Research Article

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Physical aspects of quantile residual lifetime sequence

https://doi.org/10.1515/phys-2024-0089
received May 01, 2024; accepted September 23, 2024

Abstract: The modeling of count data is found in many fields, such as statistical physics, public health, medicine, epidemiology, applied science, sociology, and agriculture. In many physical situations, it has been observed that many times in the real world, the original variables may be continuous in nature, but discrete by observation. In this study, the α-quantile residual life function for discrete lifetime models is defined and some attributes are investigated. The relation between this measure and the hazard rate function is studied. We discuss how this measure could be useful for finding the burn-in time of a lifetime dataset. Then, a new stochastic order based on the α-quantile residual life is proposed and studied.

Keywords: discrete lifetime model, α-quantile residual life, hazard rate function, stochastic orders

1 Introduction

In statistical physics, modern stochastic analysis, and traditional probability and statistics, there is a way to characterize a static or dynamic distribution using its quantile function. A direct understanding of this function offers tangible benefits that cannot be derived directly from the density function. For example, the simplest way to simulate a non-uniform random variable is to apply its quantile function to uniform deviations. Modern Monte Carlo simulation methods, techniques based on low-discrepancy sequences, and copula methods require the use of marginal distribution quantile functions. Consequently, the study of quantile functions is as important for management as many classical special functions in mathematical physics and applied analysis. In certain situations, the lifetime of an object should be measured on a discrete scale, *e.g.*, the number of power fluctuations an electrical device endures before it fails, the number of days a patient stays in the hospital, the number of days/weeks/months/years a kidney patient survives after treatment, modeling the number of times a pendulum moves before it comes to rest, the number of times a device switches on and off, and many other applications. An appropriate model for such data is the discrete lifetime model. Many authors have defined reliability measures for discrete data and investigated their properties. Salvia and Bollinger [1] introduced the hazard rate (HR) function for a discrete life model and studied its basic properties. Takahashi [2] proposed a definition of HR of nonequispaced discrete distributions. Singer and Willett [3] have used an empirical example and mathematical arguments to show how the methods of discrete-time survival analysis provide educational statisticians with an ideal framework for studying the occurrence of events. Shaked *et al.* [4] established a necessary and sufficient condition for a set of functions to be discrete multivariate conditional HR functions. Gupta *et al.* [5] have developed techniques to determine the increasing failure rate property and the decreasing failure rate property for a broad class of discrete distributions. Sandoh *et al.* [6] proposed a new modified discrete preventive maintenance policy in which failures of a system can be detected only by inspection and remedied by minimal repair. Roy and Dasgupta [7] have introduced a new discretization approach to assess the reliability of complex systems for which analytical methods do not provide a closed-form solution. Xie *et al.* [8] have introduced a different definition of the discrete failure rate function and provide the failure rate functions according to this definition for a number of useful discrete reliability functions. A thorough overview of discrete probability distributions used in reliability theory to represent the discrete lifetime of non-repairable systems was given by Bracquemond and Gaudoin [9]. To give a methodological summary of the conservation of some classes of discrete distributions under convolution and mixture, Pavlova *et al.* [10] studied some commonly used classes of discrete distributions. A failure and repair model, which is a discrete-time variant of

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the pure birth shock model, was presented by Belzunce *et al.* [11]. A novel survival tree approach for discrete-time survival data with time-varying variables is presented by Bou-Hamad *et al.* [12]. Furthermore, Eryilmaz [13] investigated a shock model in which the shocks occur according to a binomial process and determined the probability mass function and the probability-generating function of the lifetime of the system. Schmid *et al.* [14] have proposed a technique based on the result that the likelihood of a discrete survival model is equivalent to the likelihood of a regression model for binary outcome data. Li *et al.* [15] discussed a repairable system operating in dynamic regimes under the hypothesis of discrete time. Alkaff [16] proposed modeling techniques for the exact dynamic reliability analysis of systems in which the lifetimes of all components follow independent and non-identical distributions of the discrete phase type. A discrete-time version of the nonhomogeneous Poisson process has been defined and its properties were studied by Cha and Limnios [17].

As for continuous random lifetimes, the mean residual life (MRL) of discrete models has been considered by many authors. Ebrahimi [18] proposed the decreasing and increasing MRL classes of discrete lifetime distributions, Guess and Park [19] discussed some different shapes of the MRL function, Mi [20] determined the shape of the MRL function when the HR function is bathtub (BT)-shaped or upside-down bathtub (UBT)-shaped, and Salvia [21] determined some bounds for the MRL function.

Under certain circumstances, the α-quantile residual life function (α-QRL) may be preferred to the MRL, *e.g.*, if we have outliers in the data or the data are skewed or heavily censored. In addition, the concept of α-QRL for continuous random lifetimes has received considerable attention. Joe and Proschan [22] have introduced two classes of life distributions defined by the α-percentile residual life function. As Jeong and Fin [23] have shown, the quantile residual life function can be strongly influenced by competing events. Franco-Pereira and de Uña-Álvarez [24] introduced a new estimator of a percentile residual life function with censored data under a monotonicity constraint.

