Research Article

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New investigation on soliton solutions of two nonlinear PDEs in mathematical physics with a dynamical property: Bifurcation analysis

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Abstract: To comprehend nonlinear dynamics, one must have access to soliton solutions, which faithfully portray the actions of numerous physical systems and nonlinear equations. Notable nonlinear equations in relativistic physics, quantum field theory, nonlinear optics, dispersive wave phenomena, contemporary industrial applications, and plasma physics include the Klein–Gordon and Sharma–Tasso–Olver equations, which shed light on wave behavior and interactions. This study introduces a powerful approach to uncovering some novel soliton solutions for these equations, namely, the new generalized (G'/G)-expansion method. The derived soliton solutions are articulated in terms of rational, trigonometric, and hyperbolic functions, each embodying the physical implications of the equations through meticulously specified parameters. The resulting solutions encompass several waveforms, including sharp solitons, singular periodic solitons, flat kink solitons, and singular kink solitons. The results indicate that the employed method is both robust and very effective for the analysis of nonlinear evolution equations (NLEEs). It is compatible with computer algebra systems, facilitating the generation of more generalized wave solutions. The strength and versatility of the new generalized (G'/G)-expansion method suggest its potential for further research, particularly in exploring exact solutions for other NLEEs. The approach represents a significant expansion in the methodologies available for handling nonlinear wave equations, opening new avenues for theoretical and applied investigations in nonlinear science. Furthermore, the bifurcation analysis is carried out, which reveals the comprehension and precise representation of the dynamics of these two nonlinear partial differential equations. It offers the information required to build a comprehensive and significant phase portrait, including insights into solution behaviors, stability changes, and parameter dependencies.

Keywords: soliton solutions, Klein–Gordon equation, Sharma–Tasso–Olver, nonlinear equations, new generalized (G'/G)-expansion method; bifurcation analysis

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1 Introduction

The natural world exhibits a range of complex physical phenomena, from fluid dynamics and light propagation to plasma behavior and ecosystem interactions. These diverse phenomena are often effectively described by nonlinear key equations, which capture the intricate interdependencies and interactions inherent in such systems. A crucial category of these equations is the nonlinear evolution equations (NLEEs), a specific subset of partial differential equations (PDEs). NLEEs play a significant role in mathematical physics, applied mathematics, and engineering. They are particularly valuable for modeling dynamic processes in various fields: quantum mechanics, where they describe the evolution of quantum states; nonlinear optics, where they capture the propagation of light pulses through optical fibers; plasma physics, where they help understand the behavior of charged particles in electromagnetic fields; biophysics, where they model biological patterns and structures; and ecology, where they describe population dynamics and species spread [1–5].

Given their broad applicability, NLEEs are central to research in nonlinear science. They enable the modeling of phenomena where linear approximations are inadequate, particularly when interactions, feedback mechanisms, or other nonlinear effects are involved. Finding exact solutions to these equations provides critical insights into the nature and properties of physical systems, revealing stable configurations like solitons, shock waves, and traveling waves. Such solutions are essential for validating numerical simulations and providing benchmarks for approximations. Among NLEEs, the Klein–Gordon (KG) equation and the Sharma–Tasso–Olver (STO) equation are notable for their roles in nonlinear dynamics, offering significant insights into wave behavior and interactions. Investigating these exact solutions is not just a theoretical pursuit but a key aspect of advancing our understanding of complex systems and developing new technologies [6–10].

The KG equation is a fundamental relativistic wave equation in quantum field theory that characterizes scalar particles, such as the Higgs boson and specific mesons, which possess no intrinsic spin. Designed to reconcile quantum mechanics with special relativity, it guarantees the uniform behavior of particles in all inertial frames. Initially designed to represent the wave function of a single particle, it was discovered to encompass both positive and negative energy solutions, which presented conceptual difficulties. This issue was addressed by reinterpreting the equation within quantum field theory to depict fields that generate and annihilate particles, with negative energy solutions representing antiparticles. This reinterpretation corresponds with discoveries in particle physics, where each particle possesses a corresponding antiparticle. The KG equation is crucial in both particle physics and cosmology, as it delineates scalar fields pertinent to cosmic inflation and dark energy, so linking the quantum realm with the universe's large-scale structure and evolution. Due to its importance, several academics have addressed the KG equation to derive soliton solutions through diverse methodologies [11–15]. Some of the methods employed to find these solutions include the Sobolev gradients technique [11], the Jacobi elliptic function method [12], the F-expansion method [13], the Exp-function technique [14], and the Homotopy analysis technique [15], among others. Each of these techniques offers a unique pathway to derive exact or approximate solutions, enhancing our understanding of soliton behavior in nonlinear dynamics.

The STO equation is a nonlinear partial differential equation that plays a significant role in understanding wave phenomena across various scientific fields, including fluid dynamics, nonlinear optics, and plasma physics. As an integrable system, it can be solved exactly using advanced mathematical techniques such as the inverse scattering transform (IST) and Hirota's direct method. This exact solvability allows researchers to explore soliton solutions, which are stable, localized waves that retain their shape and speed even after interacting with other waves. The STO equation is particularly valuable for modeling systems where both nonlinearity and dispersion are important, such as shallow water waves, nonlinear lattice dynamics, and optical pulse propagation in fibers. Its ability to describe complex wave behavior makes it a critical tool in both theoretical studies and practical applications, offering insights into how nonlinear waves evolve, interact, and propagate in various media. The STO equation helps in predicting and controlling wave dynamics, enhancing our understanding of wave-based phenomena in both natural and engineered systems. The STO equation has recently attracted significant interest from the scientific community for its crucial role in modeling traveling-wave phenomena [16–24]. This equation is widely applied in fields such as nonlinear optics [16], plasma physics [17], dispersive wave dynamics [18], conservation laws [19], Lie symmetry analysis [20], and quantum field theory [21]. In particular, Wang [22] investigated the dynamic characteristics of the STO model and assessed the stability of its traveling wave solutions using the spectral energy estimate method. Additionally, Wang and Xu [23] employed the Lie group analysis method to find exact and explicit solutions to the STO equation, enhancing our understanding of its complex behavior. The soliton solutions of the STO equation are particularly noteworthy, as they demonstrate unique behaviors like the capacity to split into several solitons or merge into one under certain conditions [24].

