Research Article

Wael W. Mohammed* and Farah M. Al-Askar

New optical stochastic solutions for the Schrödinger equation with multiplicative Wiener process/random variable coefficients using two different methods

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Abstract: In this article, we take into consideration the stochastic Schrödinger equation (SSE) perturbed in the Itô sense by the multiplicative Wiener process. We employ an appropriate transformation to turn the SSE into another Schrödinger equation with random variable coefficients (SE-RVCs). We used the generalizing Riccati equation mapping method and the Jacobi elliptic function method to find novel hyperbolic, trigonometric, rational, and elliptic functions solutions for SE-RVCs. After that, we can acquire the SSE solutions. For the first time, in this work, we assume that the solution to the wave equation for the Schrödinger equation is stochastic, whereas all earlier studies assumed it to be deterministic. Furthermore, we give various graphs to display the effect of multiplicative Wiener process on the exact solutions to the SSE. We deduce that the multiplicative Wiener process stabilizes the solutions of the SSE.

Keywords: random variable coefficients, Schrödinger equation, stability by noise, generalizing Riccati equation mapping method

1 Introduction

Partial differential equations (PDEs) are used to model various physical phenomena in fields such as physics, engineering, and biology. Solving PDEs can be challenging due to their complexity, but there are several techniques that can be used to find solutions. Some of these methods are the F-expansion method [1], the Laplace adomian decomposition method [2,3], the generalized Adams-Bashforth-Moulton method [4], the ϕ^6-model expansion method [5], the modified generalized exponential rational function [6,7], the He's semi-inverse [8], the bilinear method [9,10], Bernoulli (G'/G)-expansion method [11,12], the $\exp(-\psi(z))$-expansion method [13], the generalized Riccati equation mapping method [14], the Lie symmetry method [15], the multivariate generalized exponential rational integral function [16], and the modified generalized Riccati equation mapping approach [17].

One of the most well-known examples of a PDE is the Schrödinger equation [18], which has a wide range applications in various fields such as nonlinear optics, heat pulses in materials, plasma physics, nonlinear acoustics, hydrodynamics, and several other nonlinear instability phenomenon [19–22]. Also, it has been used to investigate the behavior of quantum fields, for instance, the wave equation for photons in quantum electrodynamics. Scientists can obtain a better knowledge of the fundamental principles of quantum mechanics and change many disciplines of science and industry by solving the Schrödinger equation.

On the other hand, random fluctuations in the Schrödinger equation are a key aspect of quantum mechanics, emphasizing the unpredictability and probabilistic nature of quantum systems. While these fluctuations make it hard to predict a particle's exact behavior in a quantum system, they also open up new possibilities such as quantum tunneling and superposition. Harnessing these random fluctuations has resulted in technical advances in areas such as quantum computing and cryptography.

Consequently, it is important to consider the following Schrödinger equation with stochastic term:

$$i\mathcal{Z}_t - \delta_1\mathcal{Z}_{xx} + \delta_2 |\mathcal{Z}|^2\mathcal{Z} - \delta_3\mathcal{Z} = i\sigma\mathcal{Z}\mathcal{W}_t, \tag{1}$$

*** Corresponding author: Wael W. Mohammed,** Department of Mathematics, College of Science, University of Ha'il, Ha'il 2440, Saudi Arabia; Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mail: w.mohammed@uoh.edu.sa
Farah M. Al-Askar: Department of Mathematical Science, College of Science, Princess Nourah bint Abdulrahman University, P.O. Box 84428, Riyadh 11671, Saudi Arabia

where $\mathcal{Z}(x,t)$ is a complex-valued process, x and t denote the spatial coordinate and time, respectively, $i=\sqrt{-1}$, δ_1, δ_2, and δ_3 are arbitrary constants, $\mathcal{W}(t)$ is the standard Wiener process, $\mathcal{W}_t=\frac{\partial\mathcal{W}}{\partial t}$ and σ are the noise amplitude.

Because of the significance of the Schrödinger equation (1), several researchers obtain its solutions with $\sigma=0$ by using diverse approaches such as first integral method [23], $(G'/G, 1/G)$-expansion and $(1/G')$-expansion methods [24], the generalized $\exp(-\varphi(\xi))$-expansion method [25], mapping and generalized Riccati equation methods [26], mapping method [27], sine–cosine and the tanh methods [28], modified generalized Riccati equation mapping approach [29], and extended generalized Riccati equation mapping method [30]. While the analytical solutions of stochastic Schrödinger equation (SSE) have been achieved by employing different approaches such as Riccati-Bernoulli sub-ODE method and sine–cosine method [31], tanh–coth and He's semi-inverse techniques [32], unified technique [33], and modified mapping method [34]. Previous studies [31–33] assumed deterministic solutions to the wave equation, but this work assumes stochastic solutions.

Our objective of this article is to find the exact solutions to the SSE (1). To achieve this goal, we convert the SSE into another nonlinear Schrödinger equation with random variable coefficients (SE-RVCs) by using suitable transformation. After that, we obtain the exact solutions for SE-RVCs by using the generalizing Riccati equation mapping method (GREM method) and the Jacobi elliptic function method (JEF method). Finally, by using the used transformation, we can acquire the stochastic solutions for the SSE (1). In this work, we assume for the first time that the solution of the wave equation for the Schrödinger equation is stochastic, whereas all earlier studies assumed that it was deterministic. These acquired solutions are crucial in understanding several difficult physical processes due to the importance of Schrödinger equation (1) in various areas of physics, chemistry, and engineering. To see the influence of stochastic term, we provide some figures by utilizing MATLAB tools. In the end, we address the effect of noise on the obtained solutions.

