Research Article

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Analysis of a generalized proportional fractional stochastic differential equation incorporating Carathéodory's approximation and applications

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Abstract: Fractional stochastic differential equations (FSDEs) with fractional derivatives describe the anomalous diffusion processes by incorporating the memory effects and spatial heterogeneities of the porous medium. The stochastic component addresses the random nature of the fluid flow due to the variability in pore sizes and connectivity. The first objective of this research is to prove the well-posedness of a class of generalized proportional FSDEs, and we acquire the global existence and uniqueness of findings under certain settings that are coherent with the classic SDEs. The secondary purpose is to evaluate the continuity of findings in fractional-order formulations. The Carathéodory approximation is taken into account for a class of generalized proportional FSDEs, which is pivotal and provides well-known bounds on the norm of the solutions. Carathéodory's approximation aids in approximating the FSDEs governing turbulent flows, ensuring the solutions are mathematically robust and physically meaningful. As is widely documented, the existence and uniqueness of solutions to certain types of differential equations can be formed under Lipschitz and linear growth conditions. Furthermore, a class of generalized proportional FSDEs with time delays is considered according to certain new requirements. With the aid of well-known inequalities and Itô isometry technique, the Ulam–Hyers stability of the analyzed framework is addressed utilizing Lipschitz and non-Lipschitz characteristics, respectively. Additionally, we provide two illustrative examples as applications to demonstrate the authenticity of our interpretations. The demonstrated outcomes will generalize some previously published findings. Finally, this deviation from fractional Brownian motion necessitates a model that can capture the subdiffusive or superdiffusive behavior.

Keywords: generalized proportional fractional operators, stochastic generalized proportional fractional differential equations, Itô isometry formula, Carathéodry's approximation, subdiffusive or superdiffusive behavior

1 Introduction

Fractional calculus is very pertinent in meaningful manifestations due to certain similar characteristics, including memory. Numerous sorts of kernels are used in fractional formulations [1,2]. Kochubei [3] investigated a very particular form of kernel known as a general fractional integral/derivative. Luchko [4] examined such generally designed integrals and derivatives in adequate functional spaces within the structure of applied mathematics. Luchko also probed the quantifiable attributes of multiple sorts of differential equations (DEs) solutions with broad sense fractional derivatives [5–7]. Several researchers have examined the stability and performance of fractional DEs (FDEs) [8,9]. For a framework to be reliable in the context of Lyapunov, its generalized vitality is not required to decay exponentially, and previously, the Mittag–Leffler (M–L) strength and the fractional Lyapunov communicative approach for multiple kinds of FDEs were initiated [10,11].

Recently, FDEs have grown in importance in both theoretical and practical aspects, attracting a considerable amount of interest from academics [12–14]. Numerous researchers concentrate on aspects of stochastic DEs (SDEs) including solution existence and uniqueness (E-U), continuous reliance of strategies on initialization and complex behavior [15–18]. Moreover, no scholars are interested in the continuity of findings on the fractional scaling factor of this type of formula; specifically, none of them are curious about the interaction between the alternatives of classical SDEs and the fractional ones. In this work, we continued our investigation of the proportional fractional derivatives and integrals revealed in the study by Jarad *et al.* [19], which was guided by the previously mentioned investigations. We show how fractional integral operators

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affect differential operators, and *vice versa*. Furthermore, this research examined this intriguing topic of SDEs and revealed the connection among them. If the fractional order and proportional index of the generalized proportional fractional (GPF) derivative converge to one, the findings of GPF SDEs reduces to the solutions of Caputo fractional SDEs and classic SDEs, respectively.

FDEs are an essential mechanism in many disciplines of scientific domains because of their non-local feature of GPF derivatives in time [20,21]. The kernel impact or non-local feature is represented by a convolution integral with an exponential function memory kernel that also gives the GPF-DEs a greater capability in dynamic structures; this is among the primary strengths of FDEs (non-local) in correlation with traditional (local) configurations [22–24]. The analysis of revolutionary calculus and FDEs is becoming increasingly popular (see [25–27], as well as the reference materials therein). In fluid physics, stochastic FDEs incorporating Carathéodory's approximation offer a sophisticated framework for modeling complex phenomena such as turbulence, anomalous diffusion, viscoelastic fluid behavior, and particle dynamics. By ensuring the existence, uniqueness, and stability of solutions, Carathéodory's approximation makes it feasible to apply these advanced mathematical tools to real-world fluid dynamics problems, leading to deeper insights and more accurate predictions. Moreover, stochastic FDEs tracking the movement of individual fluid particles (the Lagrangian approach) provide insights into mixing, dispersion, and turbulence. The random motion of particles is influenced by both the deterministic flow field and stochastic environmental factors for Lagrangian stochastic models of fluid particles.

However, because noisy perturbations are prevalent in complicated processes, stochastic modeling has played a significant role in a variety of research and industry. Plenty of efforts on SDEs or SPDEs have been made, with numerous scientific preferences having similar properties such as well-posedness, continuity, variability, transformed manifolds, and irreducible measure. Conga *et al.* [28] investigated the existence of reliable configurations for a particular class of stochastic differential equations. Xu *et al.* [29] presented the approximation concept for SDEs *via* the Caputo fractional derivative. Wang *et al.* [30] contemplated the asymptotic dynamics of stochastic lattice frameworks involving Caputo fractional time derivatives. Doan *et al.* [31] expounded a spatial and temporal weighted norm that is employed to evaluate the asymptotic distance between two distinguishable strategies. It is interesting to note that Doan *et al.* [32] founded the Euler–Maruyama form estimation outcome for Caputo FSDEs. Carahéodory initially regarded the Carathéodory approximate solution framework for ordinary DEs, and then Bell, Mohammad, and Mao lengthened it to include the specific instances of SDEs [33]. Wang *et al.* [34] examined the continuity, and Guo *et al.* [35] applied Carathéodory's approximation for a Caputo-type FSDES.

In a presentation at Wisconsin University in 1940, Ulam suggested the reliability of systems of equations [36]. In 1941, Hyers [37] became the best person to respond to the inquiry. The Ulam–Hyers stability (U–Hs) was created as a result. In the meantime, an expanding number of individuals have been eager to look into the U–Hs. The characteristics of canonical and generalized M-L functions, as well as the U–Hs of sequential FDEs, had been demonstrated using fractional calculus and the Laplace transform technique in the study by Wang and Li [38]. Researchers examined the U–Hs, generalized U–Hs, U–H–Rassias stability, and U–H–Rassias stability of impulsive integrodifferential formulations incorporating Riemann–Liouville boundary assumptions in the study by Zada *et al.* [39]. For further investigation on U–Hs, we refer the readers to previous studies [40–42] and references cited therein.

To the highest potential of our expertise, no research has been conducted that focuses on the continuity of the findings of the GPF-SDEs in respect of fractional order, if it tends to the solution of the Caputo FSDEs and classical SDEs when the η_1 and ψ tend to 1, respectively. We begin by considering the well-posedness of generalized proportional FSDEs on the Banach space, employing various approximate techniques, and then, we deduce the global E-U of findings under certain settings that are coherent with classical SDEs. More specifically, we will analyze the well-posedness and continuity of the solutions of GPF-SDEs with the aid of the Carathéodory approximation listed in the following:

$$\begin{cases}\mathbf{D}_{\zeta}^{\eta_1;\,\psi}\mathbf{X}_{\zeta} = \mathcal{F}(t,\mathbf{X}_{\zeta})\mathrm{d}\zeta + \mathcal{G}(t,\mathbf{X}_{\zeta})\mathrm{d}\mathcal{W}_{\zeta}, \quad \zeta \geq 0,\\ \eta_1 \in (1/2, 1),\\ \mathbf{X}_0 = \mu_0 \in \mathcal{L}^2(\mho, \Delta).\end{cases} \tag{1.1}$$

Motivated by the aforesaid proclivity, in this work, we will investigate the existence and U–Hs of time-delayed generalized proportional FSDEs:

$$\begin{cases}\mathbf{D}_{\zeta}^{\eta_1;\,\psi}\mathbf{X}_{\zeta} = \mathcal{F}(\zeta,\mathbf{X}(\zeta),\mathbf{X}(\zeta-\omega))\\ \qquad\qquad + \mathcal{G}(\zeta,\mathbf{X}(\zeta),\mathbf{X}(\zeta-\omega))\dfrac{\mathrm{d}\mathcal{W}(\zeta)}{\mathrm{d}\zeta},\\ \zeta \in \mathcal{J} = [0,\Xi],\\ \mathbf{X}_0(\zeta) = \Phi(\zeta) \quad \zeta \in [0,\Xi],\end{cases} \tag{1.2}$$

where $\eta_1 \in (1/2, 1]$, $\mathcal{F} : [0,\Xi] \times \mathbb{R}^{\eth} \times \mathbb{R}^{\eth} \mapsto \mathbb{R}^{\eth}$, and $\mathcal{G} : [0,\Xi] \times \mathbb{R}^{\eth} \times \mathbb{R}^{\eth} \mapsto \mathbb{R}^{\eth\times m}$ are the measurable continuous mappings. Also, there is an m_1-dimensional

Brownian motion $\mathcal{W}(\zeta)$ defined on a complete probability space $\{\Psi, \mathfrak{F}, \mathbf{P}\}$ and a continuous mapping $\Phi(\zeta) : [-\omega, 0] \mapsto \mathbb{R}^d$ having $\mathbf{E}||\Phi(\zeta)||^2 < \infty$, where $\mathbf{E}$ is the mathematical expectation.

In comparison with the previous scientific studies [34,35], the significant achievements of this article encompass at least three components:

- In this article, we only examine the scenario of $\eta_1 \in (1/2, 1)$ and use it to investigate the well-posedness and continuity with respect to the generalized proportional FSDEs. The asymptotic behavior of solutions is then taken into account.
- The methodologies we utilize to determine the E-U of generalized proportional FSDE solutions are quite revolutionary than those of previous studies [34, 35]. Krasnoselskii's and Mönch's fixed-point hypothesis were employed in previous studies [29, 30] to investigate the E-U. However, we use Carathéodory's approximation in this article to examine the E-U.
- Several previous studies [43,44] have employed a stronger Lipschitz assumption in the investigation of multiple stabilities, E-U of FSDEs. Even so, in this article, we discussed the U-Hs of GPF-SDEs *via* weak non-Lipschitz assumptions. This is a significant step forward in the investigation of the stability of GPF-SDEs.

This work is structured as follows. In Section 2, we will present certain fundamental assumptions and outcomes. Section 3 will go over the well-posedness and continuity of GPF-SDE using the Itô isometry and well-known inequalities. A novel way the Carathéodory approximation is adopted to find the E-U of the generalized proportional FSDEs. Section 4 is dedicated to demonstrating the stability outcomes for generalized proportional FSDEs with time delays. Section 5 provides the illustrations to validate the implementation of our research results.

2 Preliminaries

This portion outlines several terminologies, interpretations, and key formulas that will be employed throughout the remainder of the article. The viewer can refer to the article [19] for its explanation and verification.

Abdeljawad [45] and Khalil *et al.* [46] provided the following limit form description of the well-known conformable derivative:

$$\mathcal{D}^{\eta_1}\mathcal{F}(\zeta) = \lim_{\varepsilon\mapsto\infty} \frac{\mathcal{F}(\zeta + \varepsilon\zeta^{1-\eta_1})}{\varepsilon}. \tag{2.1}$$

It is self-evident that when a mapping $\mathcal{F}$ is differentiable, its conformable derivative interprets

$$\mathcal{D}^{\eta_1}\mathcal{F}(\zeta) = \zeta^{1-\eta_1}\mathcal{F}'(\zeta). \tag{2.2}$$

The major disadvantage of this derivative is that the mapping $\mathcal{F}$ is not achieved when $\eta_1 = 0$ or $\eta_1 \mapsto 0$, i.e., $\mathcal{D}^0\mathcal{F} \neq \mathcal{F}$. Anderson *et al.* [47] categorized the reconfigured conformable derivative to address this issue and take advantage of the proportional derivative for process variables with two confinement specifications.

Definition 2.1. [47] For $\psi \in (0, 1]$ and assume that there are two continuous mappings $\kappa_0, \kappa_1 : [0, 1] \times \mathbb{R} \mapsto [0, \infty)$ such that $\forall \zeta \in \mathbb{R}$, we have

$$\lim_{\psi\mapsto 0^+} \kappa_1(\psi, \zeta) = 1, \quad \lim_{\psi\mapsto 0^+} \kappa_0(\psi, \zeta) = 0,$$
$$\lim_{\psi\mapsto 1^-} \kappa_1(\psi, \zeta) = 0, \quad \lim_{\psi\mapsto 1^-} \kappa_0(\psi, \zeta) = 1,$$

and $k_1(\psi, \zeta) \neq 0$, $\kappa_0(\psi, \zeta) \neq 0$, $\psi \in (0, 1]$. Then, the proportional derivative of order ψ is stated as

$$\mathcal{D}^{\psi}\mathcal{F}(\zeta) = \kappa_1(\psi, \zeta)\mathcal{F}(\zeta) + \kappa_0(\psi, \zeta)\mathcal{F}'(\zeta). \tag{2.3}$$

We recommend the viewer refer to previous studies [47,48] for more information on the control theory of the proportional derivative and its constituent mappings κ_0 and κ_1. We will confine ourselves to the particular instance where $\kappa_1(\psi, \zeta) = 1 - \psi$ and $\kappa_0(\psi, \zeta) = \psi$. Consequently, (2.3) reduces to

$$\mathcal{D}^{\psi}\mathcal{F}(\zeta) = (1 - \psi)\mathcal{F}(\zeta) + \psi\mathcal{F}'(\zeta). \tag{2.4}$$

It is simple to understand that $\lim_{\psi\mapsto 0^+}\mathcal{D}^{\psi}\mathcal{F}(\zeta) = \mathcal{F}(\zeta)$ and $\lim_{\psi\mapsto 1^-}\mathcal{D}^{\psi}\mathcal{F}(\zeta) = \mathcal{F}'(\zeta)$. And hence, the derivative (2.4) is thought to be quite comprehensive compared to the conformable derivative, which clearly does not approach the intended mappings in the same way that ψ approaches to 0.