A method for calculating the quantile residual life function that relaxes the condition of independent censoring and takes covariates into account was proposed by Noughabi *et al.* [25]. Noughabi and Kayid [26] proposed and investigated the bivariate α-quantile residual life measure. Noughabi and Franco-Pereira [27] have shown that a mixture model is bounded by its components over the quantile residual life, and they investigated how mixture models are ordered with respect to the quantile residual life function when their components are ordered. However, in the reliability literature on discrete life models, the α-QRL has not received the attention it deserves. This motivated us to define and investigate the α-QRL sequence as a reliability measure.

The remainder of this study is organized as in the following. In Section 2, we set notations and present some preliminaries. In Section 3, the α-QRL sequence is defined and studied. The relation between the α-QRL sequence and the HR sequence is discussed, and in particular, the form of it when the HR is increasing, decreasing, BT, or UBT is explored. In Section 4, a new stochastic order for discrete lifetime variables based on the α-QRL concept is defined and its connection with the HR order is investigated. Section 5 includes the conclusions and future topics.

2 Notations and preparatory contents

Let T represent a discrete random lifetime with the support $t_0, t_1, t_2, .., t_i \in \mathbb{W}$ and $\mathbb{W}$ be the set of all whole numbers, *i.e.*, $\mathbb{W} = \{0, 1, 2, 3, ..\}$. The probability mass function and the reliability function are corresponding to sequences $p_0, p_1, p_2, \ldots$ and $R_0, R_1, R_2, \ldots$ where $p_i = P(T = t_i)$ and $R_i = P(T \geq t_i)$ respectively. The support of T may be bounded from above by t_m, *i.e.*, $p_i > 0$ for $i = 0, 1, 2, \ldots, m$ and $p_i = 0$ for $i \geq m + 1$. Usually, in the reliability theory and survival analysis literature, the support of a lifetime variable is considered to be $\mathbb{W}$.

The residual life given survival up to time t_i is denoted by the sequence of conditional random variables $T^{(i)} = T - t_i \mid T \geq t_i, \quad i = 0, 1, 2, \ldots$, which guide us to the conditional reliability function

$$R^{(i)}(t) = P(T^{(i)} \geq t), \quad t \geq 0. \tag{1}$$

The following HR function was traditionally defined by Salvia and Bollinger [1]:

$$h_i = \frac{p_i}{R_i}, \quad i \in \mathbb{W}. \tag{2}$$

The HR function h_i characterizes the distribution function. Shaked *et al.* [4] investigated that a sequence h_i should satisfy one of the following necessary and sufficient conditions, based on the support of the underline model, to be an HR function:

- There exists one $m \in \mathbb{W}$ such that for every $i < m$, $h_i < 1$ and $h_m = 1$.
- For every $i \in \mathbb{W}$, $h_i < 1$ and $\sum_{i=0}^{\infty} h_i = \infty$.

Xie *et al.* [8] explored and discussed several drawbacks of the traditional definition of the HR function and redefined this concept in the following:

$$r_i = \log \frac{R_i}{R_{i+1}}, \quad i \in \mathbb{W}. \tag{3}$$

It could be checked that the sequences r_i and h_i are in a one-to-one relation:

$$r_i = -\log(1 - h_i), \quad i \in \mathbb{W}, \tag{4}$$

and implies that both have the same monotonicity attribute. Also, this relation indicates that the sequence r_i characterizes the model as h_i does.

The MRL function at t_i, which shows the mean of the remaining time to failure given survival up to t_i, is defined precisely by

$$m_i = E(T - t_i \mid T \geq t_i) = \frac{1}{R_i} \sum_{k=i}^{\infty} (t_{k+1} - t_k) R_{k+1}, \; i \in \mathbb{W}. \tag{5}$$

In the discrete case, HR, MRL, and the α-QRL and other similar functions are in fact sequences, so they are referred to as sequences too. It can be checked easily that the MRL and the traditional HR sequences are related by the following relation. Refer to Salvia [21] for a similar form:

$$m_{i+1} = \frac{m_i}{1 - h_i} - (t_{i+1} - t_i), \quad i \in \mathbb{W}. \tag{6}$$

The fact that an object or process has an increasing risk of failure due to aging or fatigue indicates that the true model follows an increasing HR sequence. In this case, the MRL sequence has a decreasing shape. However, in some situations, an object may be exposed to excessive hazards in its early-life phase, which decreases over time. Therefore, the lifetime model should naturally assume an early-life phase with decreasing and eventually increasing HR. A characteristic of such objects is that instances that pass through an early-life phase (*i.e.*, the burn-in phase) inevitably become more reliable. In such cases, the MRL sequence has an increasing and then a decreasing shape. It is obvious that the larger the MRL of the object is, the more reliable the condition it has for continuation. So, we can propose the point maximizing the MRL function as the optimal burn-in time, *i.e.*, the MRL burn-in time b^* satisfies

$$b^* = \arg\max_i m_i, \quad i \in \mathbb{W}. \tag{7}$$

An interesting topic of reliability theory and survival analysis is the study of the shape of m_i, particularly in the context of the HR sequence. Guess and Park [19] proposed a necessary and sufficient condition for which the MRL function first increases and then decreases (decreases and then increases). Mi [20] considered a discrete lifetime model and proved that if the HR sequence has a BT form, the MRL sequence is decreasing or UBT. Tang *et al.* [28] complemented this result for the models with UBT HR sequences and showed that in this case, the MRL sequence is increasing or BT. Nair *et al.* [29] presented discrete lifetime distributions with BT-HR functions.

