Research Article

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Tavis–Cummings model in the presence of a deformed field and time-dependent coupling

https://doi.org/10.1515/phys-2024-0118
received June 30, 2024; accepted November 29, 2024

Abstract: In this work, we introduce a model for a two-atom (T–A) system interacting with a field mode initially prepared in a coherent state of a Para Bose–field (P–F). The T–As are initially represented by a Bell state, and we present the quantum framework for the whole system by solving the dynamical equations. We investigate the time-dependent behavior of essential quantum resources relevant to various tasks in quantum optics and information science, including atomic population inversion, T–As entanglement, T–As-P–F entanglement, and the statistical properties of the P–F as they relate to the model parameters. Our analysis reveals how these quantum resources are affected by different parameters in the T–As-P–F model. Finally, we illustrate the evolving interdependencies among these quantum resources within the system.

Keywords: two atoms, Para–Bose field, quantum dynamics, entanglement, atomic inversion, statistical properties

1 Introduction

The interaction between light and matter is the main focus of quantum optics [1]. The primary and most widely used two models in quantum optics is the Jaynes (Tavis)-Cumming model J–CM (T–CM), which includes one atom and two atoms (T–As) in the presence of field modes [2,3]. The rotating wave approximation and dipole provided an exact solution. The J–CM (T–CM) model's integrable Hamiltonian was solved to obtain the eigenvalues and eigenvectors of the system under various extensions and quantum effects. The intensity-dependent coupling [4] is one of these extensions. A multi-level atom is another extension, there have been discussions about three-level atoms with constant coupling [5–7], four-level atoms [8,9] and five-level atoms in previous literature [10–13] where nonlinear interactions are considered under different circumstances. A variety of quantum effects are expected within the framework of the J–CM. Examples include the collapse and revival of atomic population inversion, Rabi oscillations, and photon antibunching [14], photon switching [15,16], *etc.*

Quantum correlations (QCs) are a fundamental feature of quantum systems and serve as an essential tool in the field of quantum information science [17]. Quantum entanglement (QE), one of the most prominent types of QCs, proves valuable for numerous applications, including quantum teleportation [18], quantum key distribution [19], quantum metrology [20], and several others [21,22]. QCs provide precious resources for quantum information tasks. QE had rapid expansion in the last years [23,24], which was also the time when it became an important resource for quantum information processing (QIP). When atomic motion is considered, QIP may have significant applications for the advancement of quantum systems [25]. It has been shown that the von Neumann entropy changes periodically when the time-dependent (t-d) coupling is present. These findings have also been used for the examination of the entanglement that exists between a three-level trapped ion and a laser field [26]. A further extension that may be conducted is to investigate the ways in which the field-mode structure and t-d coupling alter the dynamic characteristics of the cavity field's Wehrl entropy [27].

In recent decades, numerous researchers have explored extensions and deformations of the bosonic Fock-Heisenberg algebra (HA) to enhance various aspects of quantum field theory, with significant progress made in many areas. Various q-deformations of the simple harmonic oscillator (HO) have been proposed by several authors considering Jackson's q-calculus [28], leading to the development of novel states associated with q-deformed Lie algebras, including q-coherent, q-squeezed, and q-cat states [29–32]. Another notable adaptation of the HA is the Wigner algebra, which

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incorporates the reflection operator R and the Wigner parameter λ in bosonic commutation relations and equations of motion. This algebra includes infinite-dimensional representations for parabosons and finite-dimensional representations for parafermions [33]. Concepts of parafields and parastatistics naturally emerge from these generalizations [34,35]. When λ is associated with the Calogero coupling constant, this algebra provides a symmetry framework for the reduced two-particle Calogero-Sutherland model or the pseudo-HO, useful in solving the quantum mechanical Calogero model [36–39]. Many systems within quantum optics can be characterized by this model [40], and its SU(1,1) dynamical symmetry has been examined in the study by Perelomov [41,42]. Additionally, when $\lambda = p - 1/2$, this algebra transitions to the para-Bose oscillator algebra of order p [43,44], which offers a framework for describing particles that are neither bosons nor fermions. As $\lambda \to 0$, the algebra reduces to the standard bosonic algebra.

