Research Article

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Bilinear form and soltion solutions for (3+1)-dimensional negative-order KdV-CBS equation

https://doi.org/10.1515/phys-2025-0151
received January 14, 2025; accepted April 04, 2025

Abstract: This article investigates a significant mathematical model for multiwave interactions. For the first time, the bilinear form of the (3+1)-dimensional negative-order Korteweg–de Vries (KdV)-Calogero–Bogoyavlenskii–Schiff (CBS) equation is derived using binary Bell polynomials, and 1, 2, and 3-soliton solutions are obtained through this bilinear form. These solutions are further visualized *via* 3D and 2D plots representations. This study fills a research gap in this direction and demonstrates that the results can significantly enhance the efficiency of obtaining diverse solutions for the (3+1)-dimensional negative-order KdV-CBS equation. It is anticipated that these solutions will not only deepen our understanding of the physical phenomena associated with the equation but also reveal more complex physical behaviors, thereby advancing analytical studies on solutions to other nonlinear partial differential equations.

Keywords: higher order negative-order KdV-CBS equation, Bell polynomials, soliton solutions, Hirota method

1 Introduction

Nonlinear partial differential equations (NLPDEs) describe complex relationships between variables, capturing intricate dynamic behaviors that linear models cannot represent. Unlike linear equations, which often allow for superimposed solutions, nonlinear equations give rise to phenomena such as bifurcations, chaos, and solitons due to the interactions between variables. These characteristics make NLPDEs highly valuable in modeling a broad spectrum of physical systems across scientific and engineering disciplines.

In fluid mechanics, NLPDEs are critical for describing fluid behavior, accounting for factors like viscosity, turbulence, and nonlinear wave interactions. They provide essential insights into wave propagation and help explain extreme events like rogue waves-unexpected, large, and dangerous oceanic waves [1]. Similarly, in optical fiber communication, NLPDEs are crucial for optimizing data transmission by modeling the nonlinear effects that occur in optical fibers. The nonlinear Schrödinger equation, for example, describes light pulse propagation, helping engineers design systems to minimize signal distortion and loss. Solitons, which are stable, self-reinforcing wave packets, have proven particularly useful in improving the reliability and efficiency of optical communication systems, as they maintain their shape over long distances without dispersing, thereby preserving data integrity.

In plasma physics [2], NLPDEs are indispensable for understanding the complex behavior of plasmas, a state of matter consisting of charged particles influenced by electromagnetic forces. This understanding is vital in fusion research, where the goal is to harness nuclear fusion as a clean, virtually limitless energy source. NLPDEs are used to model plasma stability, confinement, and the interactions of plasma waves under extreme conditions, offering critical insights for the development of fusion reactors capable of sustaining energy-producing reactions.

Beyond these applications, NLPDEs are also used in meteorology, oceanography, and biology. In meteorology, they model atmospheric dynamics, improving the prediction of weather events such as storms. In oceanography, they aid in the study of wave dynamics and ocean circulation, enhancing our understanding of climate change. In biology, NLPDEs are employed to model the spread of diseases and population dynamics, assisting researchers in devising strategies for epidemic control.

In summary, NLPDEs are indispensable tools for modeling complex systems across multiple scientific and engineering fields. Their ability to describe nonlinear interactions makes them critical for advancing our understanding of fluid mechanics, optical communication, plasma physics, and beyond. As research progresses, the continued development and application of NLPDEs will likely yield deeper insights into complex systems, fostering innovation and discovery across diverse disciplines. With the

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advancement of science, scholars have discovered numerous methods for solving partial differential equations. However, no single technique has been proven universally successful in providing exact solutions for every model. In fact, a technique that performs well for one model may be ineffective for another. For instance, when studying localized solutions of NLPDEs, successfully derived various localized solutions for the Davey–Stewartson system, including dromions and rogue waves [3] using the truncated Painlevé analysis method. These solutions have demonstrated wide applications in fluid dynamics, oceanography, and nonlinear optics [4–6]. By employing symbolic computation and the Hirota method, they also successfully derived solutions for the variable coefficient higher-order Schrödinger equation, incorporating third-order dispersion, self-steepening, and stimulated Raman scattering effects. Other methods for solving partial differential equations include the tanh function method [7], Darboux transformation [8], Hirota bilinear method [9], bilinear neural network method [10], long-wave limit method [11], and Bäcklund transformation [12]. These methods can effectively aid in understanding and researching NLPDEs, enhancing our comprehension of nonlinear systems.

