Research Article

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On some solitary wave solutions of the Estevez–Mansfield–Clarkson equation with conformable fractional derivatives in time

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Abstract: In this study, a generalization of the Estevez–Mansfield–Clarkson (EMC) equation that considers the presence of conformable time-fractional derivatives is investigated analytically. The integer-order model finds applications in mathematical physics, optics, and the investigation of shape developing in liquid drops. In this study, the Sardar sub-equation method, is employed to solve the generalized EMC equation. From the Sardar sub-equation method a broad range of soliton solutions, including dark-bright, combined dark-singular and periodic singular solitons, have been obtained. Some of the results derived in this study are plotted to illustrate that the solutions are solitary waves, indeed.

Keywords: Sardar sub-equation method, Estevez–Mansfield–Clarkson equation, exact analytical solutions, solitary wave solutions

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1 Introduction

The purpose of this note is to derive a particular class of analytical solutions for some nonlinear evolution equations (NLEEs) in the form of traveling waves [1–3]. In particular, the Estevez–Mansfield–Clarkson (EMC) equation exhibits distinct characteristics compared to other well-known NLEEs, each of which plays a unique role in modeling various physical phenomena. The Korteweg–de Vries (KdV) equation, known for describing shallow water waves, is significant for its soliton solutions that emerge from the interplay of nonlinearity and dispersion. The modified KdV equation, which introduces stronger nonlinear terms, supports sharper soliton solutions, making it useful for steeper waveforms. Burgers' equation, often employed to model viscous fluid flows and turbulence, includes dissipative terms that lead to shock wave formation rather than solitons. The sine-Gordon equation, with its kink and antikink solutions, is relevant to models in field theory, crystal dislocations, and DNA dynamics. In contrast, the Sharma–Tasso–Olver equation, such as KdV, features solitonic solutions but with a modified balance between dispersion and nonlinearity, making it applicable in plasma and nonlinear wave studies. The EMC equation shares similarities with these equations but also incorporates distinctive nonlinear and dispersive terms, giving it unique solution properties that bridge gaps in classical nonlinear wave models, particularly for phenomena with complex wave interactions. Comparing these equations highlights the EMC equation's potential to model unique nonlinear dynamics across various physical systems.

In the practice, nonlinear fractional partial differential equations (NLFPDEs) are often used to mathematically formulate problems. Such scenarios require being modeled in a number of distinct situations. Fractional differential equations in engineering, mathematical physics, and dynamical systems have all been made part of numerous pieces of recent literature, by a wide range of authors in various fields of study. It is well known that fractional differential equations are an extension of classical differential equations to an arbitrary (non-integer) order. In particular, fractional differential equations involving non-local spatial and temporal interactions can be represented by power-law memory kernels. Scientific research into fractional differential equations is rising significantly due to its wide use in engineering and science. Many researchers [4–6] have examined fractional differential equations and applied several kinds of methods for solving them. In the past, fractional derivatives and integrals were thought to belong in the theoretical field of mathematics. However, a number of research investigations conducted in the previous decades have indicated that fractional phenomena may be connected to applied mathematics and engineering subjects such as control theory, water wave mechanics, plasma physics, geochemistry, vascular mechanics, fluid mechanics, and optical fibers, along with mathematics.

Only in the last few years has the use of fractional derivatives in mathematical modeling played a significant role. NLFPDEs are a more prominent way to represent any naturally occurring phenomenon. Various mathematical models are represented in a form involving different fractional derivatives, such as Atangana's conformable derivative [7], local M-derivative [8,9], p-order truncated M-derivative [10], Jumarie's fractional derivative [11], and truncated M-fractional Westervelt model [12]. The purpose of this work is to employ the Sardar sub-equation method in the investigation of traveling wave solutions of the EMC equation. That model is a non-linear system with potential applications in various fields [13–15]. In this work, we will suppose that $Z = Z(x, t)$, where $(x, t) \in \mathbb{R} \times [0, \infty)$. The function Z will satisfy the time-fractional extension of the EMC equation given by

$$D_t^{\alpha}D_{xxx}Z + aD_xZD_t^{\alpha}D_xZ + aD_{xx}ZD_t^{\alpha}Z + D_t^{2\alpha}Z = 0, \quad \forall(x, t) \in \mathbb{R} \times [0, \infty), \tag{1}$$

where t represents time, and x denotes the spatial position in one dimension. Moreover, a is a free parameter, and the operators D_x, D_{xx}, and D_{xxx} are the usual integer-order differential operators in space. On the other hand, α is a number in $(0, 1)$, and D_t^{α} and $D_t^{2\alpha}$ denote the fractional derivative operators of order α and 2α, respectively. For purposes of this report, the fractional operators will be understood in the conformable sense. We provide the definition and some properties of these operators in the following section, along with a brief description of the methodology used in this manuscript.