Here is how the remainder of the article is structured: In Section 2, we derive SE-RVCs from SSE (1) and by utilizing the GREM method and JEF method to find the exact solutions of SE-RVCs. In Section 3, we acquire the solutions of SSE (1). In Section 4, we discuss the results that we obtained and the impacts of noise. Finally, we provide the article's conclusions.

2 Schrödinger equation with RVCs and its solutions

Here, we obtain the Schrödinger equation with random variable coefficients (SE-RVCs). Using the transformation

$$\mathcal{Z}(x,t)=\mathcal{F}(x,t)\exp[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t], \quad (2)$$

and the Itô derivatives, we obtain SE-RVCs as follows:

$$\begin{aligned} &i\mathcal{F}_t-\mathcal{F}\mathcal{Y}_t-\delta_1[\mathcal{F}_{xx}+2i\mathcal{Y}_x\mathcal{F}_x+i\mathcal{Y}_{xx}\mathcal{F}-\mathcal{Y}_x^2\mathcal{F}]\\ &\quad+A(t)\mathcal{F}^3-\delta_3\mathcal{F}=0, \end{aligned} \quad (3)$$

where $\mathcal{F}$ is a stochastic real function and $A(t)=\delta_2 e^{\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t}$. Putting imaginary and real parts equal zero, we have

$$\mathcal{F}_t-2\delta_1\mathcal{Y}_x\mathcal{F}_x-\delta_1\mathcal{Y}_{xx}\mathcal{F}=0, \quad (4)$$

and

$$-\mathcal{F}\mathcal{Y}_t-\delta_1\mathcal{F}_{xx}+\delta_1\mathcal{Y}_x^2\mathcal{F}+A(t)\mathcal{F}^3-\delta_3\mathcal{F}=0. \quad (5)$$

2.1 GREM method

To find the solutions of the SE-RVCs (3), we assume the solutions of Eqs (4) and (5) in the following special forms:

$$\mathcal{F}(x,t)=\sum_{k=0}^{m}\alpha_k(t)\mathcal{X}^k(\mu),\ \mu=kx+\int_0^t\lambda(\tau)\mathrm{d}\tau, \quad (6)$$

and

$$\mathcal{Y}(x,t)=\varphi_0(t)+x\varphi_1(t), \quad (7)$$

where λ, φ_0, φ_1, $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$, and $\alpha_m\neq 0$ are functions of t, and $\mathcal{X}$ satisfies

$$\mathcal{X}'=s\mathcal{X}^2+r\mathcal{X}+p. \quad (8)$$

First, let us compute the value of m by balancing $\mathcal{F}''$ with $\mathcal{F}^3$ in Eq. (5) as follows:

$$m+2=3m\Rightarrow m=1.$$

Eq. (6) is rewritten as follows:

$$\mathcal{F}(x,t)=\alpha_0(t)+\alpha_1(t)\mathcal{X}(\mu). \quad (9)$$

Differentiating Eqs (9) and (7) with respect to t and x, we have

$$\begin{aligned}\mathcal{F}_t &= (\dot{\alpha}_0 + p\alpha_1\lambda) + (\dot{\alpha}_1 + \alpha_1 r\lambda)\mathcal{X} + s\lambda\alpha_1\mathcal{X}^2,\\ \mathcal{F}_x &= k\alpha_1[s\mathcal{X}^2 + r\mathcal{X} + p],\\ \mathcal{F}_{xx} &= k^2\alpha_1[2s^2\mathcal{X}^3 + 3sr\mathcal{X}^2 + (2sp + r^2)\mathcal{X} + rp],\\ \mathcal{F}^3 &= \alpha_1^3\mathcal{X}^3 + 3\alpha_0\alpha_1^2\mathcal{X}^2 + 3\alpha_0^2\alpha_1\mathcal{X} + \alpha_0^3,\end{aligned} \tag{10}$$

and

$$\mathcal{Y}_t = \dot{\varphi}_0 + x\dot{\varphi}_1, \quad \mathcal{Y}_x = \varphi_1, \quad \mathcal{Y}_{xx} = 0. \tag{11}$$

Eqs. (7), (9), (10), and (11) are substituted into Eqs (4) and (5). After that, by equating each coefficient of $\mathcal{X}^k$ to zero, we have

$$\begin{aligned}\mathcal{X}^0 &: \quad \dot{\alpha}_0 + p\lambda\alpha_1 - 2pk\delta_1\varphi_1\alpha_1 = 0,\\ \mathcal{X}^1 &: \quad \dot{\alpha}_1 + r\lambda\alpha_1 - 2rk\delta_1\varphi_1\alpha_1 = 0,\\ \mathcal{X}^2 &: \quad r\lambda\alpha_1 - 2s\delta_1\varphi_1\alpha_1 = 0,\end{aligned}$$

and

$$\begin{aligned}&x\mathcal{X}^0 : \ \alpha_0\dot{\varphi}_1 = 0,\\ &x\mathcal{X}^1 : \ \alpha_1\dot{\varphi}_1 = 0,\\ &\mathcal{X}^0 : \ \alpha_0\dot{\varphi}_0 + rpk^2\delta_1\alpha_1 - \varphi_1^2\delta_1\alpha_0 - A\alpha_0^3 + \delta_3\alpha_0 = 0,\\ &\mathcal{X}^1 : \ \alpha_1\dot{\varphi}_0 + (2spk^2 + r^2k^2)\delta_1\alpha_1 - \varphi_1^2\delta_1\alpha_1 - 3A\alpha_0^2\alpha_1 + \delta_3\alpha_1\\ &\qquad = 0,\\ &\mathcal{X}^2 : \ -3srk^2\delta_1\alpha_1 + 3A\alpha_0\alpha_1^2 = 0,\\ &\mathcal{X}^3 : \ -2s^2k^2\delta_1\alpha_1 + A\alpha_1^3 = 0.\end{aligned}$$