NLPDEs describe complex relationships between variables, revealing intricate dynamic behaviors. These equations are extensively employed in fields such as fluid mechanics, optical fiber communication, and plasma physics, and they play a crucial role in scientific and engineering research. By modeling phenomena such as wave propagation, turbulence, and soliton interactions, NLPDEs provide insights into the underlying mechanisms of various physical systems. Their applications range from predicting ocean waves and weather patterns to enhancing the performance of optical communication systems and understanding plasma behavior in fusion research [13–15]. Below are two important NLPDEs:

The integrable Korteweg–deVries (KdV) equation given as follows:

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1}$$

The KdV equation is a significant physical model that describes the propagation of shallow water waves. Its applications extend beyond shallow water phenomena to various other physical systems. For instance, in plasma physics, the KdV equation is predominantly employed to describe ion-acoustic waves and other nonlinear wave phenomena. In optical fiber communication, it is utilized to elucidate the propagation and interaction of optical pulses. Across these diverse physical contexts, the KdV equation and its variants offer a theoretical foundation for understanding and predicting nonlinear wave phenomena [16,17].

The integrable (2+1)-dimensional Calogero–Bogoyavlenskii–Schiff (CBS) equation given as

$$u_t + 4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy} = 0. \tag{2}$$

The nonlinear CBS equation describes the interaction between Riemann waves propagating along the y-axis and long waves propagating along the x-axis in two-dimensional space [18,19]. On the basis of Eqs (1) and (2), Wazwaz, using the negative-order hierarchy, obtained the negative-order KdV equation and the negative-order CBS equation as follows [20]:

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xx} = 0, \tag{3}$$

$$u_{xxxt} + 4u_xu_{xt} + 2u_{xx}u_t + u_{xy} = 0. \tag{4}$$

In this article, we study the (3+1)-dimensional negative-order KdV-CBS equation [21,22]:

$$u_{xt} + u_{xxxy} + 4u_xu_{xy} + 2u_{xx}u_y + \lambda u_{xx} + \mu u_{xy} + \nu u_{xz} = 0, \tag{5}$$

where $u = u(x, y, z, t)$, λ, μ, ν are arbitrary constants. Eq. (5) is derived by Wazwaz by combining Eqs (3) and (4). It has been shown to pass the Painlevé integrability test. In addition, using a simple trial function method, 1-soliton and 2-soliton solutions were obtained, but the bilinear form of the equation was not derived. Guo [23] extended this equation to derive a novel (3+1)-dimensional equation, subsequently obtaining its bilinear form. Eq. (5) is an important mathematical and physical model for studying multiwave interactions. For $\mu = \nu = 0$, Eq. (5) will be reduced to Eq. (3), and for $\lambda = \nu = 0$, Eq. (5) will be reduced to Eq. (4). This model is not only significant in traditional fields such as fluid mechanics and plasma physics but also has potential applications in emerging fields such as optics, condensed matter physics, and quantum mechanics. Therefore, the study of (5) is of great importance and has substantial research value.

Although there have been many profound studies on Eq. (5), there are still several unresolved issues persist. It is worth noting that currently, the bilinear form of Eq. (5) derived based on Bell polynomials has not been mentioned in relevant research. Therefore, in this article, we primarily investigate the bilinear form of Eq. (5) based on Bell polynomials and use this form to obtain soliton solutions. The results obtained contribute to more effectively obtaining various solutions of Eq. (5), enriching its physical significance. Moreover, these findings can be applied to a wider range of NLPDEs, thereby advancing research on exact solutions of such equations.