On the other hand, stochastic orderings have found a wide field of application in probability, statistics, and statistical decision theory. Two discrete random lifetimes T_1 and T_2, with their corresponding reliability sequences $R_{1,i}$, and $R_{2,i}$, $i \in \mathbb{W}$, could be compared in the sense of their different characteristics.

- The simplest comparison is based on the reliability function, which states that T_1 is smaller than T_2 in the usual stochastic order, $T_1 \leq_{st} T_2$, if $R_{1,i} \leq R_{2,i}$ for all i.
- T_1 is smaller than T_2 in likelihood ratio order, $T_1 \leq_{lr} T_2$, if $\frac{p_{2,i}}{p_{1,i}}$ is an increasing sequence.
- T_1 is smaller than T_2 in HR order, $T_1 \leq_{hr} T_2$, if $\frac{R_{2,i}}{R_{1,i}}$ is an increasing sequence. Equivalently, $T_1 \leq_{hr} T_2$, if $h_{1,i} \geq h_{2,i}$ for every i.
- T_1 is smaller than T_2 in MRL order, $T_1 \leq_{mlr} T_2$, if $\frac{\sum_{k=i}^{\infty}(t_{k+1} - t_k)R_{2,k+1}}{\sum_{k=i}^{\infty}(t_{k+1} - t_k)R_{1,k+1}}$ is an increasing sequence. Equivalently, $T_1 \leq_{mrl} T_2$, if $m_{1,i} \leq m_{2,i}$ for every i.

It is known that $T_1 \leq_{lr} T_2$ implies $T_1 \leq_{hr} T_2$. Also, $T_1 \leq_{hr} T_2$ implies $T_1 \leq_{st} T_2$ and $T_1 \leq_{mrl} T_2$. Refer to Dewan and Sudheesh [30].

3 α-QRL sequence

For the random lifetime T, the ith element of α-QRL sequence is the α-quantile of the remaining life $T - t_i$ given that $T \geq t_i$ and can be expressed precisely by

$$q_{\alpha,i} = \inf\{x, R^{(i)}(x) \leq \bar{\alpha}\}, \quad i \in \mathbb{W}, \tag{8}$$

where $\bar{\alpha} = 1 - \alpha$. Note that $R^{(i)}(x)$ is right continuous and has step form with jumps at integer values of x and in turn $q_{\alpha,i}$ receives just integer values. We can simplify the α-QRL sequence (8) as in the following:

$$\begin{aligned} q_{\alpha,i} &= \inf\{x : R^{(i)}(x) \leq \bar{\alpha}\} \\ &= \inf\{x : P(T \geq x + t_i \mid T \geq t_i) \leq \bar{\alpha}\} \\ &= \inf\{x : t_i + x = t_j, R_j \leq \bar{\alpha} R_i\} \\ &= \inf\{y : y = t_j, R_j \leq \bar{\alpha} R_i\} - t_i \\ &= R^{-1}(\bar{\alpha} R_i) - t_i, \quad i \in \mathbb{W}, \end{aligned} \tag{9}$$

where $R^{-1}(p) = \inf\{y : y = t_j, R_j \leq p\}$ is the inverse of the reliability function. If the support of T is bounded from above by t_m, *i.e.*, $R_i > 0$ for $i = 0, 1, 2, .., m$ and $R_i = 0$ for $i \geq m + 1$, then

$$q_{\alpha,i} = \begin{cases} R^{-1}(\bar{\alpha}R_i) - t_i > 0, & 0 \le i \le m \\ 0, & i > m. \end{cases} \quad (10)$$

To better illustrate the α- α-QRL and its applications, let T plot the days between the treatment of a tumor and its first recurrence in breast cancer patients. For treated patients who have not experienced tumor recurrence after i days, let $q_{\alpha,i}$ be the number of days from i in which tumor recurrence will occur in 100α% of these patients. Such information could be useful for hospital management when planning future patient visits. It is worth noting that Proschan [31] considered the time interval between consecutive air conditioning failures on a Boeing 720. After i days since the last maintenance, $q_{\alpha,i}$ provides the number of days from i in which the air conditioning will fail in 100α% of these aircraft. Assume that we have a random lifetime T with the support $t_0, t_1, t_2, .., t_i \in \mathbb{W}$. For computing $q_{\alpha,i}$, we apply the following steps:

- Compute $\bar{\alpha}R_i = \bar{\alpha}P(T \ge t_i)$.
- To find $R^{-1}(\bar{\alpha}R_i)$, find the smallest t_j, for that $P(T \ge t_j)$ is equal to or less than the calculated $\bar{\alpha}R_i$. Then, $q_{\alpha,i} = R^{-1}(\bar{\alpha}R_i) - t_i$.