Shehzad *et al.* worked on the NLEEs to derive bright-dark and breathers wave solutions using the $(G'/G, 1/G)$-expansion technique [16]. Nasreen *et al.* used the extended and modified rational expansion method to obtain the optical solitons for the third-order nonlinear Schrödinger equation [17]. Kai and Yin, worked on the Sharma–Tasso–Olver–Burgers equation to explore soliton structures [18]. Alam *et al.* worked on the bifurcation analysis and new exact complex solutions for the nonlinear Schrödinger equations [19]. Mathanaranjan also used the extended sinh-Gordon equation expansion method [20], new extended auxiliary equation method [21,22], modified F-expansion method [23], modified Jacobi elliptic function expansion, and the unified Riccati equation expansion [24,25]. In summary, multiple efficient methods have been efficiently developed as a way to find exact solutions for NLFPDEs. Among those methods, we can mention the generalized (G'/G)-expansion approach [26,27], the Kerr and power law of nonlinearity [28], the exp-function method [29], the extended tanh-function method [30], the extended sinh-Gordon equation method [31], the improved tanh method [32], the rational (G'/G)-expansion technique [33], the directed extended Riccati method [34,35], the first-integral method [36], the sub-equation method [37], the Riemann–Hilbert approach [38,39], the two variable $(G'/G, 1/G)$-expansion method [40,41], the modified extended tanh-function method [42], the generalized Jacobi elliptic function technique [43], and the Hirota bilinear method [44], among others. But in this study, we use the Sardar subequation method, which is a mathematical technique.

The Sardar sub-equation method is a technique used to find exact solutions of nonlinear partial differential equations (NLPDEs), including solitary wave and soliton solutions. The method is often simpler compared to other methods for solving nonlinear partial differential equations (PDEs). It reduces the NLPDE to an ordinary differential equation (ODE), which is easier to solve. It provides exact solutions, including solitary waves and soliton solutions, which are valuable for understanding the behavior of nonlinear systems in fields such as fluid dynamics, plasma physics, and optical fibers. The limitation of this method is that it does not involve any linearization or perturbation techniques, unlike other solution methods that use the two of them. This method is not applicable for all types of the NLPDEs but only for equations that provide even-order ODEs. If the ODEs include odd-order derivatives, then this method fails. It maintains the nonlinearity of the problem and changes the nonlinear terms considerably. While the method can be adopted for all the NLPDEs, it may not yield results with certain category of NLPDEs, especially those with more complicated nonlinearities. The goal of this study is to calculate exact soliton and solitary wave solutions for the conformable time-fractional

EMC equation using the Sardar sub-equation method. In this way, we will obtain a broad range of soliton solutions, including dark-bright, combined dark-singular and periodic singular solitons.

2 Preliminaries

The conformable derivative is a type of fractional derivative that generalizes the derivatives of integer orders to the fractional case, retaining certain properties of classical derivatives. Here, we provide a fresh start to this discussion by recalling the definition of conformable derivatives and some of their properties.

Definition 1. Let $g:[0,\infty)\to\mathbb{R}$ be a differentiable function, and suppose that $0<\alpha<1$. Then, the *conformable derivative* of g of order α at the point t is defined as

$$D^{\alpha}(g)(t)=\lim_{\varepsilon\to 0}\frac{g(\varepsilon t^{1-\alpha}+t)-g(t)}{\varepsilon}, \tag{2}$$

for each $t>0$ (see [12] for additional information).

Theorem 1. *Suppose that $\mu,\nu:[0,\infty)\to\mathbb{R}$ are differentiable functions, and let $\alpha\in(0,1)$. Then,*
- $D^{\alpha}(m\mu+n\nu)=mD^{\alpha}(\mu)+nD^{\alpha}(\nu)$, *for each* $m,n\in\mathbb{R}$,
- $D^{\alpha}(t^{r})=rt^{r-\alpha}$, *for each* $r\in R$,
- $D^{\alpha}(\mu\nu)=\mu D^{\alpha}(\nu)+\nu D^{\alpha}(\mu)$,
- $D^{\alpha}\left(\frac{\mu}{\nu}\right)=\frac{\nu D^{\alpha}(\mu)-\mu D^{\alpha}(\nu)}{\nu^{2}}$,
- $D^{\alpha}(m)=0$, *for each* $m\in\mathbb{R}$,
- $D^{\alpha}(\mu(t))=t^{1-\alpha}\frac{\mathrm{d}\mu(t)}{\mathrm{d}t}$, *for each* $t>0$.

In order to provide analytical solutions for Eq. (1), we will employ the Sardar sub-equation method, which consists of the following steps.

Step 1. Let us assume that the mathematical model is an NLPDE, which can be rewritten in the following form:

$$A(Z,D_t^{\alpha}Z,D_xZ,D_t^{2\alpha}Z,D_{xx}Z,\ldots)=0, \tag{3}$$

where A is a polynomial form that depends on the parameters inside the parentheses. Here, the *fractional traveling-wave transformation* $Z=Z(x,t)$ is given by $Z(x,t)=H(\phi)$, where $\phi=kx+\frac{ct^{\alpha}}{\alpha}$. The symbols k and c represent the real constants, and α is the fractional order of differentiation. Substituting this transformation into Eq. (3), we readily obtain that

$$B(H,H',H'',H''',\ldots)=0, \tag{4}$$

where B is a polynomial of H.

