Research Article

Chiranahalli Vijaya Darshan Kumar, Doddabhadrappla Gowda Prakasha*, and Nasser Bin Turki

Exploring the dynamics of fractional-order nonlinear dispersive wave system through homotopy technique

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Abstract: In this article, we study the time-dependent two-dimensional system of Wu–Zhang equations of fractional order in terms of the Caputo operator, which describes long dispersive waves that minimize and analyze the damaging effects caused by these waves. This article centers on finding soliton solutions of a non-linear (2 + 1)-dimensional time-fractional Wu–Zhang system, which has become a significant point of interest for its ability to describe the dynamics of long dispersive gravity water waves. The semi-analytical method called the q-homotopy analysis method in amalgamation with the Laplace transform is applied to uncover an analytical solution for this system of equations. The outcomes obtained through the considered method are in the form of a series solution, and they converge swiftly. The results coincide with the exact solution are portrayed through graphs and carried out numerical simulations which shows minimum residual error. This analysis shows that the technique used here is a reliable and well organized, which enhances in analyzing the higher-dimensional non-linear fractional differential equations in various sectors of science and engineering.

Keywords: fractional Wu–Zhang equation, q-homotopy analysis transform method, Caputo-fractional operator, Laplace transform

*** Corresponding author: Doddabhadrappla Gowda Prakasha**, Department of Mathematics, Davangere University, Shivagangotri, Davangere - 577007, India,
e-mail: prakashadg@davangereuniversity.ac.in, prakashadg@gmail.com
Chiranahalli Vijaya Darshan Kumar: Department of Mathematics, Davangere University, Shivagangotri, Davangere - 577007, India,
e-mail: darshancv1504@gmail.com
Nasser Bin Turki: Department of Mathematics, College of Science, King Saud University, P.O. Box - 2455, Riyadh - 11451, Saudi Arabia,
e-mail: nassert@ksu.edu.sa

1 Introduction

Nonlinear differential equations play a significant role in understanding and modeling of complex real-world phenomena. Recent studies have focused on obtaining accurate numerical solutions for nonlinear partial differential equations, which is crucial in various science and engineering disciplines. The differential equations with fractional order are called fractional differential equations (FDEs). The fractional calculus theory has been developed as a vast and continuously evolving subject. Many researchers assumed that it is a well-developed topic in mathematics and have been working on it till now. Due to the ideas of German mathematician *Leibniz* and *L-Hospital*, the theory of fractional calculus came into existence about 300 years ago [1–4]. The main benefit of studying FDE is that it gives solutions between intervals, which aids us in examining the results more understandably. The primary significance of FDEs is analyzing the behavior of complex systems with memory effects and genetic characteristics that also possess uncertainty properties. Hence, FDEs are more utilized in modeling the natural and complex phenomena in the real world such as thermodynamics [5], fluid dynamics [6], chaos behavior [7], biology [8], chemical kinetics [9], cosmology [10], financial models [11], epidemiology [12], shallow water waves [13], and many others [14–22]. Recently, there have been several modified fractional derivatives such as Riemann–Liouville (RL), Caputo, Caputo–Fabrizio, Hilfer derivative, and Ataugana–Baleanu derivative [23]. Here, we have utilized the Caputo derivative as it offers many advantages, such as it is non-local in behavior and well suitable for initial value problems. It is bounded and provides smoother behavior compared to other fractional operators. These characteristics of the Caputo operator make it easier to model systems with initial conditions. Moreover, solving non-linear FDEs is challenging and requires some computational work. As a result, many researchers have employed diverse methods to solve FDEs that arise in various phenomena. Gao et al. [24] represented the one-dimensional Cauchy problem using the

Atangana–Baleanu operator and presented the numerical solutions of a nonlinear system that arose in thermoelasticity. Alwehebi et al. [25] solved the time-fractional Burger's equation using the variational iteration method (VIM) with the aid of Maple software. Ali abd Maneea [26] investigated (1 + 1)-dimensional time-fractional coupled nonlinear Schrödinger equations to obtain approximate solitary and periodic wave solutions using the optimal homotopy analysis method (OHAM). Also, many authors have studied various FDEs using different numerical and analytical methods [27–35].

Recently, researchers have progressively focused on nonlinear wave equations due to their wide-ranging applications across various scientific fields. The classical Wu–Zhang (WZ) system is a significant nonlinear partial differential equation with more significance in obtaining soliton solutions. Soliton theory relies more on searching for accurate and numerical solutions to nonlinear equations, particularly for traveling waves. Wu and Zhang developed a trio of equations for simulating nonlinear and dispersive long gravity waves propagating in two distinct horizontal directions in shallow waters available in oceans of equal depth [36]. In this system of equations, the first equation depicts a (2 + 1)-multidimensional dispersive longitudinal wave. The other non-linear equations are crucial for marine and engineering professionals to represent in harbor and coastal development projects. The equations are presented as follows:

$$\begin{aligned}
&\frac{\partial u(\zeta,\omega,t)}{\partial t} + u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial \zeta} + v(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial \omega} + \frac{\partial w(\zeta,\omega,t)}{\partial \zeta} = 0,\\
&\frac{\partial v(\zeta,\omega,t)}{\partial t} + u(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial \zeta} + v(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial \omega} + \frac{\partial w(\zeta,\omega,t)}{\partial \omega} = 0,\\
&\frac{\partial w(\zeta,\omega,t)}{\partial t} + \frac{\partial (u(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial \zeta} + \frac{\partial (v(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial \omega} + \frac{1}{3}\left\{\frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta^3} + \frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta\partial \omega^2} + \frac{\partial^3 v(\zeta,\omega,t)}{\partial \zeta^2\partial \omega} + \frac{\partial^3 v(\zeta,\omega,t)}{\partial \omega^3}\right\} = 0.
\end{aligned} \tag{1}$$

Here, the system under consideration comprises three parameters, namely, u represents the surface velocity of water in the ζ-direction, v represents surface velocity in the direction of ω-axis, and w represents the water's elevation. $\frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta^3}$, $\frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta\partial \omega^2}$, $\frac{\partial^3 v(\zeta,\omega,t)}{\partial \zeta^2\partial \omega}$, and $\frac{\partial^3 v(\zeta,\omega,t)}{\partial \omega^3}$ are the dissipation terms that provide damping and the other nonlinear terms in Eq. (1) that acts as a stabilising agent in the propagation of dispersive long waves. The explicit solutions of this system have been beneficial for coastal and civil engineers working in the harbor and coastal design projects. Moreover, the explicit solutions along with the numerical results of this system are of fundamental interest in fluid dynamics. Several researchers studied this classical WZ system using different methods, which include reduced differential transform method [37], modified Adomian decomposition method [38], modified VIM [39], homotopy perturbation method [40], and others [41,42].

In this article, we are examining a time-dependent (2 + 1)-dimensional non-linear WZ system of equation of fractional order γ [43,45], which is given by

$$\begin{aligned}
&\mathcal{D}_t^{\gamma} u(\zeta,\omega,t) + u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial \zeta} + v(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial \omega} + \frac{\partial w(\zeta,\omega,t)}{\partial \zeta} = 0,\\
&\mathcal{D}_t^{\gamma} v(\zeta,\omega,t) + u(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial \zeta} + v(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial \omega} + \frac{\partial w(\zeta,\omega,t)}{\partial \omega} = 0,\\
&\mathcal{D}_t^{\gamma} w(\zeta,\omega,t) + \frac{\partial (u(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial \zeta} + \frac{\partial (v(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial \omega} + \frac{1}{3}\left\{\frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta^3} + \frac{\partial^3 u(\zeta,\omega,t)}{\partial \zeta\partial \omega^2} + \frac{\partial^3 v(\zeta,\omega,t)}{\partial \zeta^2\partial \omega} + \frac{\partial^3 v(\zeta,\omega,t)}{\partial \omega^3}\right\} = 0,
\end{aligned} \tag{2}$$

with initial conditions

$$\begin{aligned}
u(\zeta,\omega,0) &= \frac{d+ac}{b} + \frac{2\sqrt{3}}{3} b\tanh(b\zeta + c\omega),\\
v(\zeta,\omega,0) &= a + \frac{2\sqrt{3}}{3} c\tanh(b\zeta + c\omega),\\
w(\zeta,\omega,0) &= \frac{2}{3}(b^2+c^2)\,\mathrm{sech}^2(b\zeta + c\omega),
\end{aligned} \tag{3}$$

