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Research Article

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Abundant new interaction solutions and nonlinear dynamics for the (3+1)-dimensional Hirota–Satsuma–Ito-like equation

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Abstract: In this article, the (3+1)-dimensional Hirota–Satsuma–Ito-like equation is investigated by the modified direct method, from which some interaction solutions among lump, stripe solitons, and Jacobi elliptic function wave solutions are obtained, which are crucial in understanding complex behaviors in nonlinear systems where multiple wave types coexist and interact. The corresponding evolution and dynamics for the interaction solutions under different parameters are discussed. Such interactions are key to modeling realistic systems in which multiple phenomena coexist, such as fluid mechanics, plasma physics, and optical systems, where waves can exchange energy and form stable or unstable patterns. These results reported in this article can reveal the theoretical mechanisms of stability, energy transfer, and pattern formation in nonlinear media and may raise the possibility of related experiments and potential applications in nonlinear science fields, such as oceanography, nonlinear optics, and so on.

Keywords: soliton, lump solution, interaction solution, Hirota–Satsuma–Ito-like equation

1 Introduction

The construction of nonlinear wave solutions for soliton equations and the in-depth study of their underlying dynamic properties remains a highly active area of research in the field of integrable systems. It has been demonstrated through theoretical and experimental studies that the examination of nonlinear wave solutions is of significant importance in the elucidation of the theoretical mechanisms underlying related nonlinear phenomena across various physical fields such as Bose–Einstein condensate [1], nonlinear optics [2], oceanography [3], plasma physics [4], and even financial markets [5].

In the last few decades, various effective techniques, including the Hirota bilinear method [6], Darboux and Bäcklund transformation (BT) [7], inverse scattering transformation [8], Riemann–Hilbert problem [9], deep learning method [10–12], and so on, have been proposed to construct nonlinear wave solutions with physical meaning and analyze their corresponding evolution behavior [13–15]. At the same time, relevant theories and methods have been generalized to fractional order soliton equations, from which various physically meaningful nonlinear wave solutions and corresponding nonlinear dynamics have been studied [16–19]. Recently, lump solution, which can be considered a special nonsingular rational solution, has attracted significant interest and is commonly utilized in diverse physical fields, including oceanography and nonlinear optics. A comprehensive analysis of the interactions between lump solutions and other nonlinear wave solutions in various (2+1)- and (3+1)-dimensional evolution equations has been conducted [20–27].

In order to describe unidirectional propagation of shallow water waves, Hirota and Satsuma initially proposed a completely integrable model.

$$u_t - u_{xxt} - 3uu_t + 3u_x v_t + u_x = 0, \quad v_x = -u, \tag{1}$$

which can be solved by the inverse scattering method [28]. As integrable extension, the (2+1)-dimensional Hirota–Satsuma–Ito (HSI) equation [29]

$$w_t - u_{xxx} - 3uu_t + 3u_x v_t - \alpha u_x = 0, w_x = -u_y, v_x = -u \tag{2}$$

and (3+1)-dimensional Hirota–Satsuma–Ito-like (HSIl) equation [30]

$$cu_{yt} + u_{xxxt} + 3u_x u_{xt} + 3u_{xx} u_t + du_{zt} = 0 \tag{3}$$

have been proposed, whose nonlinear wave solutions and corresponding dynamics have also been subjected to

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investigation [31–33]. The HSIl equation is a type of nonlinear evolution equation that generalizes the behavior of wave interactions in systems where nonlinearity and dispersion play a central role, which is important in modeling physical phenomena that involve multiple interacting waves or fields with different speeds or properties. Plasma physics, HSIl equation can describe wave interactions where ions and electrons interact with different wave speeds, from which the nonlinear dynamics in plasma environments can be investigated. In fiber optics, the HSIl equation can describe the propagation of light pulses where nonlinearity and dispersion balance each other, which is useful in the study of optical solitons in nonlinear media.

The bilinear BT for the HSIl Eq. (3) has been given, and the interaction phenomena between lump waves and kink waves have also been discussed in [30]. The lump and breather solutions have been constructed and the interaction among the lump, soliton, and periodic waves have also been investigated in [31]. A natural idea is whether more interaction solutions can be constructed, such as lump and stripe solitons, stripe solitons and elliptic periodic function solutions, and so on, to characterize more practical nonlinear phenomena and to provide theoretical guidance for designing new physical experiments and predicting new physical phenomena. This is the main motivation of this study.