Given the importance and significance of NLEEs in various fields, researchers have developed and expanded numerous techniques for deriving their exact solutions over recent years. For example: the double $(G'/G,\ 1/G)$expansion method [25], the Backlund transformation method [26], the Hirota's bilinear transformation method [27], the Sardar sub-equation method [28], the modified generalized Kudryashov methods [29], the modified extended direct algebraic method [30], the modified extended tanh function method [31], the Riccati-Bernoulli sub-ODE method [32], the extended trial equation method [33], the improved Sardar sub-equation method [34], the tanh-coth method [35], the improved modified extended tanh-function method [36], the extended tanh function method [37], the extended Kudryashov-expansion technique [38], *etc.*

In a recent development, Naher *et al.* [39] introduced the new generalized (G'/G)-expansion method, which is simpler and more comprehensible for a class of NLEEs. This approach enables the discovery of a broader range of new traveling wave solutions with additional parameters. A more versatile solution ansatz, richer auxiliary equations (use of more complex, higher-order, or nonlinear differential equations for the function of G, and extended algebraic structures are some of the ways in which the new generalized (G'/G)-expansion approach surpasses earlier versions. Its versatility makes it an invaluable resource for researchers in many fields who face nonlinear differential equations. Through expanding the structure of G to incorporate higher-order and nonlinear auxiliary equations, the approach can discover new exact solutions that were previously unavailable using conventional methods. A brief comparison of this method with the other methods is presented in Table 1.

Consequently, the objective of this investigation is to investigate novel exact solutions for both the KG equation and the STO equation by employing this methodology. The study is structured as follows: initially, we present an introduction to the methods, which is followed by the application of the method to the aforementioned equations to identify soliton solutions. Subsequently, we discuss the bifurcation analysis of the dynamical system corresponding to those equations. The following section delves into the graphical behavior of the solitons that were obtained and compares our results to those of previous studies. The study concludes with a reference section that lists the sources that contributed to this research and a summary of our findings.

2 Interpretation of the method

The new generalized (G'/G)-expansion method is a mathematical tool designed to obtain exact solutions for NLEEs, particularly within the realms of mathematical physics and applied mathematics. This method builds upon the traditional (G'/G)-expansion technique, which is recognized for generating traveling wave solutions to differential equations. By introducing more general forms and additional parameters, the new method broadens the range of possible solutions, offering greater flexibility and diversity in solving complex equations [40–48].

Suppose the general form of the NLEEs is

$$F(u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown function, F is a polynomial in $u(x, t)$, x is the spatial variable, and t is temporal variable, which has the highest order derivatives and nonlinear terms and the order of this method is as follows:

Step 1: We consider the combination of real variables x and t be the variable η as follows:

$$u(x, t) = u(\eta), \eta = x \pm vt, \tag{2}$$

where v is the speed of the traveling wave. By means of Eq. (2), Eq. (1) can be converted into an ordinary differential equation (ODE) in the following form:

$$H(u, u', u'', u''', \ldots) = 0, \tag{3}$$

where H is a polynomial of u and its derivatives and the superscripts indicate the derivatives with respect to η.

Step 2: Depending on the situation, Eq. (3) can be integrated term by term, potentially multiple times, with the integration constant possibly being zero. This approach is suitable as we are specifically seeking solitary wave solutions.

Step 3: The traveling wave solution of Eq. (3) can be expressed as a polynomial of the following form:

$$u(\eta) = \sum_{k=0}^{M} a_k(d + W)^k + \sum_{k=1}^{M} b_k(d + W)^{-k}, \tag{4}$$

where either a_k or b_k may be zero but both a_k and b_k could not be zero at a time since $a_M^2 + b_M^2 \neq 0$,

Table 1: Comparison of present method with other established methods

Method	Strengths	Limitations	Advantages of present method
Hirota's method	Multi-soliton solutions	Requires bilinearization, works best for integrable equations	No need for bilinear form, works for broader equations
IST	Exact soliton solutions for integrable systems	Complex spectral analysis, applies only to integrable equations	No spectral transforms needed, applicable to non-integrable PDEs
Tanh-Coth method	Hyperbolic solutions	Limited to certain forms of PDEs	Gives more diverse function solutions
Sine-Cosine method	Trigonometric solutions	Better works mainly for periodic waves	More general wave solutions
Exp-Function method	Very simple for some PDEs	Does not always capture all solutions	More systematic algebraic structure

$a_k(k = 0, 1, 2, 3, \ldots M$, $b_k(k = 1, 2, 3, \ldots M)$, and d are arbitrary constants to be determined.

Here

$$W = (G'/G), \tag{5}$$

where $G = G(\eta)$ satisfies the following auxiliary nonlinear ODE

$$AGG'' - BGG' - EG^2 - C(G')^2 = 0, \tag{6}$$

where prime indicates the derivative with respect to η and A, B, C, E are parameters.

Step 4: To determine the positive integer M, it needs to take the homogeneous balance between the highest order nonlinear term and the highest order derivative appearing in Eq. (3).

Step 5: Inserting Eqs. (4) and (6) including Eq. (5) into Eq. (3) and utilizing the value of M obtained in step 4, we obtain polynomial in $(d + W)^M$, $(M = 0, 1, 2, \ldots)$ and $(d + W)^{-M}$, $(M = 1, 2, \ldots)$. Equating the coefficient of the resulted polynomial to zero. We obtain a class of algebraic equations for $a_k(k = 0, 1, 2, \ldots, M)$, $b_k(k = 1, 2, \ldots, M)$, d, and v.

Step 6: The general solution of Eq. (3) is known to us, substitute the value of $a_k(k = 0, 1, 2, \ldots, M)$, $b_k(k = 1, 2, \ldots, M)$, d, and v into Eq. (4), we then obtain more general type and new exact traveling wave solution of the NLEE (1).

Step 7: Using the general solution of Eq. (6), we obtain $W(\eta)$ as follows:

Family 1: Hyperbolic function solution, where $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) > 0$,

$$W(\eta) = \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\frac{l\sinh(\frac{\sqrt{Q}}{2A}\eta) + m\cosh(\frac{\sqrt{Q}}{2A}\eta)}{l\cosh(\frac{\sqrt{Q}}{2A}\eta) + m\sinh\left(\frac{\sqrt{Q}}{2A}\eta\right)}. \tag{7}$$

Family 2: Trigonometric solution, when $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) < 0$,

$$W(\eta) = \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\frac{-l\sin(\frac{\sqrt{-Q}}{2A}\cdot\eta) + m\cos(\frac{\sqrt{-Q}}{2A}\cdot\eta)}{l\cos(\frac{\sqrt{-Q}}{2A}\cdot\eta) + m\sin\left(\frac{\sqrt{-Q}}{2A}\cdot\eta\right)}. \tag{8}$$