defined on $\Omega = \{(\zeta,\omega,t) | (\zeta,\omega) \in \mathbb{R}^n, 0 \le t \le T\}$, where $\mathcal{D}_t^{\gamma}$ denotes the Caputo fractional derivative with regard to time t and $\gamma \in (0, 1]$ be the fractional order of system; a, b, c, and d are non-zero arbitrary constants. To better understand the behavior of the system, we translated the classical WZ equation into a fractional-order equation. Many scholars have explored several analytical and numerical approaches to study WZ systems by highlighting their significance. To find the exact solution to this WZ system, the exponential rational function method was employed with the help of the RL fractional derivative by Kaur and Gupta [43]. Khater et al. [44] examined the WZ system using three techniques: Adomian decomposition, quintic, and septic spline techniques. Yasmin et al. [45] proposed an effective iterative method using the Caputo operator to solve this non-linear WZ system. Tariq et al. [46] constructed traveling wave solutions to the conformable time-fractional WZ system via extended fan sub-equation method, and distinct types of graphs are presented to show the physical behavior of the solutions.

In 1992, the homotopy analysis method (HAM) was introduced by a well-known mathematician from China, namely, *Liao Shijun* [47], for solving linear and non-linear differential equations. A straightforward method is proposed for handling the characteristics of linear and non-linear equations without perturbation and linearization properties. Moreover, this method requires a lot of time to perform computational work. To overcome these limitations, a novel scheme was proposed by Singh et al. [48] called q-homotopy analysis transform method (q-HATM), which is the amalgamation q-HAM along with the Laplace transform (LT) scheme. The benefit of this technique is that it does not require properties such as linearization, extraction of any polynomial, perturbation, or inclusion of any rigid assumptions, unlike other methods. It systematically and precisely analyzes the results obtained, attracting many researchers who have implemented it in their research work. Namely, Veeresha et al. [49] studied the modified Boussinesq and approximate long wave equations of fractional order using the Caputo fractional operator that describes the properties of shallow water waves through distinct dispersion relation with the aid of q-HATM. The fractional model of the regularized long-wave equation, which explains ion acoustic waves and the shallow water waves in plasma, has been examined using this method by Kumar et al. [50]. Prakasha et al. [51] applied this efficient technique to solve the time-fractional Kaup–Kupershmidt equation, which plays a vital role in studying nonlinear dispersive waves. In this article, we study the highly-dimensional WZ system with the help of the semi-analytic technique called q-HATM. The novelty of this method lies in obtaining the solution in series form, which uses a simple algorithm to evaluate the results of this system of equations that propagates solitary wave solutions as it is a localized wave with no change in velocity and shape. With the help of auxiliary and homotopy parameters that act as a controlling parameters, the proposed method provides a swift convergence region and preserves greater accuracy, which makes it more trustworthy and efficient. $\hbar$ parameter used in this semi-analytical method aids to control the convergence region. Moreover, they offer a robust and flexible approach for finding approximating solutions and are in good agreement with the precise solution, which minimizes the residual error. To illustrate the effect of fractional order, the variations of solutions with regard to $\hbar$ and time t are shown in graphs for diverse fractional order. To our best knowledge, the proposed system is not solved using q-HATM so far.

This article has been comprised of the numerous sections that make up a clear presentation of the research as follows: Section 2 provides the basic terms and definitions related to the LT and fractional calculus. In Section 3, we proposed the q-HATM algorithm with the help of the Caputo operator. The suggested method has been implemented for the considered system, and their graphical representation and numerical simulation of the system are compared with the exact solutions of the WZ system, which is presented in Section 4. The behavior of the obtained results that portrayed in graphs and numerical evaluation were discussed in Section 5. Finally, the conclusions of our work that broadens the discussion by strengthening the technique's relevance are illustrated in Section 6.

2 Preliminaries

This section gives a summary of information about the basic definitions and properties of fractional calculus (FC) and LT. Additionally, we discuss some theorems related to the suggested method of considered system.

Definition 2.1. The RL fractional integral J^{γ} of order γ for the function $f(\tau) \in C_{\mu}(\mu \geq -1)$, [1,2] is stated as

$$J^{\gamma}f(\tau) = \frac{1}{\Gamma(\gamma)}\int_{0}^{\tau}(\tau - \vartheta)^{\gamma-1}f(\vartheta)\mathrm{d}\vartheta,$$

where Γ denotes the Gamma function.

Definition 2.2. For the function $f(\tau) \in C_{-1}^{r}$, the Caputo fractional derivative [3] is defined as

$$ {}_{0}\mathcal{D}_{\tau}^{\gamma}f(\tau) = \begin{cases} \dfrac{\mathrm{d}^{r}f(\tau)}{\mathrm{d}\tau^{r}}, & \gamma = r \in \mathbb{N}, \\ \dfrac{1}{\Gamma(r-\gamma)}\displaystyle\int_{0}^{\tau}(\tau - \theta)^{r-\gamma-1}f^{(r)}(\theta)\mathrm{d}\theta, & \gamma \in (r-1, r),\ r \in \mathbb{N}. \end{cases}$$

The notation $\mathbb{N}$ represents the set of natural numbers.

Definition 2.3. The LT of the function $f(\tau)$ is given by $\mathscr{L}[f(\tau)] = \int_{0}^{\infty}e^{-s\tau}f(\tau)\mathrm{d}\tau$. Then, the LT of $f(\tau)$ in terms of Caputo derivative [2,3] is defined as

$$\mathscr{L}[\mathcal{D}_{\tau}^{\gamma}f(\tau)] = s^{\gamma}F(s) - \sum_{k=0}^{r-1} s^{\gamma-k-1}f^{(k)}(0^{+}), \quad r-1<\gamma\le r,$$

where $F(s)$ is the LT of $f(t)$.

Theorem 2.1. (Uniqueness theorem [49]) *The solution for the considered fractional partial differential equation Eq.* (2) *obtained by q-HATM is unique, for every* $\lambda \in (0,1)$, *where* $\lambda = (K_m + \hbar) + \hbar(\delta + \varepsilon)T$.

Proof. The solution of considered Eq. (2) is

$$u(\zeta,\omega,t) = \sum_{m=0}^{\infty} u_m(\zeta,\omega,t), \tag{4}$$

where u_m is the approximate solution obtained using q-HATM. Now, suppose that u and u^* are two solutions of considered equation, then we have to show that $u = u^*$ to prove this theorem, then

$$|u-u^*| = \left|(K_m+\hbar)(u-u^*) + \hbar\mathscr{L}^{-1}\left\{\frac{1}{s^{\gamma}}\mathscr{L}[(N(u-u^*)) + (R(u-u^*))]\right\}\right|, \tag{5}$$

where N and R are the nonlinear and linear operators, respectively. By employing the convolution theorem of Laplace on Eq. (5), we have

$$\begin{aligned}|u-u^*| &= (K_m+\hbar)|u-u^*| + \hbar\int_0^t (|N(u-u^*) \\ &\quad + R(u-u^*)|)\frac{(t-\xi)^{\gamma}}{\Gamma(\gamma+1)}\mathrm{d}\xi \\ &\le (K_m+\hbar)|u-u^*| + \hbar\int_0^t (\delta|u-u^*| \\ &\quad + \varepsilon|u-u^*|)\frac{(t-\xi)^{\gamma}}{\Gamma(\gamma+1)}\mathrm{d}\xi.\end{aligned} \tag{6}$$

Applying the mean value theorem of integrals on Eq. (6), we obtain

$$\begin{aligned}|u-u^*| &\le (K_m+\hbar)|u-u^*| + \hbar(\delta^2|u-u^*| \\ &\quad + \theta^2|u-u^*| + \varepsilon|u-u^*|)\mathcal{T}, \\ |u-u^*| &\le \mathrm{k}|u-u^*|, \\ (1-k)|u-u^*| &\le 0.\end{aligned}$$