The article is structured as follows: In Section 2, we employed a special transformation to meticulously derive the bilinear form of the (3+1)-dimensional negative-order KdV-CBS equation. In Section 3, we first further derived the

N-soliton solutions of the equation *via* the Hirota bilinear method. On the basis of the multisoliton solutions, we obtained the solutions for 1, 2, and 3 soliton solutions and depicted them using 3D and 2D plots. Finally, in Section 4, we summarized our work.

2 Bilinear form of the (3+1)-dimensional negative-order KdV-CBS equation

In this section, we employ Bell polynomials to transform Eq. (5) into a bilinear equation, following the methodologies outlined in previous studies [24–26]. The use of Bell polynomials is particularly advantageous in the context of NLPDEs, as they enable a systematic approach to converting nonlinear equations into a bilinear form. This transformation allows us to simplify the original nonlinear equation, making it easier to apply various analytical techniques, such as the Hirota method, for finding exact solutions. By converting the equation into a bilinear form, we can more easily identify soliton solutions, analyze the stability of these solutions, and investigate the interactions between them. In addition, this approach facilitates the examination of the underlying structure and properties of the equation, revealing symmetries and conservation laws that may not be apparent in its original nonlinear form.

Assuming

$$u = cq_x \tag{6}$$

in (5), and substituting (6) into (5), we obtain the following result:

$$\begin{aligned} &cq_{xxt} + cq_{xxxxy} + 4c^2 q_{xx}q_{xxy} + 2c^2 q_{xxx}q_{xy} + \lambda cq_{xxx} \\ &\quad + \mu cq_{xxy} + c\nu q_{xxz} = 0, \end{aligned} \tag{7}$$

where $q = q(x, y, z, t)$ and c are arbitrary functions.

By rearranging Eq. (7), we obtain the following equation:

$$\begin{aligned} &q_{xxt} + \frac{2}{3}(q_{xxxxy} + 3cq_{xxx}q_{xy} + 3cq_{xx}q_{xxy}) + \frac{1}{3} \\ &\quad \times (6cq_{xx}q_{xxy} + q_{xxxxy}) + \lambda q_{xxx} + \mu q_{xxy} + \nu q_{xxz} = 0. \end{aligned} \tag{8}$$

By integrating (8) once with respect to the variable x, and taking the integration constant to be zero, we obtain the following result:

$$\begin{aligned} &q_{xt} + \frac{2}{3}(q_{xxxy} + 3cq_{xx}q_{xy}) + \frac{1}{3}\partial_x^{-1}(3cq_{xx}q_{xx} + q_{xxxx})_y \\ &\quad + \lambda q_{xx} + \mu q_{xy} + \nu q_{xz} = 0. \end{aligned} \tag{9}$$

In (9), let

$$3cq_{xx}q_{xx} + q_{xxxx} = q_{x\tau}, \tag{10}$$

where τ is an auxiliary variable.

When $c = 1$, (9) can be transformed into the following $\mathcal{P}$-polynomial [27–29]:

$$\begin{cases} \mathcal{P}_{xxxx} - \mathcal{P}_{x\tau} = 0, \\ \mathcal{P}_{xt} + \dfrac{2}{3}\mathcal{P}_{xxxy} + \dfrac{1}{3}\mathcal{P}_{y\tau} + \lambda\mathcal{P}_{xx} + \mu\mathcal{P}_{xy} + \nu\mathcal{P}_{xz} = 0, \end{cases} \tag{11}$$

where $\mathcal{P}_{xx} = q_{xx}$, $\mathcal{P}_{x\tau} = q_{x\tau}$, $\mathcal{P}_{xy} = q_{xy}$, $\mathcal{P}_{xz} = q_{xz}$, $\mathcal{P}_{xt} = q_{xt}$, $\mathcal{P}_{y\tau} = q_{y\tau}$, and $\mathcal{P}_{4x} = q_{4x} + 3q_{2x}^2$.