We solve these equations to obtain

$$\alpha_0(t) = r = 0, \quad \alpha_1 = \ell, \quad \varphi_1 = b, \quad \lambda(t) = \frac{b\ell^2}{ks^2}A(t),$$

and

$$\varphi_0 = -\delta_3 t + \frac{\ell^2(b^2 - 2spk^2)}{2k^2s^2}\int_0^t A(\tau)\mathrm{d}\tau, \quad \delta_1 = \frac{\ell^2 A(t)}{2k^2s^2},$$

where b and ℓ are constants. Hence, by utilizing Eq. (9), the solutions of the SE-RVCs (3) are

$$\mathcal{F}(x,t) = \ell\mathcal{X}(\mu), \quad \mu = kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau. \tag{12}$$

To determine $\mathcal{X}$, there exist several families for the solutions of Eq. (8) depending on p and s, as follows:

Family I: If $ps > 0$, then Eq. (8) has the solutions:

$$\begin{aligned}\mathcal{X}_1(\mu) &= \sqrt{\frac{p}{s}}\tan(\sqrt{ps}\,\mu),\\ \mathcal{X}_2(\mu) &= -\sqrt{\frac{p}{s}}\cot(\sqrt{ps}\,\mu),\\ \mathcal{X}_3(\mu) &= \sqrt{\frac{p}{s}}(\tan(\sqrt{4ps}\,\mu) \pm \sec(\sqrt{4ps}\,\mu)),\\ \mathcal{X}_4(\mu) &= -\sqrt{\frac{p}{s}}(\cot(\sqrt{4ps}\,\mu) \pm \csc(\sqrt{4ps}\,\mu)),\end{aligned}$$

and

$$\mathcal{X}_5(\mu) = \frac{1}{2}\sqrt{\frac{p}{s}}\left(\tan\left(\frac{1}{2}\sqrt{ps}\,\mu\right) - \cot\left(\frac{1}{2}\sqrt{ps}\,\mu\right)\right).$$

Then, SE-RVCs (3) has the trigonometric functions solutions:

$$\mathcal{F}_1(x,t) = \ell\sqrt{\frac{p}{s}}\tan\left[\sqrt{ps}\left(kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right], \tag{13}$$

$$\begin{aligned}\mathcal{F}_2(x,t) = -\ell\sqrt{\frac{p}{s}}\cot\Bigg[\sqrt{ps}\Bigg(kx\\ + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\Bigg)\Bigg],\end{aligned} \tag{14}$$

$$\begin{aligned}\mathcal{F}_3(x,t) = \ell\sqrt{\frac{p}{s}}\Bigg(\tan\Bigg[\sqrt{4ps}\Bigg(kx\\ + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\Bigg)\Bigg]\\ \pm \sec\left[\sqrt{4ps}\left(kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\Bigg),\end{aligned} \tag{15}$$

$$\begin{aligned}\mathcal{F}_4(x,t) = \ell\sqrt{\frac{p}{s}}\Bigg(\cot\Bigg[\sqrt{4ps}\Bigg(kx\\ + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\Bigg)\Bigg]\\ \pm \csc\left[\sqrt{4ps}\left(kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma W(\tau) - \frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\Bigg),\end{aligned} \tag{16}$$

and

$$\mathcal{F}_5(x,t)=\frac{1}{2}\ell\sqrt{\frac{p}{s}}\left(\tan\left[\frac{1}{2}\sqrt{ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]-\cot\left[\frac{1}{2}\sqrt{ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right). \quad (17)$$

Family II: If $ps<0$, then Eq. (8) has the solutions:

$$\mathcal{X}_6(\mu)=-\sqrt{\frac{-p}{s}}\tanh(\sqrt{-ps}\,\mu),$$
$$\mathcal{X}_7(\mu)=-\sqrt{\frac{-p}{s}}\coth(\sqrt{-ps}\,\mu),$$
$$\mathcal{X}_8(\mu)=-\sqrt{\frac{-p}{s}}(\coth(\sqrt{-4ps}\,\mu)\pm\operatorname{csch}(\sqrt{-4ps}\,\mu)),$$

and

$$\mathcal{X}_9(\mu)=\frac{-1}{2}\sqrt{\frac{-p}{s}}\left(\tanh\left(\frac{1}{2}\sqrt{-ps}\,\mu\right)+\coth\left(\frac{1}{2}\sqrt{-ps}\,\mu\right)\right).$$

Then, SE-RVCs (3) has the hyperbolic functions solution:

$$\mathcal{F}_6(x,t)=-\ell\sqrt{\frac{-p}{s}}\tanh\left[\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right], \quad (18)$$

$$\mathcal{F}_7(x,t)=-\ell\sqrt{\frac{-p}{s}}\coth\left[\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right], \quad (19)$$

$$\mathcal{F}_8(x,t)=-\ell\sqrt{\frac{-p}{s}}\left(\coth\left[\sqrt{-4ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\pm\operatorname{csch}\left[\sqrt{-4ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right), \quad (20)$$

and

$$\mathcal{F}_9(x,t)=-\frac{1}{2}\ell\sqrt{\frac{-p}{s}}\left(\tanh\left[\frac{1}{2}\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]+\coth\left[\frac{1}{2}\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right). \quad (21)$$