Since $0<k<1$, $|u-u^*| = 0 \Rightarrow u = u^*$. □

Theorem 2.2. (Convergence theorem [49]) *Let* K *be a Banach space and* $F : K \to K$ *be a non-linear mapping, then assume that*

$$\|F(v) - F(w)\| \le \delta\|v-w\|, \quad \forall v,w \in K, \quad \delta \in (0,1).$$

With the help of fixed point theory of Banach, there exists one fixed point in F and the corresponding sequence generated by the solution that has been acquired using the suggested approach that coincides with the point in F with arbitarily selecting $v_0, w_0 \in K$ *and*

$$\|w_m - w_n\| \le \frac{\delta^n}{1-\delta}\|w_1 - w_0\|.$$

Theorem 2.3. (Error analysis [49]) *If we can determine a real number* $0<\delta<1$ *that fulfills* $\|v_{i+1}(\zeta,\omega,t)\|$, *for all i. Moreover, if the series, we obtain* $\sum_{i=0}^{m} v_i(\zeta,\omega,t)$ *is the q-HATM series solution of Eq.* (2), *then the maximum absolute error was obtained*

$$\left\| v(\zeta,\omega,t) - \sum_{i=0}^{m} v_i(\zeta,\omega,t)\right\| \le \frac{\delta^m}{1-\delta}\|v_0(\zeta,\omega,t)\|.$$

3 Methodology of q-HATM

This section provides the steps involved in solving the nonlinear fractional partial differential equation using the q-HATM [48,49]. Let us consider the general form of nonlinear non-homogeneous partial differential equation of fractional order, which is given by

$$\begin{aligned}&\mathcal{D}_t^{\gamma}u(\zeta,\omega,t) + Ru(\zeta,\omega,t) + Nv(\zeta,\omega,t) \\ &\quad = f(\zeta,\omega,t), \quad 0<\gamma\le 1,\end{aligned} \tag{7}$$

where $\mathcal{D}_t^{\gamma}v(\zeta,\omega,t)$ indicates the fractional derivative of the function $v(\zeta,\omega,t)$ in terms of Caputo operator, $f(\zeta,\omega,t)$ denotes the source term, and R and N are the operators that symbolize the linear and non-linear terms, respectively. Using LT for Eq. (7),

$$\begin{aligned}&s^{\gamma}\mathscr{L}[v(\zeta,\omega,t)] - \sum_{k=0}^{r-1} s^{\gamma-k-1}v^{(k)}(\zeta,\omega,0) \\ &\quad + \mathscr{L}[Rv(\zeta,\omega,t)] + \mathscr{L}[Nv(\zeta,\omega,t)] \\ &\quad = \mathscr{L}[f(\zeta,\omega,t)].\end{aligned} \tag{8}$$

Reducing Eq. (8), we have

$$\begin{aligned}&\mathscr{L}[v(\zeta,\omega,t)] - \frac{1}{s^{\gamma}}\sum_{k=0}^{r-1} s^{\gamma-k-1}v^{k}(\zeta,\omega,0) + \frac{1}{s^{\gamma}}\{\mathscr{L}[Rv(\zeta,\omega,t)] \\ &\quad + \mathscr{L}[Nv(\zeta,\omega,t)] - \mathscr{L}[f(\zeta,\omega,t)]\} = 0.\end{aligned}$$

The non-linear operator $\mathcal{N}$ is exemplified as follows:

$$\mathcal{N}[\varphi(\zeta,\omega,t;\,q)] = \mathscr{L}[\varphi(\zeta,\omega,t;\,q)] - \frac{1}{s^{\gamma}}\sum_{k=0}^{r-1}s^{\gamma-k-1}\varphi^{(k)}(\zeta,\omega,t;\,q)(0^{+}) + \frac{1}{s^{\gamma}} \times \{\mathscr{L}[R\varphi(\zeta,\omega,t;\,q)] + \mathscr{L}[N\varphi(\zeta,\omega,t;\,q)] - \mathscr{L}[f(\zeta,\omega,t)]\}.$$

The deformation equation at $\mathcal{H}(\zeta,\omega,t) = 1$ is stated as

$$(1-rq)\mathscr{L}[\varphi(\zeta,\omega,t;\,q) - v_0(\zeta,\omega,t)] = \hbar q\mathcal{H}(\zeta,\omega,t)N[\varphi(\zeta,\omega,t;\,q)], \tag{9}$$

where $\varphi(\zeta,\omega,t;q)$ is a real unknown function and $v_0(\zeta,\omega,t)$ the starting solution, the embedding parameter is denoted by q, which falls between the interval $\left[0,\frac{1}{r}\right]$, $\mathscr{L}$ symbolizes the LT, and the auxiliary parameter is given by $\hbar \neq 0$. The terms lies in the following hold for $q = 0$ as well as $q = \frac{1}{r}$, respectively:

$$\varphi(\zeta,\omega,t;\,0) = v_0(\zeta,\omega,t),\quad \varphi\left(\zeta,\omega,t;\,\frac{1}{r}\right) = v(\zeta,\omega,t).$$

Hence, by moving the value of q in between the intervals $\left[0,\frac{1}{r}\right]$, the function $\varphi(\zeta,\omega,t;\,q)$ has solution changes from $v_0(\zeta,\omega,t)$ to the actual solution $v(\zeta,\omega,t)$. Then, using the Taylor theorem throughout q, we expand the function $\varphi(\zeta,\omega,t;\,q)$ as

$$\varphi(\zeta,\omega,t;\,q) = v_0(\zeta,\omega,t) + \sum_{m=1}^{\infty}v_m(\zeta,\omega,t)q^{m}, \tag{10}$$

with

$$v_m(\zeta,\omega,t) = \frac{1}{m!}\frac{\partial^{m}\varphi(\zeta,\omega,t;\,q)}{\partial q^{m}}\Big|_{q=0}.$$

By taking the values of $\hbar$, r, $v_0(\zeta,\omega,t)$, and $\mathcal{H}(\zeta,\omega,t)$, then at $q = \frac{1}{r}$, Eq. (10) leads to the given non-linear equation, which has the results as follows:

$$v(\zeta,\omega,t) = v_0(\zeta,\omega,t) + \sum_{m=1}^{\infty}v_m(\zeta,\omega,t)\left(\frac{1}{r}\right)^{m}.$$

Now, m-times differentiating Eq. (9) with reference to q and also dividing it by $m!$ and for $q = 0$, we can define the mth-order deformation equation as

$$\mathscr{L}[v_m(\zeta,\omega,t) - K_m v_{m-1}(\zeta,\omega,t)] = \hbar\mathcal{H}(\zeta,\omega,t)\Re_m(\vec{v}_{m-1}), \tag{11}$$

where

$$K_m = \begin{cases}0, & m \leqslant 1,\\ r, & \text{otherwise},\end{cases}$$

and

$$\Re_m(\vec{v}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}N[\varphi(\zeta,\omega,t;\,q)]}{\partial q^{m-1}|q=0},$$

and the vectors are given in the form

$$\vec{v}_m = \{v_0(\zeta,\omega,t), v_1(\zeta,\omega,t), \ldots, v_m(\zeta,\omega,t)\}.$$

Now, employing the inverse Laplace transform to Eq. (11) and for the considered non-linear differential equation, we express the recursive equation as

$$v_m(\zeta,\omega,t) = K_m v_{m-1}(\zeta,\omega,t) + \hbar\mathscr{L}^{-1}[\mathcal{H}(\zeta,\omega,t)\Re_m(\vec{v}_{m-1})]. \tag{12}$$

At last, we obtained the component-wise approximated solution for q-HATM series using Eq. (12), and we define the series as

$$v(\zeta,\omega,t) = \sum_{m=0}^{\infty}v_m(\zeta,\omega,t). \tag{13}$$

4 Solution using *q*-HATM

We consider the (2 + 1)-dimensional non-linear time-fractional Wu–Zhang system [45] to evaluate the precision of the proposed method using the Caputo fractional derivative.