The concept of coherent states (CSs) have been widely used in quantum optics and information during the last several decades. Schrödinger first introduced CSs to describe systems with a Hamiltonian HO that minimizes quantum mechanical uncertainty [45,46]. Actually, two ways are considered for the construction of CSs known as Klauder–Perelomov and Barut–Girardello CSs [42,47,48]. Moreover, several investigations based on the standard creation and annihilation operators in original J–CM are generalized in the context of f-, q-, and λ-deformed operators [49–58]. Moreover, the conventional J–CM with intensity-dependent coupling and interaction of atoms with a field in the context of Holstein–Primakoff SU (2) and SU(1,1) CSs have been examined [59–61]. Some further extensions have been proposed and studied, including transitions between two or more photons [62] and Jaynes–Cummings–Hubbard model [63,64]. Based on the above consideration, in the present we develop the model of T–As interacting with a quantum field initially prepared in a CS of Para Bose–field (P–F). We examine the dynamic behavior of essential quantum resources relevant to various tasks in quantum optics and information. This includes examining atomic population inversion, T–As entanglement, T–As-P–F entanglement, and the statistical properties of P–F, all in relation to the parameters of the quantum model.

The rest of the article is structured as follows: Section 2 provides a description of the quantum system and outlines its dynamics. In Section 3, we discuss the numerical results alongside the measures of quantumness. Finally, our conclusion is summarized in Section 4.

2 Quantum model

This section represents the Hamiltonian of the interaction of one-mode P–F with T–As of an upper (lower) state $|e_j\rangle(|g_j\rangle)$

$$\hat{H}_{\text{int}}(t) = \sum_{j=1}^{2} C_j(t)(\hat{L}|e_j\rangle\langle g_j| + \hat{L}^{+}|g_j\rangle\langle e_j|), \tag{1}$$

where $\hat{L}^{+}(\hat{L})$ is the creation (annihilation) operator of the P–F, acting on the Fock states as [65]

$$\hat{L}^{\dagger}|n\rangle = \sqrt{h(n+1)}\,|n+1\rangle, \quad \hat{L}|n\rangle = \sqrt{h(n)}\,|n-1\rangle, \tag{2}$$

where h is a positive function and is equal to $h(n) = n + 2\left(\left[\frac{n+1}{2}\right] - \left[\frac{n}{2}\right]\right)\lambda$, where λ isa real parameter, so-called deformation parameter and [.] designs the integer part. Here the P–F is described through the P–F oscillator algebra that is a pseudo HO algebra characterized by the parameter λ. The function $C_j(t)$ is considered to describe the t-d coupling strength between the P–F and each one of the T–As. In this study, we concentrate on the case of identical coupling between the T–As and P–F, that is, $C_1(t) = C_2(t) = C(t)$, such that

$$C(t) = \begin{cases} \varepsilon \sin(t) & \text{with t−d coupling} \\ \varepsilon & \text{without t−d coupling,} \end{cases} \tag{3}$$

where ε describes the coupling strength between P–F and each one of the T–As in the absence of t-d coupling.

The t-d state of the T–As-P–F system at any time $\tau > 0$, can be provided as

$$\begin{aligned} |\psi_{\text{t-d}}(\tau)\rangle = \sum_{n=0}^{\infty} & D_1(n,\tau)|n, e_1e_2\rangle + D_2(n,\tau)|n+1, e_1g_2\rangle. \\ & + D_3(n,\tau)|n+1, g_1e_2\rangle \\ & + D_4(m,\tau)|n+2, g_1g_2\rangle. \end{aligned} \tag{4}$$

Next we proceed to solve the system's equation of motion

$$-i\hbar\frac{\partial}{\partial\tau}|\psi_{\text{t-d}}(\tau)\rangle = \hat{H}_{\text{int}}(t)|\psi_{\text{t-d}}(\tau)\rangle, \tag{5}$$

with initial system state

$$\begin{aligned} |\psi(0)\rangle &= |\psi_{\text{T−As}}(0)\rangle \otimes |\psi_{\text{P−F}}(0)\rangle \\ &= (\cos(\theta)|\, e_1e_2\rangle + \sin(\theta)|\, g_1g_2\rangle) \otimes \sum_{n=0}^{\infty} R_n(\lambda)|n\rangle, \end{aligned} \tag{6}$$

where R_n is the amplitude of the superposition coefficients in the CSs of P–F and θ is a parameter characterizing the initial state of the T–As. The T–As is initially in the separable state for $\theta = 0$ while the Bell state case can be obtained for $\theta = \pi/4$.

