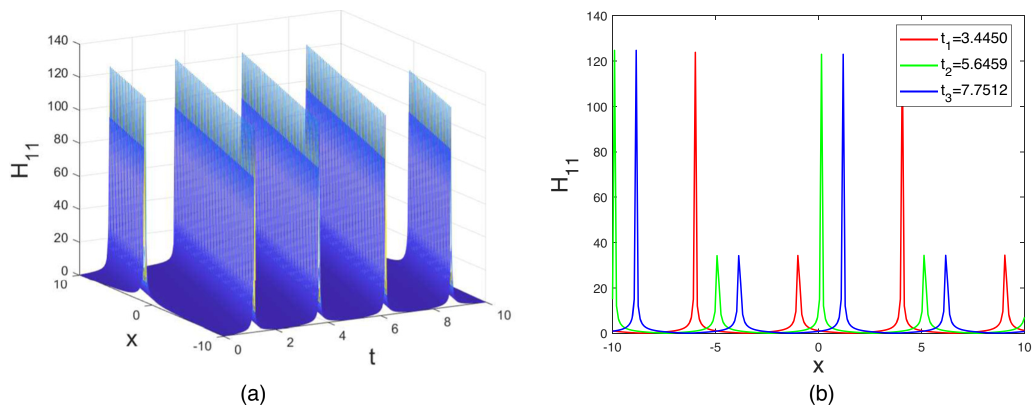

Figure 9: The figures of the solution H_{11} with $a = -2$, $b = 2$, $p_0 = 1$, $\upsilon_0 = 1$, $\tau = 0.3$, $z_\tau = 2$. (a) 3D plot and (b) 2D plot.

4 Conclusion

In this article, the $(n + 1)$-dimensional mZK equation in plasma physics is investigated. The equilibrium points and chaotic behavior of $(n + 1)$-dimensional mZK equation are studied by utilizing bifurcation theory, Lyapunov exponent, and sensitivity of initial value condition. From the evolution trajectories and Lyapunov exponents for the

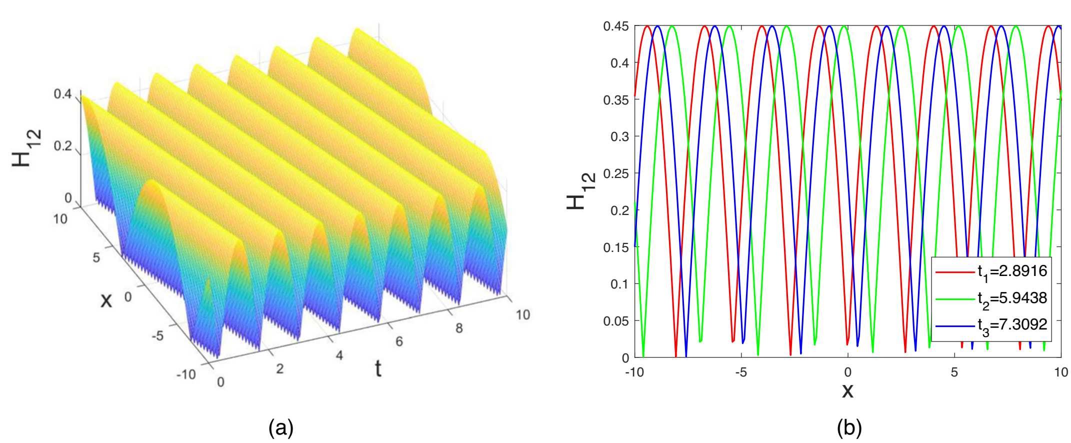

Figure 10: The figures of the solution H_{12} with $a = -2$, $b = -2$, $p_0 = 0.2$, $\upsilon_0 = -1$, $\tau = 2$, $z_\tau = 2$. (a) 3D plot and (b) 2D plot.

system under some system parameters, it can be seen that when tiny periodic disturbances are added to the system, chaotic behavior occurs. And then, some new traveling wave solutions are exploited by taking advantage of complete discrimination system for polynomial method. The structures of traveling wave solutions obtained in this study are various, such as the hyperbolic sine and cosine solutions with solitary wave, the Jacobian ellipse sine solution with periodic wave, and the Jacobi ellipse sine/cosine ratio solution with quasi-periodic wave.

The research of plasma physics equations is a complex and challenging field that covers many aspects. The above methods are reliable and effective in studying the properties and exact solutions of $(n + 1)$-dimensional mZK equation and other plasma physics equations. The expression of the traveling wave solution obtained in this study is different from the solutions obtained in other literature. Their forms are more concise, and it is easier to assign values and draw graphs. These findings will assist in the modeling and analysis of certain high-dimensional and complex nonlinear dynamic models, as well as in-depth exploration of the properties and structures of various physical equations. It should be noted that the solution method employed in this study is applicable to certain nonlinear equations in specific forms. If there are an excessive number of cross terms in the nonlinear equation, it may not be possible to use this method.

In the future, besides the dynamics and exact solutions in this work, other fields such as wave and instability, numerical simulation, and computation also need to be actively studied. We will also strive to promote the application of plasma physics equations in the fields of energy, materials, space science, and many more.

Funding information: This work was funded by Opening Fund of Geomathematics Key Laboratory of Sichuan Province (No. scsxdz2025-7).

Author contributions: Shan Zhao: methodology and writing; Jun Feng: supervision. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: All data generated or analyzed during this study are included in this published article.

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