$$\begin{aligned}
&\mathcal{D}_t^{\gamma}u(\zeta,\omega,t) + u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\zeta} + v(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\omega} + \frac{\partial w(\zeta,\omega,t)}{\partial\zeta} = 0,\\
&\mathcal{D}_t^{\gamma}v(\zeta,\omega,t) + u(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial\zeta} + v(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial\omega} + \frac{\partial w(\zeta,\omega,t)}{\partial\omega} = 0,\\
&\mathcal{D}_t^{\gamma}w(\zeta,\omega,t) + \frac{\partial(u(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\zeta} + \frac{\partial(v(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\omega} + \frac{1}{3}\left\{\frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta^3} + \frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta\partial\omega^2} + \frac{\partial^3v(\zeta,\omega,t)}{\partial\zeta^2\partial\omega} + \frac{\partial^3v(\zeta,\omega,t)}{\partial\omega^3}\right\} = 0,
\end{aligned} \tag{14}$$

subjected to initial conditions

$$\begin{aligned}
u(\zeta,\omega,0) &= \frac{d+ac}{b} + \frac{2\sqrt{3}}{3}b\tanh(b\zeta + c\omega),\\
v(\zeta,\omega,0) &= a + \frac{2\sqrt{3}}{3}c\tanh(b\zeta + c\omega),\\
w(\zeta,\omega,0) &= \frac{2}{3}(b^2 + c^2)\operatorname{sech}^2(b\zeta + c\omega).
\end{aligned} \tag{15}$$

With the help of LT on Eq. (14) along with the starting solutions in Eq. (15), we have

$$\mathscr{L}[u(\zeta,\omega,t)]-\frac{1}{s}u_0(\zeta,\omega,t)+\frac{1}{s^{\gamma}}\mathscr{L}\times\left\{u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\zeta}+v(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\omega}+\frac{\partial w(\zeta,\omega,t)}{\partial\zeta}\right\}=0,$$

$$\mathscr{L}[v(\zeta,\omega,t)]-\frac{1}{s}v_0(\zeta,\omega,t)+\frac{1}{s^{\gamma}}\mathscr{L}\times\left\{u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\zeta}+v(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\omega}+\frac{\partial w(\zeta,\omega,t)}{\partial\omega}\right\}=0,$$

$$\mathscr{L}[w(\zeta,\omega,t)]-\frac{1}{s}w_0(\zeta,\omega,t)+\frac{1}{s^{\gamma}}\mathscr{L}\times\left\{\frac{\partial(u(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\zeta}+\frac{\partial(v(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\omega}+\frac{1}{3}\left\{\frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta^3}+\frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta\partial\omega^2}+\frac{\partial^3v(\zeta,\omega,t)}{\partial\zeta^2\partial\omega}+\frac{\partial^3v(\zeta,\omega,t)}{\partial\omega^3}\right\}\right\}=0.$$

The non-linear operator $\mathcal{N}$ can be defined as

$$\begin{aligned}\mathcal{N}^1[\varphi_1(\zeta,\omega,t;q)\varphi_2(\zeta,\omega,t;q)\varphi_3(\zeta,\omega,t;q)] &= \mathscr{L}[\varphi_1(\zeta,\omega,t;q)]-\frac{1}{s}\varphi_1(\zeta,\omega,0)\\ &-\frac{1}{s^{\gamma}}\mathscr{L}\left\{\varphi_1(\zeta,\omega,t;q)\frac{\partial\varphi_1(\zeta,\omega,t;q)}{\partial\zeta}+\varphi_2(\zeta,\omega,t;q)\frac{\partial\varphi_1(\zeta,\omega,t;q)}{\partial\omega}+\frac{\partial\varphi_3(\zeta,\omega,t;q)}{\partial\zeta}\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^2[\varphi_1(\zeta,\omega,t;q)\varphi_2(\zeta,\omega,t;q)\varphi_3(\zeta,\omega,t;q)] &= \mathscr{L}[\varphi_2(\zeta,\omega,t;q)]-\frac{1}{s}\varphi_2(\zeta,\omega,0)\\ &-\frac{1}{s^{\gamma}}\mathscr{L}\left\{\varphi_1(\zeta,\omega,t;q)\frac{\partial\varphi_2(\zeta,\omega,t;q)}{\partial\zeta}+\varphi_2(\zeta,\omega,t;q)\frac{\partial\varphi_2(\zeta,\omega,t;q)}{\partial\omega}+\frac{\partial\varphi_3(\zeta,\omega,t;q)}{\partial\omega}\right\},\end{aligned}$$

$$\begin{aligned}\mathcal{N}^3[\varphi_1(\zeta,\omega,t;q)\varphi_2(\zeta,\omega,t;q)\varphi_3(\zeta,\omega,t;q)] &= \mathscr{L}[\varphi_3(\zeta,\omega,t;q)]-\frac{1}{s}\varphi_3(\zeta,\omega,0)\\ &-\frac{1}{s^{\gamma}}\mathscr{L}\left\{\frac{\partial(\varphi_1(\zeta,\omega,t;q)\varphi_3(\zeta,\omega,t;q))}{\partial\zeta}+\frac{\partial(\varphi_2(\zeta,\omega,t;q)\varphi_3(\zeta,\omega,t;q))}{\partial\omega}\right.\\ &\left.+\frac{1}{3}\left\{\frac{\partial^3\varphi_1(\zeta,\omega,t;q)}{\partial\zeta^3}+\frac{\partial^3\varphi_1(\zeta,\omega,t;q)}{\partial\zeta\partial\omega^2}+\frac{\partial^3\varphi_2(\zeta,\omega,t;q)}{\partial\zeta^2\partial\omega}+\frac{\partial^3\varphi_3(\zeta,\omega,t;q)}{\partial\omega^3}\right\}\right\}.\end{aligned}$$

With the help of the proposed algorithm, the deformation equation is defined as

$$\begin{aligned}&\mathscr{L}[u_m(\zeta,\omega,t)-K_mu_{m-1}(\zeta,\omega,t)]=\hbar\Re_{1,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}],\\ &\mathscr{L}[v_m(\zeta,\omega,t)-K_mv_{m-1}(\zeta,\omega,t)]=\hbar\Re_{2,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}],\\ &\mathscr{L}[w_m(\zeta,\omega,t)-K_mw_{m-1}(\zeta,\omega,t)]=\hbar\Re_{3,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}],\end{aligned}\tag{16}$$

where

$$\begin{aligned}\Re_{1,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}] &= \mathscr{L}[u(\zeta,\omega,t)]-\left(1-\frac{K_m}{r}\right)\frac{1}{s}u_0(\zeta,\omega,t)\\ &-\frac{1}{s^{\gamma}}\mathscr{L}\left\{u(\zeta,\omega,t)\frac{\partial u(\zeta,\omega,t)}{\partial\zeta}+v(\zeta,\omega,t)\times\frac{\partial u(\zeta,\omega,t)}{\partial\omega}+\frac{\partial w(\zeta,\omega,t)}{\partial\zeta}\right\},\\ \Re_{2,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}] &= \mathscr{L}[v(\zeta,\omega,t)]-\left(1-\frac{K_m}{r}\right)\frac{1}{s}v_0(\zeta,\omega,t)\\ &-\frac{1}{s^{\gamma}}\mathscr{L}\left\{u(\zeta,\omega,t)\frac{\partial v(\zeta,\omega,t)}{\partial\zeta}+v(\zeta,\omega,t)\times\frac{\partial v(\zeta,\omega,t)}{\partial\omega}+\frac{\partial w(\zeta,\omega,t)}{\partial\omega}\right\},\\ \Re_{3,m}[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}] &= \mathscr{L}[w(\zeta,\omega,t)]-\frac{1}{s}w_0(\zeta,\omega,t)\\ &+\frac{1}{s^{\gamma}}\mathscr{L}\left\{\frac{\partial(u(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\zeta}+\frac{\partial(v(\zeta,\omega,t)w(\zeta,\omega,t))}{\partial\omega}\right.\\ &\left.+\frac{1}{3}\left\{\frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta^3}+\frac{\partial^3u(\zeta,\omega,t)}{\partial\zeta\partial\omega^2}+\frac{\partial^3v(\zeta,\omega,t)}{\partial\zeta^2\partial\omega}+\frac{\partial^3v(\zeta,\omega,t)}{\partial\omega^3}\right\}\right\}.\end{aligned}$$