Using the relationship between the $\mathcal{P}$ polynomial and bilinear equation, the bilinear equation for (5) is as follows:

$$\begin{cases} (D_x^4 - D_xD_\tau)f\cdot f = 0, \\ \left(D_xD_t + \dfrac{2}{3}D_x^3D_y + \dfrac{1}{3}D_yD_\tau + \lambda D_x^2 + \mu D_xD_y + \nu D_xD_z\right)f \\ \qquad \cdot f = 0, \end{cases} \tag{12}$$

where $q = 2\ln f$, $f = f(x, y, z, t, \tau)$. By introducing the variable τ, the bilinear form of the equation can be fully derived.

The bilinear operator D is defined as follows:

$$\begin{aligned} &D_x^{n_1}D_y^{n_2}D_z^{n_3}D_\tau^{n_4}D_t^{n_5}(f\cdot f) \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n_1}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{n_2} \\ &\quad \times \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^{n_3}\left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial \tau'}\right)^{n_4}\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{n_5} \\ &\quad \times f(x, y, z, \tau, t)f(x', y', z', \tau', t')_{x=x',y=y',z=z',\tau=\tau',t=t'}, \end{aligned} \tag{13}$$

where n_1, n_2, n_3, n_4, and n_5 are nonnegative integers.

By using the transformation

$$q = 2\ln f \Leftrightarrow u = 2(\ln f)_x, \tag{14}$$

we transform Eq. (12) into its corresponding bilinear form:

$$\begin{aligned} &2(f_{x,x,x,x})f - 8(f_{x,x,x})(f_x) + 6(f_{x,x})^2 - 2(f_{\tau,x})f \\ &\quad + 2(f_\tau)(f_x) = 0, \\ &2(f_{t,x})f - 2(f_x)(f_t) + \frac{4(f_{x,x,x,y})f}{3} - \frac{4(f_{x,x,x})(f_y)}{3} \\ &\quad - 4(f_{x,x,y})(f_x) + 4(f_{x,x})(f_{x,y}) + \frac{2(f_{\tau,y})f}{3} - \frac{2(f_\tau)(f_y)}{3} \\ &\quad + \lambda(2(f_{x,x})f - 2(f_x)^2) + \mu(2(f_{x,y})f - 2(f_x)(f_y)) \\ &\quad + \nu(2(f_{x,z})f - 2(f_x)(f_z)) = 0. \end{aligned} \tag{15}$$

3 Soltion solutions with (3+1)-dimensional negative-order KdV-CBS equation

Wazwaz [20] employed a simple trial function method to construct the 1-soliton and 2-soliton solutions of Eq. (5). In this section, we utilize the obtained bilinear transformation to present the expression for the N-soliton solution of Eq. (5) and specifically construct the 1-soliton, 2-soliton, and 3-soliton solutions. Here, we provide the detailed expressions for them:

$$\begin{cases} f = f_{N\text{-soliton}} = \sum\limits_{r=0,1} \exp\left(\sum\limits_{\chi=1}^{N} r_\chi \xi_\chi + \sum\limits_{1\leqslant\chi<j}^{N} r_\chi r_j A_{\chi j} \right), \\ \xi_\chi = k_\chi \tau + \mu_\chi t + \eta_\chi x + \omega_\chi y + \lambda_\chi z + \eta_\chi^0, \end{cases} \tag{16}$$

where $k_\chi, \mu_\chi, \eta_\chi, \omega_\chi, \lambda_\chi$, and η_χ^0 are arbitrary constants, $r_\chi, r_j = 0, 1$.

3.1 1-soltion solution

To investigate the 1-soltion solution, setting $N = 1$, Eq. (16) can be expressed in the following manner:

$$f = 1 + e^{\xi_1}, \tag{17}$$

where $\xi_1 = k_1\tau + \mu_1 t + \eta_1 x + \omega_1 y + \lambda_1 z + \eta_0^0$.

By substituting Eq. (17) into bilinear Eq. (12), we obtain the relation among k_1 and μ_1 as follows:

$$k_1 = \eta_1^3, \mu_1 = -\eta_1^2\omega_1 - \eta_1\lambda - \mu\omega_1 - \nu\lambda_1. \tag{18}$$

By setting $\lambda = 1$, $\mu = 1$, $\nu = 1$, $\eta_1 = 1$, $\lambda_1 = 2$, $\omega_1 = 3$, and $\eta_0^0 = 0$, substituting Eq. (17) into Eq. (14), we derive the 1-soliton solution of Eq. (5).