Family 3: When $B \neq 0$, $\phi = A - C$, and $Q = B^2 + 4E(A - C) = 0$,

$$W(\eta) = \frac{B}{2\phi} + \frac{m}{l + m\eta}. \tag{9}$$

Family 4: Hyperbolic function solution, when $B = 0$, $\phi = A - C$, and $P = \phi E > 0$,

$$W(\eta) = \frac{\sqrt{P}}{\phi}\frac{l\sinh(\frac{\sqrt{P}}{A}\cdot\eta) + m\cosh(\frac{\sqrt{P}}{A}\cdot\eta)}{l\cosh(\frac{\sqrt{P}}{A}\cdot\eta) + m\sinh\left(\frac{\sqrt{P}}{A}\cdot\eta\right)}. \tag{10}$$

Family 5: Trigonometric solution, when $B = 0$, $\phi = A - C$, and $P = \phi E < 0$,

$$W(\eta) = \frac{\sqrt{-P}}{\phi}\frac{-l\sin(\frac{\sqrt{-P}}{A}\cdot\eta) + m\cos(\frac{\sqrt{-P}}{A}\cdot\eta)}{l\cos(\frac{\sqrt{-P}}{A}\cdot\eta) + m\sin\left(\frac{\sqrt{-P}}{A}\cdot\eta\right)}. \tag{11}$$

These are the general solutions of $W(\eta)$ that can be employed to establish the solutions of the KG and the STO equations.

3 Applications

In this section, we explain two applications of the mentioned method to extract abundant soliton solutions in Sections 3.1 and 3.2.

3.1 KG equation

Let us consider the KG equation of the form [11,12]

$$u_{tt} - au_{xx} + b\,u - ku^3 = 0. \tag{12}$$

Eq. (12) can be converted to the following ODE *via* the transformation $u(x,\ t) = u(\eta)$, where $\eta = x + v\,t$. Then, we attain

$$(v^2 - a)u'' + b\,u - ku^3 = 0. \tag{13}$$

Now the homogeneous balance between the highest order nonlinear term $(u)^3$ and linear term of the highest derivative u'' appears in Eq. (13), we obtain $M = 1$. As a result, the solution to Eq. (13) has the following form:

$$u(\eta) = a_0 + a_1(d + W) + b_1(d + W)^{-1}, \tag{14}$$

where $a_0,\ a_1,\ b_1$, and d are constants which are to be determined. We will determine these constants later replacing Eqs. (5) and (14) in Eq. (13), we note that the left-hand side of Eq. (13) is translated into the polynomials $(G'/G)^M$, $(M = 0,\ 1,\ 2,\ 3,\ \ldots)$ and $(G'/G)^{-M}$, $(M = 1,\ 2,\ 3,\ \ldots)$. Then, we obtain every coefficient to be zero in these generated polynomials and after obtaining the system of algebraic equations and solving the algebraic equations for the constants $a_0,\ a_1,\ b_1,\ v,\ d$, we are able to find three sets of solutions, as follows:

Set 1:

$$d=-\frac{B}{2\phi},\ a_0=0,\ a_1=\pm\frac{\sqrt{-2k(4P+B^2)b}}{4P+B^2}\phi,$$
$$b_1=\pm\frac{b(4P+B^2)}{2\sqrt{-2k(4P+B^2)b}\,\phi},\qquad(15)$$
$$v=\pm\frac{\sqrt{-(4P+B^2)(4Pa-aB^2+bA^2)}}{(4P+B^2)}.$$

Set 2:

$$a_0=0,\ a_1=\pm\frac{\sqrt{k(4P+B^2)b}}{4P+B^2}\phi,$$
$$b_1=\pm\frac{b(4P+B^2)}{4\sqrt{k(4P+B^2)b}\,\phi},\qquad(16)$$
$$v=\pm\frac{\sqrt{(4P+B^2)(8Pa+2aB^2+bA^2)}}{\sqrt{2}(4P+B^2)},\ d=-\frac{B}{2\phi}.$$

Set 3:

$$d=d,\ a_0=\pm\frac{(2d\phi+B)b}{\sqrt{k(4P+B^2)b}},\ a_1=0,$$
$$b_1=\pm\frac{2\sqrt{b(4P+B^2)k}}{k(4P+B^2)}(d^2\phi+Bd-E),$$
$$v=\pm\frac{\sqrt{(4P+B^2)(4Pa+aB^2+2bA^2)}}{(4P+B^2)}.\qquad(17)$$

Set 4:

$$a_0=\pm\frac{(2d\phi+B)b}{\sqrt{k(4P+B^2)b}},\ b_1=0,\ a_1=\pm\frac{2\sqrt{b(4P+B^2)k}}{k(4P+B^2)},$$
$$v=\pm\frac{\sqrt{(4P+B^2)(4Pa+aB^2+2bA^2)}}{(4P+B^2)}.\qquad(18)$$

Set 1: The constant values arranged in Eq. (15) should be inserted in Eq. (14), as well as Eq. (7) and simplifying, for both $l=0$ but $m\neq0$ and $m=0$, but $l\neq0$, we derive the traveling wave solutions as follows:

$$u_{11}=n_1\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{12}=n_1\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

where $n_1=\pm\sqrt{-\frac{2k(4P+B^2)\mathrm{b}}{(4P+B^2)}}\,\phi$, $n_2=\pm\frac{b(4P+B^2)}{2\sqrt{-2k(4P+B^2)b}\,\phi}$.

In a similar fashion, substituting the values of the constants arranged in Eq. (15) in (14), as well as Eqs. (8)–(11) and simplifying, for $l=0$ but $m\neq0$ and $m=0$ but $l\neq0$, we obtain the following traveling wave solutions:

$$u_{13}=n_1\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{14}=n_1\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}+n_2\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{15}=n_1\left(\frac{1}{\eta}\right)+n_2\left(\frac{1}{\eta}\right)^{-1},$$

$$u_{16}=n_1\left\{-\frac{B}{2\phi}+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}+n_2\left\{-\frac{B}{2\phi}+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{17}=n_1\left\{-\frac{B}{2\phi}+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}+n_2\left\{-\frac{B}{2\phi}+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{18}=n_1\left\{-\frac{B}{2\phi}+\frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}+n_2\left\{-\frac{B}{2\phi}+\frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{19}=n_1\left\{-\frac{B}{2\phi}-\frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}+n_2\left\{-\frac{B}{2\phi}-\frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

Set 2: Substituting the values of the constants held in Eqs (16) in (14), along with Eq. (7) and simplifying, we obtain the following traveling wave solutions for $l=0$ but $m\neq0$ and $m=0$ but $l\neq0$, which are as follows:

$$u_{21}=n_1\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{22}=n_1\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

where $n_1=\pm\frac{\sqrt{k(4P+B^2)b}}{4P+B^2}\phi$, $n_2=\pm\frac{b(4P+B^2)}{4\sqrt{k(4P+B^2)b}\,\phi}$.