Step 2. Suppose Eq. (4) has a solution of the form

$$H(\phi)=\sum_{j=0}^{m}\omega_j\eta^{j}(\phi),\quad \omega_m\neq 0, \tag{5}$$

where ω_j are the coefficients to be determined, for each $j=0,1,\ldots,m$, and $\eta(\phi)$ satisfies the ODE

$$\eta'(\phi)=\sqrt{\sigma+f\eta^{2}(\phi)+\eta^{4}(\phi)}. \tag{6}$$

Then, the solutions of Eq. (6) are given as follows.

- **Case 1.** If $\sigma=0$ and $f>0$, then

$$\eta_1^{\pm}(\phi)=\pm\sqrt{-dbf}\,\mathrm{sech}_{db}(\sqrt{f}\phi), \tag{7}$$

$$\eta_2^{\pm}(\phi)=\pm\sqrt{-dbf}\,\mathrm{csch}_{db}(\sqrt{f}\phi), \tag{8}$$

where $\mathrm{sech}_{\mathrm{db}}(\phi)=2(de^{\phi}+be^{-\phi})^{-1}$ and $\mathrm{csch}_{\mathrm{db}}(\phi)=2(de^{\phi}-be^{-\phi})^{-1}$.

- **Case 2.** If $\sigma=0$ and $f<0$, then

$$\eta_3^{\pm}(\phi)=\pm\sqrt{-dbf}\,\sec_{db}(\sqrt{-f}\phi), \tag{9}$$

$$\eta_4^{\pm}(\phi)=\pm\sqrt{-dbf}\,\csc_{db}(\sqrt{-f}\phi), \tag{10}$$

where $\sec_{\mathrm{db}}(\phi)=2(de^{i\phi}+be^{-i\phi})^{-1}$ and $\mathrm{csch}_{\mathrm{db}}(\phi)=2i(de^{i\chi}-be^{-i\phi})^{-1}$.

- **Case 3.** If $f<0$ and $\sigma=\frac{f^{2}}{4}$, then

$$\eta_5^{\pm}(\phi)=\pm\sqrt{-\frac{f}{2}}\tanh_{db}\left(\sqrt{-\frac{f}{2}}\phi\right), \tag{11}$$

$$\eta_6^{\pm}(\phi)=\pm\sqrt{-\frac{f}{2}}\coth_{db}\left(\sqrt{-\frac{f}{2}}\phi\right), \tag{12}$$

$$\eta_7^{\pm}(\phi)=\pm\sqrt{-\frac{f}{2}}(\tanh_{db}(\sqrt{-2f}\phi)+\sqrt{db}\,\mathrm{sech}_{de}(\sqrt{-2f}\phi)), \tag{13}$$

$$\eta_8^{\pm}(\phi)=\pm\sqrt{-\frac{f}{2}}(\coth_{db}(\sqrt{-2f}\phi)+\sqrt{db}\,\mathrm{csch}_{db}(\sqrt{-2f}\phi)), \tag{14}$$

$$\eta_9^{\pm}(\phi)=\pm\sqrt{-\frac{f}{2}}\left(\tanh_{db}\left(\sqrt{-\frac{f}{2}}\phi\right)+\coth_{db}\left(\sqrt{-\frac{f}{2}}\phi\right)\right), \tag{15}$$

where $\tanh_{\mathrm{db}}(\phi)=(de^{\phi}-be^{-\phi})(be^{\chi}+be^{-\phi})^{-1}$ and $\coth_{\mathrm{db}}(\phi)=(de^{\phi}+be^{-\phi})(de^{\phi}-be^{-\phi})^{-1}$.

- **Case 4.** If $f>0$ and $\sigma=\frac{f^{2}}{4}$, then

$$\eta_{10}^{\pm}(\phi)=\pm\sqrt{\frac{f}{2}}\tan_{db}\left(\sqrt{\frac{f}{2}}\phi\right), \tag{16}$$

$$\eta_{11}^{\pm}(\phi)=\pm\sqrt{\frac{f}{2}}\cot_{db}\left(\sqrt{\frac{f}{2}}\phi\right), \tag{17}$$

$$\eta_{12}^{\pm}(\phi) = \pm\sqrt{\frac{f}{2}}(\tan_{db}(\sqrt{2f}\phi) + \sqrt{db}\sec_{de}(\sqrt{2f}\phi)), \quad (18)$$

$$\eta_{13}^{\pm}(\phi) = \pm\sqrt{\frac{f}{2}}(\cot_{db}(\sqrt{2f}\phi) + \sqrt{db}\csc_{de}(\sqrt{2f}\phi)), \quad (19)$$

$$\eta_{14}^{\pm}(\phi) = \pm\sqrt{\frac{f}{2}}\left(\tan_{db}\left(\sqrt{\frac{f}{2}}\phi\right) + \cot_{db}\left(\sqrt{\frac{f}{2}}\phi\right)\right), \quad (20)$$

where $\tan_{\mathrm{de}}(\phi) = -i(de^{\phi} - be^{-\phi})(de^{\phi} + be^{-\phi})^{-1}$ and $\coth_{\mathrm{db}}(\phi) = -i(de^{\phi} + be^{-\phi})(de^{\phi} - be^{-\phi})^{-1}$.