Family III: If $p=0$ and $s\neq 0$, then the solution of Eq. (8) is:

$$\mathcal{X}_{10}(\mu)=\frac{-1}{s\mu}.$$

Then, the solution of SE-RVCs (3) is

$$\mathcal{F}_{10}(x,t)=\frac{-\ell}{s\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)}. \quad (22)$$

2.2 JEF method

Supposing the solutions of Eqs (4) and (5), with $m=1$, has the form

$$\mathcal{F}(x,t)=a_0(t)+a_1(t)J(\mu), \quad (23)$$

where $J(\mu)$ is one of the following elliptic functions $sn(\omega\mu,\check{n})$, $cn(\omega\mu,\check{n})$, or $dn(\omega\mu,\check{n})$. Differentiating Eq. (23) with respect to t and x, we have

$$\begin{aligned}&\mathcal{F}_t=\dot{a}_0+\dot{a}_1J+\omega\lambda a_1J', \quad \mathcal{F}_x=\omega k a_1J',\\ &\mathcal{F}_{xx}=a_1k^2\omega^2J''=a_1B_1J+a_1B_2J^3,\\ &\mathcal{F}^3=a_1^3J^3+3a_0a_1^2J^2+3a_0^2a_1J+a_0^3,\end{aligned} \quad (24)$$

where B_1 and B_2 are constants depending on $\omega,\check{n},k$, and they will be defined later. Eqs. (24) and (11) are plugged into Eqs (4) and (5). After that, by equating each coefficient of J^k to zero, we have

$$\begin{aligned}J^0:&\quad \dot{a}_0=0,\\ J^1:&\quad \dot{a}_1=0,\\ J':&\quad \omega\lambda a_1-2\delta_1\varphi_1 k\omega a_1=0,\end{aligned}$$

and

$$
\begin{aligned}
xJ^0: &\quad a_0\dot{\varphi}_1 = 0,\\
xJ^1: &\quad a_1\dot{\varphi}_1 = 0,\\
J^0: &\quad -a_0\dot{\varphi}_0 + \varphi_1^2\delta_1 a_0 + Aa_0^3 - \delta_3 a_0 = 0,\\
J^1: &\quad -a_1\dot{\varphi}_0 + \varphi_1^2\delta_1 a_1 - \delta_1 a_1 B_1 + 3Aa_0^2 a_1 - \delta_3 a_1 = 0,\\
J^2: &\quad 3Aa_0a_1^2 = 0,\\
J^3: &\quad -\delta_1 B_2 a_1 + Aa_1^3 = 0.
\end{aligned}
$$

We solve these equations to obtain

$$a_0(t) = 0, \quad a_1 = \hbar, \quad \delta_1 = \frac{\hbar^2 A(t)}{B_2}, \quad \varphi_1 = \varepsilon,$$

and

$$\varphi_0 = -\delta_3 t + \frac{\hbar^2(\varepsilon^2 - B_1)}{B_2}\int_0^t A(\tau)\mathrm{d}\tau, \quad \lambda(t) = \frac{2\varepsilon k\hbar^2}{B_2}A(t),$$

where ε and $\hbar$ are constants. Hence, by utilizing Eq. (23), the solutions of the SE-RVCs (3) are as follows:

$$\mathcal{F}(x,t) = \hbar J(\mu), \quad \mu = kx + \frac{2\varepsilon k\hbar^2\delta_2}{B_2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau. \tag{25}$$

Let us now define $J(\mu)$ as follows:

Set 1: If $J(\mu) = sn(\omega\mu, \check{n})$, then Eq. (25) takes the form

$$\mathcal{F}(x,t) = \hbar\left[sn\left(\omega kx + \frac{2\varepsilon\omega k\hbar^2\delta_2}{B_2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau, \check{n}\right)\right], \tag{26}$$

where

$$B_1 = -k^2\omega^2(1+\check{n}^2) \quad \text{and} \quad B_2 = 2k^2\omega^2\check{n}^2.$$

Set 2: If $J(\mu) = cn(\omega\mu, \check{n})$, then Eq. (25) takes the form

$$\mathcal{F}(x,t) = \hbar\left[cn\left(\omega kx + \frac{2\varepsilon\omega k\hbar^2\delta_2}{B_2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau, \check{n}\right)\right], \tag{27}$$

where

$$B_1 = k^2\omega^2(1-2\check{n}^2) \quad \text{and} \quad B_2 = -2k^2\omega^2\check{n}^2.$$

Set 3: If $J(\mu) = dn(\omega\mu, \check{n})$, then Eq. (25) takes the form

$$\mathcal{F}(x,t) = \hbar\left[dn\left(\omega kx + \frac{2\varepsilon\omega k\hbar^2\delta_2}{B_2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau, \check{n}\right)\right], \tag{28}$$

where

$$B_1 = k^2\omega^2(2-\check{n}^2) \quad \text{and} \quad B_2 = -2k^2\omega^2.$$

3 Exact solutions of SSE

By substituting Eq. (12) into Eq. (2), we attain the solution of SSE (1) as follows:

$$\mathcal{Z}(x,t) = \mathcal{F}(x,t)\exp\left[i\mathcal{Y}(x,t) + \sigma\mathcal{W}(t) - \frac{1}{2}\sigma^2 t\right]. \tag{29}$$