The integral and derivative of the GPF are described in the following.

Definition 2.2. [19] For $\psi \in (0, 1]$, $\eta_1 \in \mathbb{C}$, and $\Re(\eta_1) > 0$, then there is a GPF integral of order η_1, which is stated as

$$\begin{aligned} {}_{a_1}\mathbf{I}^{\eta_1,\psi}\mathcal{F}(\zeta) &= \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \int_{a_1}^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\omega)}(\zeta - \omega)^{\eta_1-1}\mathcal{F}(\omega)\mathrm{d}\omega \\ &= \psi^{-\eta_1} e^{\frac{\psi-1}{\psi}\zeta}\left({}_{a_1}\mathbf{I}^{\eta_1}\left(e^{\frac{1-\psi}{\psi}\zeta}\mathcal{F}(\zeta)\right)\right). \end{aligned} \tag{2.5}$$

Definition 2.3. [19] For $\psi \in (0, 1]$, $\eta_1 \in \mathbb{C}$, and $\Re(\eta_1) \geq 0$, then there is a GPF derivative of order η_1, which is stated as

$$
{}_{a_1}\mathcal{D}^{\eta_1,\psi}\mathcal{F}(\zeta) = (\mathcal{D}^{\mathfrak{n},\eta_1}{}_{a_1}\mathbf{I}^{\eta_1,\psi}\mathcal{F})(\zeta) = \frac{\mathcal{D}_{\zeta}^{\mathfrak{n},\psi}}{\psi^{\mathfrak{n}-\eta_1}\Gamma(\mathfrak{n}-\eta_1)}\int_{a_1}^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\omega)}(\zeta-\omega)^{\mathfrak{n}-\eta_1-1} \mathcal{F}(\omega)\mathrm{d}\omega, \quad \mathfrak{n} = [\Re(\eta_1)] + 1. \tag{2.6}
$$

Definition 2.4. [19] Let $\psi \in (0,1]$. Then, the fractional operator

$$
{}_{a_1}\mathcal{D}^{\eta_1,\psi}\mathcal{F}(\zeta) = \frac{1}{\psi^{\mathfrak{n}-\eta_1}\Gamma(\mathfrak{n}-\eta_1)}\int_{a_1}^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\omega)}(\zeta-\omega)^{\mathfrak{n}-\eta_1-1}(\mathcal{D}^{\mathfrak{n},\psi}\mathcal{F}) \times (\omega)\mathrm{d}\omega, \quad \mathfrak{n} = [\Re(\eta_1)] + 1 \tag{2.7}
$$

is the left-sided GPF derivative of the function $\mathcal{F}$ in the context of Caputo of order η_1, where $\mathfrak{n} = [\eta_1] + 1$.

By letting $\psi = 1$ in Definitions 2.3 and 2.4, then we obtain the left Riemann–Liouville (R-L) and Caputo fractional derivatives. Furthermore, it is noticeable that

$$
\lim_{\eta_1 \mapsto 0}(\mathcal{D}^{\eta_1,\psi}\mathcal{F})(\zeta) = \mathcal{F}(\zeta) \quad \text{and} \quad \lim_{\eta_1 \mapsto 1}(\mathcal{D}^{\eta_1,\psi}\mathcal{F})(\zeta) = (\mathcal{D}^{\eta_1}\mathcal{F})(\zeta).
$$

Furthermore, we implement the respective approaches to ensure the solution's global E-U. Assume that Δ signifies the Hilbert space and $\|.\|$ represents its norm. For the purposes of clarity, we will suppose that the functions $k_\iota(\zeta)$ have the identical upper estimate k_1 for $\iota = 1, 2$.

($\mathbf{A}_1$): Lipschitz condition: $\forall \zeta \geq 0$, $\exists$ a bounded mapping $k_1(\zeta)$ such that $\forall \mu, \nu \in \Delta$

$$
|\mathcal{F}(\zeta,\mu) - \mathcal{F}(\zeta,\nu)|^2 + |\mathcal{G}(\zeta,\mu) - \mathcal{G}(\zeta,\nu)|^2 \leq k_1(\zeta)|\mu-\nu|^2.
$$

($\mathbf{A}_2$): Growth condition: $\forall \zeta \geq 0$, $\exists$ a bounded mapping $k_2(\zeta)$ such that $\forall \mu \in \Delta$

$$
|\mathcal{F}(\zeta,\mu)|^2 + |\mathcal{G}(\zeta,\mu)|^2 \leq k_2(\zeta)(1 + |\mu|^2).
$$

To cope with FDEs, we require the following generalized Gronwall's lemma for exponential function-type kernel (see [49]) as follows:

Lemma 2.1. *For $\eta_1 > 0, \psi \in (0,1]$ and suppose that there are two non-negative locally integrable mappings defined on $[0,\Xi]$ and $\bar{b}(\zeta)$ is a positive, increasing, and continuous mapping on $\zeta \in [0,\Xi)$ having*

$$
\bar{u}(\zeta) \leq \bar{a}(\zeta) + \psi^{\eta_1}\Gamma(\eta_1)\bar{b}(\zeta)({}_0\mathbf{I}^{\eta_1,\psi}\bar{u})(\zeta), \tag{2.8}
$$

then

$$
\bar{u}(\zeta) \leq \bar{a}(\zeta) + \int_0^{\zeta}\left[\sum_{\mathfrak{n}=1}^{\infty}\frac{(\bar{b}(\zeta)\Gamma(\eta_1))^{\mathfrak{n}}}{\Gamma(\mathfrak{n}\eta_1)e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}}(\zeta-\varphi)^{\mathfrak{n}\eta_1-1}\bar{a}(\varphi)\right]\mathrm{d}\varphi, \quad \zeta \in [0,\Xi), \tag{2.9}
$$

where $\Gamma(.)$ is the Euler-Gamma function.

3 Main results

In this section, we will present the well-posedness and the continuity of the solution of (1.1).

3.1 Well-posedness

First, we surmise the E-U of mild solutions for the subsequent equation using the aforementioned hypotheses ($\mathbf{A}_1$) and ($\mathbf{A}_2$):

$$
\begin{cases}
\mathbf{D}_{\zeta}^{\eta_1;\,\psi}\mathbf{X}_{\zeta} = \mathcal{F}(t,\mathbf{X}_{\zeta})\mathrm{d}\zeta + \mathcal{G}(t,\mathbf{X}_{\zeta})\mathrm{d}\mathcal{W}_{\zeta}, \quad \zeta \geq 0,\\
\eta_1 \in (1/2,1),\\
\mathbf{X}_0 = \mu_0 \in \mathcal{L}^2(\mho,\Delta),
\end{cases} \tag{3.1}
$$

where $\mathcal{W}_{\zeta}$ denotes the Brownian evolution and $\mathcal{F}$ and $\mathcal{G}$ are Δ-valued mappings.

Definition 3.1. Suppose there is an Δ-valued $\mathfrak{F}_{\zeta}$-adapted stochastic technique $\mathbf{X}_{\zeta}$, $\zeta \in [0,\Xi]$ is termed as the mild solutions of the initial value problem (2.6), if $\mathbf{X}_{\zeta} \in \mathfrak{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ and fulfills the subsequent integral formulation:

$$
\mathbf{X}_{\zeta} = \mu_0 + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_{\varphi})\mathrm{d}\varphi + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_{\varphi})\mathrm{d}\mathcal{W}_{\varphi}. \tag{3.2}
$$

Theorem 3.1. *Under the suppositions ($\mathbf{A}_1$) and ($\mathbf{A}_2$), for each $\mu_0 \in \mathcal{L}^2(\mho,\Delta)$, then (3.1) has only one mild solution $\mathbf{X}_{\zeta} \in \mathfrak{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ such that*

$$
\sup_{\zeta \in [0,\Xi]} \mathbf{E}\,|\mathbf{X}_{\zeta}|^2 < \infty.
$$

Proof. The proof of the theorem will be classified into three cases.

Case I: Here, we intend to consider the simplistic form of arithmetic variant

$$
|a_1 + a_2 + a_3|^2 \leq |a_1|^2 + |a_2|^2 + |a_3|^2. \tag{3.3}
$$

We have

$$
\begin{aligned}
\mathbf{E}\,|\mathbf{X}_\zeta|^2 &\le 3\mathbf{E}\,|\mu_0|^2 \\
&\quad + 3\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2 \\
&\quad + \left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\mathcal{W}_\varphi\right|^2 \\
&:= \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
\end{aligned}
$$

In view of Cauchy–Schwarz inequality and assumption ($\mathbf{A}_2$), utilizing them for $\mathcal{J}_2$, we have

$$
\mathcal{J}_2 \le \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{2(\eta_1-1)}(1+\mathbf{E}\,|\mathbf{X}_\varphi|^2)\mathrm{d}\varphi,
$$

since $|e^{2(\frac{\psi-1}{\psi}\zeta)}| < 1$, we have

$$
\begin{aligned}
\mathcal{J}_2 &\le \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left\{\frac{\zeta^{2\eta_1-1}}{2\eta_1-1} + \int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{2(\eta_1-1)}\mathbf{E}\,|\mathbf{X}_\varphi|^2\mathrm{d}\varphi\right\} \\
&\le \frac{\kappa\Xi^{2\eta_1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} \\
&\quad + \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{2(\eta_1-1)}\mathbf{E}\,|\mathbf{X}_\varphi|^2\mathrm{d}\varphi.
\end{aligned}
$$

For $\mathcal{J}_3$, utilizing the Itô's symmetry technique and assumption ($\mathbf{A}_2$), we have

$$
\begin{aligned}
\mathcal{J}_3 &\le \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{2(\eta_1-1)}(1+\mathbf{E}\,|\mathbf{X}_\varphi|^2)\mathrm{d}\varphi \\
&\le \frac{\kappa\Xi^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} \\
&\quad + \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{2(\eta_1-1)}\mathbf{E}\,|\mathbf{X}_\varphi|^2\mathrm{d}\varphi.
\end{aligned}
$$

Thus, we have

$$
\mathbf{E}\,|\mathbf{X}_\zeta|^2 \le \bar{r}_1 + \bar{r}_2\int_0^\zeta e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{(2\eta_1-1)-1}\mathbf{E}\,|\mathbf{X}_\varphi|^2\mathrm{d}\varphi,
$$

letting

$$
\bar{r}_1 = 3\mathbf{E}\,|\mu_0|^2 + 3\frac{\kappa\Xi^{2\eta_1-1}(\Xi+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} \tag{3.4}
$$

and

$$
\bar{r}_2 = \frac{3\kappa(\Xi+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}. \tag{3.5}
$$

Using the fact of Lemma 2.1 yields

$$
\begin{aligned}
&\mathbf{E}\,|\mathbf{X}_\zeta|^2 \\
&\le \bar{r}_1\left\{1 + \int_0^\zeta\sum_{\mathrm{n}=1}^\infty\frac{(\bar{r}_2\Gamma(2\eta_1-1))^{\mathrm{n}}}{\Gamma(2\mathrm{n}\eta_1-\mathrm{n})}e^{\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}(\zeta-\varphi)^{\mathrm{n}(2\eta_1-1)-1}\mathrm{d}\varphi\right\} \\
&\le \bar{r}_1\left\{1 + \sum_{\mathrm{n}=1}^\infty\frac{(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})^{\mathrm{n}}}{\Gamma(2\mathrm{n}\eta_1-\mathrm{n}+1)}\right\} \\
&\le \bar{r}_1\{1+\mathcal{E}_{2\eta_1-1,1}(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})\} < \infty, \quad \forall\zeta\in[0,\Xi],
\end{aligned}
$$

where $\mathcal{E}_{2\eta_1-1,1}(.)$ represents the Mittag–Leffler function of two parametric form [10,11]. Thus, we have

$$
\sup_{\zeta\in[0,\Xi]}\mathbf{E}\,|\mathbf{X}_\zeta|^2 < \infty,
$$

which predicts $\mathbf{X}_\zeta \in \mathcal{L}^\infty([0,\Xi],\mathcal{L}^2(\mho;\Delta))$.

Case II. In this case, we will illustrate that $\mathbf{X}_\zeta \in \mathcal{L}^\infty([0,\Xi],\mathcal{L}^2(\mho;\Delta))$,

$$
\begin{aligned}
&\mathbf{E}\,|\mathbf{X}_\zeta - \mathbf{X}_{\zeta_0}|^2 \\
&\le 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \\
&\quad\times\left|\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi - \int_0^{\zeta_0} e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}(\zeta_0-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2 \\
&\quad + 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\mathcal{W}_\varphi - \int_0^{\zeta_0} e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}(\zeta_0-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\mathcal{W}_\varphi\right|^2 \\
&:= 2(\Theta_1+\Theta_2).
\end{aligned} \tag{3.6}
$$

Furthermore, we intend to prove that Θ_1 and Θ_2 are bounded for every provided ε, whenever the factors of ζ_0 are close to ζ. In general, suppose that $0 < \zeta_0 \le \zeta < \Xi$ and the case for $0 < \zeta \le \zeta_0 < \Xi$ is analogous.