This article employs the generalized direct method to investigate the HSIl Eq. (3). The following is a description of the organization of this article. In Section 2, the interaction solutions between lump and stripe solitons for the HSIl Eq. (3) are presented, and the corresponding fusion phenomena are discussed. In Section 3, the interaction solutions between stripe solitons and Jacobi elliptic function waves for the HSIl Eq. (3), whose dynamics are investigated. In Section 4, by combining positive quadratic functions and Jacobi elliptic functions, the interaction solutions between lump and Jacobi elliptic function waves for the HSIl Eq. (3) are obtained. In Section 5, the three mixed-action solutions are subjected to analysis. The conclusion and discussion are presented in Section 6.

2 Interaction solutions between lump and stripe solitons

Using the dependent variable transformation $u = (\ln f)_x$, the Hirota bilinear form for the Eq. (3) can be given as follows:

$$(D_x^3 D_y + cD_y D_t + dD_z D_t) f \cdot f = 0, \tag{4}$$

where the differential operator D [6] is defined as follows:

$$\begin{aligned} &D_x^{n_1} D_y^{n_2} D_z^{n_3} D_t^{n_4} (f \cdot g) \\ &= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n_1} \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^{n_2} \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right)^{n_3} \\ &\times \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^{n_4} f(x, y, z, t) g(x', y', z', t')\Big|_{x'=x, y'=y, z'=z, t'=t}. \end{aligned} \tag{5}$$

Generally, the N-soliton solutions can be constructed by taking $f = \sum_{\mu=0,1} \exp\left(\sum_{i<j}^{N} \mu_i \mu_j A_{ij} + \sum_{i=1}^{N} \mu_i \eta_i\right)$. However, these soliton solutions in the high-dimensional situations do not have a tendency to zero everywhere in space, and they are usually referred to as line soliton solutions. To obtain solutions in which the entire space tends to zero, that is the lump solutions, the long-wave limit method [34] has been widely used, but this technique is usually effective in (2+1) dimensions and sometimes fails in (3+1) dimensions. In 2015, the positive quadratic function method for constructing the lump solutions was proposed [20]. In 2017, the method for finding interaction solutions between rogue wave and strip solitons was proposed [27]. Subsequently, the interaction solutions between various types of nonlinear wave solutions were obtained. Here, we take a more generalized approach, extending the spatial variable t as a function of t, which allows the construction of nonlinear wave solutions with greater physical significance and demonstrates a richer range of nonlinear dynamics, although it greatly increases the computational complexity and difficulty.

In the study of nonlinear local waves, lump and stripe waves are two distinct types of waves that describe different physical phenomena. In physics, lump waves are localized, nontraveling waveforms that typically decay to zero in all spatial directions. Stripe waves describe wave structures that are infinite in one dimension and periodic or localized in the perpendicular direction, which represent spatially extended patterns, often with a periodic structure along one axis, resembling "stripes" in their shape. The objective of this section is to construct an interaction solution between lump and one-stripe solitons for HSIl Eq. (3), the function f, is expressed as follows:

$$f = \xi_1^2 + \xi_2^2 + k e^{(k_1 x + k_2 y + k_3 z + k_4)} + a_9(t), \tag{6}$$

with

$$\begin{aligned} \xi_1 = a_1 x + a_2 y + a_3 z + a_4(t), \xi_2 = a_5 x + a_6 y + a_7 z \\ + a_8(t), \end{aligned} \tag{7}$$

where a_i $(1 \leq i \leq 3, 5 \leq i \leq 7)$, k_i $(1 \leq i \leq 4)$, and k are real parameters to be determined, and $a_4(t)$ and $a_8(t)$ are arbitrary functions of the only variable t. Generally, the lump solutions for HSIl Eq. (3) can be derived under the case $k = 0$, while the interaction solutions between lump and