By substituting the t-d state of the system into the motion equation and using the initial conditions, the functions $D_{\mathrm{j}}(n, \tau)$ are obtained to be

$$\begin{aligned} D_1(n,\tau) = {} & \frac{R_n(\lambda)}{h(n+2,\lambda) + h(n+1,\lambda)}[\cos\theta\{h(n+2,\lambda) \\ & + h(n+1,\lambda)\cos(d(\tau)\xi_n(\lambda))\} \\ & + \sin\theta\sqrt{h(n+1,\lambda)h(n+2,\lambda)} \\ & \times \{\cos(d(\tau)\xi_n(\lambda)) - 1\}], \end{aligned} \tag{7}$$

$$\begin{aligned} D_2(n,\tau) = {} & \frac{-iR_n(\lambda)\sin(d(\tau)\xi_n(\lambda))}{\xi_n(\lambda)}[\cos\theta\sqrt{h(n+1,\lambda)} \\ & + \sin\theta\sqrt{h(n+2,\lambda)}], \end{aligned} \tag{8}$$

$$\begin{aligned} & D_4(n,\tau) \\ & = \frac{R_n(\lambda)}{h(n+2,\lambda) + h(n+1,\lambda)}[\sqrt{h(n+1,\lambda)h(n+2,\lambda)} \\ & \times \cos\theta\{\cos(d(\tau)\xi_n(\lambda)) - 1\} \\ & + \sin\theta\{h(n+2,\lambda)\cos(d(\tau)\xi_n(\lambda)) + h(n+1,\lambda)\}], \end{aligned} \tag{9}$$

with $D_3(n, \tau) = D_2(n, \tau)$ and that

$$\xi_n(\lambda) = \sqrt{2h(n+2,\lambda) + 2h(n+1,\lambda)}, \tag{10}$$

and

$$d(\tau) = \int_0^\tau C(t)\mathrm{d}t.$$

The CSs corresponding to the P–F are introduced as [65]

$$\begin{aligned} |Z,\lambda\rangle & = \sum_{n=0}^{\infty} R_n(\lambda)|n\rangle \\ & = \frac{\exp\left(-\frac{|Z|^2}{2}\right)4^{\lambda}\sqrt{(\lambda!)^3}}{\sqrt{(2\lambda)!}} \sum_{n=0}^{\infty} Z^n \sqrt{\frac{\left(\lambda + \left[\frac{n}{2}\right]\right)!}{\left[\frac{n}{2}\right]!(2\lambda+n)!}} \\ & \times L_{\lambda}^{\left[\frac{n-1}{2}\right]+\frac{1}{2}}\left(\frac{|Z|^2}{2}\right)|n\rangle, \end{aligned} \tag{11}$$

where $L_{\lambda}^{m}(\ldots)$ is the associated Laguerre polynomials. In the limit $\lambda = 0$, the CSs of P–F tends to the case of Glauber CSs.