For Θ_1, we obtain

$$
\begin{aligned}
\Theta_1 &\le 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\times\left|\int_{\zeta_0}^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2 \\
&\quad + 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta_0} e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}[(\zeta-\varphi)^{\eta_1-1} - (\zeta_0-\varphi)^{\eta_1-1}]\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2 \\
&:= 2\Theta_{11} + 2\Theta_{12}.
\end{aligned} \tag{3.7}
$$

Again, using the fact of the Cauchy–Schwarz inequality, we provide a bound to Θ_{11} as follows:

$$\begin{aligned}\Theta_{11} \le& \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_{\zeta_0}^{\zeta}\left|e^{2\left(\frac{\psi-1}{\psi}(\zeta-\varphi)\right)}\right|(\zeta-\varphi)^{2\eta_1-2}\mathrm{d}\varphi \\ &\times \int_{\zeta_0}^{\zeta}\mathbf{E}\,|\mathcal{F}(\varphi,\mathbf{X}_\varphi)|^2 \\ \le& \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}(\zeta-\zeta_0)^{2\eta_1-1} \\ &\times \int_{\zeta_0}^{\zeta}[1+\mathbf{E}\,|\mathbf{X}_\varphi|^2]\mathrm{d}\varphi \\ \le& \frac{\mathcal{M}\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}(\zeta-\zeta_0)^{2\eta_1} \\ &\times \left\{\text{since}\left|e^{2\left(\frac{\psi-1}{\psi}\zeta\right)}\right|<1\right\}.\end{aligned} \tag{3.8}$$

It is not challenging to illustrate that $\exists\lambda_1$ such that $\forall 0<\zeta-\zeta_0<\lambda_1$, we have $\Theta_{11}<\frac{\varepsilon}{8}$. For Θ_{12}, as a result, we have the foregoing:

$$\begin{aligned}\Theta_{12} =& \mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta_0}e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}[(\zeta-\varphi)^{\eta_1-1}\right. \\ &\left.-(\zeta_0-\varphi)^{\eta_1-1}]\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2 \\ \le& \frac{\kappa_1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta_0}\left|e^{(\frac{\psi-1}{\psi}(\zeta_0-\varphi))}\right|^2[(\zeta-\varphi)^{\eta_1-1} \\ &-(\zeta_0-\varphi)^{\eta_1-1}]^2\mathrm{d}\varphi\int_0^{\zeta_0}[1+\mathbf{E}\,|\mathbf{X}_\varphi|^2]\mathrm{d}\varphi \\ \le& \frac{\mathcal{M}\Xi\kappa_1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta_0}[(\zeta-\varphi)^{2\eta_1-2}-(\zeta_0-\varphi)^{2\eta_1-2}]^2\mathrm{d}\varphi \\ =& \frac{\mathcal{M}\Xi\kappa_1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left\{\frac{(\zeta-\zeta_0)^{2\eta_1-1}}{2\eta_1-1}+\frac{\zeta_0^{2\eta_1-1}}{2\eta_1-1}-\frac{\zeta^{2\eta_1-1}}{2\eta_1-1}\right\} \\ \le& \frac{\mathcal{M}\Xi\kappa_1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\frac{(\zeta-\zeta_0)^{2\eta_1-1}}{2\eta_1-1},\end{aligned} \tag{3.9}$$

using the fact that $\left|e^{\frac{\psi-1}{\psi}\zeta_0}\right|^2<1$, and observe that $\frac{\zeta_0^{2\eta_1-1}}{2\eta_1-1}-\frac{\zeta^{2\eta_1-1}}{2\eta_1-1}\le 0$. Choosing appropriately λ_2, given that $\zeta\in[0,\Xi]$ and $0<\zeta-\zeta_0<\lambda_2$, we can obtain the bound $\Theta_{12}<\varepsilon/8$. For Θ_2, employing Itô's isometry technique, it makes a bound that is equivalent to Θ_1. So, we intend to demonstrate that $\exists\lambda_3>0$ such that for $\zeta\in[0,\Xi]$ and $0<\zeta-\zeta_0<\lambda_3$, we express as

$$\Theta_2 \le \frac{\mathcal{M}\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\frac{(\zeta-\zeta_0)^{2\eta_1-1}}{2\eta_1-1}<\frac{\varepsilon}{4}. \tag{3.10}$$

As a result of the foregoing analysis for Θ_1 and Θ_2, choosing $\lambda=\min\{\lambda_\iota\}$, $\iota=1,2,3$ and incorporating all of the previous assumptions yields the result, for all $\zeta\in[0,\Xi]$ and $0<\zeta-\zeta_0<\lambda$

$$\Rightarrow \mathbf{E}\,|\mathbf{X}(\zeta)-\mathbf{X}(\zeta_0)|^2<\varepsilon.$$

Case III. Here, we employ the Banach f_p hypothesis, and we will demonstrate that (2.6) has only one result in $\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))\forall\Xi<\infty$. Now, introducing the functional $\Phi(.)$ on $\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ as follows:

$$\begin{aligned}(\Phi\mathbf{X})(\zeta) =& \mu_0+\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi \\ &+\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\mathcal{W}_\varphi.\end{aligned} \tag{3.11}$$

Cases I and II demonstrate that the function $\phi(.)$ from $\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ to $\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ is well defined.-furthermore, the Banach fixed point theorem will then be utilized to demonstrate the E-U of solutions for all $\zeta\in[0,\infty)$. For $\mathbf{X}_\zeta,\mathbf{Y}_\zeta\in\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ having $\mu_0=\nu_0$, the norm of $\mathbb{C}([0,\Xi];\mathcal{L}^2(\mho,\Delta))$ is defined by

$$|\mathcal{F}(\zeta)|_{\varrho}=\sup_{\zeta\in[0,\Xi]}\mathbf{E}\,|\mathcal{F}(\zeta)|^2<\infty. \tag{3.12}$$

Therefore,

$$\begin{aligned}&\mathbf{E}\,|(\Phi\mathbf{X})(\zeta)-(\Phi\mathbf{Y})(\zeta)|^2 \\ &\le 2\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}[\mathcal{F}(\varphi,\mathbf{X}_\varphi)\right. \\ &\qquad\left.-\mathcal{F}(\varphi,\mathbf{Y}_\varphi)]\mathrm{d}\varphi\right|^2 \\ &\le 2\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}[\mathcal{G}(\varphi,\mathbf{X}_\varphi)\right. . \\ &\qquad\left.-\mathcal{G}(\varphi,\mathbf{Y}_\varphi)]\mathrm{d}\mathcal{W}_\varphi\right|^2\end{aligned} \tag{3.13}$$

Choosing $\eta_2=2\eta_1-1>0$, making use of Cauchy–Schwarz variant, and Itô's isometry technique, then we have

$$\begin{aligned}&\mathbf{E}\,|(\Phi\mathbf{X})(\zeta)-(\Phi\mathbf{Y})(\zeta)|^2 \\ &\le \frac{2\kappa(1+\Xi)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}(\zeta-\varphi)\eta_2-1\mathbf{E}\,|\mathbf{X}_\varphi-\mathbf{Y}_\varphi|^2\mathrm{d}\varphi,\end{aligned} \tag{3.14}$$

using the fact that $\left|e^{\frac{\psi-1}{\psi}\zeta}\right|^2<1$ and we assert that

$$\begin{aligned}&\mathbf{E}\,|(\Phi^{\mathfrak{n}}\mathbf{X})(\zeta)-(\Phi^{\mathfrak{n}}\mathbf{Y})(\zeta)|^2\\&\le\frac{(\Gamma(\eta_2))^{\mathfrak{n}}}{\eta_2\Gamma(\mathfrak{n}\eta_2)}\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}}\zeta^{\mathfrak{n}\eta_2}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}.\end{aligned}\tag{3.15}$$

For $\mathfrak{n}=1$, we obtain

$$\mathbf{E}\,|(\Phi\mathbf{X})(\zeta)-(\Phi\mathbf{Y})(\zeta)|^2\le\frac{\zeta^{\eta_2}}{\eta_2}\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}.\tag{3.16}$$

If we make the assumption that (3.15) is true for $\mathfrak{n}=\ell$, we can conclude that it is indeed valid for $\mathfrak{n}=\ell+1$,

$$\begin{aligned}&\mathbf{E}\,|(\Phi^{\ell+1}\mathbf{X})(\zeta)-(\Phi^{\ell+1}\mathbf{Y})(\zeta)|^2\\&\le\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{\eta_2-1}\\&\quad\times\mathbf{E}\,|(\Phi^{\ell}\mathbf{X})(\varphi)-(\Phi^{\ell}\mathbf{Y})(\varphi)|^2d\varphi\\&\le\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^\zeta\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{\eta_2-1}\frac{(\Gamma(\eta_2))^{\ell}}{\eta_2\Gamma(\ell\eta_2)}\\&\quad\times\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\ell}\varphi^{\ell\eta_2}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}d\varphi\\&\le\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\ell+1}\frac{(\Gamma(\eta_2))^{\ell}}{\eta_2\Gamma(\ell\eta_2)}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}\\&\quad\times\int_0^\zeta(\zeta-\varphi)^{\eta_2-1}\varphi^{\ell\eta_2}d\varphi.\end{aligned}\tag{3.17}$$

Using the fact that $\left|e^{\frac{\psi-1}{\psi}\zeta}\right|^2<1$ and to obtain our assertion for $\mathfrak{n}=\ell+1$, we simply have to evaluate the aforementioned integral $\int_0^\zeta(\zeta-\varphi)^{\eta_2-1}\varphi^{\ell\eta_2}d\varphi$. Utilizing the supposition $\varphi=\zeta\nu$, then

$$\begin{aligned}\int_0^\zeta(\zeta-\varphi)^{\eta_2-1}\varphi^{\ell\eta_2}d\varphi&=\int_0^1(1-\nu)^{\eta_2-1}\zeta^{\eta_2+\ell\eta_2}\nu^{\ell\eta_2}d\nu\\&=\zeta^{(\ell+1)\eta_2}\int_0^1(1-\nu)^{\eta_2-1}\nu^{\ell\eta_2}d\nu\\&=\zeta^{(\ell+1)\eta_2}\frac{\Gamma(\eta_2)\Gamma(\ell\eta_2+1)}{\Gamma((\ell+1)\eta_2+1)}.\end{aligned}\tag{3.18}$$

Merging the aforesaid identity with (3.17), then we have

$$\begin{aligned}&\mathbf{E}\,|(\Phi^{\ell+1}\mathbf{X})(\zeta)-(\Phi^{\ell+1}\mathbf{Y})(\zeta)|^2\\&\le\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\ell+1}\frac{(\Gamma(\eta_2))^{\ell}}{\eta_2\Gamma(\ell\eta_2)}\\&\quad\times|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}\zeta^{(\ell+1)\eta_2}\frac{\Gamma(\eta_2)\Gamma(\ell\eta_2+1)}{\Gamma((\ell+1)\eta_2+1)}\\&=\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\ell+1}\frac{(\Gamma(\eta_2))^{\ell+1}}{\eta_2}\\&\quad\times|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}\zeta^{(\ell+1)\eta_2}\frac{\Gamma(\ell\eta_2+1)}{\Gamma(\ell\eta_2)\Gamma((\ell+1)\eta_2+1)}\\&\le\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\ell+1}\frac{(\Gamma(\eta_2))^{\ell+1}}{\eta_2}\\&\quad\times\frac{\zeta^{(\ell+1)\eta_2}}{\Gamma((\ell+1)\eta_2)}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}.\end{aligned}\tag{3.19}$$

Then, for all $\mathfrak{n}$, we come to the respective approximation:

$$\begin{aligned}&\mathbf{E}\,|(\Phi^{\mathfrak{n}}\mathbf{X})(\zeta)-(\Phi^{\mathfrak{n}}\mathbf{Y})(\zeta)|^2\\&\le\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}}\frac{(\Gamma(\eta_2))^{\mathfrak{n}}}{\eta_2}\frac{\Xi^{\mathfrak{n}\eta_2}}{\Gamma(\mathfrak{n}\eta_2)}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|_{\varrho}.\end{aligned}\tag{3.20}$$

Now, simply to demonstrate the respective evidence to apply the Banach fixed point theorem and deduce the E-U of solutions, for this, we have

$$\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}}\frac{(\Gamma(\eta_2))^{\mathfrak{n}}}{\eta_2}\frac{\Xi^{\mathfrak{n}\eta_2}}{\Gamma(\mathfrak{n}\eta_2)}\mapsto0,\quad\text{as }\mathfrak{n}\mapsto\infty.$$

Then, all that remains is to demonstrate

$$\sum_{\mathfrak{n}=1}^{\infty}\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}}\frac{(\Gamma(\eta_2))^{\mathfrak{n}}}{\eta_2}\frac{\Xi^{\mathfrak{n}\eta_2}}{\Gamma(\mathfrak{n}\eta_2)}<\infty.$$

Moreover, employing the D'Alembert discriminant technique, we simply have to prove the foregoing:

$$\lim_{\mathfrak{n}\mapsto\infty}\frac{\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}+1}\frac{(\Gamma(\eta_2))^{\mathfrak{n}+1}}{\eta_2}\frac{\Xi^{(\mathfrak{n}+1)\eta_2}}{\Gamma((\mathfrak{n}+1)\eta_2)}}{\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^{\mathfrak{n}}\frac{(\Gamma(\eta_2))^{\mathfrak{n}}}{\eta_2}\frac{\Xi^{\mathfrak{n}\eta_2}}{\Gamma(\mathfrak{n}\eta_2)}}<1.$$

After simplification,

$$\lim_{\mathfrak{n}\mapsto\infty}\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\frac{\Gamma(\eta_2)\Xi^{\eta_2}\Gamma(\mathfrak{n}\eta_2)}{\Gamma((\mathfrak{n}+1)\eta_2)}<1.$$