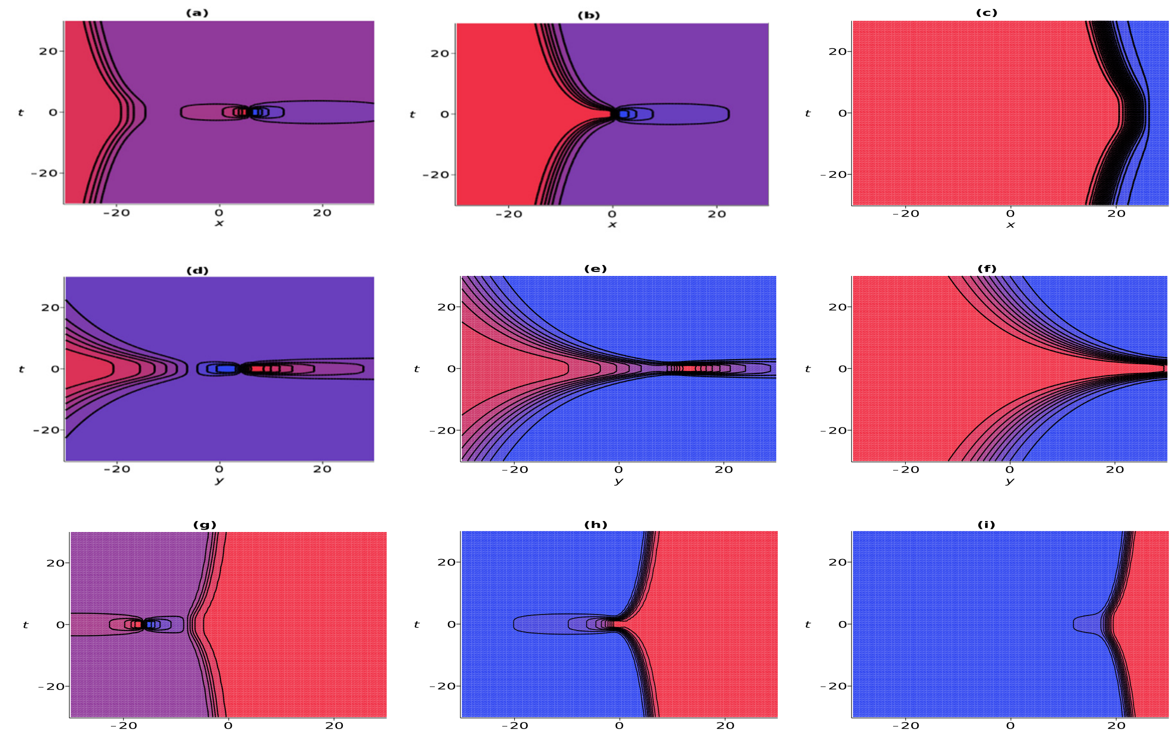

Figure 1: The contour propagation for the solution (10) with $a_3 = 1, a_5 = 1, a_7 = 1, a_8 = t^2, a_9 = 1, k = 1, k_1 = -1, k_3 = 2, c = 3, d = 1, p_1 = 1, y = 1$, and (a) $y = 1, z = -6$; (b) $y = 1, z = 0$; (c) $y = 1, z = 15$; (d) $x = 1, z = 0$; (e) $x = 1, z = 2$; (f) $x = 1, z = 6$; (g) $x = -1, y = -50$; (h) $x = -1, y = 0$; and (i) $x = -1, y = 100$.

one-stripe solitons can be obtained by taking $a_4(t) = a_{40}t + a_{41}$, $a_8(t) = a_{80}t + a_{81}$, and $a_9(t) = a_{90}$. Here, we consider the more general case, that is, $a_4(t)$, $a_8(t)$, and $a_9(t)$ are arbitrary functions, which to the best of our knowledge has not yet been investigated.

By substituting Eq. (6) into Eq. (3), these wave parameters can be determined by direct and tedious calculations as follows:

$$a_1 = \frac{a_3a_5}{a_7}, \quad a_2 = -\frac{a_3d}{c}, \quad a_4(t) = -\frac{a_7a_8(t)}{a_3} + p_1,$$
$$a_6 = -\frac{a_7d}{c}, \quad k_2 = -\frac{k_1^3 + k_3d}{c},$$
$$k_4 = \ln\left(-\frac{3k_1a_5(a_3^2 + a_7^2)}{a_7^2}\right), \tag{8}$$

under the constraint condition $a_1a_7 = a_3a_5 \neq 0$, $k_1 < 0$, where p_1 is the integral constant. Then, the function f can be obtained as follows:

$$f = \left(\frac{a_3a_5}{a_7}x - \frac{a_3d}{c}y + a_3z - \frac{a_7a_8(t)}{a_3} + p_1\right)^2 + \left(a_5x - \frac{a_7d}{c}y + a_7z + a_8(t)\right)^2 + ke^{\left(k_1x - \frac{k_1^3+k_3d}{c}y + k_3z + \ln\left(-\frac{3k_1a_5(a_3^2+a_7^2)}{a_7^2}\right)\right)} + a_9(t), \tag{9}$$

from which the interaction solution for HSIl Eq. (3) can be given as follows:

$$u = \frac{4a_5a_3^2\left(-\frac{3}{2}kk_1^2ca_5f_1 + (a_5x + a_7z) - da_7y\right)(a_3^2 + a_7^2)c}{-3kk_1c^2a_3^2a_5^2(a_3^2 + a_7^2)f_1 + c^2a_7^2(a_3^2 + a_7^2)a_8(t)^2 + a_3^2(c^2a_7^2a_9 + (a_3^2 + a_7^2)((a_5x + a_7z)c - da_7y)^2)}, \tag{10}$$

where

$$f_1 = e^{\left(\frac{-k_1^3y + k_1cx + k_3cz - k_3dy}{c}\right)}. \tag{11}$$

Various exact interaction solutions between lump and one-stripe solitons for HSIl Eq. (3) can be obtained by choosing different values of parameters. As a concrete example, the contour plots under the parameters $a_3 = 1$, $a_5 = 1$, $a_7 = 1$, $a_8 = t^2$, $a_9 = 1$, $k = 1$, $k_1 = -1$, $k_3 = 2$, $c = 3$, $d = 1$, $p_1 = 1$, from which the fusion phenomena can be observed, are shown in Figure 1 to demonstrate the evolution of the interaction solution (10) in different spaces. Figure 1(a)–(c) shows the contours for the solution (10) in x–t space, representing the interaction process of the stripe and lump solitons, *i.e.*, from the juxtaposition of the stripe and lump solitons (cf. Figure 1(a)) to the absorption of the lump soliton by the stripe soliton (cf. Figure 1(c)). Figure 1(d)–(f) shows the contours for the solution (10) in y–t space, representing the similar interaction process as x–t space. Figure 1(g)–(i) demonstrates the contour evolution for the solution (10) in y–t space and also exhibit the collision between process of the stripe and lump solitons, except that the lump velocity is faster than the stripe soliton velocity; thus, it is the process of the lump soliton chasing the stripe soliton and being absorbed.

3 Interaction solutions between stripe solitons and Jacobi elliptic function waves

This section is primarily concerned with the interaction between stripe solitons and Jacobi elliptic function waves. To this end, the function f is taken as follows:

$$f = k_1 + k_2 e^{g} + k_3 \mathrm{JacobiSN}(h, m), \tag{12}$$

where

$$g = a_1 x + a_2 y + a_3 z + a_4(t),\ h = a_5 x + a_6 y + a_7 z + a_8(t), \tag{13}$$

where k_i $(1 \le i \le 3)$, a_i $(1 \le i \le 3)$, a_i $(5 \le i \le 7)$ are real parameters to be determined, and $a_4(t)$, $a_8(t)$ are arbitrary functions of variable t.

By substituting Eq. (12) with Eqs. (13) into Eq. (3), the parameters can be obtained as follows:

$$a_2 = -\frac{a_1^3 + a_3 d}{c},\ a_5 = 0,\ a_6 = -\frac{a_7 d}{c}, \tag{14}$$

from which the function f can be written as follows:

$$f = k_1 + k_2 e^{\left(a_1 x - \frac{a_1^3 + a_3 d}{c} y + a_3 z + a_4(t)\right)} + k_3 \mathrm{JacobiSN}\left(-\frac{a_7 d}{c} y + a_7 z + a_8(t), m\right), \tag{15}$$

Then, the interaction solutions between stripe solitons and Jacobi elliptic function can be derived as follows:

$$u = \frac{2 k_2 a_1 e^{\left(\frac{-a_1^3 y + a_1 c x + a_3 c z - a_3 d y + a_4(t) c}{c}\right)}}{k_1 + k_2 e^{\left(\frac{-a_1^3 y + a_1 c x + a_3 c z - a_3 d y + a_4(t) c}{c}\right)} + k_3 \mathrm{JacobiSN}\left(\frac{a_7 c z - a_7 d y + a_8(t) c}{c}, m\right)}. \tag{16}$$