Using the inverse Laplace transformation on Eq. (16), we obtain

$$\begin{aligned}u_m(\zeta,\omega,t)&=K_mu_{m-1}(\zeta,\omega,t)+\hbar\mathscr{L}^{-1}\{\Re_m[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}]\},\\ v_m(\zeta,\omega,t)&=K_mv_{m-1}(\zeta,\omega,t)+\hbar\mathscr{L}^{-1}\{\Re_m[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}]\},\\ w_m(\zeta,\omega,t)&=K_mw_{m-1}(\zeta,\omega,t)+\hbar\mathscr{L}^{-1}\{\Re_m[\vec{u}_{m-1},\vec{v}_{m-1},\vec{w}_{m-1}]\}.\end{aligned}\tag{17}$$

On simplifying Eq. (17), systematically using the given initial conditions, we obtain the following:

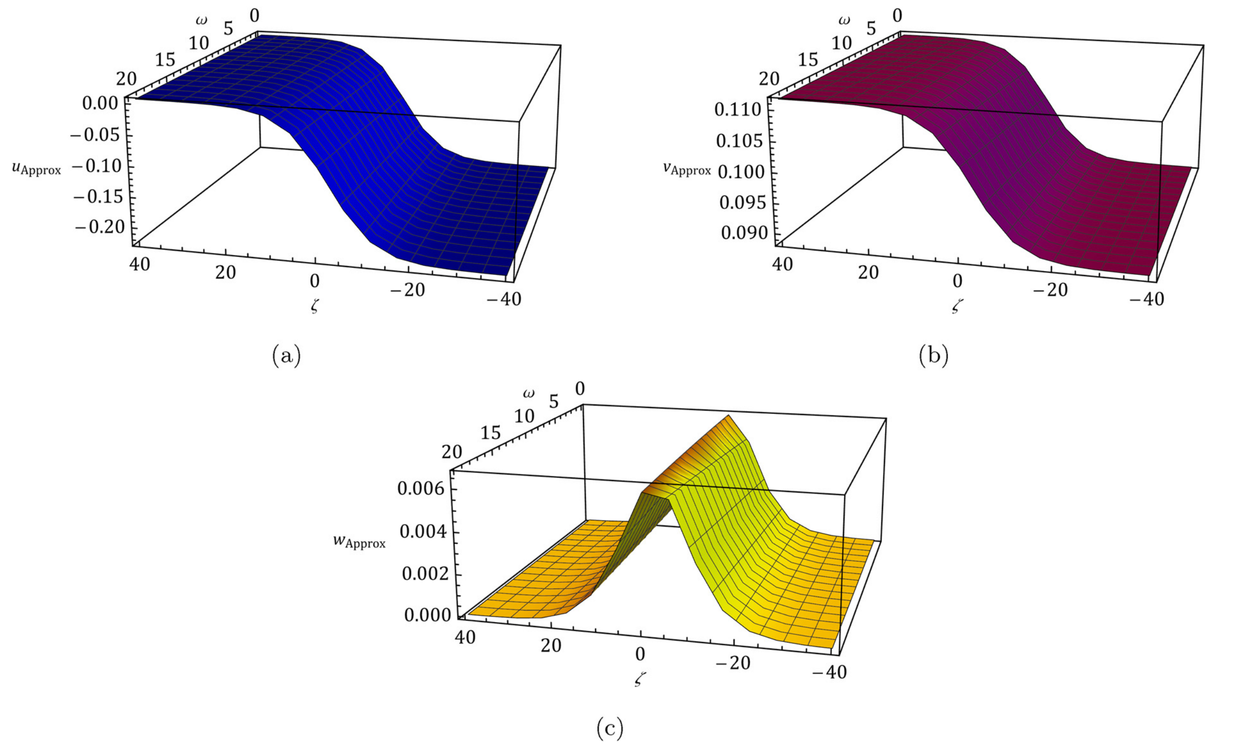

Figure 1: Nature of achieved surfaces of q-HATM series solution of (a) $u(\zeta,\omega,t)$, (b) $v(\zeta,\omega,t)$ and (c) $w(\zeta,\omega,t)$ at $a = b = 0.1$, $c = d = 0.01$, $t = 1$, $r = 1$, and $\hbar = -1$.

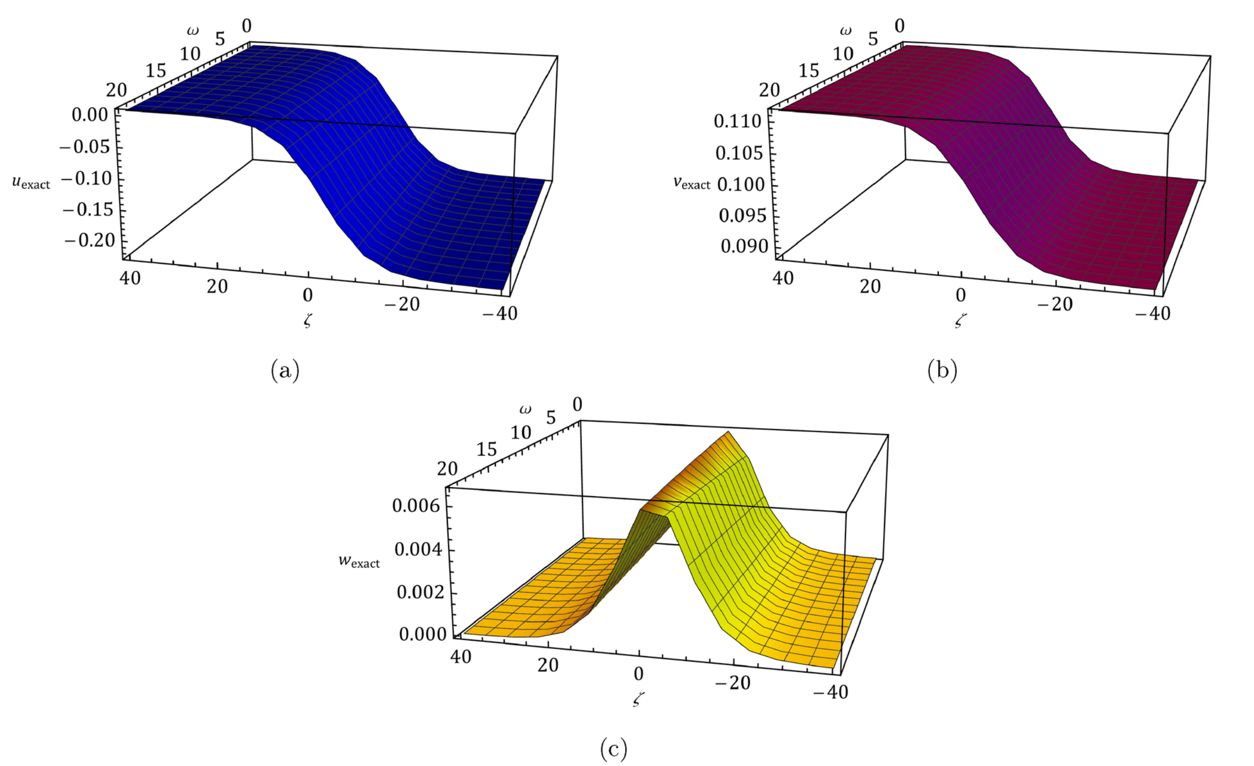

Figure 2: 3D plots of exact solution of (a) $u(\zeta,\omega,t)$, (b) $v(\zeta,\omega,t)$ and (c) $w(\zeta,\omega,t)$ for Eq. (14) at $a = b = 0.1$, $c = d = 0.01$, $t = 1$, $r = 1$, and $\hbar = -1$.