Similarly, by substituting the values of the constants contained in Eq. (16) in (14), together with Eqs. (8)–(11) and simplifying, we obtain respectively the following traveling wave solutions for $l=0$ but $m\neq0$ and $m=0$ but $l\neq0$ as follows:

$$u_{23}=n_1\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}+n_2\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{24} = n_1\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + n_2\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{25} = n_1\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + n_2\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{26} = n_1\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + n_2\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{27} = n_1\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + n_2\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{28} = n_1\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + n_2\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

Set 3: Inserting the values of the constants from Eq. (17) in (14), as well as Eqs. (7)–(11) and simplifying, we obtain the following traveling wave solutions for $l = 0$ but $m \neq 0$ and $m = 0$ but $l \neq 0$ as follows:

$$u_{31} = r_1 + n_2\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{32} = r_1 + n_2\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{33} = r_1 + n_2\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{34} = r_1 + n_2\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{35} = r_1 + n_2\left\{d + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{36} = r_1 + n_2\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{2A}\eta\right)\right\}^{-1},$$

$$u_{37} = r_1 + n_2\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{38} = r_1 + n_2\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

where $r_1 = \pm\frac{(2d\phi + B)b}{\sqrt{k(4P + B^2)b}}$, $n_2 = \pm\frac{2\sqrt{b(4P + B^2)k}}{k(4P + B^2)}(d^2\phi + Bd - E)$.

Set 4: In a similar fashion, replacing the values of the parameters from Eq. (18) in Eq. (14), including Eqs. (7)–(11) and simplifying, we obtain the following traveling wave solutions for $l = 0$ but $m \neq 0$ and $m = 0$ but $l \neq 0$, respectively.

$$u_{41} = r_1 + n_1\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{42} = r_1 + n_2\left\{d + \frac{B}{2\phi} + \frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{43} = r_1 + n_2\left\{d + \frac{B}{2\phi} + \frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{44} = r_1 + n_1\left\{d + \frac{B}{2\phi} - \frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{45} = r_1 + n_1\left\{d + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{46} = r_1 + n_1\left\{d + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{47} = r_1 + n_1\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{48} = r_1 + n_1\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\},$$

where $r_1 = \pm\frac{(2d\phi + B)b}{\sqrt{k(4P + B^2)b}}$, $n_1 = \pm\frac{2\sqrt{b(4P + B^2)k}}{k(4P + B^2)}$.

3.2 STO equation

The STO equation comprises evenly the double nonlinear terms $(u^3)_x$ and $(u^2)_{xx}$ and linear dispersive term u_{xxx}. This demonstrated that the equation is one of the evolution equations that have infinitely many symmetries. This is a widely used model equation for nonlinear dispersive wave transmission in inhomogeneous mediums. The STO equation attracted a significant amount of research work by both physicists and mathematicians due to its appearance in scientific applications. Consider the STO equation of the following form [16,49]:

$$u_t + \alpha(u^3)_x + \frac{3}{2}\alpha(u^2)_{xx} + \alpha u_{xxx} = 0. \quad (19)$$

Now, we reduce Eq. (19) into an ODE by choosing the transformation $u(x, t) = u(\eta)$, where $\eta = x - vt$. Then, we obtain

$$-v(u)' + \alpha(u^3)' + \frac{3}{2}\alpha(u^2)'' + \alpha u''' = 0. \quad (20)$$

Thus, integrating Eq. (20) with respect to η gives

$$-vu + \alpha\, u^3 + 3u\alpha u' + \alpha\, u'' + c = 0, \quad (21)$$

where c is an integrating constant that is to be determined. Now taking the homogeneous balance between the highest order nonlinear term u^3 and linear term of the highest derivative u'' occurring in Eq. (21), we obtain $M = 1$. Therefore, the solution of Eq. (21) is of the following form:

$$u(\eta) = a_0 + a_1(d + W) + b_1(d + W)^{-1}, \quad (22)$$

where a_0, a_1, b_1, and d are constants which are determined. Substitute Eqs. (5) and (22) in Eq. (21), then the left-hand side of Eq. (21) is translated into polynomial in $(G'/G)^M$, $(M = 0, 1, 2, 3, \ldots)$ and $(G'/G)^{-M}$, $(M = 0, 1, 2, 3, \ldots)$ and we collect each coefficient of this resulting polynomials equal to zero. We attain a class of algebraic equations and solve the algebraic equations for the constants a_0, a_1, b_1, v, d and find nine sets of solutions as follows:

$$\text{Set 1:} \quad d = -\frac{B}{2\phi},\ c = 0, \quad v = \frac{4\alpha Q}{A^2},\ a_0 = 0, \quad a_1 = \frac{2\phi}{A},\ b_1 = \frac{Q}{2A\phi}. \quad (23)$$

$$\text{Set 2:} \quad d = -\frac{B}{2\phi},\ c = 2(a_0^2A^2 - Q)\alpha a_0, \quad v = \frac{(3a_0^2A^2 + Q)\alpha}{A^2},\ a_0 = a_0,\ a_1 = \frac{\phi}{A},\ b_1 = \frac{Q}{2A\phi}. \quad (24)$$

$$\text{Set 3:} \quad d = d,\ c = 0, \quad v = \frac{\alpha Q}{A^2},\ a_0 = 0, \quad a_1 = \frac{\phi}{A},\ b_1 = \frac{Q}{4\phi A}. \quad (25)$$

$$\text{Set 4:} \quad d = d,\ c = 0, \quad v = \frac{\alpha Q}{A^2},\ a_0 = \frac{2\phi d + B}{A}, \quad a_1 = \frac{2\phi}{A},\ b_1 = 0. \quad (26)$$

$$\text{Set 5:} \quad d = d,\ c = 0, \quad v = \frac{\alpha Q}{A^2},\ a_0 = \frac{2\phi d + B}{A}, \quad a_1 = 0, \quad b_1 = -\frac{2(d^2\phi + dB - E)}{A}. \quad (27)$$

$$\text{Set 6:}\ d = \frac{-3B \pm \sqrt{B^2 + 4P}}{6\phi}, \quad c = \frac{20\alpha(8AER + 12ABE - 8ER + 2B^2R - 12BCE + 3B^3)}{27A^3}, \quad v = \frac{7\alpha(4P + B^2)}{3A^2},$$

$$a_0 = -\frac{-3B \pm \sqrt{B^2 + 4P} + 3B}{3A},\ a_1 = \frac{2\phi}{A}, \quad b_1 = \frac{2(4P + B^2)}{9A\phi}, \quad (28)$$

where $R = -\frac{-3B \pm \sqrt{B^2 + 4P}}{2}$.