It is worth noting that when $\mu\mapsto\infty$, the correlation between the Gamma function and the Stirling approximation generates the following:

$$\Gamma(\mu) \approx \sqrt{2\pi}\,\exp(-\mu)\mu^{\mu-0.5}.$$

Therefore,

$$\begin{aligned}
&\lim_{\mathfrak{n}\mapsto\infty} \frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\frac{\Gamma(\eta_2)\Xi^{\eta_2}\Gamma(\mathfrak{n}\eta_2)}{\Gamma((\mathfrak{n}+1)\eta_2)}\\
&= \lim_{\mathfrak{n}\mapsto\infty}\left(\frac{2\kappa\Gamma(\eta_1+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\right)^2\Gamma(\eta_2)\Xi^{\eta_2}\exp(\eta_2)\\
&\times\sqrt{\frac{\mathfrak{n}+1}{\mathfrak{n}}}\left(\frac{\mathfrak{n}}{\mathfrak{n}+!}\right)^{\mathfrak{n}\eta_2}\frac{1}{(\mathfrak{n}\eta_2+\eta_2)^{\eta_2}} = 0.
\end{aligned}$$

For sufficiently large $\mathfrak{n}$, then $\left(\frac{\Gamma\kappa+\kappa}{\psi^{\eta_1}\Gamma(\eta_1)}\right)^{\mathfrak{n}}\frac{(\Gamma(\eta_2))^{\mathfrak{n}}\Xi^{\mathfrak{n}\eta_2}}{\eta_2\Gamma(\mathfrak{n}\eta_2)} < 1$, this indicates that $\Phi(.)$ is a contraction mapping on $\mathbb{C}([0,\Xi];\ \mathcal{L}^2(\mho,\Delta))$, $\forall\Xi<\infty$. The confirmation is now complete. □

Theorem 3.2. *Suppose the hypothesis of Theorem* 3.1 *satisfies. Then, for each* $\Xi\in[0,\infty)$, $\exists$ *a constant* $\mathcal{M}(\eta_1;\ \psi,\Xi)$ *such that*

$$\sup_{\zeta\in[0,\Xi]}\mathbf{E}\,|\mathbf{X}_\zeta-\mathbf{Y}_\zeta|^2 \le \mathcal{M}(\eta_1;\ \psi,\Xi)\mathbf{E}\,|\mu_0-\nu_0|^2, \tag{3.21}$$

where μ_0 *and* ν_0 *are the ICs for* $\mathbf{X}_\zeta$ *and* $\mathbf{Y}_\zeta$, *respectively.*

Proof. Considering inequality (3.3) and implementing on the following expression:

$$\begin{aligned}
&\mathbf{E}\,|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)|^2\\
&\le 3\mathbf{E}\,|\mu_0-\nu_0|^2\\
&+3\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\varphi\right|^2\\
&+\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi)\mathrm{d}\mathcal{W}_\varphi\right|^2\\
&\coloneqq \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.
\end{aligned}$$

Adopting the same technique of Theorem 3.1 and Lemma 2.1, we can establish the proof. Therefore, we omitted the details. □

3.2 Continuity *via* GPF

Here, we contemplate the continuity of the system of (1.1) on the GPF order for SDEs, i.e., when $\eta_1\mapsto\eta_2$, the correlation exists between the following:

$$\begin{cases}\mathbf{D}_\zeta^{\eta_1;\ \psi}\mathbf{X}_\zeta = \mathcal{F}(t,\mathbf{X}_\zeta)\mathrm{d}\zeta+\mathcal{G}(t,\mathbf{X}_\zeta)\mathrm{d}\mathcal{W}_\zeta, & \zeta\ge 0,\\ \eta_1\in(1/2,1),\\ \mathbf{X}_0=\mu_0\in\mathcal{L}^2(\mho,\Delta),\end{cases} \tag{3.22}$$

and

$$\begin{cases}\mathbf{D}_\zeta^{\eta_2;\ \psi}\mathbf{X}_\zeta = \mathcal{F}(t,\mathbf{X}_\zeta)\mathrm{d}\zeta+\mathcal{G}(t,\mathbf{X}_\zeta)\mathrm{d}\mathcal{W}_\zeta, & \zeta\ge 0,\\ \eta_2\in(1/2,1),\\ \mathbf{X}_0=\mu_0\in\mathcal{L}^2(\mho,\Delta).\end{cases} \tag{3.23}$$

Specifically, letting $\eta_2=1$, then we explore a correlation of findings between the FSDEe and the classical SDE. For this, we have the following important consequence.

Theorem 3.3. *Under the hypothesis* $(\mathbf{A}_1)$ *and* $(\mathbf{A}_2)$, *then the solution of* (3.22) *tends to the solution of* (3.23) *in the context of* $\mathbb{C}([0,\Xi];\ \mathcal{L}^2(\mho,\Delta))$, *when* $\eta_1\mapsto\eta_2$, *for* $0.5<\eta_1\le\eta_2\le 1$.

Proof. By means of inequality (3.3), we have

$$\begin{aligned}
&\mathbf{E}\,|\mathbf{X}_\zeta^{\eta_1}-\mathbf{Y}_\zeta^{\eta_2}|^2\\
&\le 2\mathbf{E}\left|\int_0^\zeta\left\{\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi^{\eta_1})\right.\right.\\
&\left.\left.-\frac{1}{\psi^{\eta_2}\Gamma(\eta_2)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_2-1}\mathcal{F}(\varphi,\mathbf{X}_\varphi^{\eta_2})\right\}\mathrm{d}\varphi\right|^2\\
&+2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^\zeta\left\{\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi^{\eta_1})\right.\right.\\
&\left.\left.-\frac{1}{\psi^{\eta_2}\Gamma(\eta_2)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_2-1}\mathcal{G}(\varphi,\mathbf{X}_\varphi^{\eta_2})\right\}\mathrm{d}\mathcal{W}_\varphi\right|^2\\
&\coloneqq 2\Upsilon_1+2\Upsilon_2.
\end{aligned}$$

Taking the following bound for Υ_1, we have

$$\Upsilon_1 \le 2\Xi\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mathcal{F}(\varphi,\mathbf{X}_\varphi^{\eta_1})-\mathcal{F}(\varphi,\mathbf{X}_\varphi^{\eta_2})|^2 d\varphi$$
$$+ 2\Xi\int_0^\zeta\left\{\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1} - \frac{1}{\psi^{\eta_2}\Gamma(\eta_2)}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_2-1}\right\}^2\mathbf{E}\,|\mathcal{F}(\varphi,\mathbf{X}_\varphi^{\eta_2})|^2 d\varphi$$
$$\le 2\kappa\Xi\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi$$
$$+ 2\mathcal{M}\Xi\int_0^\zeta\left\{\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2} - \frac{2}{\psi^{\eta_1+\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{\eta_1+\eta_2-2} + \frac{1}{\psi^{2\eta_2}(\Gamma(\eta_2))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_2-2}\right\}d\varphi$$
$$= 2\kappa\Xi\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2}\mathbf{E} \times |\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + 2\mathcal{M}\Xi\left\{\frac{\zeta^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} - \frac{2\zeta^{\eta_1+\eta_2-1}}{\psi^{\eta_1+\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)(\eta_1+\eta_2-1)} + \frac{\zeta 2\eta_2-1}{\psi^{2\eta_2}(\Gamma(\eta_2))^2(2\eta_2-1)}\right\}$$
$$=: 2\kappa\Xi\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2} \times \mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + \varepsilon_1(\eta_1,\eta_2),$$

where we have used the fact that $|e^{\frac{\psi-1}{\psi}\zeta}|^2 < 1$ and $\varepsilon_1(\eta_1,\eta_2) = 2\mathcal{M}\Xi\left\{\frac{\zeta^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} - \frac{2\zeta^{\eta_1+\eta_2-1}}{\psi^{\eta_1+\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)(\eta_1+\eta_2-1)} + \frac{\zeta 2\eta_2-1}{\psi^{2\eta_2}(\Gamma(\eta_2))^2(2\eta_2-1)}\right\}$. In view of the continuity of the gamma function, we have $\lim_{\eta_1\mapsto\eta_2}\varepsilon_1(\eta_1,\eta_2)\mapsto 0$. Then, for every $\varepsilon>0, \exists\lambda_1>0$ such that for $0<\eta_2-\eta_1<\lambda_1$, we have $\varepsilon_1(\eta_1,\eta_2)\in(0,\varepsilon/2)$, which indicates

$$\Upsilon_1 \le 2\kappa\Xi\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2} \times \mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + \frac{\varepsilon}{2},\ \text{when } 0<\eta_2-\eta_1<\lambda_1.$$

For Υ_2, using Itô’s isometry technique and a analogous bound as Υ_1, yields

$$\Upsilon_2 \le 2\kappa\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + 2\mathcal{M}\left\{\frac{\zeta^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} - \frac{2\zeta^{\eta_1+\eta_2-1}}{\psi^{\eta_1+\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)(\eta_1+\eta_2-1)} + \frac{\zeta 2\eta_2-1}{\psi^{2\eta_2}(\Gamma(\eta_2))^2(2\eta_2-1)}\right\}$$
$$=: 2\kappa\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2} \times \mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + \varepsilon_2(\eta_1,\eta_2),$$

where $\varepsilon_2(\eta_1,\eta_2) = 2\mathcal{M}\left\{\frac{\zeta^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)} - \frac{2\zeta^{\eta_1+\eta_2-1}}{\psi^{\eta_1+\eta_2}\Gamma(\eta_1)\Gamma(\eta_2)(\eta_1+\eta_2-1)} + \frac{\zeta 2\eta_2-1}{\psi^{2\eta_2}(\Gamma(\eta_2))^2(2\eta_2-1)}\right\}$. Moreover, for any $\varepsilon>0, \exists\lambda_2>0$ such that for $0<\eta_2-\eta_1<\lambda_2$, we have $\varepsilon_2(\eta_1,\eta_2)\in(0,\varepsilon/2)$, which means that

$$\Upsilon_2 \le 2\kappa\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + \frac{\varepsilon}{2},\ \text{when } 0<\eta_2-\eta_1<\lambda_2.$$

Mingling the bounds of Υ_1 and Υ_2 and selecting $\lambda = \min\{\lambda_1,\lambda_2\}$, then for any $\varepsilon>0, \exists\lambda>0$ such that for $0<\eta_2-\eta_1<\lambda$, we have

$$\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 \le 2\kappa(\Xi+1)\int_0^\zeta \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{(2\eta_1-1)-1}\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 d\varphi + \varepsilon.$$

Using the fact that $\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 < 1$ and making use of Lemma 2.1, we have

$$\mathbf{E}\,|\mathbf{X}_\varphi^{\eta_1}-\mathbf{X}_\varphi^{\eta_2}|^2 \le \varepsilon\left(1+\mathcal{E}_{2\eta_1-1,1}\left(\frac{\kappa(\Xi+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\Gamma(2\eta_1-1)\Xi^{2\eta_1-1}\right)\right) \mapsto 0,$$

as $\varepsilon\mapsto 0$. This is the desired outcome. □

3.3 Carathéodory's approximation

Here, the CA for SDE is discussed in this portion. We attempt to define Carathéodory's approximate findings for SFDEs in the same way that we did for SDEs. For each integer $\mathfrak{n} \geq 1$, we describe $\mu_{\mathfrak{n}}(\zeta) = \mu_0$ for $\zeta \in [-1, 0]$ and

$$\mu_{\mathfrak{n}}(\zeta) = \mu_0 + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \times \int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\varphi + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G} \times \left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\mathcal{W}_{\varphi}, \quad \forall \zeta \in (0, \Xi].$$

It is worth noting that $\mu_{\mathfrak{n}}(\zeta)$ can be calculated for $\zeta \in [0, 1/\mathfrak{n}]$ by

$$\mu_{\mathfrak{n}}(\zeta) = \mu_0 + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \times \int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi, \mu_0)\mathrm{d}\varphi + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi, \mu_0)\mathrm{d}\mathcal{W}_{\varphi},$$

then for $\zeta \in [1/\mathfrak{n}, 2/\mathfrak{n}]$, we have

$$\mu_{\mathfrak{n}}(\zeta) = \mu_0\left(\frac{1}{\mathfrak{n}}\right) + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \times \int_{\frac{1}{\mathfrak{n}}}^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\varphi + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_{\frac{1}{\mathfrak{n}}}^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G} \times \left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\mathcal{W}_{\varphi},$$

and henceforth. Using this method, we can evaluate $\mu_{\mathfrak{n}}(\zeta)$ one by one on the intervals $[0, 1/\mathfrak{n}]$, $(1/\mathfrak{n}, 2/\mathfrak{n}], \ldots$.