Here, we take $a_4(t) = t^2$ as an example to demonstrate the evolution and dynamic properties of the interaction solution (16). It is not hard to verify that the the interaction solution (16) eventually tends to the plane wave as time develops, that is,

$$\lim_{t \to +\infty} u = \lim_{t \to +\infty} \frac{2 k_2 a_1 e^{\left(\frac{-a_1^3 y + a_1 c x + a_3 c z - a_3 d y + a_4(t) c}{c}\right)}}{k_1 + k_2 e^{\left(\frac{-a_1^3 y + a_1 c x + a_3 c z - a_3 d y + a_4(t) c}{c}\right)} + k_3 \mathrm{JacobiSN}\left(\frac{a_7 c z - a_7 d y + a_8(t) c}{c}, m\right)} = 2. \tag{17}$$

Figure 2 presents the 3D evolution and density plots for the interaction solution (16) in the x–z space, from which it can be seen the interaction between the stripe solitons and the periodic solution. There are similar evolution states in other spaces, which are omitted here for the sake of simplicity.

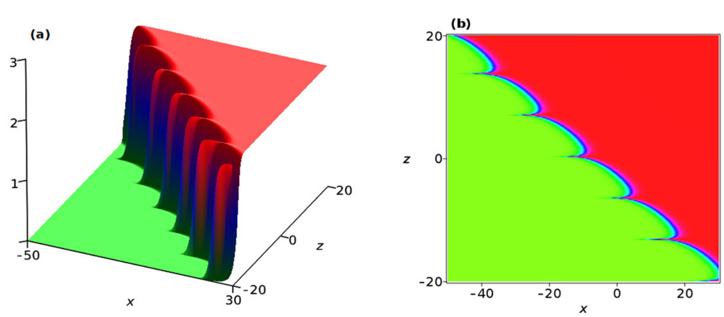

Figure 2: The evolution for the interaction solution (16) with parameters $a_1 = 1, a_3 = 2, a_4 = t^2, a_8 = \cos(t), k_1 = 1, k_2 = 1, k_3 = 1, c = 1, d = 1, m = 0.5, y = 1, t = 3$: (a) 3D plot and (b) density plot.

4 Interaction solutions between lump and Jacobi elliptic function waves

The objective of this section is to obtain the interaction solutions between lump and Jacobi elliptic function waves for the HSIl Eq. (3). Here, we mainly give two construction methods.

Case 1

In this case, the function f is defined as the combination of a positive quadratic function and a Jacobi elliptic function, that is,

$$f = k_1 + k_2 g^2 + k_3 \text{JacobiSN}(h, m), \tag{18}$$

with

$$g = a_1 x + a_2 y + a_3 z + a_4, h = a_5 x + a_6 y + a_7 z + a_8(t), \tag{19}$$

where k_i $(1 \le i \le 3), a_i$ $(1 \le i \le 7)$ are real parameters to be determined, and $a_8(t)$ is the function of variable t.

By substituting function f (12) into Eq. (3) and computing directly, we have

$$a_2 = -\frac{a_3 d}{c}, a_5 = 0, a_6 = -\frac{a_7 d}{c}, \tag{20}$$

from which the function f can be obtained as follows:

$$\begin{aligned} f = k_1 + k_2 \left(a_1 x - \frac{a_3 d}{c} y + a_3 z + a_4\right)^2 \\ + k_3 \text{JacobiSN}\left(-\frac{a_7 d}{c} y + a_7 z + a_8(t), m\right), \end{aligned} \tag{21}$$

Then, the interaction between the lump and Jacobi elliptic function waves for the HSIl Eq. (3) can be given as follows:

$$u = \frac{4 c a_1 k_2 f_2}{k_3 c^2 f_1 + k_2 f_2^2 + k_1 c^2}, \tag{22}$$

where

$$\begin{aligned} f_1 &= \text{JacobiSN}\left(\frac{a_7 c z - a_7 \text{d} y + a_8(t) c}{c}, m\right), \\ f_2 &= (c(a_1 x + a_3 z + a_4) - a_3 \text{d} y), \end{aligned} \tag{23}$$

It is obvious that the nonsingularity of solution (22) can be guaranteed by taking suitable values for the parameters k_1, k_2, and k_3, since the Jacobi function f_1 is periodic with a range $[-1, 1]$.