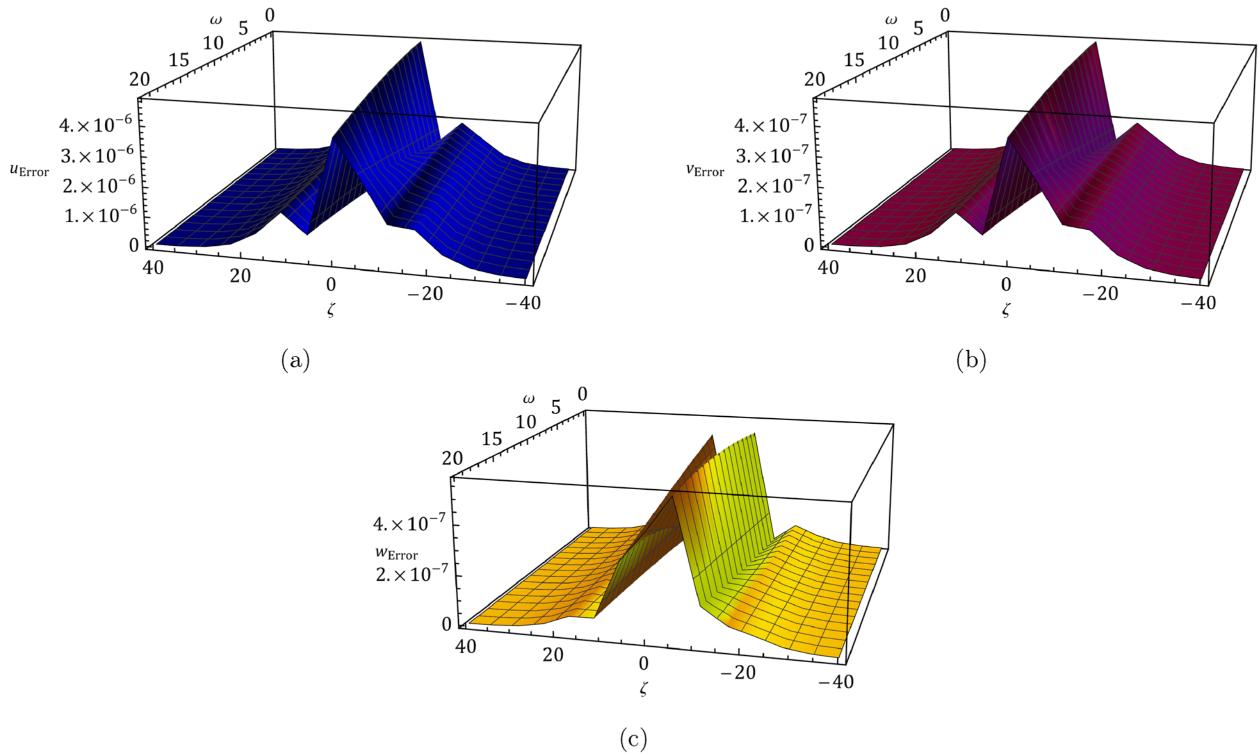

Figure 3: Nature of absolute error for Eq. (14) of (a) $u(\zeta, \omega, t)$, (b) $v(\zeta, \omega, t)$, and (c) $w(\zeta, \omega, t)$, respectively.

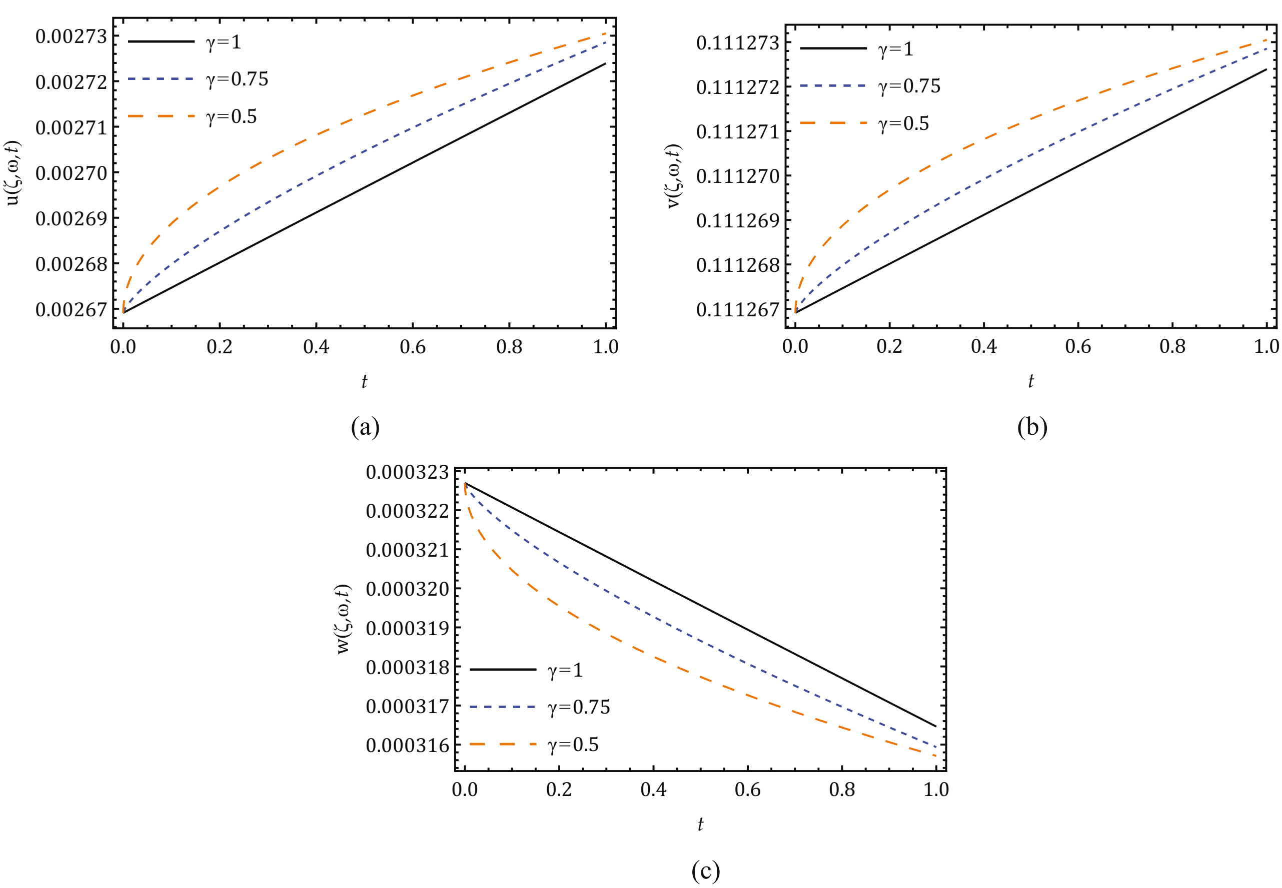

Figure 4: Nature of approximate solution with respect to t for Eq. (14) of (a) $u(\zeta, \omega, t)$, (b) $v(\zeta, \omega, t)$, and (c) $w(\zeta, \omega, t)$, respectively, with distinct γ values at $r = 1$, $\hbar = -1$, $\zeta = 20$, and $\omega = 20$.