$$\text{Set 7:}\ d = \frac{-3B \pm \sqrt{B^2 + 4P}}{6\phi}, \quad c = -\frac{20\alpha(8AER + 12ABE - 8ER + 2B^2R - 12BCE + 3B^3)}{27A^3},$$

$$v = \frac{7\alpha(4P + B^2)}{3A^2},\ a_0 = \frac{-3B \pm \sqrt{B^2 + 4P} + 3B}{3A}, \quad a_1 = \frac{\phi}{A},\ b_1 = \frac{2(4P + B^2)}{9A\phi}, \quad (29)$$

where $R = -\frac{-3B \pm \sqrt{B^2 + 4P}}{2}$.

$$\text{Set 8}:d = d,\ c = \frac{\alpha}{A^3}\{(A^2 + C^2)(3d^2B - 2dE) + \phi(B^2d - BE - 6d^3AC) + 6A^3a_0d^2 + 6A^3a_0^2d - 2A^2a_0E + 3A^2a_0^2BAa_0B^2 + 2a_0ACE - 12a_0A^2Cd^2 + 6a_0AC^2d^2 + 6A^2a_0Bd - 6A^2a_0^2Cd + 4dACE + 6ABCd^2 - 6a_0ABCd + 2d^3A^3 - 2C^3d^3 + 2a_0^3A^3\},$$

$$v = \frac{\alpha(3d^2\phi^2 + 6a_0Ad\phi + 3Bd\phi + P + 3a_0AB + 3a_0^2A^2 + B^2)}{A^2},$$

$$a_0 = a_0, a_1 = \frac{\phi}{A}, b_1 = 0. \quad (30)$$

$$\text{Set 9}:\ c = -\frac{\alpha}{A^3}\{(A^2 + C^2)(3d^2B - 2dE) + \phi(B^2d - BE - 6d^3AC) - 6a_0d^2\phi^2 + 2d^3(A^3 - C^3) + 6A^3a_0^2d + 2a_0A^2E + 3a_0^2A^2B - Aa_0B^2 - 2a_0ACE - 6A^2a_0Bd - 6a_0^2A^2Cd + 4ACEd - 6ABCd^2 + 6a_0ABCd - 2a_0^3A^3\},$$

$$v = \frac{\alpha\{3d^2(A^2 + C^2) - 6a_0Ad\phi + P + 3Bd\phi - 6ACd^2 + 3a_0^2A^2 - 3a_0AB\}}{A^2}, \quad (31)$$

$$a_0 = a_0, a_1 = 0, b_1 = -\frac{d^2\phi + dB - E}{A},$$

where $\phi = A - C$, $P = \phi E$, $Q = B^2 + 4E(A - C)$, and A, B, C, E are parameters.

Similarly, for set 1:

$$u_{51} = \frac{2\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{52} = \frac{2\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1}.$$

Substituting the values of the constants provided in Eq. (23) in (22), including Eqs. (8)–(11) and simplifying, we obtain the following traveling wave solutions for $l = 0$ but $m \neq 0$ and $m = 0$ but $l \neq 0$, respectively.

$$u_{53} = \frac{2\phi}{A}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{54} = \frac{2\phi}{A}\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{55} = \frac{2\phi}{A}\left(\frac{1}{\eta}\right) + \frac{Q}{2A\phi}\left(\frac{1}{\eta}\right),$$

$$u_{56} = \frac{2\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{57} = \frac{2\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{58} = \frac{2\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{59} = \frac{2\phi}{A}\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

Similarly, for set 2:

$$u_{61} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{62} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{63} = a_0 + \frac{\phi}{A}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{64} = a_0 + \frac{\phi}{A}\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{65} = a_0 + \frac{\phi}{A}\left(\frac{1}{\eta}\right) + \frac{Q}{2A\phi}\left(\frac{1}{\eta}\right)^{-1},$$

$$u_{66} = a_0 + \frac{\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{67} = a_0 + \frac{\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{68} = a_0 + \frac{\phi}{A}\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{69} = a_0 + \frac{\phi}{A}\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\} + \frac{Q}{2A\phi}\left\{-\frac{B}{2\phi} - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

Similarly, for set 3:

$$u_{71}=\frac{\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{72}=\frac{\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{73}=\frac{\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{74}=\frac{\phi}{A}\left\{d+\frac{B}{2\phi}-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{B}{2\phi}-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{75}=\frac{\phi}{A}\left(d+\frac{B}{2\phi}+\frac{1}{\eta}\right)+\frac{Q}{4A\phi}\left(d+\frac{B}{2\phi}+\frac{1}{\eta}\right)^{-1},$$

$$u_{76}=\frac{\phi}{A}\left\{d+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{77}=\frac{\phi}{A}\left\{d+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{78}=\frac{\phi}{A}\left\{d+\frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d+\frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{79}=\frac{\phi}{A}\left\{d-\frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}+\frac{Q}{4A\phi}\left\{d-\frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1}.$$

Similarly, for set 4:

$$u_{81}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{82}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\},$$

$$u_{83}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{B}{2\phi}+\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{84}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{B}{2\phi}-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\},$$

$$u_{85}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left(d+\frac{B}{2\phi}+\frac{1}{\eta}\right),$$

$$u_{86}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{87}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\},$$

$$u_{88}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d+\frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\},$$

$$u_{89}=\frac{2\phi d+B}{A}+\frac{2\phi}{A}\left\{d-\frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}.$$

Similarly, for set 5:

$$u_{91}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\coth\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{92}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{B}{2\phi}+\frac{\sqrt{Q}}{2\phi}\tanh\left(\frac{\sqrt{Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{93}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{B}{2\phi}+\frac{\sqrt{-Q}}{2\phi}\cot\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{94}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{B}{2\phi}-\frac{\sqrt{-Q}}{2\phi}\tan\left(\frac{\sqrt{-Q}}{2A}\eta\right)\right\}^{-1},$$

$$u_{95}=\frac{2\phi d+B}{A}+n_1\left(d+\frac{B}{2\phi}+\frac{1}{\eta}\right)^{-1},$$

$$u_{96}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{\sqrt{P}}{\phi}\coth\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{97}=\frac{2\phi d+B}{A}+n_1\left\{d+\frac{\sqrt{P}}{\phi}\tanh\left(\frac{\sqrt{P}}{A}\eta\right)\right\}^{-1},$$

$$u_{98} = \frac{2\phi d + B}{A} + n_1\left\{d + \frac{\sqrt{-P}}{\phi}\cot\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

$$u_{99} = \frac{2\phi d + B}{A} + n_1\left\{d - \frac{\sqrt{-P}}{\phi}\tan\left(\frac{\sqrt{-P}}{A}\eta\right)\right\}^{-1},$$

where, $n_1 = -\frac{2(d^2\phi + dB - E)}{A}$.