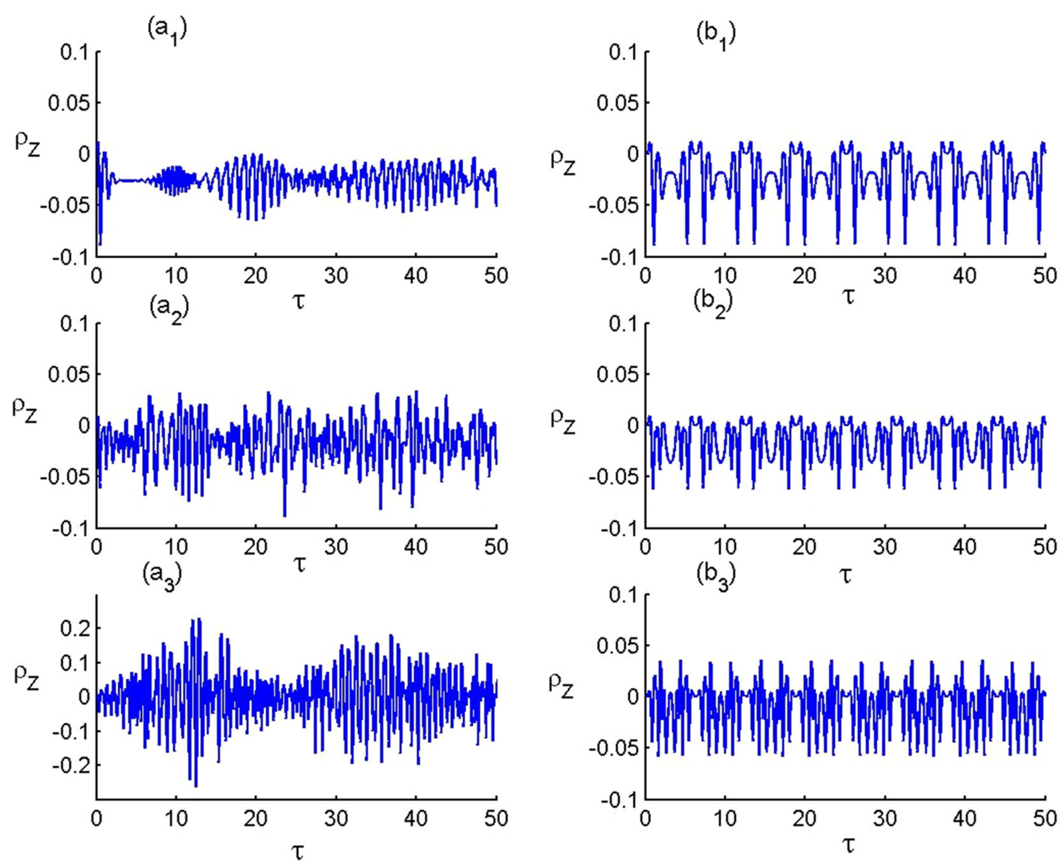

Figure 1: Time evolution of ρ_z for T–As in the presence of P–F with $z = 3$ and $\theta = \pi/4$. Panels (a_i) and (b_i) represent the cases of constant t-d coupling ($C(t) = \varepsilon$) and oscillating t-d coupling ($C(t) = \varepsilon\ \sin(t)$), respectively. Panels (a1, b1), (a2, b2), and (a3, b3) correspond to λ values of 0, 1, and 5, respectively.

The reduced density matrices of subsystems are obtained from the T–As-P–F density matrix, $\rho_{\text{t-d}}(\tau) = |\psi_{\text{t-d}}(\tau)\rangle\langle\Psi_{\text{t-d}}(\tau)|$, as

$$\rho_{\text{T-As}}(\tau) = \text{Tr}_{\text{P-F}}\{\rho_{\text{t-d}}(\tau)\} = \sum_{j=1}^{4}\sum_{n=1}^{4}\rho_{jn}|j\rangle\langle n|, \tag{12}$$

$$\rho_{\text{P-F}}(\tau) = \text{Tr}_{T-A}\{\rho_{\text{t-d}}(\tau)\} = \sum_{k}^{\infty}\rho_{k}|k\rangle\langle k|. \tag{13}$$

3 Quantum measures and their dynamical behavior

This section displays different measures for quantumness with their dynamical behavior discussion through the effect of t-d coupling ($C(t)$) and P–F deformation parameter (λ).

3.1 Atomic inversion

We begin by defining the population inversion for the T–As in terms of the functions $D_j(n,\tau)$ as

$$\rho_z(\tau) = \sum_{m=0}^{\infty}(\lceil D_1(m,\tau)\rceil^2 + \lceil D_2(m,\tau)\rceil^2 - \lceil D_3(m,\tau)\rceil^2 - \lceil D_4(m,\tau)\rceil^2). \tag{14}$$

In the case where the two atoms are identically coupled, we have $D_2(m,\tau) = D_3(m,\tau)$. Under this condition, the expression for the atomic inversion simplifies to

$$\rho_z(\tau) = \sum_{m=0}^{\infty}(\lceil D_1(m,\tau)\rceil^2 - \lceil D_4(m,\tau)\rceil^2).$$

The population inversion, ρ_z of T–As with $\lambda = 0$, $\lambda = 1$, and $\lambda = 5$ is shown in the Figure 1. In general, it is evident that the parameter λ in the absence and presence of t–d coupling has an impact on the dynamics of the function ρ_z.