Step 3. Use the balancing rule to determine the value of m in Eq. (5).
Step 4. Next, substitute Eqs (5) and (6) into (4) to determine the equations in power of $\eta(\phi)$.
Step 5. After obtaining nonzero solutions and setting all the coefficients of $\eta(\phi)$ equal to zero to produce the system of algebraic equations, we find out solutions by solving this system.

3 Results

To start with, we transform the PDE in Eq. (1) using the fractional traveling-wave transformation $\phi = kx + \frac{ct^{\alpha}}{\alpha}$. Substituting this transformation into Eq. (1) and letting $H' = V$, we readily obtain the ODE

$$c^2V + ack^2V^2 + ck^3V'' = 0. \quad (21)$$

Using now the balancing procedure on Eq. (21) yields that $m = 2$. Substituting this value into Eq. (5), we obtain that

$$H(\phi) = \omega_0 + \omega_1\eta(\phi) + \omega_2\eta(\phi)^2, \quad (22)$$

where ω_0, ω_1, and ω_2 are the constants. Substituting now Eq. (22) and their derivatives into Eq. (21), we reach the system of algebraic equations

$$\begin{aligned} ack^2\omega_0^2 + c^2\omega_0 + 2ck^3\sigma\omega_2 &= 0,\\ 2ack^2\omega_0\omega_1 + c^2\omega_1 + cgk^3\omega_1 &= 0,\\ ack^2\omega_1^2 + 2ack^2\omega_0\omega_2 + c^2\omega_2 + 4cgk^3\omega_2 &= 0,\\ ack^2\omega_1^2 + 2ack^2\omega_0\omega_2 + c^2\omega_2 + 4cgk^3\omega_2 &= 0,\\ ack^2\omega_2^2 + 6ck^3\omega_2 &= 0. \end{aligned} \quad (23)$$

It is possible to check now that the solution of this system is given by

$$\omega_0 = -\frac{\sqrt{c^2 + 48k^6\sigma} + c}{2ak^2}, \quad \omega_1 = 0, \quad g = \frac{\sqrt{c^2 + 48k^6\sigma}}{4k^3}, \quad \omega_2 = -\frac{6k}{a}. \quad (24)$$

Now, we are in a position to derive exact solitary wave solutions for our mathematical model. More precisely, the solutions are obtained from Eqs (6) and (22) when we substitute Eq. (24) and any of the functions in the cases presented in the previous section. After performing those substitutions and reducing algebraically, we obtain the following functions:

$$Z_1(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{6dbfk}{a}\operatorname{sech}^2\left(\sqrt{f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right), \quad (25)$$

$$Z_2(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{6dbfk}{a}\operatorname{csch}^2\left(\sqrt{f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right), \quad (26)$$

$$Z_3(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{6dbfk}{a}\sec^2\left(\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right), \quad (27)$$

$$Z_4(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{6dbfk}{a}\csc^2\left(\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right). \quad (28)$$

$$Z_5(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{3fk}{a}\tanh^2\left(\frac{\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)}{\sqrt{2}}\right). \quad (29)$$

$$Z_6(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{3fk}{a}\coth^2\left(\frac{\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)}{\sqrt{2}}\right), \quad (30)$$

$$Z_7(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{3fk}{a}\left[\sqrt{db}\operatorname{sech}\left(\sqrt{2}\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right) + \tanh\left(\sqrt{2}\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right)\right]^2, \quad (31)$$

$$Z_8(x,t) = -\frac{1}{2ak^2}(\sqrt{c^2 + 48k^6\sigma} + c) + \frac{3fk}{a}\left[\sqrt{db}\operatorname{csch}\left(\sqrt{2}\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right) + \coth\left(\sqrt{2}\sqrt{-f}\left(\frac{ct^{\alpha}}{\alpha} + kx\right)\right)\right]^2, \quad (32)$$

$$Z_9(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c\right. + \frac{3fk}{a}\left[\tanh\left(\frac{\sqrt{-f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right) + \coth\left(\frac{\sqrt{-f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right)\right]^2\Bigg), \tag{33}$$

$$Z_{10}(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c - \frac{3fk}{a}\tan^2\left(\frac{\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right)\right), \tag{34}$$

$$Z_{11}(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c - \frac{3fk}{a}\cot^2\left(\frac{\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right)\right), \tag{35}$$

$$Z_{12}(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c - \frac{3fk}{a}\left[\sqrt{db}\sec\left(\sqrt{2}\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)\right) + \tan\left(\sqrt{2}\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)\right)\right]^2\right), \tag{36}$$

$$Z_{13}(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c - \frac{3fk}{a}\left[\sqrt{db}\csc\left(\sqrt{2}\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)\right) + \cot\left(\sqrt{2}\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)\right)\right]^2\right), \tag{37}$$

$$Z_{14}(x,t) = -\frac{1}{2ak^2}\left(\sqrt{c^2+48k^6\sigma}+c - \frac{3fk}{a}\left[\tan\left(\frac{\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right) + \cot\left(\frac{\sqrt{f}\left(\frac{ct^\alpha}{\alpha}+kx\right)}{\sqrt{2}}\right)\right]^2\right). \tag{38}$$