Lemma 3.1. *Under the hypothesis* $(\mathbf{A}_2)$, $\forall \mathfrak{n} \leq 1$, *then*

$$\sup_{\zeta\in[0,\Xi]} \mathbf{E}\,|\mu_{\mathfrak{n}}(\zeta)|^2 \leq \mho := \bar{r}_1(1 + \mathcal{E}_{2\eta_1-1,1}(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})) < \infty, \tag{3.24}$$

where $\bar{r}_1$ *and* $\bar{r}_2$ *are defined in* (3.4) *and* (3.5), *respectively, and* $\mathcal{E}_{2\eta_1-1,1}$ *denotes the Mittag–Leffler function containing two parameters.*

Proof. By means of the hypothesis described in (3.3), we have

$$\mathbf{E}\,|\mu_{\mathfrak{n}}(\zeta)|^2 \leq 3\mathbf{E}|\mu_0| + 3\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \times \int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\varphi\right|^2 + 3\mathbf{E}\left|\frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G} \times \left(\varphi, \mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right)\mathrm{d}\mathcal{W}_{\varphi}\right|^2 := 3U_1 + 3U_2 + 3U_3.$$

In view of Cauchy–Schwarz variant and supposition $(\mathbf{A}_2)$, the component U_2 can be estimated as follows:

$$\begin{aligned} U_2 &\leq \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2(\eta_1-1)}\left(1 + \mathbf{E}\left|\mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right|^2\right)\mathrm{d}\varphi \\ &\leq \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left\{\frac{\zeta^{2\eta_1-1}}{2\eta_1-1} + \int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2(\eta_1-1)}\left(1 + \mathbf{E}\left|\mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right|^2\right)\mathrm{d}\varphi\right\} \\ &\leq \frac{\kappa\Xi^{2\eta_1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2} + \frac{\Xi\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \times \int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{2(\eta_1-1)}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathfrak{n}}(r_1)|^2\mathrm{d}\varphi. \end{aligned}$$

Using the fact of $|e^{\frac{\psi-1}{\psi}\zeta}|^2 < 1$. Accordingly, with the aid of Itô's isometry methodology and assumption $(\mathbf{A}_2)$, we can approximate the stochastic integral factor as

$$\begin{aligned} U_3 &\leq \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times (\zeta-\varphi)^{2(\eta_1-1)}\left(1 + \mathbf{E}\left|\mu_{\mathfrak{n}}\left(\varphi - \frac{1}{\mathfrak{n}}\right)\right|^2\right)\mathrm{d}\varphi \\ &\leq \frac{\kappa\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2} + \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \times \int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{2(\eta_1-1)}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathfrak{n}}(r_1)|^2\mathrm{d}\varphi. \end{aligned}$$

Taking into account (3.4) and (3.5), then using the terms U_1, U_2, and U_3, we have

$$\mathbf{E}\,|\mu_{\mathrm{n}}(\zeta)|^2 \le \bar{r}_1 + \bar{r}_2\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{(2\eta_1-1)-1} \times \sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2\mathrm{d}\varphi.$$

Observe that, for $\zeta_2 \ge \zeta$, we have

$$\int_0^{\zeta_2}\left|e^{\frac{\psi-1}{\psi}(\zeta_2-\varphi)}\right|^2 (\zeta_2-\varphi)^{(2\eta_1-1)-1}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2\mathrm{d}\varphi \ge \int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{(2\eta_1-1)-1}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2\mathrm{d}\varphi.$$

Therefore, we have

$$\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2 \le \bar{r}_1 + \bar{r}_2\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{(2\eta_1-1)-1}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2\mathrm{d}\varphi.$$

Utilizing the fact that $|e^{\frac{\psi-1}{\psi}\zeta}| < 1$, and considering Lemma 2.1, we can achieve straightforwardly

$$\begin{aligned}\sup_{r_1\in[0,\varphi]}\mathbf{E}\,|\mu_{\mathrm{n}}(r_1)|^2 &\le \bar{r}_1\left(1 + \int_0^{\zeta}\sum_{\mathrm{n}=1}^{\infty}\frac{(\bar{r}_2\Gamma(2\eta_1-1))^{\mathrm{n}}}{\Gamma(\mathrm{n}(2\eta_1-1))}(\zeta-\varphi)^{\mathrm{n}(2\eta_1-1)-1}\mathrm{d}\varphi\right)\\ &\le \bar{r}_1\left\{1 + \sum_{\mathrm{n}=1}^{\infty}\frac{(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})^{\mathrm{n}}}{\Gamma(\mathrm{n}(2\eta_1-1)+1)}\right\}\\ &\le \bar{r}_1\{1+\mathcal{E}_{2\eta_1-1,1}(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})\} < \infty,\quad \forall\zeta\in[0,\Xi],\end{aligned}$$

which is the desired result. □

Lemma 3.2. *Under the supposition* $(\mathbf{A}_2)$, $\forall\mathrm{n}\ge 1$ *and* $0\le\zeta_0<\zeta\le\Xi$ *having* $\zeta-\zeta_0\le 1$, *then*

$$\mathbf{E}\,|\mu_{\mathrm{n}}(\zeta)-\mu_{\mathrm{n}}(\zeta_0)|^2 \le \mathcal{M}(\zeta-\zeta_0)^{2\eta_1-1}.$$

Proof. By the given hypothesis, we observe that

$$\begin{aligned}\mathbf{E}\,|\mu_{\mathrm{n}}(\zeta)-\mu_{\mathrm{n}}(\zeta_0)|^2 &\le 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\varphi - \int_0^{\zeta_0}e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}(\zeta_0-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\varphi\right|^2\\ &\quad + 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\mathcal{W}_{\varphi} - \int_0^{\zeta_0}e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}(\zeta_0-\varphi)^{\eta_1-1}\mathcal{G}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\mathcal{W}_{\varphi}\right|^2\\ &=: 2\mathcal{R}_1+2\mathcal{R}_2.\end{aligned}$$

For $\mathcal{R}_1$, we have

$$\begin{aligned}\mathcal{R}_1 &\le 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \times \left|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\varphi\right|^2\\ &\quad + 2\mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta_0}e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}[(\zeta-\varphi)^{\eta_1-1} - (\zeta_0-\varphi)^{\eta_1-1}]\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\varphi\right|^2\\ &=: 2\mathcal{R}_{11}+2\mathcal{R}_{12}.\end{aligned}$$

By means of the Cauchy–Schwarz inequality, $\zeta-\zeta_0\le 1$, we provide the subsequent bounds for $\mathcal{R}_{11}$ as

$$\begin{aligned}\mathcal{R}_{11} &\le \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2}\mathrm{d}\varphi\right| \times \int_{\zeta_0}^{\zeta}\mathbf{E}\left|\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\right|^2\mathrm{d}\varphi\\ &\le \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}(\zeta-\zeta_0)^{2\eta_1-1} \times \int_{\zeta_0}^{\zeta}\left\{1+\mathbf{E}\left|\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right|^2\right\}\mathrm{d}\varphi\\ &\le \frac{(\mho+1)\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}(\zeta-\zeta_0)^{2\eta_1-1}, \times\left(\text{Since }\left|e^{\frac{\psi-1}{\psi}(\zeta-\zeta_0)}\right|^2<1\right),\end{aligned}$$

where $\mho = \bar{r}_1\{1+\mathcal{E}_{2\eta_1-1,1}(\bar{r}_2\Gamma(2\eta_1-1)\Xi^{2\eta_1-1})\}$ has been stated in Lemma 3.1.

As a consequence, for $\mathcal{R}_{12}$, we have the aforementioned:

$$\begin{aligned}\mathcal{R}_{12} &= \mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left|\int_0^{\zeta_0}e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}[(\zeta-\varphi)^{\eta_1-1} - (\zeta_0-\varphi)^{\eta_1-1}]\mathcal{F}\left(\varphi,\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right)\mathrm{d}\varphi\right|^2\\ &\le \mathbf{E}\frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta_0}\left|e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}\right|^2[(\zeta-\varphi)^{\eta_1-1} - (\zeta_0-\varphi)^{\eta_1-1}]\mathrm{d}\varphi\int_0^{\zeta_0}\left\{1+\mathbf{E}\left|\mu_{\mathrm{n}}\left(\varphi-\frac{1}{\mathrm{n}}\right)\right|^2\right\}\mathrm{d}\varphi\\ &\le \frac{\kappa\mathcal{M}\Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta_0}\left|e^{\frac{\psi-1}{\psi}(\zeta_0-\varphi)}\right|^2[(\zeta-\varphi)^{2\eta_1-2} - (\zeta_0-\varphi)^{2\eta_1-2}]\mathrm{d}\varphi\\ &\le \frac{\kappa\mathcal{M}\Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left\{\frac{(\zeta-\zeta_0)^{2\eta_1-1}}{2\eta_1-1}+\frac{\zeta_0^{2\eta_1-1}}{2\eta_1-1} - \frac{\zeta^{2\eta_1-1}}{2\eta_1-1}\right\}\left(\text{since }\left|e^{\frac{\psi-1}{\psi}\zeta_0}\right|^2<1\right),\\ &\le \frac{\kappa\mathcal{M}\Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\frac{(\zeta-\zeta_0)^{2\eta_1-1}}{2\eta_1-1}.\end{aligned}$$

For $\mathcal{R}_2$, employing the Itô isometry strategy and assumption ($\mathbf{A}_2$), and equivalent analytical approaches to $\mathcal{R}_1$, it can be demonstrated that

$$\mathcal{R}_2 \leq \frac{\kappa \mathcal{M} \Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \frac{(\zeta - \zeta_0)^{2\eta_1 - 1}}{2\eta_1 - 1}.$$

Mingling all the concluding mathematical expressions yields

$$\mathbf{E}\,|\mu_n(\zeta) - \mu_n(\zeta_0)|^2 \leq \mathcal{M}(\zeta - \zeta_0)^{2\eta_1 - 1}.$$

This yields the desired outcome. □

Theorem 3.4. *Under the hypothesis of* ($\mathbf{A}_1$) *and* ($\mathbf{A}_2$), *assume that there is a unique solution* $\mu(\zeta)$ *of* (2.6). *Then, for* $n \geq 1$,

$$\sup_{\zeta \in [0, \Xi]} \mathbf{E}\,|\mu(\zeta) - \mu_n(\zeta)|^2 \leq \mathcal{M} n^{1 - 2\eta_1}.$$

Proof. Observe that

$$\begin{aligned}
&\mu(\zeta) - \mu_n(\zeta) \\
&= \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \left\{ \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}(\varphi, \mu(\varphi)) d\varphi \right. \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\varphi \right\} \\
&\quad + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \left\{ \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}(\varphi, \mu(\varphi)) d\mathcal{W}_\varphi \right. \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\mathcal{W}_\varphi \right\}.
\end{aligned}$$

So, applying the well-known inequality defined in (3.3), we have

$$\begin{aligned}
&\mathbf{E}\,|\mu(\zeta) - \mu_n(\zeta)|^2 \\
&\leq 2\mathbf{E} \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \left| \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}(\varphi, \mu(\varphi)) d\varphi \right. \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\varphi \right|^2 \\
&\quad + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)} \left| \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}(\varphi, \mu(\varphi)) d\mathcal{W}_\varphi \right. \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\mathcal{W}_\varphi \right|^2 \\
&=: 2(Q_1 + Q_2).
\end{aligned}$$

For Q_1, we have

$$\begin{aligned}
Q_1 &\leq 2\mathbf{E} \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \left| \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}(\varphi, \mu(\varphi)) d\varphi \right. \\
&\quad - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}(\varphi, \mu_n(\varphi)) d\varphi \\
&\quad + 2\mathbf{E} \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}(\varphi, \mu_n(\varphi)) d\varphi \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{F}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\varphi \right|^2 \\
&=: 2(Q_{11} + Q_{12}).
\end{aligned}$$

Taking into consideration the Cauchy–Schwarz variant and assumption ($\mathbf{A}_2$), we have the aforementioned interpretation for Q_{11}:

$$\begin{aligned}
Q_{11} &\leq \frac{\kappa \Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \int_0^\zeta \\
&\quad \times \left| e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{2\eta_1 - 2} \right|^2 \mathbf{E}\,|\mu(\varphi) - \mu_n(\varphi)|^2 d\varphi.
\end{aligned}$$

Analogously, for Q_{12}, we obtain

$$\begin{aligned}
Q_{12} &\leq \frac{\kappa \Xi^{2\eta_1 - 1}}{(2\eta_1 - 1)\psi^{2\eta_1}(\Gamma(\eta_1))^2} \\
&\quad \times \int_0^\zeta \mathbf{E} \left| \mu_n(\varphi) - \mu_n\left(\varphi - \frac{1}{n}\right) \right|^2 d\varphi.
\end{aligned}$$

Furthermore, we can categorize Q_2 into two components as follows:

$$\begin{aligned}
Q_2 &\leq 2\mathbf{E} \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \left| \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}(\varphi, \mu(\varphi)) d\varphi \right. \\
&\quad - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}(\varphi, \mu_n(\varphi)) d\mathcal{W}_\varphi \\
&\quad + 2\mathbf{E} \frac{1}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}(\varphi, \mu_n(\varphi)) d\varphi \\
&\quad \left. - \int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} (\zeta - \varphi)^{\eta_1 - 1} \mathcal{G}\left(\varphi, \mu_n\left(\varphi - \frac{1}{n}\right)\right) d\mathcal{W}_\varphi \right|^2 \\
&=: 2(Q_{21} + Q_{22}).
\end{aligned}$$

Utilizing the Itô's isometry formulation, we have

$$Q_{21} \leq \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2} \int_0^\zeta \left| e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} \right|^2 (\zeta - \varphi)^{2\eta_1 - 2} \mathbf{E}\,|\mu(\varphi) - \mu_n(\varphi)|^2 d\varphi$$

and

$$Q_{22} \leq \frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 (\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 d\varphi.$$

Incorporating the assumptions for Q_1 and Q_2, it is calculated that

$$\begin{aligned}&\mathbf{E}\,|\mu(\zeta)-\mu_n(\zeta)|^2\\ &\leq \frac{\kappa(\Xi+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times(\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mu(\varphi)-\mu_n(\varphi)|^2 d\varphi\\ &\quad+\frac{\kappa\Xi^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}\int_0^{\zeta}\mathbf{E}\times\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 d\varphi\\ &\quad+\frac{\kappa}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times(\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 d\varphi.\end{aligned}\tag{3.25}$$

Utilizing Lemma 3.2, if $\varphi \geq 1/n$, then

$$\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 \leq \mathcal{M}n^{1-2\eta_1},$$

On the other hand, if $\varphi \in [0, 1/n)$,

$$\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 = \mathbf{E}\,|\mu_n(\varphi)-\mu_n(0)|^2 \leq \mathcal{M}n^{1-2\eta_1}.$$

From (3.25), we have

$$\begin{aligned}&\mathbf{E}\,|\mu(\zeta)-\mu_n(\zeta)|^2\\ &\leq \frac{\kappa(\Xi+1)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times(\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mu(\varphi)-\mu_n(\varphi)|^2 d\varphi\\ &\quad+\frac{\kappa\Xi^{2\eta_1-1}}{\psi^{2\eta_1}(\Gamma(\eta_1))^2(2\eta_1-1)}\times\int_0^{\zeta}\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 d\varphi\\ &\quad+\frac{\Xi}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times(\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\left|\mu_n(\varphi)-\mu_n\left(\varphi-\frac{1}{n}\right)\right|^2 d\varphi\\ &\leq \frac{\kappa(1+\Xi)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2 \times(\zeta-\varphi)^{2\eta_1-2}\mathbf{E}\,|\mu(\varphi)-\mu_n(\varphi)|^2 d\varphi\\ &\quad+\frac{\kappa\Xi^{2\eta_1-1}(1+\Xi)}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}n^{1-2\eta_1}\\ &=: \bar{r}_3\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2}\\ &\quad\mathbf{E}\,|\mu(\varphi)-\mu_n(\varphi)|^2 d\varphi+\bar{r}_4.\end{aligned}\tag{3.26}$$

In view of Lemma 2.1, we find

$$\begin{aligned}\mathbf{E}\,|\mu(\zeta)-\mu_n(\zeta)|^2 &\leq \bar{r}_4(1+\mathcal{E}_{2\eta_1-1,1}(\bar{r}_3\Gamma(2\eta_1-1)\Xi^{2\eta_1-1}))\\ &=: \mathcal{M}n^{1-2\eta_1}.\end{aligned}\tag{3.27}$$

This yields the desired outcome. □

Some remarkable results of the aforesaid findings are as follows:

Remark 3.1. **(i)** Letting $\psi = 1$, then (1.1) reduces to the Caputo-type FSDE (2.6), the convergent rate of the framework in Theorem 3.4 corresponds to the widely recognized convergent rate of the fractional Carathéodory's findings [35].