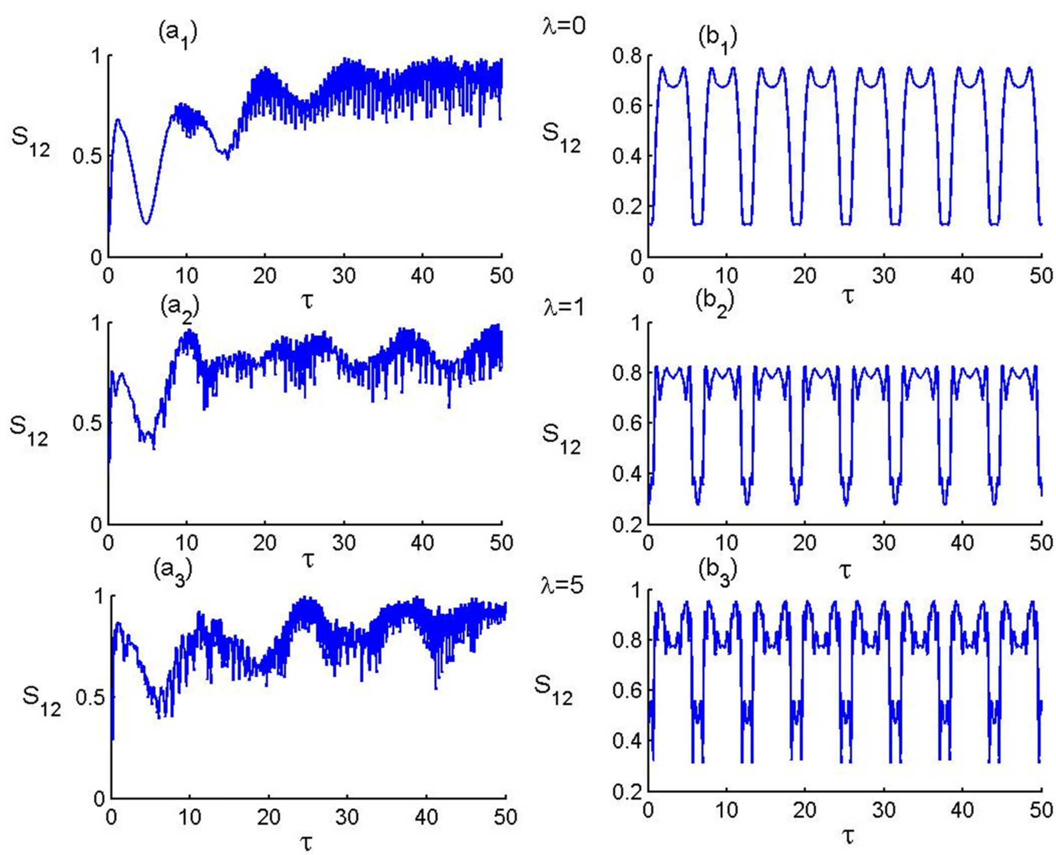

Figure 2: Time evolution of T–As-P–F entanglement in the presence of P–F with $z = 3$ and $\theta = \pi/4$. Panels (a_i) and (b_i) represent the cases of constant t-d coupling ($C(t) = \varepsilon$) and oscillating t-d coupling ($C(t) = \varepsilon \ \sin(t)$), respectively. Panels (a1, b1), (a2, b2), and (a3, b3) correspond to λ values of 0, 1, and 5, respectively.

We can see that the function ρ_z exhibits rapid oscillations with collapse and revival events in the case of $C(t) = \varepsilon$ and $\lambda = 0$. Here $\lambda = 0$ corresponds to the case of ordinary CSs. When the t-d effect is present, function ρ_z oscillations take on a regular and periodic character. Furthermore, we can see that the collapse's period doesn't change during the dynamics. On the other hand, in both the absence and presence of the t-d coupling effect, the existence of the deformed field (P–F with $\lambda = 1$ and $\lambda = 5$) enhances the oscillations of the atomic inversion with the increase in their amplitude.

3.2 T–As-P–F entanglement

To analyze the dynamics of the T–As-P–F entanglement, the Neumann entropy of the subsystems, T–A or P–F, is considered as a measure

$$E_{\mathrm{P-F(T-A)}} = -\mathrm{Tr}\{\rho_{\mathrm{P-F(T-A)}} \ln[\rho_{\mathrm{P-F(T-A)}}]\}. \tag{15}$$

Here $\rho_{\mathrm{P-F(T-A)}}$ represents the P – F(T – A) density operator given by Eqs. (14) and (15). The function $E_{\mathrm{F(T-A)}}$ takes the form

$$E_{\mathrm{P-F(T-A)}} = -\sum_{l=1}^{\infty(4)} \eta_l^{\mathrm{P-F}(T-A)} \ln \eta_l^{\mathrm{P-F(T-A)}}, \tag{16}$$

where $\eta_l^{\mathrm{P-F(T-A)}}$ are the eigenvalues of the state $\rho_{\mathrm{P-F(T-A)}}$.