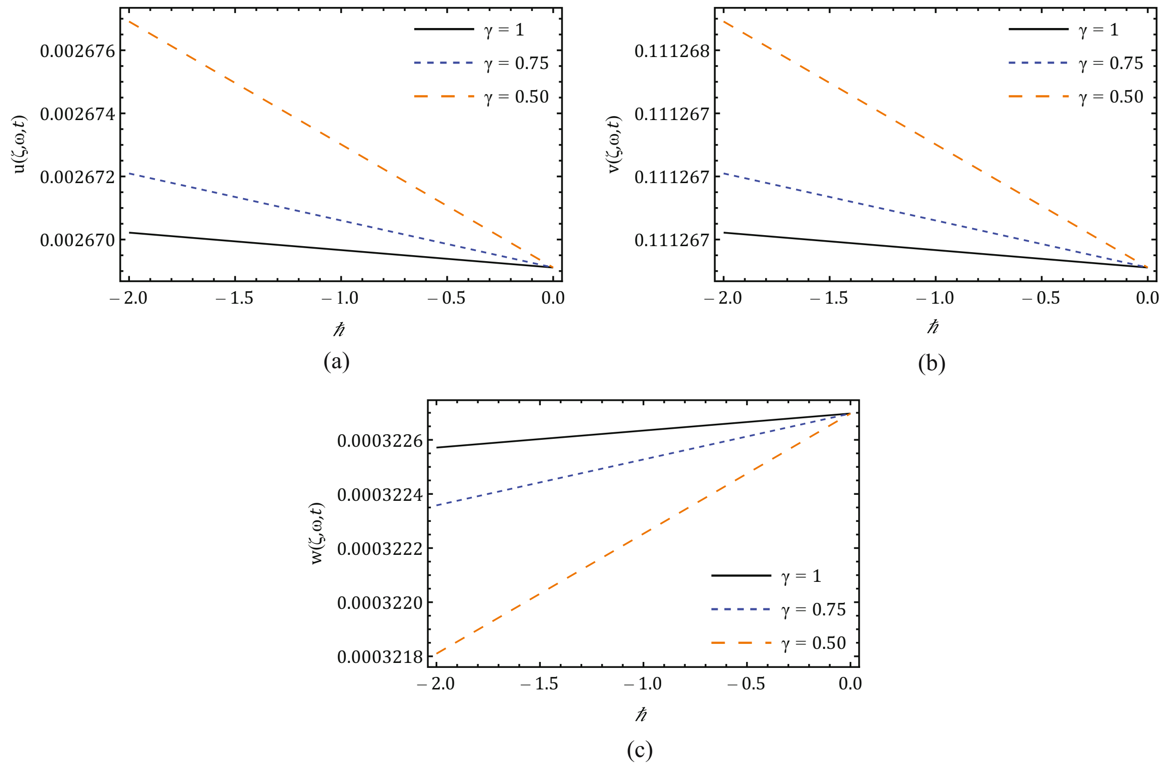

Figure 5: Nature of response of $\hbar$-curves of (a) $u(\zeta, \omega, t)$, (b) $v(\zeta, \omega, t)$, and (c) $w(\zeta, \omega, t)$, respectively, for Eq. (14) when $r = 1$ at $\zeta = 20, t = 0.01, \omega = 20$ for distinct fractional order γ.

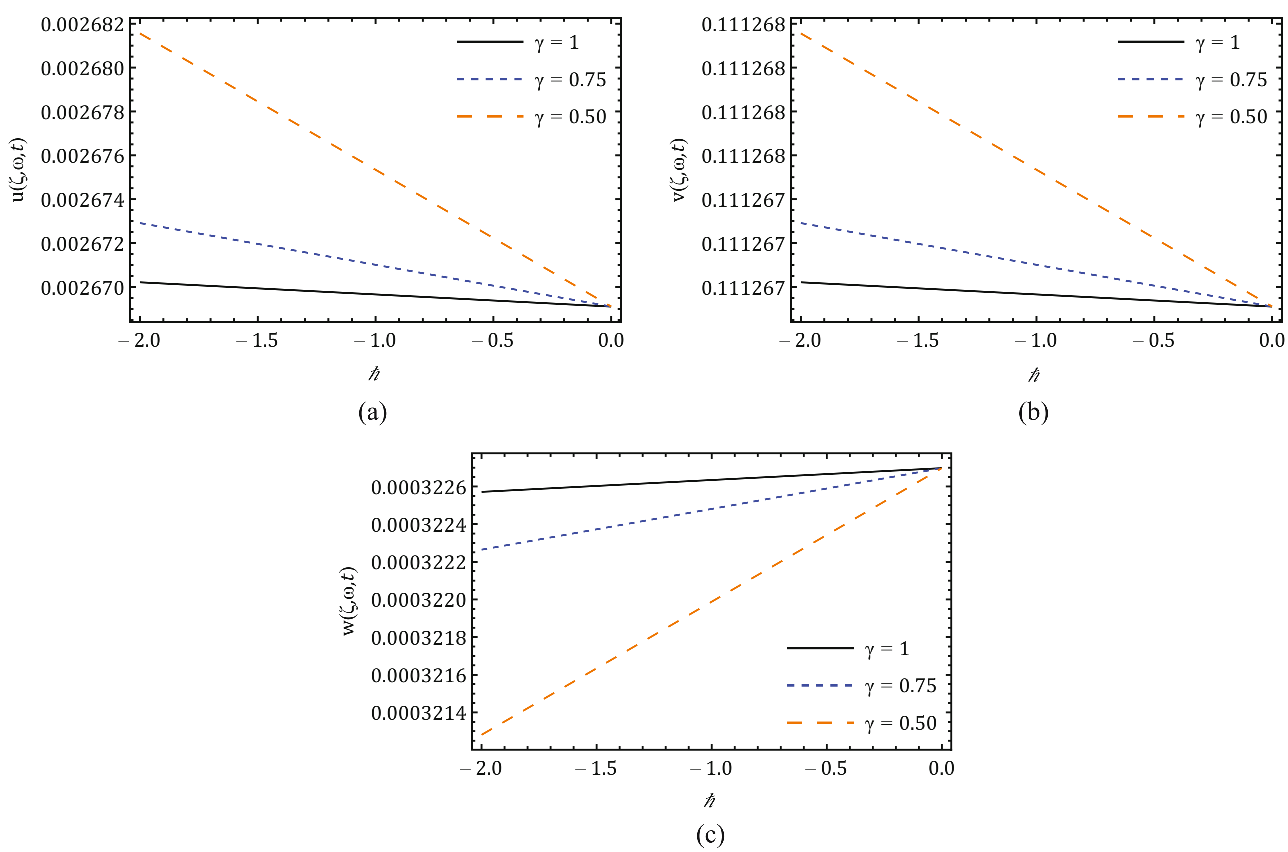

Figure 6: Nature of response of $\hbar$-curves of (a) $u(\zeta, \omega, t)$, (b) $v(\zeta, \omega, t)$, and (c) $w(\zeta, \omega, t)$, respectively, for Eq. (14) for $r = 2$ at $\zeta = 20, t = 0.01, \omega = 20$, for distinct values of γ.

Table 1: q-HATM, MVIM results, exact solution, and absolute error of surface velocity of water $u(\zeta, \omega, t)$ for Eq. (14) at $a = b = 0.1$, $c = d = 0.01$, $\omega = 20$, $t = 5$, $n = 1$, $\hbar = -1$, and $\alpha = 1$

ζ	Exact solution (w_{exact})	MVIM solution ($w_{\text{MV IM}}$) [39]	q-HATM solution ($w_{q\text{-HATM}}$)	Absolute error $\lvert w_{\text{exact}} - w_{\text{MV IM}}\rvert$ [39]	Absolute error $\lvert w_{\text{exact}} - w_{q\text{-HATM}}\rvert$
−50	-2.25453×10^{-1}	-2.25453×10^{-1}	-2.25453×10^{-1}	1.6451825×10^{-6}	5.14451×10^{-9}
−40	-2.25342×10^{-1}	-2.25342×10^{-1}	-2.25342×10^{-1}	1.2148126×10^{-5}	3.78834×10^{-8}
−30	-2.2453×10^{-1}	-2.2453×10^{-1}	-2.2453×10^{-1}	8.9315718×10^{-5}	2.72929×10^{-7}
−20	-2.18701×10^{-1}	-2.18702×10^{-1}	-2.18699×10^{-1}	6.3605735×10^{-4}	1.66699×10^{-6}
−10	-1.83341×10^{-1}	-1.83356×10^{-1}	-1.83339×10^{-1}	3.6024380×10^{-3}	1.79946×10^{-6}
00	-8.17192×10^{-2}	-8.17021×10^{-2}	-8.17275×10^{-2}	6.4447985×10^{-3}	8.256×10^{-6}
10	-1.20486×10^{-2}	-1.20416×10^{-2}	-1.20455×10^{-2}	1.9107842×10^{-4}	3.19039×10^{-6}
20	2.93273×10^{-3}	2.93247×10^{-3}	2.93359×10^{-3}	2.9196656×10^{-4}	8.65501×10^{-7}
30	5.12337×10^{-3}	5.12331×10^{-3}	-5.1235×10^{-3}	4.0172038×10^{-5}	1.27743×10^{-7}
40	5.42307×10^{-3}	5.42307×10^{-3}	5.42309×10^{-3}	$5.44887493 \times 10^{-6}$	1.74906×10^{-8}
50	5.46369×10^{-3}	5.46369×10^{-3}	5.4637×10^{-3}	$7.37648418 \times 10^{-7}$	2.37082×10^{-9}

$$u(\zeta, \omega, t) = u_0(\zeta, \omega, t) + \sum_{m=1}^{\infty} u_m(\zeta, \omega, t)\left(\frac{1}{r}\right)^m,$$
$$v(\zeta, \omega, t) = v_0(\zeta, \omega, t) + \sum_{m=1}^{\infty} v_m(\zeta, \omega, t)\left(\frac{1}{r}\right)^m,$$
$$w(\zeta, \omega, t) = w_0(\zeta, \omega, t) + \sum_{m=1}^{\infty} w_m(\zeta, \omega, t)\left(\frac{1}{r}\right)^m.$$

By putting values of $\gamma = 1, \hbar = -1$ and $r = 1$, we have the solution $\sum_{m=1}^{N} v_m(\zeta, \omega, t)\left(\frac{1}{r}\right)^m$, which converges with exact solutions

$$u(\zeta, \omega, t) = \frac{d + ac}{b} + \frac{2\sqrt{3}}{3} b \tanh(b\zeta + c\omega + \mathrm{d}t),$$
$$v(\zeta, \omega, t) = a + \frac{2\sqrt{3}}{3} c \tanh(b\zeta + c\omega + \mathrm{d}t),$$
$$w(\zeta, \omega, t) = \frac{2}{3}(b^2 + c^2) \operatorname{sech}^2(b\zeta + c\omega + \mathrm{d}t),$$

which is of the classical Wu–Zhang system as $N \to \infty$.