4 Bifurcation analysis

Bifurcation analysis is an important tool for understanding the qualitative changes in nonlinear dynamical systems that support physical events. It investigates how changes in system characteristics cause shifts in equilibrium, stability, or the creation of complex behaviors like oscillations or chaos. These changes, known as bifurcations, are essential for modeling and studying critical physical processes such as phase transitions, fluid instabilities, and wave propagation. Physicists can predict and define system behavior under different conditions by finding critical points and categorizing bifurcations. Additionally, bifurcation analysis simplifies the study of intricate physical models by exposing common patterns among seemingly unrelated systems. It is essential for gaining understanding of theoretical models and directing experimental investigations of the related phenomenon because it offers a rigorous framework for investigating the relationships between stability, symmetry, and nonlinear interactions [50–53]. A brief bifurcation analysis of the KG and STO equations is given in this section. It is regarded as a planner dynamical system in this sense and can have the following forms:

4.1 KG equation

Eq. (13) is used to define the planner dynamical system in this instance as follows:

$$\left.\begin{aligned} u' &= N \\ N' &= B_1u^3 - B_2u \end{aligned}\right\}, \tag{32}$$

where $B_1 = \frac{k}{(v^2 - a)}$, and $B_2 = \frac{k}{(v^2 - a)}$.

To analyze the phase portrait of this system, it is crucial to first determine the equilibrium points by setting the derivatives $u' = 0$ and $N' = 0$. By solving these equations, we obtain three equilibrium points: $e_1(0, 0)$, $e_2\left(\sqrt{\frac{b}{k}}, 0\right)$, and $e_3\left(-\sqrt{\frac{b}{k}}, 0\right)$. These equilibrium points are dependent on the system's parameters, namely, a, b, k, and v. The location and nature of these equilibrium points play a key role in understanding the system's dynamic behavior depending on $v^2 - a$.

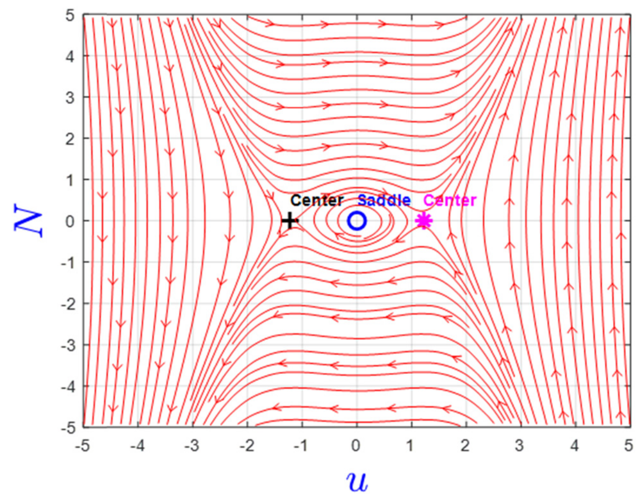

Figure 1: Phase behavior of Eq. (32) when $v^2 - a > 0$, where $k = 2$, $b = 3$, $a = 1$, and $v = 2$.

Case I: When $v^2 - a > 0$, (Figure 1) the eigenvalues of the Jacobian matrix are real and have opposite signs, indicating a saddle point. This means one eigenvalue is positive (unstable) and the other negative (stable), creating both stable and unstable directions in the system. Depending on the exact relationship between v and v, the eigenvalues could also be purely imaginary, leading to center-like behavior with closed orbits, representing periodic or oscillatory motion.

Case II: When $v^2 - a < 0$, (Figure 2), the system behaves in an oscillatory or spiral fashion, and the

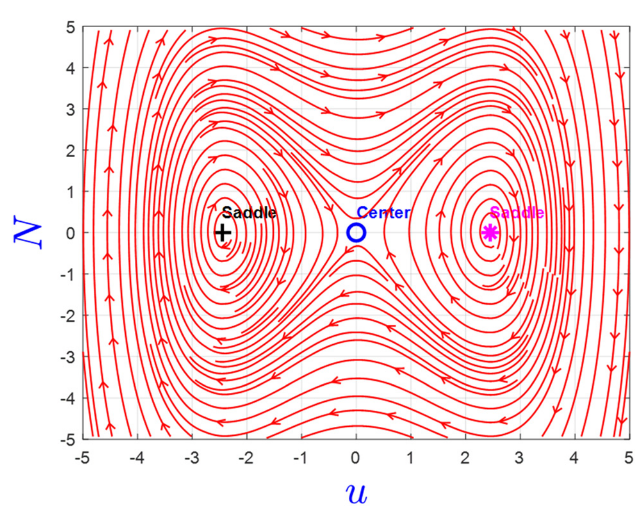

Figure 2: Phase behavior of Eq. (32) when $v^2 - a < 0$, where $k = 0.5$, $b = 3$, $a = 2$, and$v = 1$.

Jacobian is a complex conjugate. Trajectories spiral inward when the real part is negative, signifying stable spirals (stable focus). Trajectories exhibit unstable spirals (unstable focus) when the real portion is positive. The system exhibits periodic motion with closed orbits around the equilibrium point if the eigenvalues are entirely imaginary.

Case III: When $v^2 - \alpha = 0$, in this scenario, the system undergoes a bifurcation as the eigenvalues of the Jacobian matrix become degenerate. This can lead to a dramatic change in the nature of the equilibrium point, causing a qualitative shift in the system's dynamics. The system may transition from stable to unstable behavior or change from one type of equilibrium (*e.g.*, stable point) to another, such as a limit cycle.

4.2 STO equation

Eq. (13) serves as the basis for defining the planar dynamical system in this case considering that the integrating constant is zero that gives

$$\left.\begin{aligned} u' &= N \\ N' &= C_1u^2 - u^3 - 3uN \end{aligned}\right\}, \tag{33}$$

where $C_1 = \frac{v}{\alpha}$.

Similarly, the equilibrium points for this system are determined as $f_1(0,0), f_2\left(\sqrt{\frac{v}{\alpha}}, 0\right)$ and $f_3\left(-\sqrt{\frac{v}{\alpha}}, 0\right)$. These points depend on the system's parameters, α and v. The position and characteristics of these equilibrium points are crucial for analyzing and interpreting the dynamic behavior of the system.