In Figure 2, we illustrate the dynamic behavior of quantum entropy in relation to the parameter values of the physical model, using the same parameter settings as in Figure 1. The figure highlights significant physical scenarios based on the values of $C(t)$ and λ. The $E_{\mathrm{P-F(T-A)}}$ function shows t-d growth and oscillates, maintaining a steady oscillatory pattern over extended periods when $\lambda \to 0$ limit and $C(t) = \varepsilon$. The QE of the T–As-P–F state stabilizes within a specific time range, mirroring the behavior of population inversion, and the T–As-P–F entanglement remains trapped by the P–F for large values of scaled time τ. On the other hand, with the t-d coupling effect, $E_{\mathrm{P-F(T-A)}}$ oscillates periodically with period $\tau = 2\pi$, which

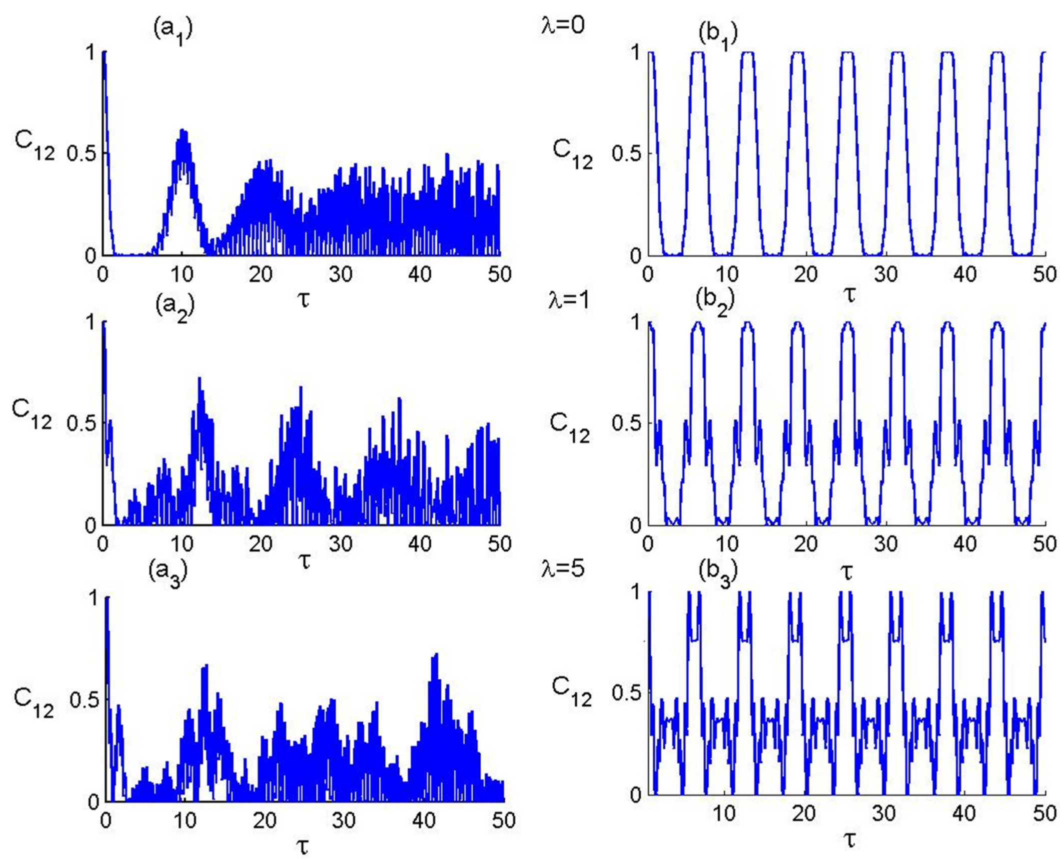

Figure 3: Time evolution of T–As entanglement in the presence of P–F with $z = 3$ and $\theta = \pi/4$. Panels (a_*i*) and (b_*i*) represent the cases of constant t-d coupling ($C(t) = \varepsilon$) and oscillating t-d coupling ($C(t) = \varepsilon \sin(t)$), respectively. Panels (a1, b1), (a2, b2), and (a3, b3) correspond to λ values of 0, 1, and 5, respectively.