It can be checked easily from (6) that $m_{i+1} - m_i \ge t_i - t_{i+1}$. The following lines show that a similar property holds for α-QRL:

$$\begin{aligned} q_{\alpha,i+1} - q_{\alpha,i} &= \inf\{y : y = t_j, R_j \le \bar{\alpha}R_{i+1}\} - t_{i+1} \\ &\quad -\inf\{y : y = t_j, R_j \le \bar{\alpha}R_i\} + t_i \\ &\ge t_i - t_{i+1}. \end{aligned}$$

The reliability function could be expressed in terms of HR by

$$R_i = \prod_{k=0}^{i-1} \bar{h}_k,$$

where $\bar{h}_k = 1 - h_k$. Then, $q_{\alpha,i}$ can be expressed as in the following:

$$\begin{aligned} q_{\alpha,i} &= \inf\{x : t_i + x = t_j, R_j \le \bar{\alpha}R_i\} \\ &= \inf\left\{x : t_i + x = t_j, \prod_{k=i}^{j-1} \bar{h}_k \le \bar{\alpha}\right\}. \end{aligned} \quad (11)$$

Note that $j \ge i$. Now, by (4), we have

$$q_{\alpha,i} = \inf\left\{x : t_i + x = t_j, \sum_{k=i}^{j-1} r_k \ge -\log\bar{\alpha}\right\}. \quad (12)$$

Eqs. (11) and (12) reveal the close relationship between the HR and the α-QRL sequences and give the main clue in proving the following results:

Theorem 1.

i. *If h_i is an increasing sequence, then $q_{\alpha,i}$ is decreasing.*
ii. *If h_i is a decreasing sequence, then $q_{\alpha,i}$ is increasing.*

Proof. i. Let h_i be increasing and $i \in \mathbb{W}$ be arbitrary and fixed. By (11), we have

$$\prod_{k=i}^{j-1} \bar{h}_k \le \bar{\alpha}, \quad (13)$$

where $t_j = t_i + q_{\alpha,i}$. We must show that $q_{\alpha,i+1} \le q_{\alpha,i}$. Applying (11) again, we can write

$$q_{\alpha,i+1} = \inf\left\{x : t_s = t_{i+1} + x, \prod_{k=i+1}^{s-1} \bar{h}_k \le \bar{\alpha}\right\}. \quad (14)$$

It is sufficient to show that

$$\prod_{k=i+1}^{s-1} \bar{h}_k \le \bar{\alpha}, \quad (15)$$

where $t_s = t_{i+1} + q_{\alpha,i}$. Now, since h_i is increasing, $\bar{h}_i$ is decreasing and

$$\prod_{k=i+1}^{s-1} \bar{h}_k = \frac{\bar{h}_j}{\bar{h}_i} \prod_{k=j+1}^{s-1} \bar{h}_i \prod_{k=i}^{j-1} \bar{h}_k \le \bar{\alpha}, \quad (16)$$

where the inequality is true by (13) and the fact that $j \ge i$ and $0 \le \bar{h}_i \le 1$ for any i. It proves part i. The proof of part ii is completely similar.

Here, we state BT and UBT versions of sequences (functions) that are suitable for describing the behavior of α-QRL in the discrete case. □

Definition 1. A sequence g_i is called BT with change points $0 < i_0 \le i_1 < \infty$, if g_i is decreasing for $0 \le i \le i_0$, $g_i > g_{i_0}$ for every $0 \le i < i_0$, $g_i = g_{i_0}$ for $i_0 \le i \le i_1$, and for every $i \ge i_1$, g_i is increasing.

A sequence g_i is called UBT with change points $0 < i_0 \le i_1 < \infty$ if $-g_i$ is BT with change points $0 < i_0 \le i_1 < \infty$.

Theorem 2.

i. *Let HR sequence h_i be BT with change points $0 < k_0 \le k_1 < \infty$, then $q_{\alpha,i}$ is decreasing for $k \ge k'_0$, where $k'_0 \le k_0$.*
ii. *Let HR sequence h_i be UBT with change points $0 < k_0 \le k_1 < \infty$, then $q_{\alpha,i}$ is increasing for $k \ge k'_0$, where $k'_0 \le k_0$.*

Proof.

i. Similar to the proof of Theorem 1, we can show that $q_{\alpha,i}$ is decreasing for $k \ge k_0$. Thus, there exists $k'_0 \le k_0$ such that $q_{\alpha,i}$ is decreasing for $k \ge k'_0$.
ii. The proof is similar to part i. □

When lifetime data exhibit a BT HR model, we can define the burn-in time $b^* = t_j$ maximizing the α-QRL, specially the median residual life. Theorem 2 shows that the burn-in time b^* is not greater than the first change point of the HR sequence.

Example 1. Lai and Wang [32] studied the attributes of a discrete BT HR model with the following density function:

$$p_i = \frac{i^a}{\sum_{k=1}^{N} k^a}, \quad i = 1, 2, \ldots, N, \quad a \in \mathbb{R}, N = 1, 2, 3, \ldots \tag{17}$$

The reliability and HR sequences are

$$R_i = \frac{\sum_{k=i}^{N} k^a}{\sum_{k=1}^{N} k^a}, \quad i = 1, 2, \ldots, N, \tag{18}$$

and

$$h_i = \frac{i^a}{\sum_{k=i}^{N} k^a}, \quad i = 1, 2, \ldots, N. \tag{19}$$

Also, the α-QRL sequence is

$$q_{\alpha,i} = \inf\left\{x : x \geq i, x = j, \sum_{k=j}^{N} k^a \leq \bar{\alpha} \sum_{k=i}^{N} k^a\right\} - i, \quad i = 1, 2, \ldots, N. \tag{20}$$

Lai and Wang [32] proved that the HR is increasing for $a \geq 0$, and BT for $a < 0$. Thus, by Theorem 1, when $a \geq 0$, the α-QRL is decreasing. When $a < 0$, the α-QRL is eventually decreases, and it has at least one maximum. Figure 1 shows the density and HR sequences of the MP model for some different parameters. For $a = -1, -2, -3$, the HR sequences take their minimums at $i = 8, 10, 12$, respectively. Also, the median residual life sequences are maximized at $i = 5, 6, 6$, respectively, for $a = -1, -2, -3$.