(ii) Letting $\psi = \eta_1 = 1$, then (2.6) reduces to the SDE (2.6), the convergent rate of the framework in Theorem 3.4 corresponds to the widely recognized convergent rate of the Carathéodory's findings [33].

4 Ulam–Hyers stability

Definition 4.1. Suppose there is $\mathbb{R}^d$-value stochastic process $\{\mathbf{X}(\zeta)\}_{\zeta\in[-\omega,\Xi]}$ is termed as a solution to (1.2) if it meets the addresses criteria:

(i) $\{\mathbf{X}(\zeta)\}$ is ζ-continuous and $\mathfrak{F}_\zeta$ adapted.

(ii) $\{\mathcal{F}(\zeta, \mathbf{X}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)))\} \in \mathcal{L}([0,\Xi]\times\mathbb{R}^\eth\times\mathbb{R}^\eth;\mathbb{R}^\eth)$ and $\{\mathcal{G}(\zeta, \mathbf{X}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)))\} \in \mathcal{L}^2([0,\Xi]\times\mathbb{R}^\eth\times\mathbb{R}^\eth;\mathbb{R}^{\eth\times m_1})$.

(iii) For $\forall\zeta\in[-\omega,\Xi]$,

$$\mathbf{X}(\zeta) = \begin{cases} \Phi_0 + \dfrac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\displaystyle\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1} \mathcal{F}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega))\mathrm{d}\varphi \\ + \dfrac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\displaystyle\int_0^\zeta e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\mathcal{G}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi - \omega))\mathrm{d}\mathcal{W}(\varphi), \zeta\in\mathcal{J}, \\ \Phi(\zeta), \zeta\in[-\omega,0], \end{cases} \tag{4.1}$$

where $\mathbf{E}\left(\int_{-\omega}^{\Xi}\|\mathbf{X}(\zeta)\|^2\mathrm{d}\zeta\right) < \infty$.

(iv) For every other solution $\tilde{\mathbf{X}}(\zeta)$, we find $\mathbf{P}\{\mathbf{X}(\zeta) = \tilde{\mathbf{X}}(\zeta), \zeta\in[-\omega,\Xi]\} = 1$.

Definition 4.2. Assume that system (1.2) is U-Hs if $\exists$ a real number $\lambda > 0$ such that $\forall\varepsilon > 0$ and for every differentiable mapping $\mathbf{Y}(\zeta)\in([0,\Xi],\mathbb{R}^\eth)$:

$$\mathbb{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|{}^c\mathcal{D}_{0^+}^{\eta_1;\psi}\mathbf{Y}(\zeta) - \mathcal{F}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega)) - \mathcal{G}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega))\frac{\mathrm{d}\mathcal{W}(\zeta)}{\mathrm{d}\zeta}\right\|^2\right) \le \varepsilon, \tag{4.2}$$

and $\exists$ a solution $\mathbf{X}(\zeta)\in([0,\Xi],\mathbb{R}^\eth)$ of (1.2) satisfying

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{Y}(\zeta) - \mathbf{X}(\zeta)\|^2\right) < \varepsilon\lambda. \tag{4.3}$$

Remark 4.1. Suppose there is a mapping $\mathbf{Y}(\zeta)\in([0,\Xi],\mathbb{R}^\eth)$ is a solution of (4.2) if and only if $\exists$ a mapping $\hbar(\zeta)\in([0,\Xi],\mathbb{R}^\eth)$ such that

(i) $\mathbf{E}(\sup\|\hbar(\zeta)\|^2_{\zeta\in[0,\Xi]}) \le \varepsilon$,

(ii) ${}^c\mathcal{D}_{0^+}^{\eta_1;\psi}\mathbf{Y}(\zeta) = \mathcal{F}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega)) + \mathcal{G}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega))\frac{\mathrm{d}\mathcal{W}(\zeta)}{\mathrm{d}\zeta} + \hbar(\zeta)$.

($\mathbf{B}_1$) (**Lipschitz assumption**) For every $\mathcal{F}, \mathcal{G}\in\mathbb{R}^\eth$, there is a fixed $\mathcal{M} > 0$ such that $\forall\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2\in\mathbb{R}^\eth$, $\zeta\in[0,\Xi]$,

$$\begin{aligned}&\|\mathcal{F}(\zeta, \mathbf{X}_1, \mathbf{Y}_1) - \mathcal{F}(\zeta, \mathbf{X}_2, \mathbf{Y}_2)\| \\ &\vee\|\mathcal{G}(\zeta, \mathbf{X}_1, \mathbf{Y}_1) - \mathcal{G}(\zeta, \mathbf{X}_2, \mathbf{Y}_2)\| \\ &\le \mathcal{L}(\|\mathbf{X}_1 - \mathbf{X}_2\| + \|\mathbf{Y}_1 - \mathbf{Y}_2\|),\end{aligned} \tag{4.4}$$

where $\mathcal{F}$ and $\mathcal{G}$ are uniformly continuous mappings and $\vee$ signifies as $\mathbf{X}_1\vee\mathbf{X}_2 = \max\{\mathbf{X}_1, \mathbf{X}_2\}$.

($\mathbf{B}_2$) (**Non-Lipschitz condition**) A mapping $\Lambda(\zeta, \mathcal{U}_1, \mathcal{U}_2) : [0,+\infty)\times\mathbb{R}^+\times\mathbb{R}^+\mapsto\mathbb{R}^+$ such that **(a)** $\forall\mathbf{X}_1, \mathbf{X}_2, \mathbf{Y}_1, \mathbf{Y}_2\in\mathbb{R}^\eth$ and $\zeta\in[0,\Xi]$

$$\begin{aligned}&\|\mathcal{F}(\zeta, \mathbf{X}_1, \mathbf{Y}_1) - \mathcal{F}(\zeta, \mathbf{X}_2, \mathbf{Y}_2)\|^2 \\ &\vee\|\mathcal{G}(\zeta, \mathbf{X}_1, \mathbf{Y}_1) - \mathcal{G}(\zeta, \mathbf{X}_2, \mathbf{Y}_2)\|^2 \\ &\le \Lambda(\zeta, \|\mathbf{X}_1 - \mathbf{X}_2\|^2, \|\mathbf{Y}_1 - \mathbf{Y}_2\|^2),\end{aligned} \tag{4.5}$$

where $\mathcal{F}$ and $\mathcal{G}$ are continuous and bounded mappings. Also, $\Lambda(\zeta, \mathcal{U}_1, \mathcal{U}_2)$ is monotone, increasingly continuous, and concave mapping having $\Lambda(\zeta, 0, 0) = 0$, $\zeta\ge 0$.

(b) For each $\zeta\in\mathbb{R}^+$ and every positive mapping $\mathbf{Y}(\zeta)$ such that

$$\mathbf{Y}(\zeta) \le m_1\int_0^\zeta\Lambda(\varphi, \mathbf{Y}(\varphi))\mathrm{d}\varphi, \tag{4.6}$$

where $m_1 > 0$ is a constant and $\Lambda(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi)) = \Lambda(\varphi, \mathbf{Y}(\varphi))$, we have $\mathbf{Y}(\zeta) = 0$.

($\mathbf{B}_3$)$\exists$ three mappings $\bar{a}(\zeta)$, $\bar{b}(\zeta)$, and $\bar{q}(\zeta)$ such that

$$\Lambda(\zeta, \mathcal{U}_1, \mathcal{U}_2) \le \bar{a}(\zeta) + \bar{b}(\zeta)\mathcal{U}_1 + \bar{q}(\zeta)\mathcal{U}_2, \mathcal{U}_1, \mathcal{U}_2 > 0,$$

$$\int_0^\Xi\bar{a}(\zeta)\mathrm{d}\zeta < \infty, \int_0^\Xi\bar{b}(\zeta)\mathrm{d}\zeta < \infty, \int_0^\Xi\bar{q}(\zeta)\mathrm{d}\zeta < \infty. \tag{4.7}$$

In order to find the solution $\mathbf{X}(\zeta)$, $\zeta\in[0,\Xi]$ of system (1.2) is U-Hs and investigate the stability of the findings of FSDEs (1.2) considering Lipchitz and non-Lipschitz assumptions.

Theorem 4.1. *Under assumption* ($\mathbf{B}_1$) *and* $12\mathcal{M}^2\Xi^{2\eta_1-1/2} < (4\eta_1-3)^{1/2}\psi^{2\eta_1}(\Gamma(\eta_1))^2$, *then the GPF-SDE* (1.2) *is U-Hs at* $[0,\Xi]$.

Proof. By means of Definition 4.1 and Remark 4.1, we write

$$\begin{aligned}\mathbf{X}(\zeta) = \mu_0 + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)} \\ \times (\zeta-\varphi)^{\eta_1-1}\mathcal{F}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega))\mathrm{d}\varphi \\ + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1} \\ \times \mathcal{G}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega))\mathrm{d}\mathcal{W}_\varphi \\ + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\hbar(\varphi)\mathrm{d}\varphi.\end{aligned} \tag{4.8}$$

In view of Definition 4.1 and utilizing (4.8), we have

$$\begin{aligned}\mathbf{X}(\zeta) - \mathbf{Y}(\zeta) = \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1} \\ \times (\mathcal{F}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega)) \\ - \mathcal{F}(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi-\omega)))\mathrm{d}\varphi \\ + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1} \\ \times (\mathcal{G}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega)) \\ - \mathcal{G}(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi-\omega)))\mathrm{d}\mathcal{W}_\varphi \\ + \frac{1}{\psi^{\eta_1}\Gamma(\eta_1)}\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\hbar(\varphi)\mathrm{d}\varphi.\end{aligned}$$

Making use of Jensen’s variant, we find

$$\begin{aligned}&\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right) \\ &\le \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}(\mathcal{F}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega)) - \mathcal{F}(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi-\omega)))\mathrm{d}\varphi\right\|^2\right) \\ &\quad + \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}(\mathcal{G}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega)) - \mathcal{G}(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi-\omega)))\mathrm{d}\mathcal{W}_\varphi\right\|^2\right) \\ &\quad + \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta} e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\hbar(\varphi)\mathrm{d}\varphi\right\|^2\right) \\ &=: \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3.\end{aligned}$$

Employing the Hölder inequality and assumption ($\mathbf{B}_1$), one can find

$$\begin{aligned}\mathcal{T}_1 &\le \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2(\zeta-\varphi)^{2\eta_1-2}\mathrm{d}\varphi\right\|\right) \times \mathbf{E}\int_0^{\Xi}\|\mathcal{F}(\varphi, \mathbf{X}(\varphi), \mathbf{X}(\varphi-\omega)) - \mathcal{F}(\varphi, \mathbf{Y}(\varphi), \mathbf{Y}(\varphi-\omega)\|^2\mathrm{d}\varphi \\ &\le \frac{3\mathcal{M}^2\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\int_0^{\Xi}(\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\| + \|\mathbf{X}(\varphi-\omega)-\mathbf{Y}(\varphi-\omega)\|)^2\mathrm{d}\varphi \\ &\le \frac{6\mathcal{M}^2\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}\int_0^{\Xi}(\mathbf{E}\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2 + \mathbf{E}\|\mathbf{X}(\varphi-\omega)-\mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi \\ &= \bar{r}_5\Xi^{2\eta_1-1}\int_0^{\Xi}(\mathbf{E}\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2)\mathrm{d}\varphi + \bar{r}_5\Xi^{2\eta_1-1}\int_0^{\Xi}(\mathbf{E}\|\mathbf{X}(\varphi-\omega)-\mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi,\end{aligned} \tag{4.9}$$

where we have used the fact that $\left|e^{\frac{\psi-1}{\psi}\Xi}\right|^2 < 1$ and $\bar{r}_5 = \frac{6\mathcal{M}^2}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}$.