4 Discussion

In this section, we discuss the graphical results for the fractional-order EMC model. The solutions for different wave structures have been numerically examined in three-dimensional, two-dimensional, and contour forms using suitable values of the parameters. We showed that the soliton solutions appear as bright soliton, dark soliton, singular soliton, and solitary wave. We are convinced that the results of this study are novel. Indeed, to the best of our knowledge, there are no previous results that use the analytical methodology employed in this work on the fractional-order EMC model. Moreover, it is important to note that our analytical findings can be related to those derived from other approaches, such as the Riccati ODE expanding techniques, the F-expansion method, the tanh-method, and other related approaches, within specific limitations. Our work might be interpreted as a generalization of existing techniques, providing a more comprehensive framework from which these particular techniques can be drawn or comprehended.

It is important to mention that the solutions were plotted by means of specialized mathematical software. In this section, we plot the graphs of some of the solutions. As we mentioned previously, we plot three-dimensional, two-dimensional, and contour plots in each case. Observe

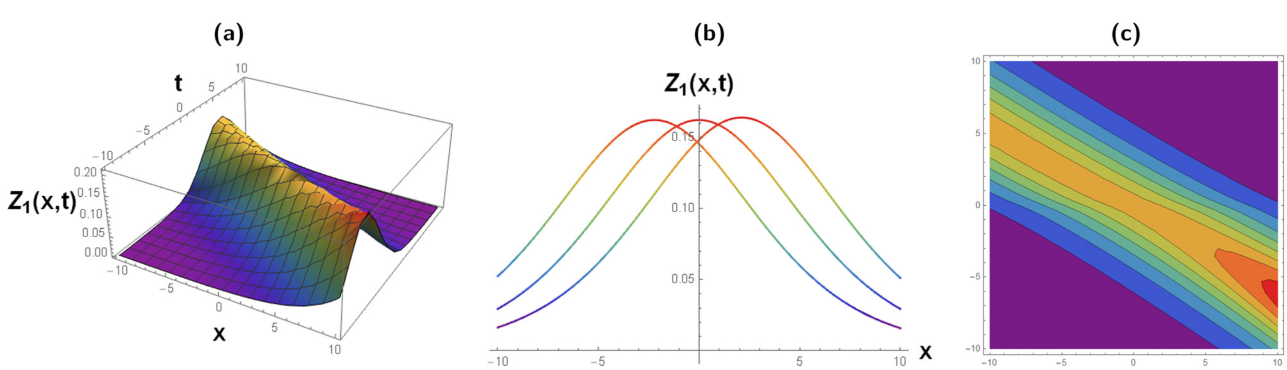

Figure 1: (a) Three-dimensional, (b) two-dimensional, and (c) contour plots showing the graphical behavior of Eq. (25) with parameters $a = 1.5$, $\alpha = 0.9$, $c = -1$, $d = -0.9$, $b = -0.4$, $f = 0.09$, $k = -0.5$, $l = -1.7$, and $\sigma = 0$.

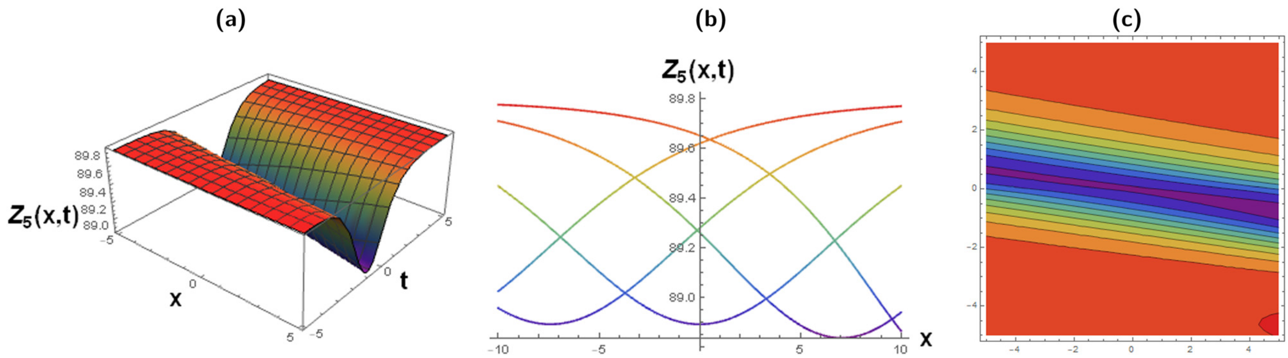

Figure 2: (a) Three-dimensional, (b) two-dimensional, and (c) contour plots showing the graphical behavior of Eq. (29) with parameters $a = 0.5$, $\alpha = 0.9$, $c = 1$, $d = 0.9$, $b = 0.4$, $f = -1$, $k = 0.15$, $l = 0.7$, and $\sigma = 0.25$.