Example 2. A very flexible model which is suitable in situations we deal with monotone, BT, and UBT HR functions is the competing risk model with the following reliability function:

$$R_i = 2^{-(ai)^b - (ai)^{\frac{1}{b}}}, \quad i = 0, 1, 2, \ldots, \quad a, b > 0. \tag{21}$$

Table 1 shows the number of cycles to failure for 60 electrical appliances in a life test. This dataset was reported by Lawless [33] and analyzed by many authors, *e.g.*, Bebbington *et al.* [34]. Shafaei *et al.* [35] showed that the model (21) could be a good candidate for this dataset. The maximum-likelihood estimate of the parameters is reported to be $\hat{a} = 0.000311$ and $\hat{b} = 0.5954$. Figure 2 shows the density and FR function of the estimated competing risk at the data points. These plots exhibit a BT form for the HR function and a unimodal form for the median residual life. The HR is minimized at $i = 776$ and the median residual life is maximized at $i = 234$, and this point could be considered as a burn-in time for the appliances.

Theorem 3. *Suppose that the support of T is regular and the corresponding $q_{\alpha,i}$ is strictly decreasing in i for all $\alpha \in \{\bar{h}_0, \bar{h}_1, \bar{h}_2, \ldots\}$. Then, h_i is strictly increasing in i.*

Proof. Let $\bar{\alpha}_r = \bar{h}_r$ for all possible r indices. Assume that h_i is not strictly increasing. Then, there exists i such that $\bar{h}_i \leq \bar{h}_{i+1}$. Thus,

$$\bar{h}_i \leq \bar{h}_{i+1} = \bar{\alpha}_{i+1}.$$

Then,

$$q_{\alpha_{i+1},i} = \inf\left\{x : t_j = t_i + x, \prod_{k=i}^{j-1} \bar{h}_k \leq \bar{\alpha}_{i+1}\right\} = t_{i+1} - t_i \tag{22}$$

and

$$q_{\alpha_{i+1},i+1} = \inf\left\{x : t_j = t_{i+1} + x, \prod_{k=i+1}^{j-1} \bar{h}_k \leq \bar{\alpha}_{i+1}\right\} = t_{i+2} - t_{i+1}. \tag{23}$$

Since the support of T is regular, (22) and (23) indicate that $q_{\alpha_{i+1},i} = q_{\alpha_{i+1},i+1}$, which contradicts with the assumptions. So, the HR must be strictly increasing. □

Definition 2. A lifetime random variable T is new better than used in α-QRL (α-NBUQ) if $q_{\alpha,0} \geq q_{\alpha,i}$ for every $i > 0$. Similarly, T is new worse than used in α-QRL (α-NWUQ) if $q_{\alpha,0} \leq q_{\alpha,i}$ for every $i > 0$.

Theorem 1. *shows that if h_i is increasing (decreasing), then the corresponding T is α-NBUQ (α-NWUQ).*

Assume a discrete lifetime T with reliability sequence $R_i, i \in \mathbb{W}$. Now, if this object is exposed to stress, shock, or environmental factors, its corresponding reliability sequence may be expressed by

$$R_i^* = R_i^{\theta}, \theta > 0, \tag{24}$$

and based on the fact that the affecting factor causes smaller or bigger lifetime, $\theta > 1$ or $\theta < 1$. The corresponding HR, defined by Xie *et al.* [8], of this model is

$$r_i^* = \log\frac{R_i^*}{R_{i+1}^*} = \theta \log\frac{R_i}{R_{i+1}} = \theta r_i \quad \theta > 0, \tag{25}$$

and due to this relation, the model (24) is called the proportional HR model. Let T be a discrete random lifetime and T^* correspond to one proportional HR model of it. By the fact that the forms of h_i and r_i are the same, it follows that a discrete random lifetime T is IHR if and only if T^* is.