Now, utilizing the Itô isometry and the Hölder inequality, we find

$$\mathcal{T}_2 \le \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left\|\int_0^{\Xi} e^{\frac{\psi-1}{\psi}(\Xi-\varphi)}(\Xi-\varphi)^{\eta_1-1}(\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega))\mathrm{d}\mathcal{W}_\varphi\right\|^2$$
$$= \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\int_0^{\Xi}\left\|e^{\frac{\psi-1}{\psi}(\Xi-\varphi)}(\Xi-\varphi)^{\eta_1-1}(\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega))\right\|^2\mathrm{d}\mathcal{W}_\varphi\right.$$
$$\le \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left(\int_0^{\Xi}\left|e^{\frac{\psi-1}{\psi}(\Xi-\varphi)}\right|^4(\Xi-\varphi)^{4\eta_1-4}\mathrm{d}\varphi\right)^{1/2}$$
$$\times\ \mathbf{E}\left(\int_0^{\Xi}\|\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega)\|^4\mathrm{d}\mathcal{W}_\varphi\right)^{1/2}$$
$$\le \frac{3\Xi^{2\eta_1-3/2}}{(4\eta_1-3)^{1/2}\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\int_0^{\Xi}\|\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega)\|^4\mathrm{d}\mathcal{W}_\varphi\right)^{1/2}$$
$$= \bar{r}_6\Xi^{2\eta_1-3/2}\mathbf{E}\left(\int_0^{\Xi}\|\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega)\|^4\mathrm{d}\mathcal{W}_\varphi\right)^{1/2},$$

where we have used the fact that $\left|e^{\frac{\psi-1}{\psi}\Xi}\right|^2 < 1$ and $\bar{r}_6 = \frac{3}{(4\eta_1-3)^{1/2}\psi^{2\eta_1}(\Gamma(\eta_1))^2}$. Since $\|\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega)) - \mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega))\|^4$ is a continuous mapping on $[0,\Xi]$, making use of mean value theorem for integrals $\exists\ \tilde{y}\in[0,\Xi]$ such that

$$\begin{aligned}\int_0^{\Xi}&\|\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))\\ &-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega))\|^4\mathrm{d}\varphi\\ =\ &\Xi\|\mathcal{G}(\tilde{y},\mathbf{X}(\tilde{y}),\mathbf{X}(\tilde{y}-\omega))\\ &-\mathcal{G}(\tilde{y},\mathbf{Y}(\tilde{y}),\mathbf{Y}(\tilde{y}-\omega))\|^4.\end{aligned}\tag{4.10}$$

Under assumption ($\mathbf{B}_1$) and Jensen's inequality, we find

$$\begin{aligned}\mathcal{T}_2 &\le \bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}\|\mathcal{G}(\tilde{y},\mathbf{X}(\tilde{2y}),\mathbf{X}(\tilde{y}-\omega))\\ &\quad-\mathcal{G}(\tilde{y},\mathbf{Y}(\tilde{y}),\mathbf{Y}(\tilde{y}-\omega))\|^2\\ &\le 2\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}(\mathbf{E}\|\mathbf{X}(\tilde{y})-\mathbf{Y}(\tilde{y})\|^2+\mathbf{E}\|\mathbf{X}(\tilde{y}-\omega)\\ &\quad-\mathbf{Y}(\tilde{y}-\omega)\|^2).\end{aligned}\tag{4.11}$$

Thus, applying the Cauchy–Schwarz variant and Remark 4.1 to produce

$$\begin{aligned}\mathcal{T}_3 &\le \frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left\{\sup_{\zeta\in[0,\Xi]}\left(\int_0^{\zeta}\left|e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}\right|^2\right.\right.\\ &\quad\times\ (\zeta-\varphi)^{2\eta_1-2}\mathrm{d}\varphi\left.\left.\right)\int_0^{\zeta}\|\hbar(\varphi)\|^2\mathrm{d}\varphi\right\}\\ &\le \frac{3\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}\left\{\int_0^{\Xi}\mathbf{E}(\sup_{\varphi\in[0,\Xi]}\|\hbar(\varphi)\|^2)\right\}\mathrm{d}\varphi\\ &\le \frac{3\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}\varepsilon\\ &= \bar{r}_7\Xi^{2\eta_1}\varepsilon,\end{aligned}\tag{4.12}$$

where $\bar{r}_7 = \frac{3}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}$.

It follows that

$$\begin{aligned}&\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right)\\ &\le\ \bar{r}_5\Xi^{2\eta_1-1}\left(\int_0^{\Xi}\mathbf{E}(\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2)\mathrm{d}\varphi+\int_0^{\Xi}\mathbf{E}(\|\mathbf{X}(\varphi-\omega)\right.\\ &\quad-\mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi)+2\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\{\mathbf{E}\|\mathbf{X}(\tilde{y})\\ &\quad-\mathbf{Y}(\tilde{y})\|^2+\mathbf{E}\|\mathbf{X}(\tilde{y}-\omega)-\mathbf{Y}(\tilde{y}-\omega)\|^2\}+\bar{r}_7\Xi^{2\eta_1}\varepsilon\\ &\le\ \bar{r}_5\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}\left(\sup_{\upsilon_1\in[0,\varphi]}\|\mathbf{X}(\upsilon_1)-\mathbf{Y}(\upsilon_1)\|^2\right)\mathrm{d}\varphi\\ &\quad+\bar{r}_5\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}\left(\sup_{\upsilon_1\in[0,\varphi]}\|\mathbf{X}(\upsilon_1-\omega)-\mathbf{Y}(\upsilon_1-\omega)\|^2\right)\mathrm{d}\varphi\\ &\quad+2\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}\left(\sup_{\tilde{y}_1\in[0,\tilde{y}]}\|\mathbf{X}(\tilde{y}_1)-\mathbf{Y}(\tilde{y}_1)\|^2\right)\\ &\quad+2\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}\left(\sup_{\tilde{y}_1\in[0,\tilde{y}]}\|\mathbf{X}(\tilde{y}_1-\omega)-\mathbf{Y}(\tilde{y}_1-\omega)\|^2\right)\\ &\quad+\bar{r}_7\Xi^{2\eta_1}\varepsilon.\end{aligned}$$

In contrast to the methodology of interacting with the delay [50,51], we acquire

$$\mathcal{U}_2(\Xi) = \mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right),$$

also,

$$\mathbf{E}\left(\sup_{\zeta\in[-\omega,0]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right) = 0,$$

and then we attain

$$\mathbf{E}\left(\sup_{\upsilon_1\in[0,\varphi]}\|\mathbf{X}(\varphi-\omega)-\mathbf{Y}(\varphi-\omega)\|^2\right)=\mathcal{U}_2(\varphi-\omega).$$

So that

$$\mathcal{U}_2(\Xi)\leq\bar{r}_5\Xi^{2\eta_1-1}\left(\int_0^{\Xi}\mathcal{U}_2(\varphi)\mathrm{d}\varphi+\int_0^{\Xi}\mathcal{U}_2(\varphi-\omega)\mathrm{d}\varphi\right)$$
$$+\,2\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}(\mathcal{U}_2(\tilde{y}\,+\mathcal{U}_2(\tilde{y}_1-\omega)))+\bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Define a set $\mathcal{U}_1(\Xi)=\sup_{\varpi\in[-\omega,\Xi]}\mathcal{U}_2(\varpi)$, then $\mathcal{U}_2(\varphi)\leq\mathcal{U}_1(\varphi)$ and $\mathcal{U}_2(\varphi-\omega)\leq\mathcal{U}_1(\varphi-\omega)$. Therefore,

$$\mathcal{U}_2(\Xi)\leq2\bar{r}_5\Xi^{2\eta_1-1}\int_0^{\Xi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi+4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\mathcal{U}_1(\tilde{y})+\bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

For $\varpi\in[0.\ \Xi]$, we find

$$\mathcal{U}_2(\varpi)\leq2\bar{r}_5\varpi^{2\eta_1-1}\int_0^{\varpi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi+4\mathcal{M}^2\bar{r}_6\varpi^{2\eta_1-1/2}\mathcal{U}_1(\tilde{y})$$
$$+\,\bar{r}_7\varpi^{2\eta_1}\varepsilon$$
$$\leq2\bar{r}_5\Xi^{2\eta_1-1}\int_0^{\varpi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi+4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\mathcal{U}_1(\tilde{y})+\bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Therefore, we find

$$\mathcal{U}_1(\Xi)=\sup_{\varpi\in[-\omega,\Xi]}\mathcal{U}_2(\varpi)$$
$$\leq\max\left\{\sup_{\varpi\in[-\omega,0]}\mathcal{U}_2(\varpi),\ \sup_{\varpi\in[0,\Xi]}\mathcal{U}_2(\varpi)\right\}$$
$$\leq2\bar{r}_5\Xi^{2\eta_1-1}\int_0^{\varpi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi+4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}\mathcal{U}_1(\Xi)+\bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Thus,

$$\mathcal{U}_1(\Xi)\leq\frac{2\bar{r}_5\Xi^{2\eta_1-1}}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\int_0^{\Xi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi+\frac{\bar{r}_7\Xi^{2\eta_1}\varepsilon}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}.$$

According to Lemma 2.1, we find

$$\mathcal{U}_1(\Xi)\leq\frac{\bar{r}_7\Xi^{2\eta_1}\varepsilon}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\mathcal{E}\left(\frac{2\bar{r}_5\Xi^{2\eta_1}}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\right).$$

Clearly, we see that

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right)$$
$$\leq\frac{\bar{r}_7\Xi^{2\eta_1}\varepsilon}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\mathcal{E}\left(\frac{2\bar{r}_5\Xi^{2\eta_1}}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\right).$$

Finally, for every $\varepsilon>0$, $\exists\ \lambda=\frac{\bar{r}_7\Xi^{2\eta_1}\varepsilon}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\mathcal{E}\left(\frac{2\bar{r}_5\Xi^{2\eta_1}}{1-4\mathcal{M}^2\bar{r}_6\Xi^{2\eta_1-1/2}}\right)$ such that

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right)\leq\varepsilon\lambda.$$

As a result, this theorem is established. □

Theorem 4.2. *Under the supposition* ($\mathbf{B}_2$) *and* ($\mathbf{B}_3$) *satisfy,* $6\tilde{\mathcal{M}}\Xi^{2\eta_1-1/2}<(4\eta_1-3)^{1/2}\psi^{2\eta_1}(\Gamma(\eta_1))^2$, $\mathcal{M}=\max\{\sup_{\zeta\in[0,\Xi]}\bar{b}(\zeta),\sup_{\zeta\in[0,\Xi]}\bar{q}(\zeta)\}$, *and* $\exists$ *a constant* $\Im$ *fulfilling* $(3(4\eta_1-3)^{1/2}\Xi^{2\eta_1}+3(2\eta_1-1)\Xi^{2\eta_1-1/2}/(2\eta_1-1)(4\eta_1-3)^{1/2}\psi^{2\eta_1}(\Gamma(\eta_1))^2-6(2\eta_1-1)\mathcal{M}\Xi^{2\eta_1-1/2}\sup_{\zeta\in[0,\Xi]})\leq\Im\varepsilon$, $\eta_1\in(0.75,1)$. *The GPF-FSDE is U-Hs at* $[0,\Xi]$.

Proof. By means of the variant 4.9, we find

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right)$$
$$\leq\frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}(\mathcal{F}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{F}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega)))\mathrm{d}\varphi\right\|^2\right)$$
$$+\frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}(\mathcal{G}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega))-\mathcal{G}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega)))\mathrm{d}\mathcal{W}_\varphi\right\|^2\right)$$
$$+\frac{3}{\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\left\|\int_0^{\zeta}e^{\frac{\psi-1}{\psi}(\zeta-\varphi)}(\zeta-\varphi)^{\eta_1-1}\hbar(\varphi)\mathrm{d}\varphi\right\|^2\right)$$
$$=:H_1+H_2+H_3.$$

Attempting to take variant (4.9) and supposition ($\mathbf{B}_2$) into consideration,

$$H_1 \leq \frac{3\Xi^{2\eta_1-1}}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}\mathbf{E}\left(\int_0^{\Xi}\|\mathcal{F}(\varphi,\mathbf{X}(\varphi),\mathbf{X}(\varphi-\omega)) - \mathcal{F}(\varphi,\mathbf{Y}(\varphi),\mathbf{Y}(\varphi-\omega))\|^2 d\varphi\right)$$
$$\leq \bar{r}_8\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}\Lambda(\varphi,\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2,\|\mathbf{X}(\varphi-\omega) - \mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi,$$

where $\bar{r}_8 = \frac{3}{(2\eta_1-1)\psi^{2\eta_1}(\Gamma(\eta_1))^2}$.