To demonstrate the evolution and interaction properties of the solution (22), we take parameters $a_1 = 1, a_3 = 1, a_4 = 1, a_7 = -1, k_1 = 1, c = 1, d = 1$ as an example, under which the solution (22) can be simplified as follows:

$$u = \frac{4(x - y + z + 1)}{k_3 \text{JacobiSN}(a_8(t) + z - y, m) + (x - y + z + 1)^2 + 1}, \tag{24}$$

which clearly has no singularity under condition $k_3 < 1$. At the same time, Eq. (24) implies that the nonlinear wave solution is localized in all spatial directions and periodic in the t direction, whose periodicity is determined by the function $a_8(t)$ and parameter m. Figure 3 shows the evolution and contour plots for the solution (24) in x–t space, which illustrates the interaction between lump and periodic wave solutions.

Case 2

In this case, the function f is taken as the combination of two positive quadratic functions and a Jacobi elliptic function, that is,

$$f = g^2 + h^2 + k \text{JacobiSN}(l, m) + a_9, \tag{25}$$

with

$$\begin{aligned} g &= a_1 x + a_2 y + a_3 z + a_4(t), \\ h &= a_5 x + a_6 y + a_7 z + a_8(t), \\ l &= k_1 + k_2 y + k_3 z + k_4(t), \end{aligned} \tag{26}$$

where $k_i (1 \le i \le 3), a_i (1 \le i \le 7)$ $(i \ne 4)$ are real parameters to be determined, and $a_4(t)$, $a_8(t)$, and $k_4(t)$ are the functions of variable t. In the same way as in case 1, by symbolic calculation, we obtain

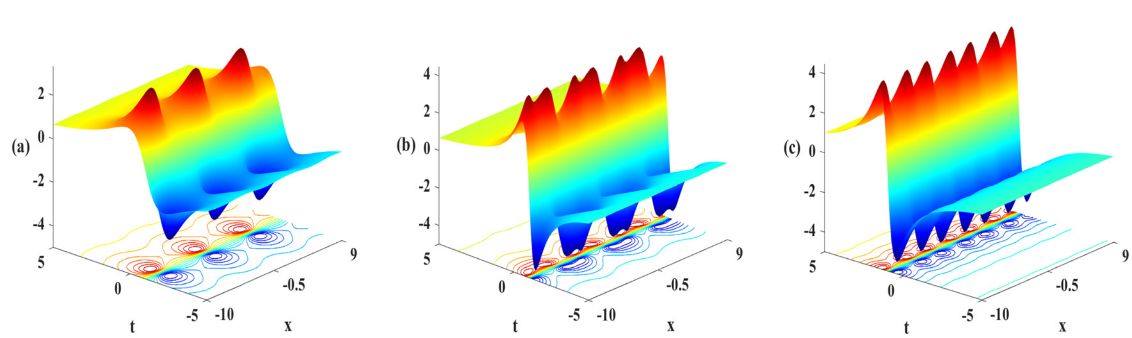

Figure 3: The 3D and contour propagation for the interaction solution (24) with parameters $a_8 = \sin(t)$, $m = 0.5$, $z = 1$, and (a) $y = 1$, (b) $y = 2$, and (c) $y = 3$.

$$a_2 = \frac{a_3k_2}{k_3}, \quad a_4(t) = -\frac{a_5, a_8(t)}{a_1} + p,$$
$$a_6 = \frac{a_3k_2}{k_3}, \quad k_1 = 0, \quad c = -\frac{dk_3}{k_2}, \tag{27}$$

where p is the integral constant. Then, the function f can be simplified as follows:

$$\begin{aligned} f = &\left(a_1x + \frac{a_3k_2y}{k_3} + a_3z - \frac{a_8(t)a_5}{a_1} + p\right)^2 \\ &+ \left(a_5x + \frac{a_7k_2y}{k_3} + a_7z + a_8\right)^2 \\ &+ k\text{JacobiSN}(k_2y + k_3z + k_4, m) + a_9, \end{aligned} \tag{28}$$

from which the interaction solution between lump and Jacobi elliptic function waves for Eq. (3) can be obtained by substituting Eq. (28) into $u = (\ln f)_x$. To demonstrate the corresponding dynamic properties, we take the parameters $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = -t$, $a_5 = 1$, $a_6 = 1$, $a_7 = 1$, $a_8 = t$, $a_9 = 2$, $k_1 = 0$, $k_2 = 1$, $k_3 = 2$, $k_4 = t^2$, $k = 1$, $c = -1$, $m = 0.5$, $p = 0$, and the interaction solution can be derived as follows:

$$u = \frac{4(2x + y + 2z)}{2(x + y + z)^2 + 2t^2 + \text{JacobiSN}(t^2 + y + 2z, 0.5) + 2}. \tag{29}$$