Case I: When $v > 0$, but $\alpha < 0$ or $v < 0$, but $\alpha > 0$, the eigenvalues at f_1 correspond to a saddle point, exhibiting real values with opposing signs, which is indicative of saddle behavior (Figure 3). However, for f_2 and f_3, the Jacobian determinant is zero, suggesting that these are non-hyperbolic equilibria. A more detailed nonlinear analysis is necessary to fully understand their dynamics. Although nonlinear terms may influence their behavior, potentially giving rise to phenomena like limit cycles or other complex dynamics, non-hyperbolic points with a zero determinant typically correspond to degenerate equilibrium points.

Case II: When $v > 0$, and $\alpha > 0$ or $v < 0$, and $\alpha < 0$, Thus, for $\alpha = 1$ and $v = 3$, the system exhibits a saddle point at f_1, a stable node at f_2, and an unstable node at f_3 (Figure 4). The nature of the system's dynamics will depend on the initial conditions. The equilibrium points play a crucial role in identifying the types of attractors and repellents present in the system, helping to categorize the behavior of trajectories and how they evolve over time.

5 Discussion and graphical representations of the solutions

The solutions obtained through this method are straightforward and easier to interpret, effectively capturing the mechanisms underlying complex nonlinear physical phenomena. Graphical representations play a crucial role in

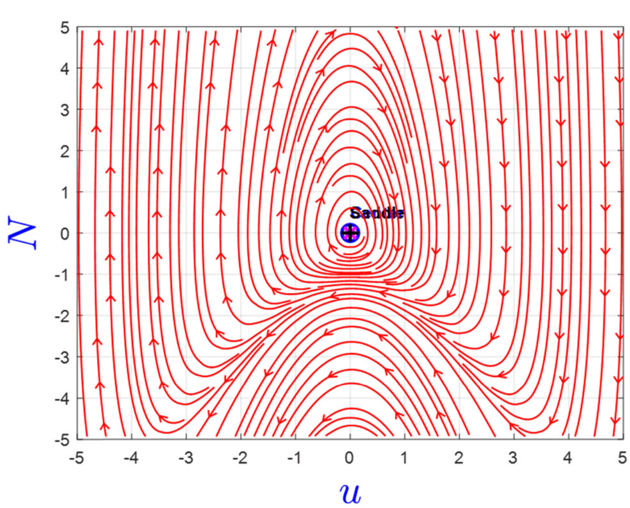

Figure 3: Phase behavior of Eq. (33) when $v > 0$, but $\alpha < 0$ or $v < 0$, but $\alpha > 0$, where $\alpha = -0.1$, $v = 0.3$.

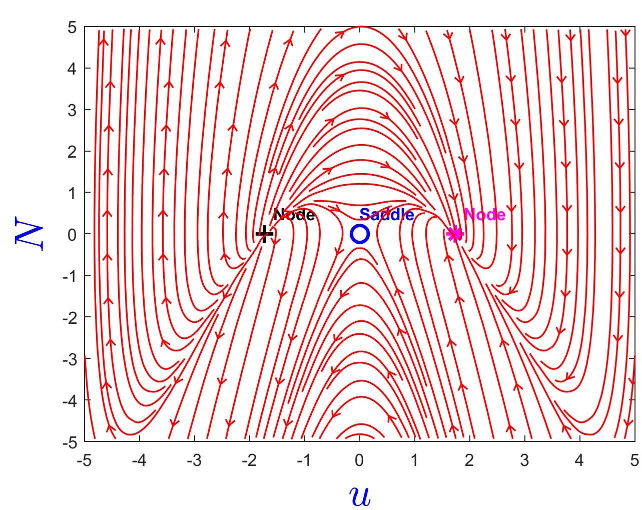

Figure 4: Phase behavior of Eq. (33) when $v > 0$, but $\alpha > 0$ or $v < 0$, but $\alpha < 0$, where $\alpha = 1$, $v = 3$.

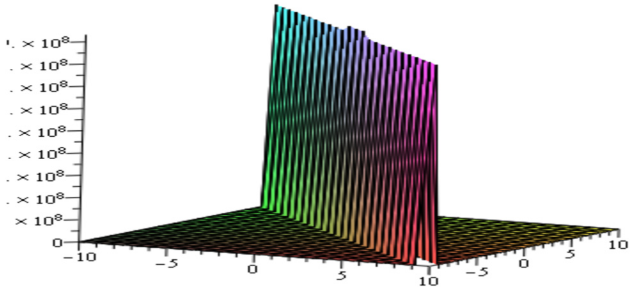

Figure 5: Shape of sharp soliton of u_{11} for the values of the parameters $k = -1$, $b = 5$, $A = 2$, $B = 1$, $C = 1$, $E = 3$, and $v = 1$ within the range $-10 \leq x, t \leq 10$.

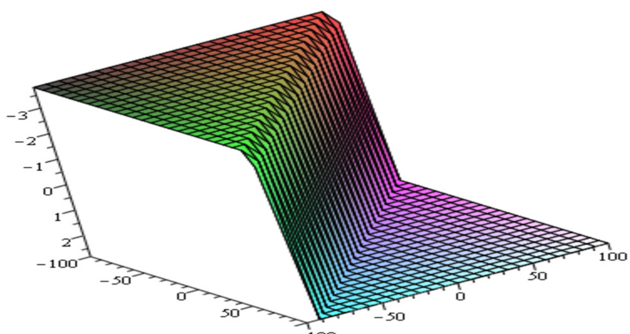

Figure 6: Shape of kink soliton of u_{17} for the values of the parameters, $k = -1$, $b = 1$, $A = 2$, $B = 1$, $C = 1$, $E = 3$, and $v = 1$ within the interval $-100 \leq x, t \leq 100$.

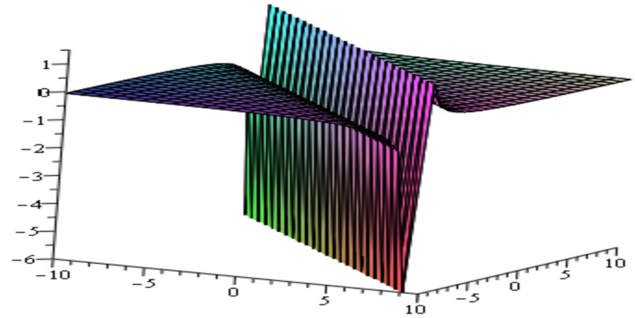

Figure 8: Shape of the singular kink soliton of u_{55} for the values of the parameters, $l = 20$, $m = 30$, $b = 5$, $A = 2$, $B = 2$, $C = 1$, $E = -1$, and $v = 1$ within the interval $-10 \leq x, t \leq 10$.