As shown in Figure 1, the 1-soliton propagates in the positive direction along the Y-axis. During propagation, the soliton can maintain its shape and speed without significant attenuation or deformation, even over long distances. The soliton model can help understand and predict the behavior of catastrophic water waves, such as tsunamis.

3.2 2-soltion solution

To investigate the 2-soltion solution, setting $N = 2$, Eq. (16) can be expressed in the following manner:

$$f = 1 + e^{\xi_1} + e^{\xi_2} + A_{12} e^{\xi_1+\xi_2}, \tag{19}$$

where $\xi_i = k_i\tau + \mu_i t + \eta_i x + \omega_i y + \lambda_i z + \eta_i^0$.

By substituting Eq. (19) into bilinear Eq. (12), we obtain the relation among k_i and μ_i as follows:

$$k_i = \eta_i^3, \mu_i = -\eta_i^2\omega_i - \eta_i\lambda - \mu\omega_i - \nu\lambda_i, A_{12} = \frac{(\eta_1-\eta_2)^2}{(\eta_1+\eta_2)^2}, \tag{20}$$

where $i = 1, 2$.

Letting the parameters $\lambda = 1$, $\mu = 2$, $\nu = 2$, $\eta_1 = 0.5$, $\eta_2 = 1$, $\lambda_1 = 0.8$, $\lambda_2 = 0.5$, $\omega_1 = -0.8$, $\omega_2 = 1$, and $\eta_i^0 = 0$, substituting Eq. (19) into Eq. (14), we obtain the 2-soliton solution of Eq. (5).

As shown in Figure 2, at $t = -15$, two solitons propagate independently, each preserving its distinct shape. When they intersect, they undergo intense nonlinear interactions, which cause notable changes in their waveforms and may lead to overlapping and the formation of intricate patterns. By $t = 15$, solitons continue their travel, returning to their original shapes as they move apart. This dynamic

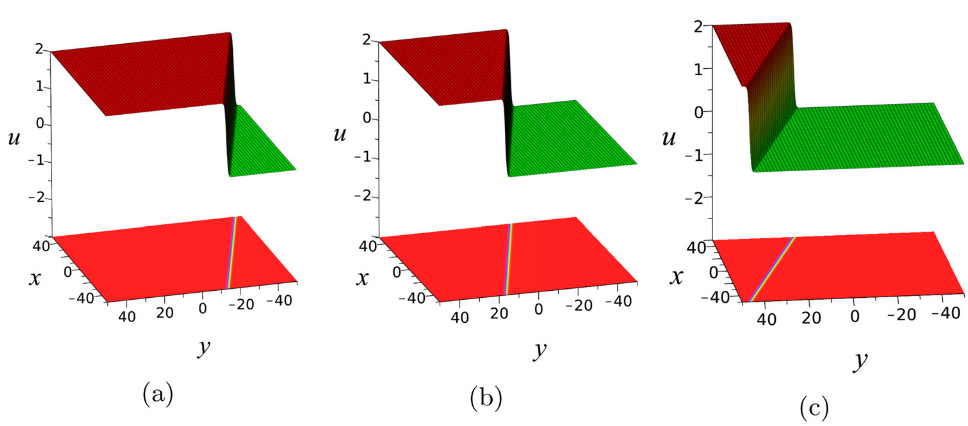

Figure 1: 1-soliton wave solution. (a)–(c) show the three-dimensional plot and density plot of the (x, y) plane for $t = -15$, 0, and 15, respectively.

behavior highlights the solitons' resilience and interaction characteristics.

3.3 3-soltion solution

To investigate the 3-soltion solution, setting $N = 3$, Eq. (16) can be expressed in the following manner:

$$f = 1 + e^{\xi_1} + e^{\xi_2} + e^{\xi_3} + A_{12}e^{\xi_1+\xi_2} + A_{13}e^{\xi_1+\xi_3} + A_{23}e^{\xi_2+\xi_3} + A_{12}A_{13}A_{23}e^{\xi_1+\xi_2+\xi_3}, \quad (21)$$

where $A_{ij} = \frac{(\eta_i - \eta_j)^2}{(\eta_i + \eta_j)^2}$.