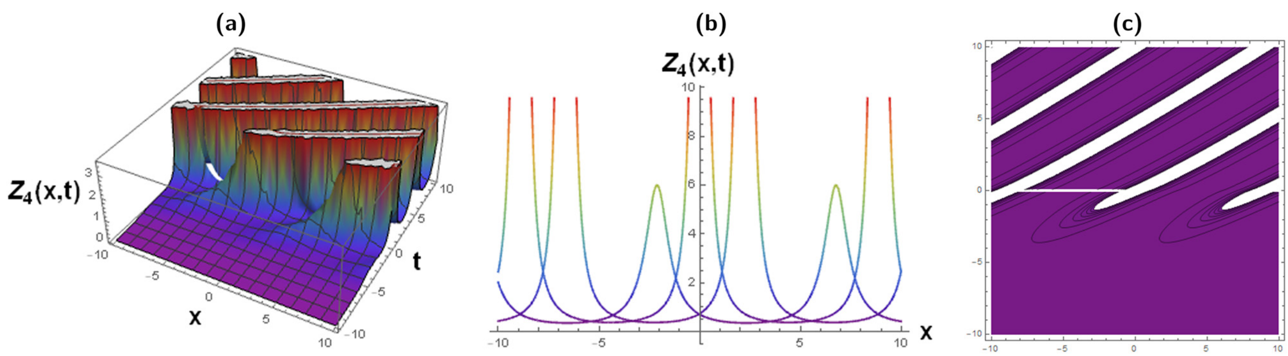

Figure 3: (a) Three-dimensional, (b) two-dimensional, and (c) contour plots showing the graphical behavior of Eq. (28) with parameters $a = 1.5$, $\alpha = 0.9$, $c = -1$, $d = 0.9$, $b = 0.4$, $f = -0.5$, $k = 0.5$, $l = 1.7$, and $\sigma = 0$.

that the solutions in Figure 1 represent the bright solitons, and Figure 2 represent dark soliton solutions. Meanwhile, Figures 3–7 show the presence of multiwave structures. In each case, the figures were obtained by assigning suitable values to the model and solution parameters. For convenience, those parameter values are provided in the caption of each figure. Finally, Figures 8 and 9 are obtained for various values of the fractional order $\alpha = 0.3, 0.6$, and 0.9, which shows the impact of the fractional differentiation order on the behavior of the solitons. In the former, we chose the solution (25), while the latter uses (29).

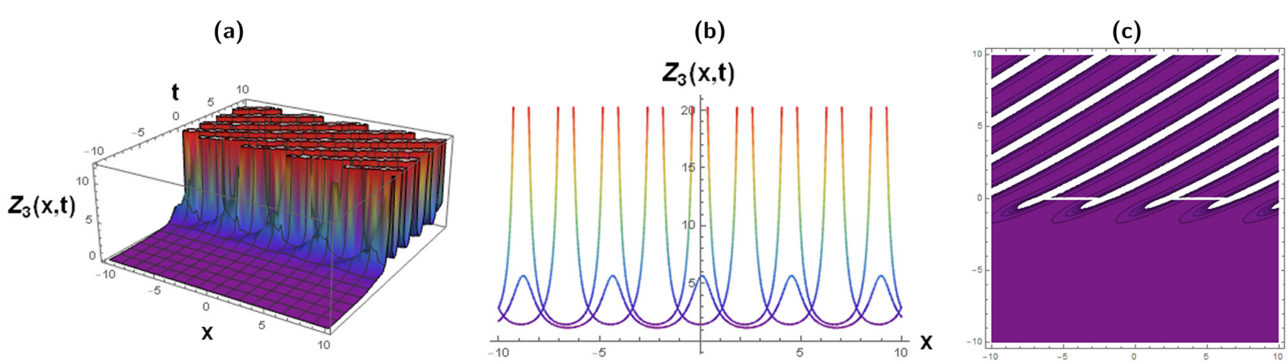

Figure 4: (a) Three-dimensional, (b) two-dimensional, and (c) contour plots showing the graphical behavior of Eq. (27) with parameters $a = 1.5$, $\alpha = 0.9$, $c = -1$, $d = 0.9$, $b = 0.4$, $f = -2$, $k = 0.5$, $l = 1.7$, and $\sigma = 0$.