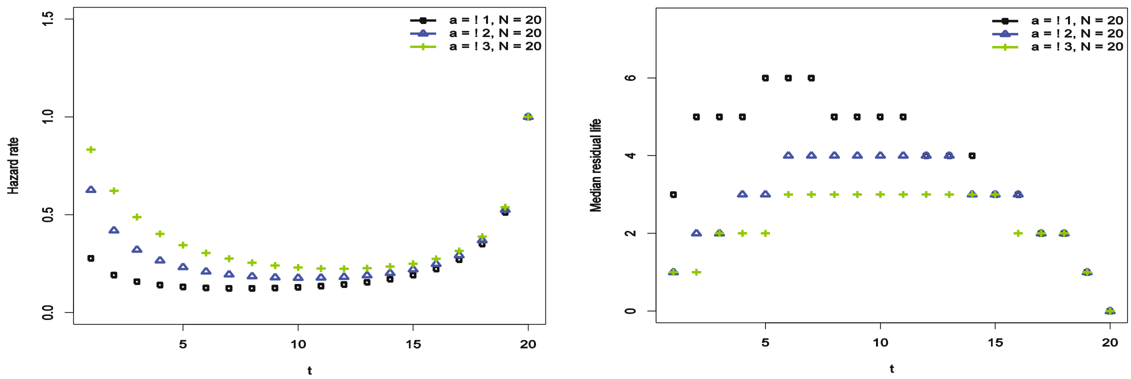

Figure 1: Density and HR sequences of the MP model for some values of parameters.

Table 1: Number of cycles to failure for 60 electrical appliances

0014	0034	0059	0061	0069	0080	0123	0142	0165	0210
0381	0464	0479	0556	0574	0839	0917	0969	0991	1,064
1,088	1,091	1,174	1,270	1,275	1,355	1,397	1,477	1,578	1,649
1,702	1,893	1,932	2,001	2,161	2,292	2,326	2,337	2,628	2,785
2,811	2,886	2,993	3,122	3,248	3,715	3,790	3,857	3,912	4,100
4,106	4,116	4,315	4,510	4,584	5,267	5,299	5,583	6,065	9,701

Dewan and Sudheesh [30] showed that T is decreasing mean residual life (increasing mean residual life) if and only if T^* is. Assume that $q^*_{\alpha,i}$ is the α-QRL of T^*. Then,

$$\begin{aligned} q_{\alpha,i} &= \inf\{y : y = t_j, R_j \le \bar{\alpha} R_i\} - t_i \\ &= \inf\{y : y = t_j, R_j^\theta \le \bar{\alpha}^\theta R_i^\theta\} - t_i \\ &= q^*_{\beta,i}, \quad i = 0, 1, 2, .. \end{aligned} \tag{26}$$

where $\beta = 1 - \bar{\alpha}^\theta$. The following result follows directly from (26).

Corollary 1. *Let* $\beta = 1 - \bar{\alpha}^\theta$.

i. T *is* α-decreasing quantile residual life (α-increasing quantile residual life) *if and only if* T^* *is* β-decreasing quantile residual life (β-increasing quantile residual life).
ii. T *is* α-NBUQ (α-NWUQ) *if and only if* T^* *is* β-NBUQ (β-NWUQ).

4 α-QRL order

Let T_1 and T_2 represent two discrete random lifetimes with common support $t_0, t_1, t_2, ..$ and the α-QRL sequences of T_1 and T_2 be denoted by $q_{1,\alpha,i}$ and $q_{2,\alpha,i}$, respectively.

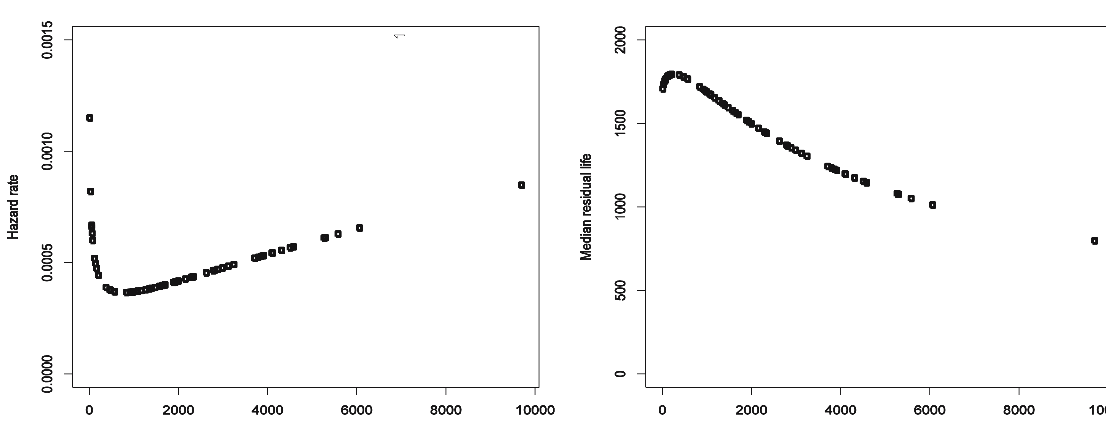

Figure 2: Density and FR function of the estimated competing risk model (21).

Definition 3. T_1 is smaller than T_2 in α-QRL order, $T_1 \leq_{\alpha-\mathrm{QRL}} T_2$, if for every i in the support, $q_{1,\alpha,i} \leq q_{2,\alpha,i}$.

From (10), it is clear that if $T_1 \leq_{\alpha-\mathrm{QRL}} T_2$ and the support of T_2 is bounded from above by t_m, then the support of T_1 is bounded from above by $t'_m \leq t_m$.