3.1 GREM method

Plugging Eqs (13)–(22) into (29), then the SSE has the solutions:

$$
\begin{aligned}
\mathcal{Z}_1(x,t) = \ell\sqrt{\frac{p}{s}}\tan\left[\sqrt{ps}\left(kx\right.\right. &\\
\left.\left. + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]&\\
\times e^{[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]},&
\end{aligned} \tag{30}
$$

$$
\begin{aligned}
\mathcal{Z}_2(x,t) = -\ell\sqrt{\frac{p}{s}}\cot[\sqrt{ps}(kx &\\
+ \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau)] & e^{[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]},
\end{aligned} \tag{31}
$$

$$
\begin{aligned}
\mathcal{Z}_3(x,t) = \ell\sqrt{\frac{p}{s}}\left(\tan\left[\sqrt{4ps}\left(kx\right.\right.\right. &\\
\left.\left. + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right] &\\
\left. - \sec\left[\sqrt{ps}\left(kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right) &\\
\times e^{[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]},&
\end{aligned} \tag{32}
$$

$$
\begin{aligned}
\mathcal{Z}_4(x,t) = \ell\sqrt{\frac{p}{s}}\left(\cot\left[\sqrt{4ps}\left(kx\right.\right.\right. &\\
\left.\left. + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right] &\\
\left. + \csc\left[\sqrt{4ps}\left(kx + \frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right) &\\
\times e^{[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}&
\end{aligned} \tag{33}
$$

$$\mathcal{Z}_5(x,t)=\frac{1}{2}\ell\sqrt{\frac{p}{s}}\left(\tan\left[\frac{\sqrt{ps}}{2}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]-\cot\left[\frac{\sqrt{ps}}{2}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right)\times e^{[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}, \tag{34}$$

for $ps > 0$,

$$\mathcal{Z}_6(x,t)=\ell\sqrt{\frac{-p}{s}}\tanh\left[\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]e^{\left[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t\right]}, \tag{35}$$

$$\mathcal{Z}_7(x,t)=\ell\sqrt{\frac{-p}{s}}\coth\left[\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]e^{\left[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t\right]}, \tag{36}$$

$$\mathcal{Z}_8(x,t)=\ell\sqrt{\frac{-p}{s}}\left(\coth\left[\sqrt{-4ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\mp\operatorname{csch}\left[\sqrt{-4ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right)e^{\left[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t\right]}, \tag{37}$$

$$\mathcal{Z}_9(x,t)=-\frac{1}{2}\ell\sqrt{\frac{-p}{s}}\left(\tanh\left[\frac{1}{2}\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]+\coth\left[\frac{1}{2}\sqrt{-ps}\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right]\right)e^{\left[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t\right]}, \tag{38}$$

for $ps < 0$, and

$$\mathcal{Z}_{10}(x,t)=\left(\frac{-1}{s\left(kx+\frac{b\ell^2\delta_2}{ks^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)}\right)e^{\left[i\mathcal{Y}+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t\right]}\quad \text{for}\quad s\neq 0, \tag{39}$$

where

$$\mathcal{Y}(x,t)=bx-\delta_3 t+\frac{\delta_2\ell^2(b^2-2spk^2)}{2k^2s^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau.$$

Remark 1. Putting $\sigma = 0$ (*i.e.*, no noise) and $\delta_3 = 0$ in Eqs (32), (33), and (37), we have the same results that stated in [24].

Remark 2. Putting $\sigma = 0$ and $\delta_3 = 0$ in Eqs (30) and (35), we have the same results that stated in [23].

3.2 JEF method

Substituting Eqs (26)–(28) into (29), then the SSE has the solutions:

$$\mathcal{Z}(x,t)=\hbar\left(sn\left(\omega kx+\frac{\varepsilon\hbar^2\delta_2}{k\omega\check{n}^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau,\check{n}\right)\right)\times e^{[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}, \tag{40}$$

$$\mathcal{Z}(x,t)=\hbar\left(cn\left(\omega kx-\frac{\varepsilon\hbar^2\delta_2}{k\omega\check{n}^2}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau,\check{n}\right)\right)\times e^{[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}, \tag{41}$$

and

$$\mathcal{Z}(x,t)=\hbar\left(dn\left(\omega kx-\frac{\varepsilon\hbar^2\delta_2}{k\omega}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau,\check{n}\right)\right)\times e^{[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}, \tag{42}$$

where

$$\mathcal{Y}=\varepsilon x-\delta_3 t+\frac{\hbar^2(\varepsilon^2-B_1)}{B_2}\int_0^t A(\tau)\mathrm{d}\tau.$$