agrees with the period of population inversion oscillations in the $\lambda \to 0$ limit with the presence of t-d coupling effect. In the existence of the t-d effect, we discover that the T–As-P–F state approaches the maximum value of entanglement for $\lambda \to 5$. This means that the increase in the value of parameter λ results in an increase in the maximum amount of entanglement. Whereas, the behavior of the $E_{\mathrm{P-F(T-A)}}$ function does not vary much over time in the absence of t-d coupling effect and the presence of field deformation ($\lambda \neq 0$). The obtained results illustrate that the manipulation of the T–As-P–F entanglement in the considered model can be achieved by convenable choice on the coupling function and deformation parameter.

3.3 T–As entanglement

To analyze the nonlocal correlation between the T–As, we utilize the concurrence that is given by [66]

$$C_{12} \coloneqq \max\{0, \vartheta_1 - \vartheta_2 - \vartheta_3 - \vartheta_4\}, \tag{17}$$

where ϑ_j represents the eigenvalues of the matrix $\rho_{\text{t-d}}\widetilde{\rho}_{\text{t-d}}$, arranged in descending order, and $\widetilde{\rho}_{\text{t-d}}$ denotes the density matrix associated with the Pauli matrix σ_Y by

$$\widetilde{\rho}_{\text{t-d}} = (\sigma_Y \otimes \sigma_Y)\rho^*_{\text{t-d}}(\sigma_Y \otimes \sigma_Y). \tag{18}$$

Here $\rho^*_{\text{t-d}}$ is the complex conjugate of $\rho_{\text{t-d}}$. For $C_{12} = 0$, the T–As state is separable, while for $C_{12} = 1$, the T–As state is maximally entangled.

Figure 3 shows the time variation of concurrence with regard to the values of the T–As-P–F parameters. In general, it is evident that $C(t)$ and λ have a significant impact on the dynamic behavior of the function C_{12}. In the absence of t-d coupling and field deformation effects, the value of function C_{12} ranges between 0 and 1 illustrating that the QE of the T–As state oscillates between a maximum and minimum value and cannot reach its maximum value at $\tau = 0$. Indeed, we observe that the function C_{12} begins at its maximum value of 1, undergoes random oscillations, and then gradually decreases over time, eventually settling into a steady state of fluctuating oscillations. The function C_{12} has a peculiar behavior during evolution, where it takes