Figure 4 shows the interaction process between the lump and Jacobi elliptic function waves at different times. It can be observed that as time t increase, the effect of the interaction between the two waves gradually weakens, which is consistent with the physical phenomenon of energy collision.

5 Mixed interaction solutions

In this section, we attempt to construct the mixed interaction solutions combined with Jacobian, exponential, and integral quadratic functions. To this end, the function f is taken as follows:

$$f = \xi_1^2 + \xi_2^2 + \alpha e^g + \beta\text{JacobiSN}(h, m) + a_9, \tag{30}$$

with

$$\begin{aligned} \xi_1 &= a_1x + a_2y + a_3z + a_4(t), \\ \xi_2 &= a_5x + a_6y + a_7z + a_8(t), \\ g &= k_1x + k_2y + k_3z + k_4, \\ h &= k_5x + k_6y + k_7z + k_8(t), \end{aligned} \tag{31}$$

where $a_4(t)$, $a_8(t)$, and $k_8(t)$ are the functions of variable t. Besides them, $k_i (1 \le i \le 7)$, $a_j (1 \le j \le 9)(j \ne 4, 8)$, α, β are real parameters to be determined.

By substituting Eq. (30) for Eq. (3), we have

$$a_2 = \frac{a_3k_6}{k_7}, \quad a_4(t) = -\frac{a_5a_8(t)}{a_1} + p, \quad a_6 = \frac{a_7k_6}{k_7},$$
$$k_2 = \frac{k_6(k_1^2 + dk_3)}{dk_7}, \quad k_5 = 0, \quad c = -\frac{dk_7}{k_6}, \tag{32}$$

where p is integral constant, from which the function f can be derived as follows:

$$\begin{aligned} f = &\left(a_1x + \frac{a_3k_6}{k_7}y + a_3z + -\frac{a_5a_8(t)}{a_1} + p\right)^2 \\ &+ \left(a_5x + \frac{a_7k_6}{k_7}y + a_7z + a_8(t)\right)^2 \\ &+ \alpha e^{k_1x + \frac{k_6(k_1^2 + dk_3)}{dk_7}y + k_3z + k_4} + \beta\text{JacobiSN}(k_6y \\ &+ k_7z + k_8(t), m). \end{aligned} \tag{33}$$

Similar to the previous cases, the mixed solution u can be obtained by directly substituting. Figure 5 shows the evolution behavior of the mixed solution by taking parameter values $a_1 = 1$, $a_2 = 1$, $a_3 = 1$, $a_4 = -t$, $a_5 = 1$, $a_6 = 1$, $a_7 = 1$, $a_8 = t$, $a_9 = 2$, $k = 1$, $k_1 = 1$, $k_2 = 3/2$, $k_3 = 1$, $k_4 = 1$, $k_5 = 0$, $k_6 = 1$, $k_7 = 1$, $k_8 = t$, $d = 2$, $c = -1$, $m = 0.5$, $p = 0$, $\alpha = 1$, $\beta = 1$, under which the mixed solution can be simplified as follows:

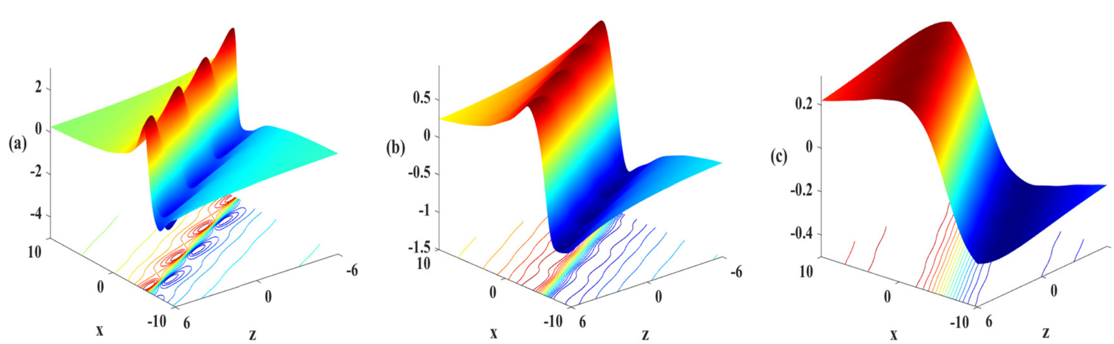

Figure 4: The 3D and contour propagation for the interaction solution (29) with parameters $m = 0.5$, $y = -0.1$ and (a) $t = 0$, (b) $t = 2$, and (c) $t = 6$.

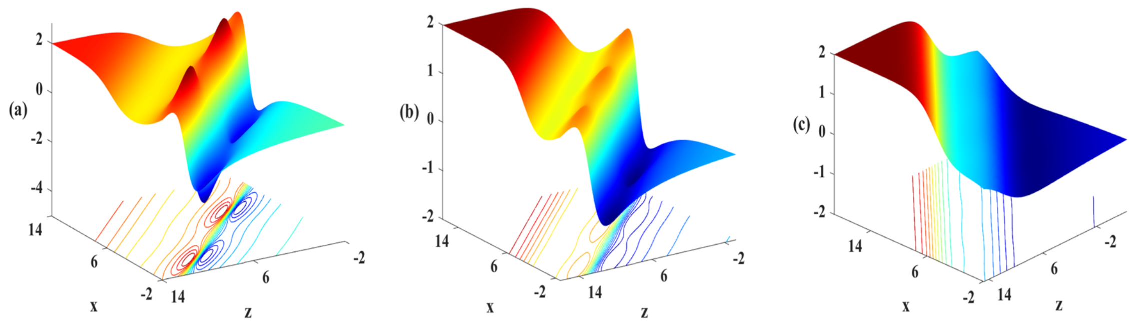

Figure 5: The 3D and contour propagation for the interaction solution (34) with parameters $m = 0.5, y = -10$ and (a) $t = 0$, (b) $t = 1.5$, and (c) $t = 5$.

$$u = \frac{2\left(4x + 4y + 4z + e^{z+1+x+\frac{3y}{2}}\right)}{2t^2 + 2(x + y + z)^2 + e^{z+1+x+\frac{3y}{2}}} + \mathrm{JacobiSN}(y + z + t, 0.5) + 2 \quad (34)$$

In contrast to the case 2 figure presented in Section 4, Figure 5 depicts the presence of an additional striped soliton, which is more readily discernible from the expression of function f (30). It can be observed from Figure 5 that interactions between the three types of waves over time exhibit similarities to those observed in previous sections.

6 Conclusion and discussion

The present article examines the interactions between three distinct types of solutions, namely, lump, stripe solitons, and Jacobi elliptic function solutions. The (3+1)-dimensional HSIL equation has been constructed based on a modified direct method, with an accompanying discussion of its associated evolution and dynamics. In contrast to the independent variables of existing methods, which are typically represented by linear functions of spatial and temporal variables, the variables in this study represent a combination of linear functions of spatial variables and arbitrary spatial functions, thereby exhibiting a more intricate relationship with the dependent variables. As a result, a number of fascinating dynamical properties of the interaction solutions for the HSIl equation have been identified, which may provide theoretical insight into the underlying mechanisms of related nonlinear physical phenomena.

The study of interaction solutions enriches the structure of nonlinear local wave solutions for soliton equations, provides theoretical models for characterizing complex nonlinear phenomena in different physical fields, including nonlinear optics, plasma and oceanography, and provides theoretical guidance for predicting new nonlinear phenomena and designing new physical experiments. At the same time, it provides an information source for studying complex nonlinear waves using numerical simulation and deep learning method. Of course, there are still many issues that need further investigation, such as how to improve this method to obtain more physically meaningful interaction solutions and nonlinear dynamical properties? How to extend this method to more nonlinear evolution equations with practical physical significance? How to effectively combine numerical simulation and deep learning method to study the nonlinear dynamics of the interaction solutions in depth? These are also our upcoming studies in the near future.

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