When $\check{n}\to 1$, then Eqs (40)–(42) tend to

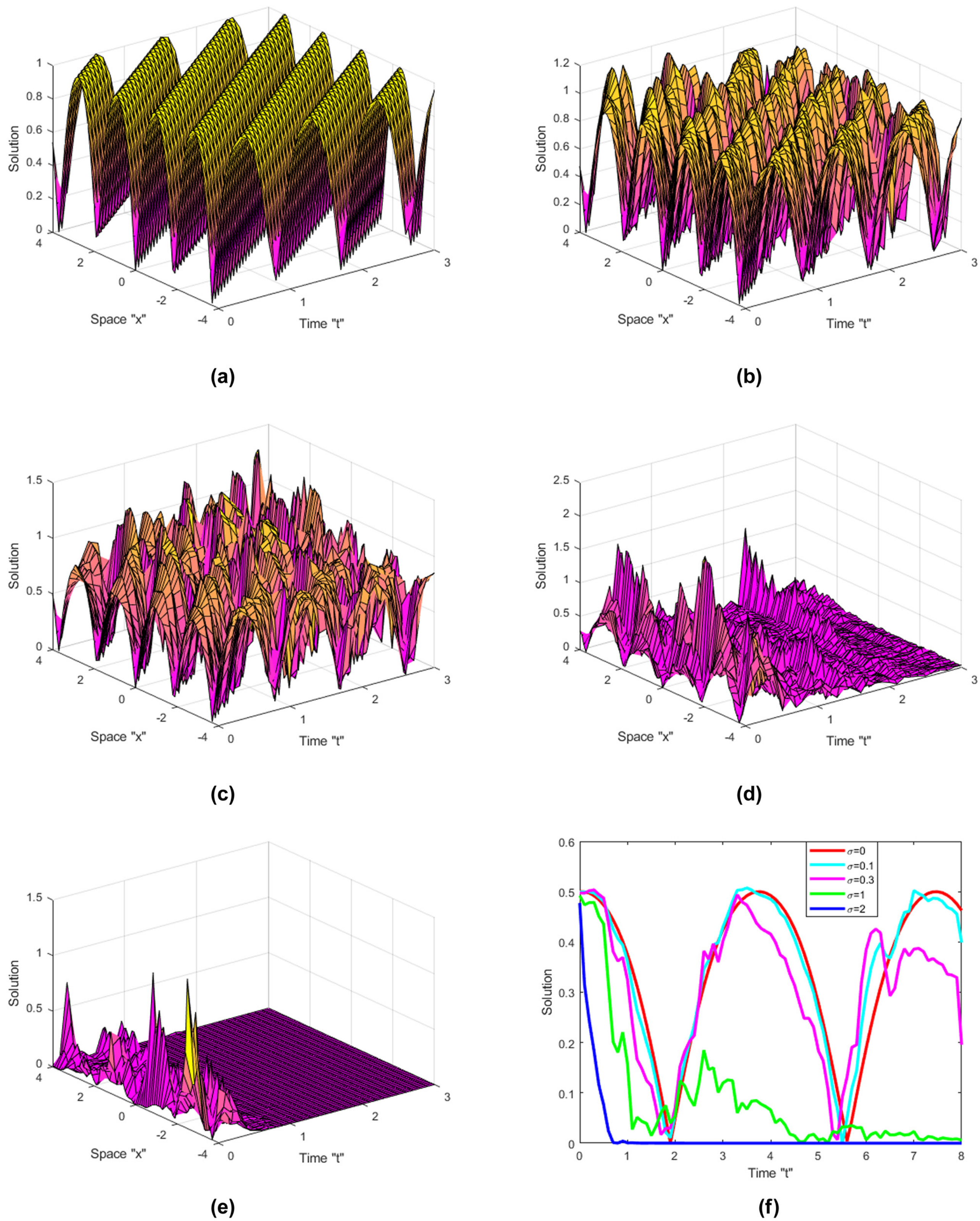

Figure 1: (a)–(e) Exhibit 3D-profile of $|\mathcal{Z}(x, t)|$ described in Eq. (41) with $\hbar = \check{n} = 0.5, \delta_2 = 2, \varepsilon = 1, \omega = k = 1, x \in [4, 4]$, and $t \in [0, 4]$, (f) 2D-profile of Eq. (41) with $\sigma = 0, 0.1, 0.3, 1$, and 2. (a) $\sigma = 0$, (b) $\sigma = 0.1$, (c) $\sigma = 0.3$, (d) $\sigma = 1$, (e) $\sigma = 2$, (f) $\sigma = 0, 0.1, 0.3, 1, 2$.