Theorem 4. *$T_1 \leq_{\mathrm{hr}} T_2$ implies $T_1 \leq_{\alpha-\mathrm{QRL}} T_2$.*

Proof. Since $T_1 \leq_{\mathrm{hr}} T_2$, $\bar{h}_{1,i} \leq \bar{h}_{2,i}$ for every i in the support. Using (11), the α-QRL of T_2 at i is

$$q_{2,\alpha,i} = \inf\left\{x : t_i + x = t_j, \prod_{k=i}^{j-1} \bar{h}_{2,k} \leq \bar{\alpha}\right\}.$$

Now, for a specific j that $\prod_{k=i}^{j-1} \bar{h}_{2,k} \leq \bar{\alpha}$, we have

$$\prod_{k=i}^{j-1} \bar{h}_{1,k} \leq \prod_{k=i}^{j-1} \bar{h}_{2,k} \leq \bar{\alpha},$$

which proves that $q_{1,\alpha,i} \leq q_{2,\alpha,i}$ for every i and completes the proof. □

Theorem 5. *Let $T_1 \leq_{\alpha-\mathrm{QRL}} T_2$ for all $\alpha \in \{\bar{h}_{1,0}, \bar{h}_{1,1}, \bar{h}_{1,2}, ..\}$, then $T_1 \leq_{\mathrm{hr}} T_2$.*

Proof. We should show that $h_{1,i} \geq h_{2,i}$ for all i. Let i be arbitrary and denote $\bar{\alpha}_i = \bar{h}_{1,i}$. Then □

$$q_{1,i,\alpha_i} = \inf\left\{x : t_i + x = t_j, \prod_{k=i}^{j-1} \bar{h}_{1,i} \leq \bar{\alpha}_i\right\} = t_{i+1} - t_i. \quad (27)$$

Since $T_1 \leq_{\alpha_i-\mathrm{QRL}} T_2$, we have

$$q_{2,i,\alpha_i} = \inf\left\{x : t_i + x = t_j, \prod_{k=i}^{j-1} \bar{h}_{2,i} \leq \bar{\alpha}_i\right\} \geq t_{i+1} - t_i. \quad (28)$$

Then, comparing (27) and (28) shows that $\bar{h}_{2,i} \geq \bar{h}_{1,i}$ or equivalently $h_{1,i} \geq h_{2,i}$, which completes the proof.

Assume that T_1^* and T_2^* correspond to the proportional HR models of T_1 and T_2, respectively. The following theorem shows that the $\alpha - QRL$ order implies an order for the proportional HR models. The proof follows from (26) directly.

Corollary 2. *$T_1 \leq_{\alpha-\mathrm{QRL}} T_2$ if and only if $T_1^* \leq_{\beta-\mathrm{QRL}} T_2^*$, where $\beta = 1 - \bar{\alpha}^{\theta}$.*

5 Conclusion

The α-QRL sequence for discrete data was defined, and its relationship to the HR sequence was investigated. An increasing (decreasing) HR sequence implies a decreasing (increasing) α-QRL. If the HR sequence has a BT (UBT) form, the α-QRL usually has a UBT (BT) form. If the HR has a BT form, the point that maximizes the α-QRL can be used as a suitable burn-in time point. The α-QRL order in the discrete context is defined, and it is proved that this order is weaker than the HR order. In addition, the basic properties of α-QRL in the proportional HR model are discussed. Some interesting topics that can be considered in future research are listed in the following:

- To the best of our knowledge, the α-QRL for a value of α does not clearly characterize the base model. Therefore, the question of how the model can be characterized by α-QRL sequences remains an open problem.
- Problems in estimating the proposed α-QRL sequence can be extended to estimate monotonic α-QRL sequences.
- For discrete models, the α-quantile past sequence can be defined and analyzed in a similar way.
- The concept of α-QRL can be extended to multivariate relationships when two or more dependent lifetime random variables are involved. In this context, the problem of estimating the α-QRL function in a multivariate context is interesting.
- Although all observations are effectively discrete, they can have a very different observational resolution/accuracy if the observations are actually related to a continuous variable. For example, an underlying time variable can be observed with the unit one of a week or with the unit one of an hour. The relationship between these two resolutions based on the quintile residual life sequence is an interesting and remains an open problem.

Acknowledgments: The authors would like to thank the editor and the two anonymous reviewers for their suggestions and constructive comments, which have improved the presentation and readability of the article.

Funding information: This work was supported by Researchers Supporting Project Number (RSP2024R464), King Saud University, Riyadh, Saudi Arabia.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

Data availability statement: The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