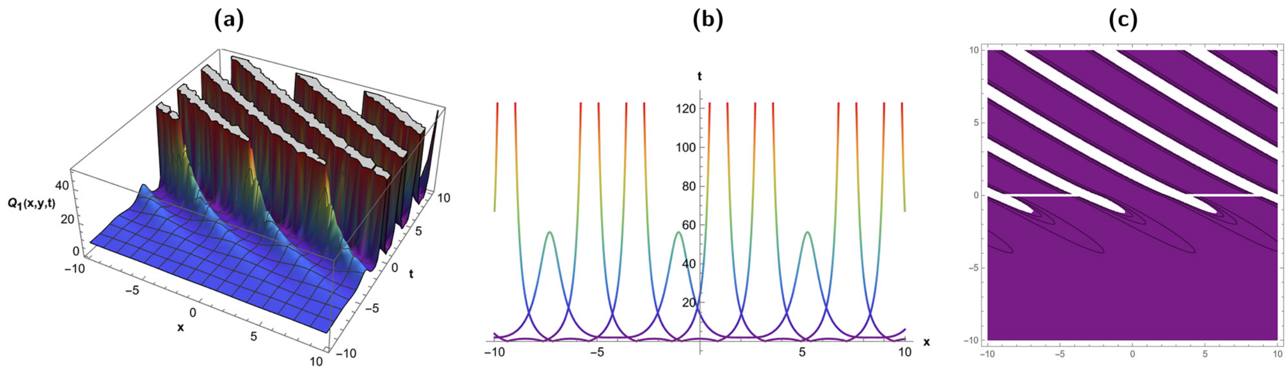

Figure 5: (a) Three-dimensional, (b) two-dimensional, and (c) contour plots showing the graphical behavior of Eq. (34) with parameters $a = -0.5$, $\alpha = 0.9$, $c = -1$, $d = 0.9$, $b = 0.4$, $f = 2$, $k = -0.5$, $l = 0.7$, and $\sigma = 1$.

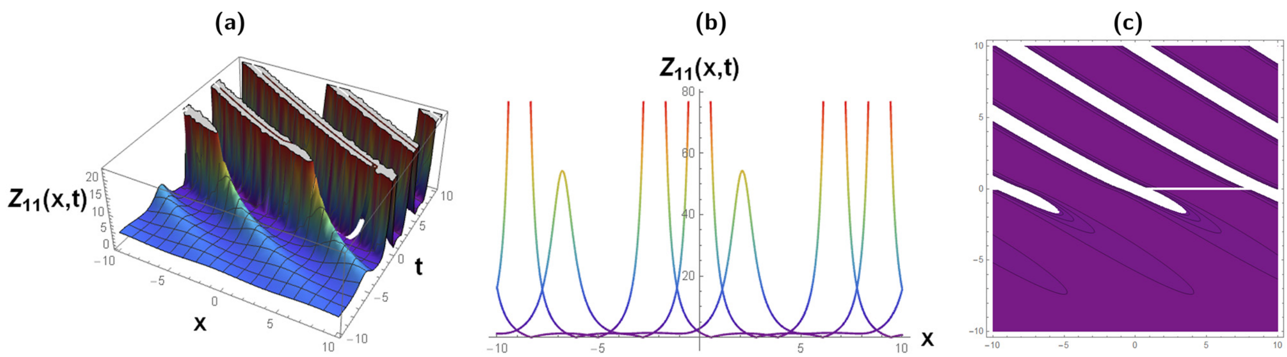

Figure 6: (a) Three-dimensional, (b) two-dimensional and (c) contour plots showing the graphical behavior of Eq. (35) with parameters $a = -0.5$, $\alpha = 0.9$, $c = -1$, $d = 0.9$, $b = 0.4$, $f = 1$, $k = -0.5$, $l = 0.7$, and $\sigma = 1$.

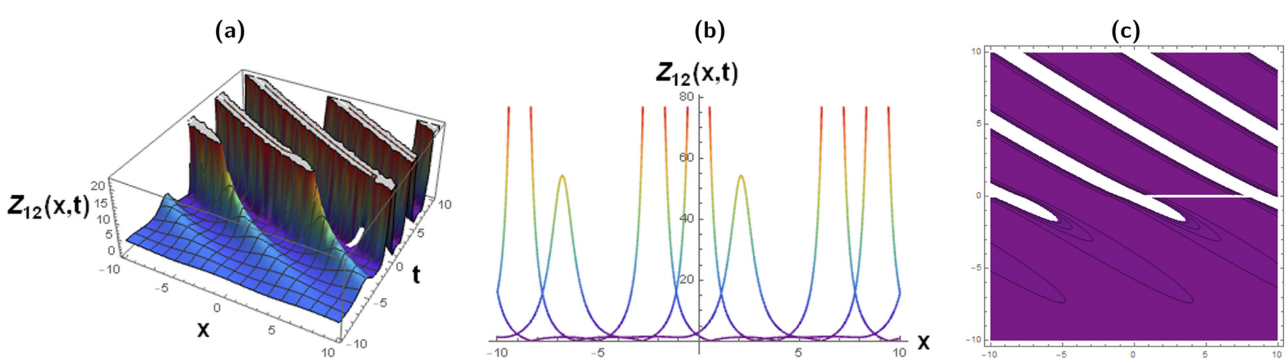

Figure 7: (a) Three-dimensional, (b) two-dimensional and (c) contour plots showing the graphical behavior of Eq. (36) with parameters $a = -0.5$, $\alpha = 0.9$, $c = -1$, $d = 0.9$, $b = 0.4$, $f = 0.8$, $k = -0.5$, $l = 0.7$, and $\sigma = 1$.