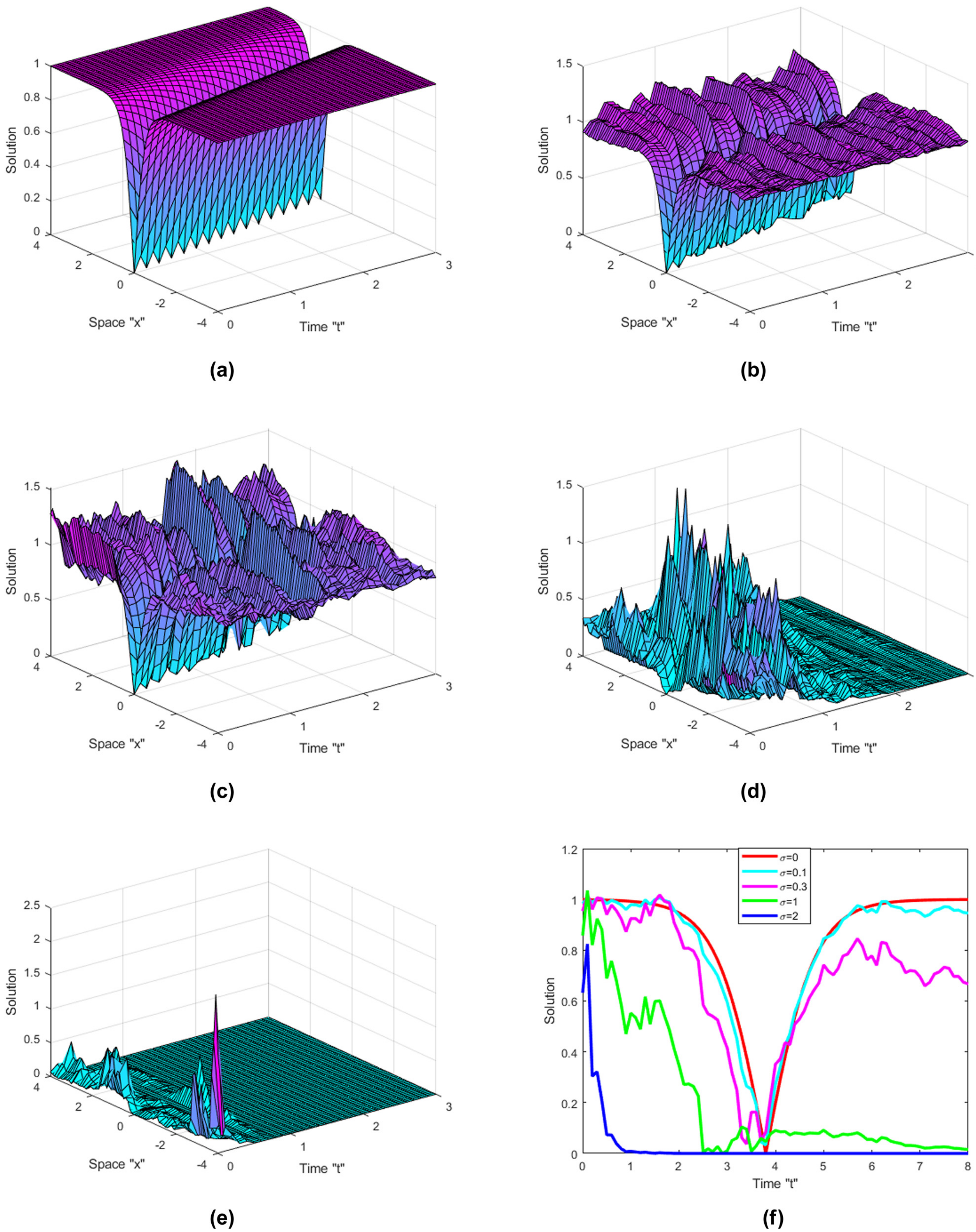

Figure 2: (a)–(e) Exhibit 3D-profile of $|\mathcal{Z}(x,t)|$ introduced in Eq. (43) with $\hbar = \check{n} = 1$, $\delta_2 = 2$, $\varepsilon = 1$, $\omega = k = 1$, $x \in [4, 4]$, and $t \in [0, 4]$ (f) displays 2D-profile of Eq. (43) with $\sigma = 0, 0.1, 0.3, 1$, and 2. (a) $\sigma = 0$, (b) $\sigma = 1$, (c) $\sigma = 0.3$, (d) $\sigma = 1$, (e) $\sigma = 2$, and (f) $\sigma = 0, 0.1, 0.3, 1, 2$.