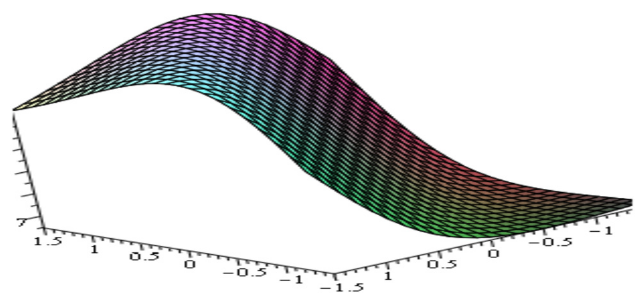

Figure 9: Shape of flat kink soliton of u_{57} for the values of the parameters, $A = 2$, $B = 5$, $C = 1$, $E = 3$, and $v = 1$ within the interval $-1.5 \leq x, t \leq 1.5$.

simplifying and clarifying these solutions, providing a visual means to understand the dynamics involved in such phenomena. The graphical illustrations of some of the derived traveling wave solutions are presented in Figures 5–10, generated using computational tools like Maple. These visualizations help to clearly demonstrate the behavior of the solutions over time and space, offering valuable insights into their properties, such as amplitude, frequency, and wave speed. By employing graphical tools, the intricate nature of the solutions becomes more accessible, allowing for a deeper understanding of the nonlinear dynamics they represent.

The graphical analysis reveals that when the velocity of the traveling wave is set to 1, the resulting graphs of the obtained solutions clearly exhibit characteristics of an exact kink soliton, a singular kink soliton, and singular periodic solutions. However, as the velocity of the traveling wave deviates from this value, the shapes of the graphs

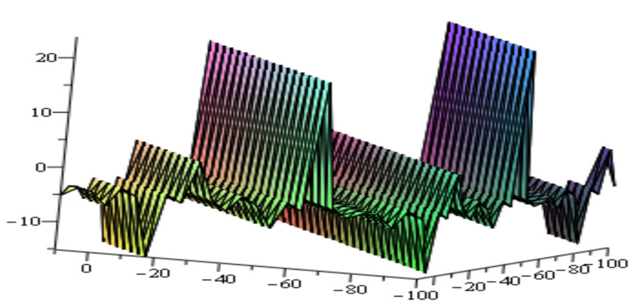

Figure 7: Shape of the singular periodic soliton of u_{18} for the values of the parameters, $k = -1$, $b = 1$, $A = 2$, $B = 5$, $C = 3$, $E = 3$, and $v = 1$ within the interval $-100 \leq x, t \leq 10$.

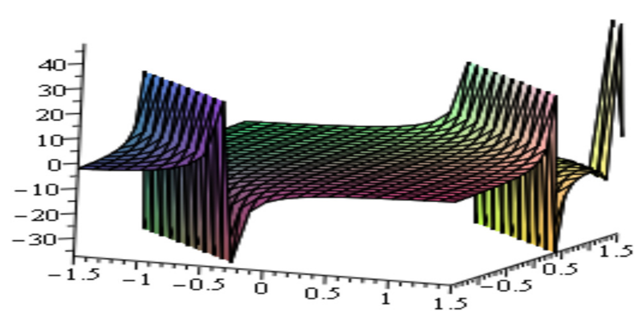

Figure 10: Shape of periodic kink soliton of u_{59} for the values of the parameters, $A = 2$, $B = 5$, $C = 1$, $E = -3$, and $v = 1$ within the interval $-1.5 \leq x, t \leq 1.5$.

begin to deform, indicating changes in the structure and behavior of the solutions.

In soliton dynamics, these soliton solutions provide important insights and have practical applications across multiple disciplines. Sharp solitons play a critical role in optical fibers by preserving the shape of light pulses over long distances, which is essential for high-speed data transmission [34,54]. They also aid in understanding solitary waves in shallow water systems. Kink solitons, characterized by their smooth transitions, are important in field theory and particle physics for representing domain walls, and they also describe gradual changes in light intensity in nonlinear optics [4,55]. Singular periodic solitons, which feature periodic patterns with singularities, are valuable for studying wave behavior in media with periodic structures and defects, such as optical lattices and plasma environments [2,56]. Singular kink solitons, which combine kink-like transitions with singular features, are useful for analyzing shock waves and boundary interfaces. Flat kink solitons, exhibiting constant amplitude with sharp transitions, are applied in examining phase changes in materials and stable light pulses in nonlinear optics [8,9]. Finally, periodic kink solitons, with their periodic structures and kink-like transitions, are employed to investigate wave propagation in periodic media and lattice systems, enhancing the understanding of phenomena in condensed matter physics [28,30]. Each soliton type contributes to a comprehensive understanding of complex systems and supports the advancement of both theoretical research and practical applications.

6 Conclusion

In conclusion, this study highlights the effectiveness of the new generalized (G'/G) expansion method for obtaining soliton solutions of the KG and STO equations, both of which are pivotal in numerous fields, including relativistic physics, quantum field theory, nonlinear optics, dispersive wave phenomena, and plasma physics. The method is shown to generate a wide range of soliton solutions, including rational, trigonometric, and hyperbolic forms, each reflecting distinct physical properties of the nonlinear systems being studied. A comparative analysis indicates that several of the obtained soliton solutions, particularly the sharp soliton and periodic soliton, are novel contributions to the literature. The study demonstrates the robustness and adaptability of the (G'/G) expansion method, suggesting that it is a powerful technique for exploring soliton dynamics in a variety of NLEEs. Additionally, its compatibility with computer algebra systems makes it highly practical for both theoretical analysis and real-world applications. This approach broadens the toolkit available for researchers in nonlinear dynamics, opening new directions for future exploration and discovery in this area. Furthermore, bifurcation analysis offers a detailed and accurate understanding of the dynamics associated with these two nonlinear PDEs. It sheds light on solution patterns, shifts in stability, and how the system responds to changes in parameters. This information is crucial for developing a thorough and well-represented phase portrait. Additionally, such analyses pave the way for applying these mathematical models effectively to address real-world challenges, highlighting their practical significance. This study provides significant insights into soliton dynamics within these equations; however, it possesses several limitations. Its usefulness is predominantly restricted to idealized situations, and extending its use to non-integrable systems poses difficulties. Moreover, the practical uses are constrained. Furthermore, empirical confirmation of theoretical findings is frequently absent. Subsequent research must confront these limitations to improve practical applicability and substantiate theoretical conclusions.

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