References

[1] Salvia A, Bollinger RC. On discrete hazard functions. IEEE Trans Reliab. 1982;R-31:458–9.
[2] Takahashi K. Hazard rates of nonequispaced discrete distributions. IEEE Trans Reliab. 1988;37(5):556–7.
[3] Singer JD, Willett JB. It's about time: Using discrete-time survival analysis to study duration and the timing of events. J Educ Stat. 1993;18(2):155–95.
[4] Shaked MJ, Shanthikumar G, Valdez-Torres JB. Discrete hazard rate functions. Comput Oper Res. 1995;22(4):391–402. doi: 10.1016/0305-0548(94)00048-D.
[5] Gupta PL, Gupta RC, Tripathi RC. On the monotonic properties of discrete failure rates. J Stat Plan Inference. 1997;65(2):255–68. doi: 10.1016/S0378-3758(97)00064-5.
[6] Sandoh H, Hirakoshi H, Nakagawa T. A new modified discrete preventive maintenance policy and its application to hard disk management. J Qual Maint Eng. 1998;4(4):284–90. doi: 10.1108/13552519810234001.
[7] Roy D, Dasgupta T. A discretizing approach for evaluating reliability of complex systems under stress-strength model. Comput Oper Res. 2001;50(2):145–50.
[8] Xie M, Gaudoin O, Bracquemond C. Redefining failure rate function for discrete distributions. Int J Reliab Qual Saf Eng. 2002;09(03):275–85.
[9] Bracquemond C, Gaudoin O. A survey on discrete lifetime distributions. Int J Reliab Qual Saf Eng. 2003;10(01):69–98. doi: 10.1142/S0218539303001007.
[10] Pavlova K, Willmot GC. The preservation of classes of discrete distributions under convolution and mixing. Insur: Math Econ. 2006;38(2):391–405. doi: 10.1016/j.insmatheco.2005.10.001.
[11] Belzunce F, Ortega EM, Ruiz JM. Aging properties of a discrete-time failure and repair model. IEEE Trans Reliab. 2009;58(1):161–71.
[12] Bou-Hamad I, Larocque D, Ben-Ameur H. Discrete-time survival trees and forests with time-varying covariates: application to bankruptcy data. Stat Model. 2011;11(5):429–46.
[13] Eryilmaz S. On the lifetime behavior of a discrete time shock model. J Comput Appl Math. 2013;237(1):384–8. doi: 10.1016/j.cam.2012.06.008.
[14] Schmid M, Küchenhoff H, Hoerauf A, Tutz G. A survival tree method for the analysis of discrete event times in clinical and epidemiological studies. Stat Med. 2016;35(5):734–5.
[15] Li Y, Cui L, Lin C. Modeling and analysis for multi-state systems with discrete-time Markov regime-switching. Reliab Eng Syst. 2017;166:41–9. doi: 10.1016/j.ress.2017.03.024.
[16] Alkaff A. Discrete time dynamic reliability modeling for systems with multistate components. Reliab Eng Syst. 2021;209:107462.
[17] Cha JH, Limnios N. Discrete time minimal repair process and its reliability applications under random environments. Stoch Models. 2022;1–22. doi: 10.1080/15326349.2021.2018336.
[18] Ebrahimi N. Classes of discrete decreasing and increasing mean residual life distributions. IEEE Trans Reliab. 1986;R-35:403–5.
[19] Guess F, Park DH. Modeling discrete bathtub and upside-down bathtub mean residual-life functions. IEEE Trans Reliab. 1988;R-37(5):545–9.
[20] Mi J. Discrete bathtub failure rate and upside-down bathtub mean residual life. Nav Res Logist. 1993;40:361–71.
[21] Salvia A. Some results on discrete mean residual life. IEEE Trans Reliab. 1996;45(3):359–61.
[22] Joe H, Proschan F. Percentile residual life functions. Oper Res. 1984;32:668–78.
[23] Jeong JH, Fine JP. Nonparametric inference on cause-specific quantile residual life. Biometrical J. 2013;55:68–81.
[24] Franco-Pereira AM, de Uña-Álvarez J. Estimation of a monotone percentile residual life function under random censorship. Biometrical J. 2013;55:52–67.
[25] Lin C, Zhang L, Zhou Y. Inference on quantile residual life function under right-censored data. J Nonparametr Stat. 2016;28:617–43.
[26] Noughabi MS, Kayid M. Bivariate quantile residual life: a characterization theorem and statistical properties. Stat Pap. 2019;60:2001–12. doi: 10.1007/s00362-017-0905-9.
[27] Noughabi MS, Franco-Pereira AM. Estimation of two ordered quantile residual life functions based on mixtures. J Stat Comput Simul. 2021;91(18):3792–813. doi: 10.1080/00949655.2021.1947277.
[28] Tang LC, Lu Y, Chew EP. Mean residual life of lifetime distributions. IEEE Trans Reliab. 1999;48(1):73–8.
[29] Nair N, Unnikrishnan, Ramesh, NP, Sankaran, PG, Discrete. distributions with bathtub-shaped hazard rates. S Afr Stat j. 2017;51:1. doi: 10.10520/EJC-677191b10.
[30] Dewan I, Sudheesh K. On proportional (reversed) hazard model for discrete data. Technical report. Indian Statistical Institute. New Delhi, India: 2009.
[31] Proschan F. Theoretical explanation of observed decreasing failure rate. Technometrics. 1963;5:375–83.
[32] Lai CD, Wang DQ. A finite range discrete life distribution. Int J Reliab Qual Safety Eng. 1995;2(2):147–60. doi: 10.1142/S0218539395000125.
[33] Lawless JF. Statistical models and methods for lifetime data. 2nd edn. New York: John Wiley & sons; 2003.
[34] Bebbington M, Lai CD, Zitikis R. Useful periods for lifetime distributions with bathtub shaped hazard rate functions. IEEE Trans Reliab. 2006;55(2):245–51.
[35] Shafaei Noughabi M, Rezaei Roknabadi AH, Mohtashami Borzadaran GR. Some discrete lifetime distributions with bathtub-shaped hazard rate functions. Qual Eng. 2013;25(3):225–36. doi: 10.1080/08982112.2013.769055.