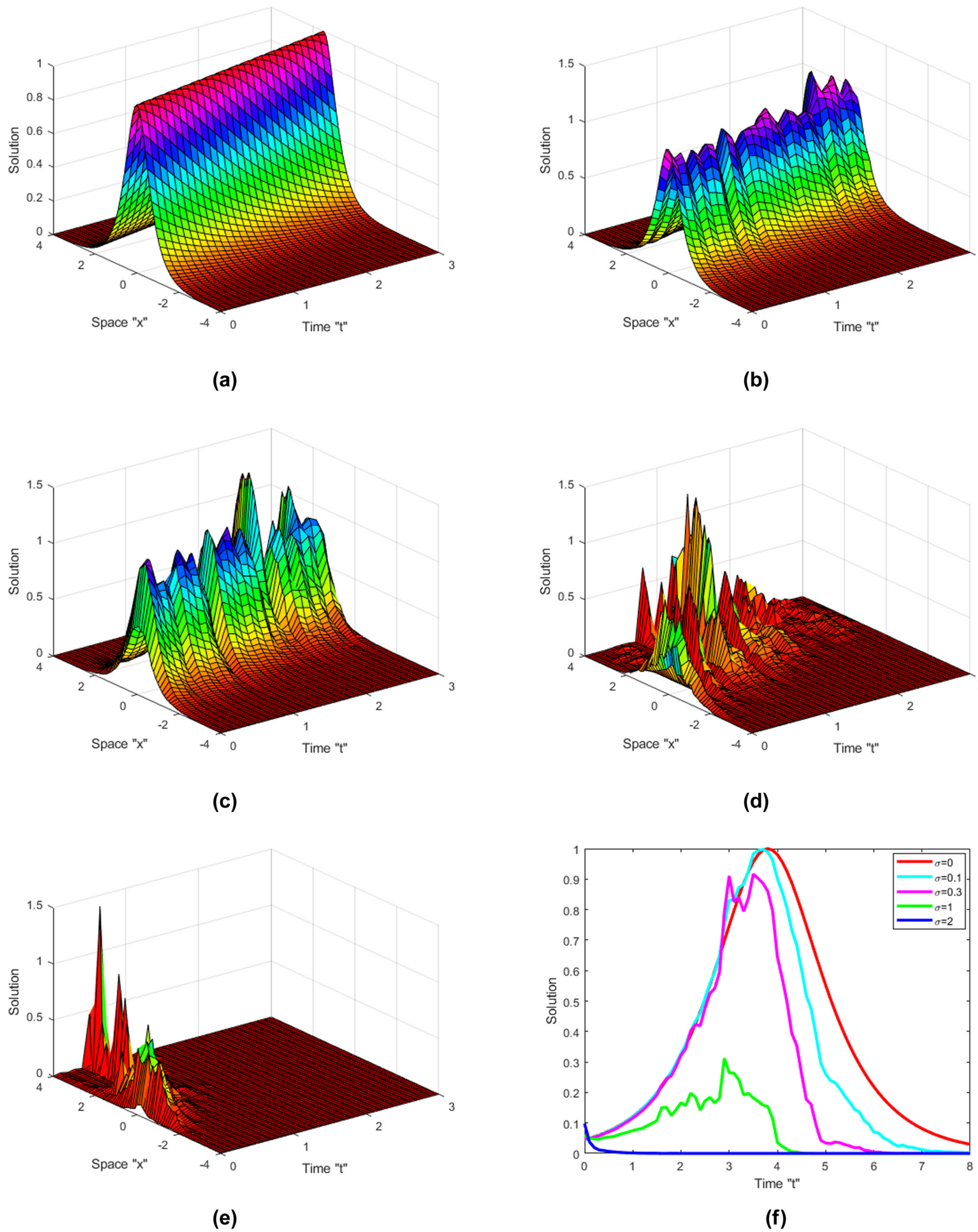

Figure 3: (a)–(e) Exhibit 3D-profile of $|\mathcal{Z}(x,t)|$ introduced in Eq. (44) with $\hbar = \check{n} = 0.5, \delta_2 = 2, \quad \varepsilon = 1, \omega = k = 1, x \in [4, 4]$, and $t \in [0, 3]$ (f) displays 2D-profile of Eq. (44) with $\sigma = 0$, 0.1, 0.3, 1, and 2. (a) $\sigma = 0$, (b) $\sigma = 0.1$, (c) $\sigma = 0$, (d) $\sigma = 1$, (e) $\sigma = 2$, and (f) $\sigma = 0$, 0.1, 0.3, 1, 2.

$$\mathcal{Z}(x,t)=\hbar\left(\tanh\left(\omega kx+\frac{\varepsilon\hbar^2\delta_2}{k\omega}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right)\times e^{[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]},\tag{43}$$

and

$$\mathcal{Z}(x,t)=\hbar\left(\operatorname{sech}\left(\omega kx-\frac{\varepsilon\hbar^2\delta_2}{k\omega}\int_0^t e^{\sigma\mathcal{W}(\tau)-\frac{1}{2}\sigma^2\tau}\mathrm{d}\tau\right)\right)\times e^{[i\mathcal{Y}(x,t)+\sigma\mathcal{W}(t)-\frac{1}{2}\sigma^2 t]}.\tag{44}$$

4 Physical meaning and effect of noise

Physical meaning: The SSE provides a powerful tool for studying the behavior of quantum systems in the presence of stochastic influences. By incorporating random fluctuations into the evolution of quantum states, this equation offers a more realistic and nuanced understanding of quantum phenomena. Through its ability to capture the physical implications of noise and uncertainty in quantum systems, the stochastic Schrödinger equation helps to deepen our understanding of the inherently probabilistic nature of the quantum world. Here, we obtained the exact stochastic solutions of the SSE (1). We utilized two methods including the GREM method and the JEF method. There are many kinds of solutions including singular periodic solutions, dark solutions, and light solutions. Singular solitons are important because of their special properties that allow for effective information transfer and the investigation of complex wave dynamics in many different fields of science. In nonlinear optics, singular solitons play an essential role in light transmission over optical fibers. Solitons, unlike regular light waves, do not scatter over long distances due to their self-focusing qualities, allowing for the effective transport of information at high rates. This makes them crucial in telecommunications, where data transfer over vast distances is critical for preserving network connectivity and speed.

Effect of noise: Now, we address the impact of multiplicative white noise on the exact solutions of the SSE (1). Numerous numerical simulations of various solutions with different intensity of noise are shown. Figures 1–3 display the solutions $\mathcal{Z}(x,t)$ described in Eqs (41), (43) and (44) for different intensity of noise σ as follows:

Figures 1–3 show that when noise is ignored (*i.e.*, $\sigma = 0$), different kinds of solutions appear, including periodic solutions, singular solutions, optical light solutions, dark solutions, and so on. When noise is introduced at $\sigma = 0.1, 0.3, 1, 2$, the surface flattens after some transit patterns. This result shows how multiplicative white noise affects the SSE (1) solutions, stabilizing them around zero.

5 Conclusion

In this article, we considered the SSE (1) driven by multiplicative Wiener process in the Itô sense. By using appropriate transformation, we converted the SSE to another Schrödinger equation with random variable coefficients (3) (SE-RVCs). By using the GREM method and JEF method, we obtained a new stochastic exact solutions for SE-RVCs in the type of trigonometric, hyperbolic, rational, and elliptic functions. After that, we acquired the obtained solutions of SSE (1). Moreover, we generalized some previous solutions such as the results reported in previous studies [23,24]. Because of the importance of Schrödinger equation in nonlinear waves in a liquid-filled elastic tube, solitary wave and nonlinear instability problems, plasma waves and hydromagnetic, heat transfer in a solid, nonlinear optics, and propagation in the piezoelectric semi- conductors, the acquired solutions are crucial in understanding several difficult physical processes. Finally, some graphics were included to demonstrate the effect of stochastic term on the stochastic exact solutions of the SSE. We deduced that the multiplicative Wiener process stabilized the solutions of SSE (1). In the future work, we can address the Schrödinger equation (1) with additive noise.

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