5 Results and discussion

In the present framework, we mainly focused on solutions obtained by the highly-dimensional fractional Wu–Zhang system that exhibits long dispersive wave phenomenon and explored the dynamics of various fractional order. We have utilized a more reliable method called q-HATM, to ensure the accuracy of our analysis and examined the solution using Caputo fractional derivative. In our analysis, we illustrated the approximation solutions of Eq. (14) in Figure 1, which helps in assessing the feasibility and effectiveness of the suggested approach. Figure 2 provides a comprehensive explanation of the behavior of precise solution of considered fractional WZ system in the 3D plots. Absolute error between q-HATM series solution and exact solution is portrayed in Figure 3. The nature of variation of water particle velocity $u(\zeta, \omega, t)$ and $v(\zeta, \omega, t)$ and elevation $w(\zeta, \omega, t)$ with reference to t is displayed in Figure 4.

Table 2: q-HATM, MVIM results, exact solution, and absolute error of surface velocity of water $v(\zeta, \omega, t)$ for Eq. (14) at $a = b = 0.1$, $c = d = 0.01$, $\omega = 20$, $t = 5$, $n = 1$, $\hbar = -1$, and $\alpha = 1$

ζ	Exact solution (w_{exact})	MVIM solution ($w_{\text{MV IM}}$) [39]	q-HATM solution ($w_{q\text{-HATM}}$)	Absolute error $\lvert w_{\text{exact}} - w_{\text{MV IM}}\rvert$ [39]	Absolute error $\lvert w_{\text{exact}} - w_{q\text{-HATM}}\rvert$
−50	0.0884547	0.0884547	0.0884547	$8.08571164 \times 10^{-9}$	2.67189×10^{-10}
−40	0.0884658	0.0884657	0.0884658	$5.96388798 \times 10^{-8}$	1.96727×10^{-9}
−30	0.088547	0.0885466	0.088547	$4.34887678 \times 10^{-7}$	1.41588×10^{-8}
−20	0.0891299	0.089127	0.0891298	$2.91626141 \times 10^{-6}$	8.58106×10^{-8}
−10	0.0926659	0.0926551	0.0926659	$1.07971370 \times 10^{-5}$	8.0558×10^{-8}
00	0.102828	0.102834	0.102828	$5.87519681 \times 10^{-6}$	3.99429×10^{-7}
10	0.109795	0.109802	0.109795	$7.18185473 \times 10^{-6}$	1.58615×10^{-7}
20	0.111293	0.111295	0.111293	$1.30807366 \times 10^{-6}$	4.18528×10^{-8}
30	0.111512	0.111513	0.111512	$1.84431598 \times 10^{-7}$	6.15812×10^{-9}
40	0.111542	0.111542	0.111542	$2.50992772 \times 10^{-8}$	8.42821×10^{-10}
50	0.111546	0.111546	0.111546	$3.39937593 \times 10^{-9}$	1.14237×10^{-10}

Table 3: q-HATM, MVIM results, exact solution, and absolute error of elevation of water $w(\zeta, \omega, t)$, for Eq. (14) at $a = b = 0.1$, $c = d = 0.01$, $\omega = 20$, $t = 5$, $n = 1$, $\hbar = -1$, and $\alpha = 1$

ζ	Exact solution (w_{exact})	MVIM solution ($w_{MV\,IM}$) [39]	q-HATM solution ($w_{q\text{-HATM}}$)	Absolute error $\lvert w_{exact} - w_{MV\,IM}\rvert$ [39]	Absolute error $\lvert w_{exact} - w_{q\text{-HATM}}\rvert$
−50	2.015707×10^{-6}	1.6451825×10^{-6}	2.0154×10^{-6}	$3.705249242 \times 10^{-7}$	3.11436×10^{-10}
−40	0.000014879	0.0000121481	0.0000148777	$2.731814611 \times 10^{-6}$	2.2849×10^{-9}
−30	0.000109176	0.0000893157	0.00010916	$1.986047907 \times 10^{-5}$	1.60091×10^{-8}
−20	0.000766334	0.0006360573	0.000766258	$1.302774181 \times 10^{-4}$	7.6713×10^{-8}
−10	0.004017011	0.0036024380	0.00401731	$4.145730832 \times 10^{-4}$	2.95485×10^{-7}
00	0.006329433	0.0061447985	0.00632901	$1.153652836 \times 10^{-4}$	4.19451×10^{-7}
10	0.001888126	0.0019107842	0.00188816	$2.265746793 \times 10^{-5}$	2.9078×10^{-8}
20	0.000292663	0.0002919665	0.000292708	$6.972254850 \times 10^{-7}$	4.40576×10^{-8}
30	0.000040371	0.0000401720	0.0000403782	$1.991078199 \times 10^{-7}$	7.08854×10^{-9}
40	5.477852×10^{-6}	4.488749×10^{-6}	5.47883×10^{-6}	$2.87710754 \times 10^{-8}$	9.81212×10^{-10}
50	7.416075×10^{-7}	7.3764841×10^{-7}	7.41741×10^{-7}	$3.95110044 \times 10^{-9}$	1.33197×10^{-10}

As time increases, the surface velocity of water $u(\zeta, \omega, t)$ and $v(\zeta, \omega, t)$ increases and water's elevation $w(\zeta, \omega, t)$ decreases. In Figures 5 and 6, we depicted the 2D plots of $\hbar$-curves when $r = 1$ and $r = 2$, respectively and that exhibits the behavior of Eq. (14) for distinct values of γ, which clearly demonstrates the suggested method's convergence ability. Moreover, this $\hbar$ parameter used in this semi-analytical method specifies and controls the convergence region. The numerical simulation of q-HATM solution in comparison with modified VIM and exact solution of Eq. (14) at $a = b = 0.1$, $c = d = 0.01, t = 5, \zeta = 20, \omega = 20$ has been demonstrated for $u(\zeta, \omega, t)$, $v(\zeta, \omega, t)$, and $w(\zeta, \omega, t)$ along with absolute error in Tables 1–3 respectively. All these graphical representations and numerical solutions are achieved with the assistance of Mathematica software. Notably, the outcomes obtained are more consistent and agrees with the exact solution, which preserves the efficiency of the technique used. This shows that the proposed approach can be applied to vast areas of FDEs, mainly which have initial conditions in various divisions of science and technology. This method is quite trustworthy, most dependable, and suitable for various research studies that deal with complex FDEs.

6 Conclusion

We have discussed (2 + 1)-dimensional non-linear time-fractional Wu–Zhang system of equations using a semi-analytical technique called q-HATM. The suggested approach is a powerful method for solving time-fractional WZ system, and Mathematica software is used effectively to implement this method. We derived the series solution of high-dimensional non-linear WZ system and demonstrated the graphical analysis of surfaces, which elucidates the characteristics of the system. This study had provided more valuable insights into the dynamics of the dispersive water waves and depicted a crucial merging region, which determines the effectiveness of the method. The results demonstrated using q-HATM have exceptional accuracy and consistency, which converge quickly. The system's action is also dependent on both time instant and time history, which can be easily analysed by the fractional calculus concept. Moreover, our analysis confirms that the proposed method is exceptionally effective, resilient, and successfully resolves non-linear partial differential equations encountered in science and engineering.

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