Substituting Eq. (21) into bilinear Eq. (12), letting the parameters $\eta_1 = 1.4$, $\eta_3 = 0.4$, $\eta_2 = 1.2$, $\omega_1 = 1$, $\omega_3 = 2$, $\omega_2 = 2.5$, $\lambda_1 = 1$, $\lambda_2 = 2.5$, $\lambda_3 = 3$, $\lambda = 1$, $\mu = 1$, $\nu = 2$, and $\eta_i^0 = 0$, and substituting Eq. (21) into Eq. (14), we obtain the 3-soliton solution of Eq. (5).

According to Figure 3, the dynamic behavior of the 3-soliton solution closely resembles that of the 2-soliton solution. The solitons maintain their shapes before and after interaction, demonstrating stability and persistence in their form throughout the process. This behavior illustrates the characteristic resilience of solitons in maintaining their structure despite complex interactions.

4 Conclusions

In this article, we examined the (3+1)-dimensional negative-order KdV-CBS equation, a widely used mathematical and physical model for describing multiwave interactions with significant physical implications. This equation generalizes the KdV equation, extending its applicability to more complex systems, including interactions in higher

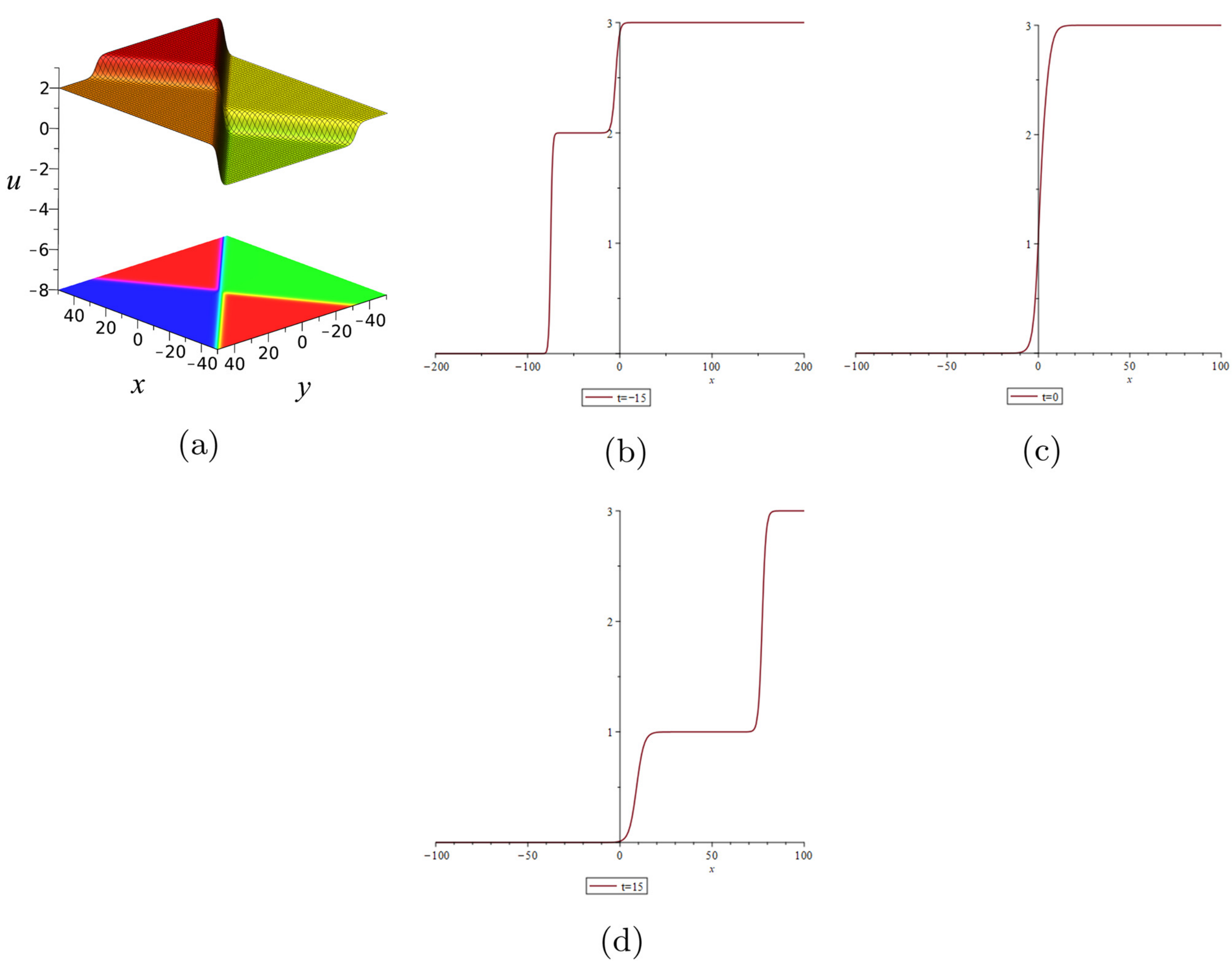

Figure 2: 2-soliton wave solution. (a) show the three-dimensional plot and density plot of the (x, y) plane. (b)–(d) show the two-dimensional plot of the x plane for $t = -15$, 0, and 15, respectively.