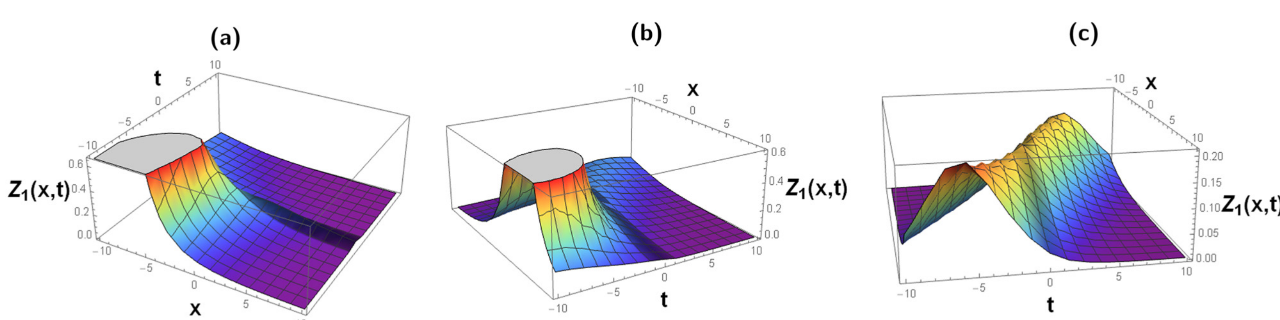

Figure 8: Three-dimensional plots of the impact of fractional order on the solution from Eq. (25) with (a) $\alpha = 0.3$, (b) $\alpha = 0.6$, and (c) $\alpha = 0.9$.

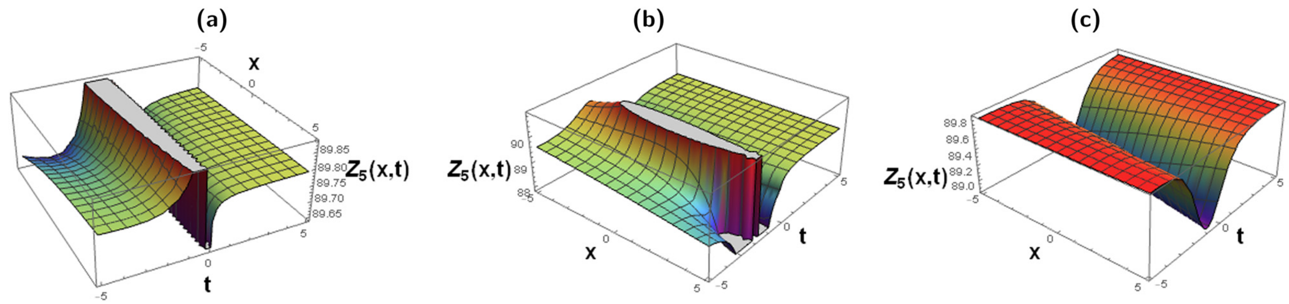

Figure 9: Three-dimensional plots of the impact of fractional order on the solution from Eq. (25) with (a) α = 0.3, (b) α = 0.6, and (c) α = 0.9.

5 Conclusion

In this work, we employed the Sardar sub-equation method to derive exact solitary wave solutions of a generalized EMC equation. The mathematical model investigated here is a PDE, which considers the presence of fractional-order derivatives in time, which are understood in the conformable sense. To start with, the mathematical model is transformed into an ODE using the fractional traveling-wave transformation. The Sardar sub-equation method is applied then to derive various solutions in exact form. The associated solutions are in terms of generalized hyperbolic and generalized trigonometric functions. In that way, we derive the solutions for combined dark-bright, singular, periodic singular, dark, bright, and combined dark-singular soliton. Some figures are provided in this work, and they provide evidence of the different natures of the solutions.

In the physical sciences, and especially in fluid mechanics, solitary waves describe the stable, localized waveforms that maintain their shape while propagating, as observed in shallow water or internal waves in stratified fluids. These solutions are invaluable for modeling tsunamis and coastal waves, where understanding wave stability and interaction is crucial for predicting wave behaviors and designing protective infrastructure. In optical fiber systems, solitary wave (or soliton) solutions play an essential role in the stable transmission of data. Solitons, arising from the balance between dispersion and nonlinearity in the medium, prevent signal distortion over long distances, making them ideal for high-speed, long-range communication. As a future direction of investigation after this study, the authors will consider nonlinear dynamical systems under time-fractional derivatives of different types, including Caputo-type fractional derivatives. Moreover, the authors intend to take into account the presence of noise in NLEEs, as it was done in the study by Macías-Díaz et al. [45].

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