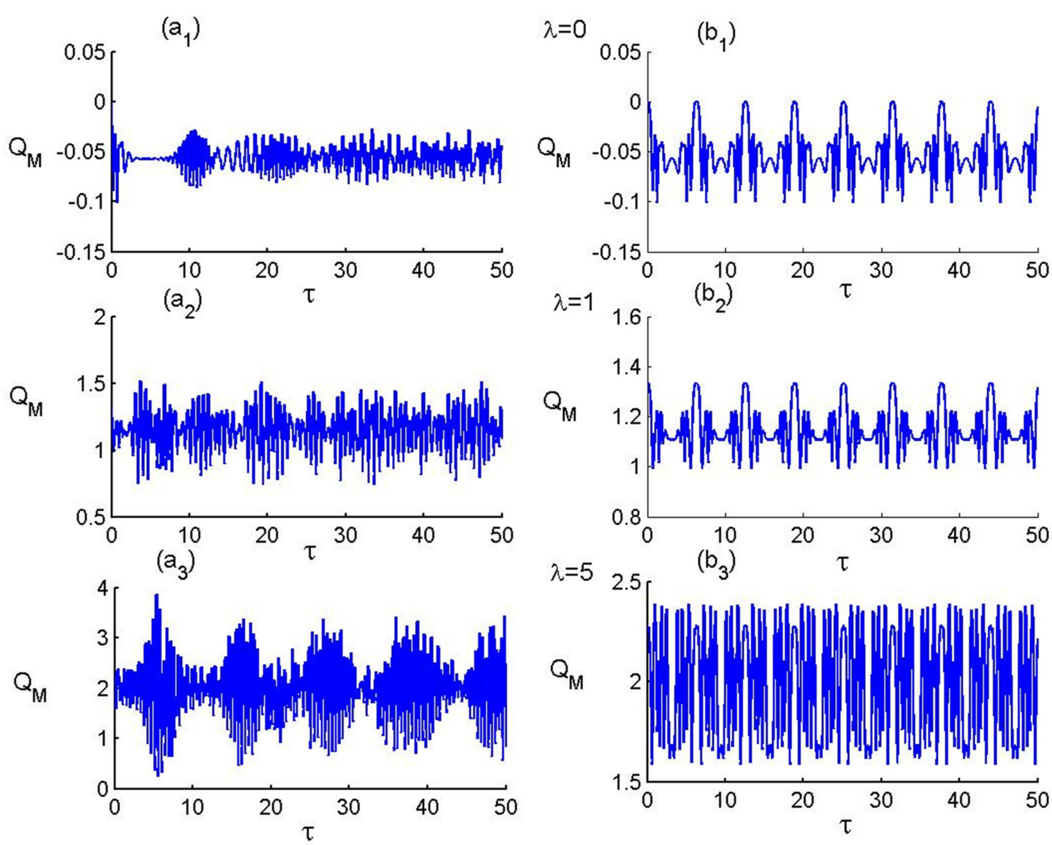

Figure 4: The Mandel parameter for the P–F with $z = 3$ and $\theta = \pi/4$. Panels (a_i) and (b_i) represent the cases of constant t-d coupling ($C(t) = \varepsilon$) and oscillating t-d coupling ($C(t) = \varepsilon \sin(t)$), respectively. Panels (a1, b1), (a2, b2), and (a3, b3) correspond to λ values of 0, 1, and 5, respectively.

unexpectedly zero values for a brief time interval, exhibiting sudden birth and death phenomena of QE. The same behavior of the measure of QE can be obtained in the absence of t-d coupling with different values of λ. The periods of the sudden death phenomena remain constant throughout the dynamics when the t-d coupling effect is included, and the concurrence oscillations become regular, exhibiting periodic behavior. The sudden death phenomena of QE will disappear with a regular dynamic of concurrence in the presence of field deformation and t-d coupling effect. This finding suggests that the interaction between the T–As and the P–F in the present model is crucial for maintaining and controlling the QE of the T–As state, and that the dependence on the field's deformation is considerable with respect to the t-d coupling effect.

3.4 Statistical properties of the P–F

Now, we analyze the distribution of P–F photons using the Mandel parameter that is defined by [67,68]

$$Q_{\mathrm{M}} = \frac{1}{\mathrm{Tr}(\hat{L}^{\dagger}\hat{L})}\{\mathrm{Tr}(\hat{L}^{\dagger}\hat{L})^2 - [\mathrm{Tr}(\hat{L}^{\dagger}\hat{L})]^2 - \mathrm{Tr}(\hat{L}^{\dagger}\hat{L})\}. \quad (19)$$

The P–F exhibits the Poissonian statistics for $Q_{\mathrm{M}} = 0$, while P–F has sub-Poissonian statistics for $Q_{\mathrm{M}} < 0$, and governed by a super-Poissonian statistics when $Q_{\mathrm{M}} > 0$. Figure 4 shows the dynamics of the Mandel parameter with regard to the values of the T–As-P–F parameters. It is clear that the parameter $Q_{\mathrm{M}} < 0$ and $Q_{\mathrm{M}} > 0$ during the dynamics for different choice of $C(t)$ and λ, exhibiting a sub-Poissonian and super-Poissonian photon distribution. Notably, selecting suitable values for $C(t)$ and λ can reduce the value of Q_{M}.

4 Conclusion

In this article, we have developed a model of T–As system that interacts with a field mode initially defined in a CS of P–F. We have considered that the T–As are initially described by a Bell state and display the quantum model of the whole system with the solution of the dynamics equation in the absence and presence of t-d coupling effect. We have examined the t-d behavior of essential quantum resources relevant to various tasks in quantum optics and information science, including atomic population inversion, T–As entanglement, T–As-P–F entanglement, and the statistical properties of the P–F as they relate to the model parameters. In this context, our analysis revealed how these quantum resources are affected by different parameters in the T–As-P–F model. Finally, we have illustrated the evolving interdependencies among these quantum resources within the quantum system.

Funding information: Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R225), Princess Nourah bint Abdulrahman University, Riyadh, Saudi Arabia.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: All data generated or analyzed during this study are included in this published article.

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