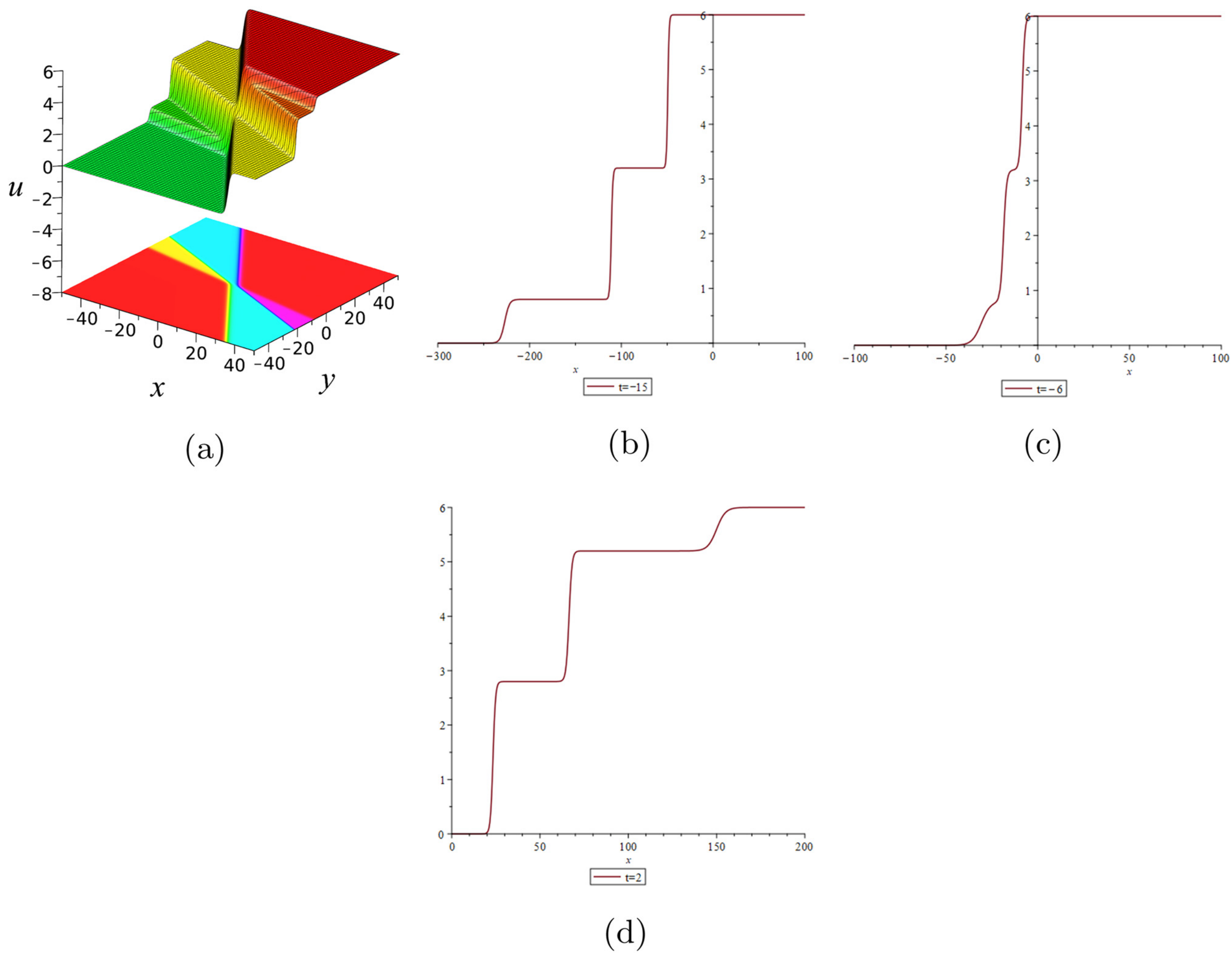

Figure 3: 3-soliton wave solution. (a) show the three-dimensional plot and density plot of the (x, y) plane. (b)–(d) show the two-dimensional plot of the x plane for $t = -15$, -6, and 2, respectively.

dimensions. It provides a valuable framework for understanding phenomena such as wave propagation, soliton behavior, and nonlinear interactions in fields like fluid dynamics, plasma physics, and optical systems.

We observed that for specific parameter values, the equation reduces to well-known forms: when $\mu = \nu = 0$, (5) simplifies to (3), and when $\lambda = \nu = 0$, (5) simplifies to (4). These reductions offer insights into the relationships between different forms of the equation and allow for a clearer analysis of how various terms influence the system's dynamics.

A major contribution of this work is the application of binary Bell polynomials to the (3+1)-dimensional negative-order KdV-CBS equation. This approach enables the transformation of the equation into its bilinear form, following a detailed derivation process. The bilinear form is especially beneficial as it simplifies the nonlinear terms, facilitating analysis and solution finding. Through this transformation, we not only derive explicit solutions but also gain a deeper understanding of the equation's structure and behavior, thus opening new avenues for studying complex nonlinear equations.

By using the bilinear form, we successfully derived 1-soliton, 2-soliton, and 3-soliton solutions. Solitons, which are stable wave packets that maintain their shape during propagation, play a crucial role in many physical systems. The soliton solutions obtained demonstrate the intricate behaviors that arise from the equation, including interactions between solitons and the patterns they form. Studying soliton solutions is particularly important as solitons appear in various fields, ranging from water waves to optical fibers. Understanding their interactions in higher dimensions can offer valuable insights into more complex systems, such as wave interactions in multidimensional media.

The results of this study contribute to a broader understanding of nonlinear wave equations and soliton theory.

By applying binary Bell polynomials to this class of equations, we demonstrated a systematic approach to their analysis and solution. This method can be extended to other nonlinear equations, particularly those in higher dimensions, where traditional solution methods may prove less effective.

Future research could expand upon these findings by exploring the application of binary Bell polynomials to more complex equations, such as those involving higher-order derivatives or additional nonlinear terms. Furthermore, the soliton solutions derived in this article could be further studied to understand their stability, interactions, and potential for describing real-world phenomena. For instance, researchers might examine how solitons behave under perturbations or how they evolve over longer timescales.

In conclusion, this work presents a thorough analysis of the (3+1)-dimensional negative-order KdV-CBS equation using binary Bell polynomials, contributing to the growing body of knowledge on multiwave interactions and soliton theory. We hope that the results obtained here will be valuable to researchers interested in nonlinear wave equations and their applications across various scientific and engineering fields. The methods and solutions presented in this article pave the way for future studies on higher-dimensional systems and more complex mathematical models.

Funding information: The author states no funding involved.

Author contributions: The author has accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The author states no conflict of interest.

Data availability statement: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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