Combining (4.10)–(4.12) and assumption ($\mathbf{B}_2$), we find

$$H_2 \leq \bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}\|\mathcal{G}(\tilde{y},\mathbf{X}(\tilde{y}),\mathbf{X}(\tilde{y}-\omega)) - \mathcal{G}(\tilde{y},\mathbf{Y}(\tilde{y}),\mathbf{Y}(\tilde{y}-\omega))\|^2$$
$$\leq \bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}\Lambda(\tilde{y},\|\mathbf{X}(\tilde{y})-\mathbf{Y}(\tilde{y})\|^2,\|\mathbf{X}(\tilde{y}-\omega) - \mathbf{Y}(\tilde{y}-\omega)\|^2).$$

It is simple to achieve employing supposition ($\mathbf{B}_3$), we find

$$H_1 + H_2 \leq \bar{r}_8\Xi^{2\eta_1-1/2}\int_0^{\Xi}\mathbf{E}\Lambda(\varphi,\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2,\|\mathbf{X}(\varphi-\omega) - \mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi + \bar{r}_6\Xi^{2\eta_1-1/2}\int_0^{\Xi}\mathbf{E}\Lambda(\tilde{y},\|\mathbf{X}(\tilde{y}) - \mathbf{Y}(\tilde{y})\|^2,\|\mathbf{X}(\tilde{y}-\omega)-\mathbf{Y}(\tilde{y}-\omega)\|^2)$$
$$\leq \bar{r}_8\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}(\bar{a}(\varphi)+\bar{b}(\varphi)\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2 + \bar{q}(\varphi)\|\mathbf{X}(\varphi-\omega)-\mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi + \bar{r}_6\Xi^{2\eta_1-1/2}\mathbf{E}(\bar{a}(\tilde{y})+\bar{b}(\tilde{y})\|\mathbf{X}(\tilde{y})-\mathbf{Y}(\tilde{y})\|^2 + \bar{q}(\tilde{y})\|\mathbf{X}(\tilde{y}-\omega)-\mathbf{Y}(\tilde{y}-\omega)\|^2)$$
$$\leq (\bar{r}_8\Xi^{2\eta_1}+\bar{r}_6\Xi^{2\eta_1-1/2})\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\Xi}(\mathbf{E}\|\mathbf{X}(\varphi)-\mathbf{Y}(\varphi)\|^2 + \mathbf{E}\|\mathbf{X}(\varphi-\omega) - \mathbf{Y}(\varphi-\omega)\|^2)\mathrm{d}\varphi + \bar{r}_6\Xi^{2\eta_1-1/2}(\mathbf{E}\|\mathbf{X}(\tilde{y})-\mathbf{Y}(\tilde{y})\|^2 + \mathbf{E}\|\mathbf{X}(\tilde{y}-\omega)-\mathbf{Y}(\tilde{y}-\omega)\|^2).$$

Utilizing the variant (4.12), we have

$$H_3 \leq \bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Then,

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right) \leq (\bar{r}_8\Xi^{2\eta_1}+\bar{r}_6\Xi^{2\eta_1-1/2})\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}\left(\sup_{\upsilon_1\in[0,\varphi]}\|\mathbf{X}(\upsilon_1)-\mathbf{Y}(\upsilon_1)\|^2\right)\mathrm{d}\varphi + \bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\Xi}\mathbf{E}\left(\sup_{\upsilon_1\in[0,\varphi]}\|\mathbf{X}(\upsilon_1-\omega)-\mathbf{Y}(\upsilon_1-\omega)\|^2\right)\mathrm{d}\varphi + \bar{r}_6\tilde{\mathcal{M}}\Xi^{2\eta_1-1/2}\mathbf{E}\left(\sup_{\upsilon_2\in[0,\tilde{y}]}\|\mathbf{X}(\upsilon_2)-\mathbf{Y}(\upsilon_2)\|^2\right) + \bar{r}_6\tilde{\mathcal{M}}\Xi^{2\eta_1-1/2}\mathbf{E} \times \left(\sup_{\upsilon_2\in[0,\tilde{y}]}\|\mathbf{X}(\upsilon_2-\omega)-\mathbf{Y}(\upsilon_2-\omega)\|^2\right) + \bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Define a set $\mathcal{U}_2(\Xi) = \mathbf{E}(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2)$ and $\mathbf{E}(\sup_{\zeta\in[-\omega,0]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2 = 0$, we can find

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\upsilon_2-\omega)-\mathbf{Y}(\upsilon_2-\omega)\|^2\right) = \mathcal{U}_2(\varphi-\omega).$$

Evidently, we can draw the conclusion that

$$\mathcal{U}_2(\Xi) \leq \bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\left(\int_0^{\Xi}\mathcal{U}_2(\varphi)\mathrm{d}\varphi + \int_0^{\Xi}\mathcal{U}_2(\varphi-\omega)\mathrm{d}\varphi\right) + \bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1/2}(\mathcal{U}_2(\tilde{y})+\mathcal{U}_2(\tilde{y}-\omega)) + (\bar{r}_8\Xi^{2\eta_1}+\bar{r}_6\Xi^{2\eta_1-1/2})\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Choosing $\mathcal{U}_1(\Xi) = \sup_{\varpi\in[-\omega,\Xi]}\mathcal{U}_2(\varpi)$, then $\mathcal{U}_2(\varphi) \leq \mathcal{U}_1(\varphi)$ and $\mathcal{U}_2(\varphi-\omega) \leq \mathcal{U}_1(\varphi)$. Therefore,

$$\mathcal{U}_2(\Xi) \leq 2\bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\Xi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi + 2\bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\mathcal{U}_1(\tilde{y}) + (\bar{r}_8\Xi^{2\eta_1}+\bar{r}_6\Xi^{2\eta_1-1/2})\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

For every $\varpi \in [0,\Xi]$, we attain that

$$\mathcal{U}_2(\varpi) \leq 2\bar{r}_8\tilde{\mathcal{M}}\varpi^{2\eta_1-1}\int_0^{\varpi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi + 2\bar{r}_8\tilde{\mathcal{M}}\varpi^{2\eta_1-1}\mathcal{U}_1(\tilde{y}) + (\bar{r}_8\varpi^{2\eta_1}+\bar{r}_6\varpi^{2\eta_1-1/2})\sup_{\varphi\in[0,\varpi]}\bar{a}(\varphi) + \bar{r}_7\varpi^{2\eta_1}\varepsilon$$
$$\leq 2\bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\Xi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi + 2\bar{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\mathcal{U}_1(\tilde{y}) + (\bar{r}_8\Xi^{2\eta_1}+\bar{r}_6\Xi^{2\eta_1-1/2})\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \bar{r}_7\Xi^{2\eta_1}\varepsilon.$$

Furthermore, we have

$$\begin{aligned}\mathcal{U}_1(\Xi) &= \sup_{\varpi\in[-\omega,\Xi]} \mathcal{U}_2(\varpi)\\ &\le \max\left\{\sup_{\varpi\in[-\omega,0]} \mathcal{U}_2(\varpi), \sup_{\varpi\in[0,\Xi]} \mathcal{U}_2(\varpi)\right\}\\ &\le 2\overline{r}_8\tilde{\mathcal{M}}\Xi^{2\eta_1-1}\int_0^{\varpi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi + 2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}\mathcal{U}_1(\Xi)\\ &\quad + (\overline{r}_8\Xi^{2\eta_1} + \overline{r}_2\Xi^{2\eta_1-1/2}) \sup_{\varphi\in[0,\Xi]} \bar{a}(\varphi) + \overline{r}_7\Xi^{2\eta_1}\varepsilon.\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{U}_1(\Xi) &\le \frac{2\tilde{\mathcal{M}}\overline{r}_8\Xi^{2\eta_1-1}}{1-4\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\\ &\quad\times\int_0^{\Xi}\mathcal{U}_1(\varphi)\mathrm{d}\varphi + \frac{\overline{r}_8\Xi^{2\eta_1} + \overline{r}_6\Xi^{2\eta_1-1/2}}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\\ &\quad\times\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi) + \frac{\overline{r}_7\Xi^{2\eta_1}\varepsilon}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}.\end{aligned}$$

According to Lemma 2.1, we find

$$\begin{aligned}\mathcal{U}_1(\Xi) \le \Bigg(&\frac{\overline{r}_8\Xi^{2\eta_1} + \overline{r}_6\Xi^{2\eta_1-1/2}}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi)\\ &+ \frac{\overline{r}_7\Xi^{2\eta_1}\varepsilon}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\Bigg)\mathcal{E}\left(\frac{2\tilde{\mathcal{M}}\overline{r}_8\Xi^{2\eta_1}}{1-4\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right).\end{aligned}$$

Clearly, we see that

$$\begin{aligned}&\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right)\\ &\le \Bigg(\frac{\overline{r}_8\Xi^{2\eta_1} + \overline{r}_6\Xi^{2\eta_1-1/2}}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\sup_{\varphi\in[0,\Xi]}\bar{a}(\varphi)\\ &\quad + \frac{\overline{r}_7\Xi^{2\eta_1}\varepsilon}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\Bigg)\mathcal{E}\left(\frac{2\tilde{\mathcal{M}}\overline{r}_8\Xi^{2\eta_1}}{1-4\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right)\\ &\le \left(\mathfrak{I} + \frac{\overline{r}_7\Xi^{2\eta_1}\varepsilon}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right)\varepsilon\mathcal{E}\left(\frac{2\tilde{\mathcal{M}}\overline{r}_8\Xi^{2\eta_1+1}}{1-4\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right),\end{aligned}$$

Finally, for every $\varepsilon > 0$, $\exists\lambda = \left(\mathfrak{I} + \frac{\overline{r}_7\Xi^{2\eta_1}\varepsilon}{1-2\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right)\mathcal{E}\left(\frac{2\tilde{\mathcal{M}}\overline{r}_8\Xi^{2\eta_1+1}}{1-4\tilde{\mathcal{M}}\overline{r}_6\Xi^{2\eta_1-1/2}}\right)$ such that

$$\mathbf{E}\left(\sup_{\zeta\in[0,\Xi]}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\|^2\right) \le \varepsilon\lambda.$$

As a result, this theorem is established. □

5 Examples

In this section, we will present illustrative examples for the previous findings.

Example 5.1. Assume the U-Hs and E-U of the result to the subsequent equation:

$$\begin{aligned}{}^{c}\mathcal{D}_{0^+}^{\eta_1;\psi}\mathbf{X}(\zeta) &= \frac{1}{15}\sin\mathbf{X}(\zeta) + \frac{2}{23}\cos\mathbf{X}(\zeta-\omega)\\ &\quad + \left\{\frac{\Gamma(1/2)}{24\pi}\cos^2\mathbf{X}(\zeta) + \frac{\ln 1.5}{\pi e^4}\mathbf{X}(\zeta-\omega)\right\}\frac{\mathrm{d}\mathcal{W}(\zeta)}{\mathrm{d}\zeta},\end{aligned}\tag{5.1}$$

where $\eta_1 = 0.8$, $\psi = 1$, and $\zeta \in [0, 8]$. Since the mappings $\mathcal{F}$ and $\mathcal{G}$ are uniformly continuous.

Now, we have

$$\begin{aligned}&\mathcal{F}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega))\\ &= \frac{1}{15}\sin\mathbf{X}(\zeta) + \frac{2}{23}\cos\mathbf{X}(\zeta-\omega),\\ &\mathcal{G}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega))\\ &= \frac{\Gamma(1/2)}{24\pi}\cos^2\mathbf{X}(\zeta) + \frac{\ln 1.5}{\pi e^4}\mathbf{X}(\zeta-\omega).\end{aligned}\tag{5.2}$$

Thus, we have

$$\begin{aligned}&\|\mathcal{F}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)) - \mathcal{F}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega))\|\\ &= \left\|\frac{1}{15}\sin\mathbf{X}(\zeta) + \frac{2}{23}\cos\mathbf{X}(\zeta-\omega) - \frac{1}{15}\sin\mathbf{Y}(\zeta) + \frac{2}{23}\cos\mathbf{Y}(\zeta-\omega)\right\|\\ &\le \frac{1}{15}\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\| + \frac{2}{23}\|\mathbf{X}(\zeta-\omega)-\mathbf{Y}(\zeta-\omega)\|\\ &\le \frac{2}{23}(\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\| + \|\mathbf{X}(\zeta-\omega)-\mathbf{Y}(\zeta-\omega)\|).\end{aligned}$$

Analogously, we have

$$\begin{aligned}&\|\mathcal{G}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)) - \mathcal{G}(\zeta, \mathbf{Y}(\zeta), \mathbf{Y}(\zeta-\omega))\|\\ &\le \frac{\Gamma(0.5)}{12\pi}\|\cos\mathbf{X}(\zeta) - \cos\mathbf{Y}(\zeta)\| + \frac{\ln 3.5}{\pi e^4}\|\mathbf{X}(\zeta-\omega) - \mathbf{Y}(\zeta-\omega)\|\\ &\le \frac{2}{23}(\|\mathbf{X}(\zeta)-\mathbf{Y}(\zeta)\| + \|\mathbf{X}(\zeta-\omega)-\mathbf{Y}(\zeta-\omega)\|),\end{aligned}$$

$\mathcal{F}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega))$ and $\mathcal{G}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega))$ fulfills assumption $(\mathbf{B}_1)$, $3^{1.5}\Xi^{2\eta_1-1}\mathcal{M} = 3^{1.5}\times 2\times 5^{0.6}\times(2/23)^2 \approx 9/10 < (\eta_1-0.75)^{0.5}(\Gamma(\eta_1))^2 = \sqrt{0.05}\Gamma^2(0.75) \approx 3/10$ and $12\mathcal{M}^2\Xi^{2\eta_1-1/2} = 12(2/23)^2\times 5^{1.1} \approx 27/50 < (4\eta_1-3)^{0.5}\Gamma^2(\eta_1) = \sqrt{0.2}\Gamma^2(0.8) \approx 3/5$. As a consequence of Remark 4.1 and

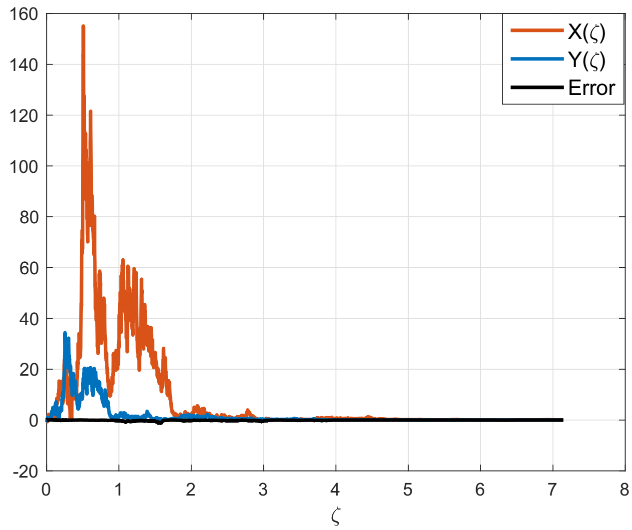

Figure 1: $\mathbf{X}(0) = 1$, $\hbar = \varepsilon^{1/2}$, $\mathbf{E}(\sup_{\zeta\in[0,8]}\|\hbar(\zeta)\|^2) = \mathbf{E}(\sup_{\zeta\in[0,8]}\|\varepsilon^{1/2}\|^2) = 0.001$.

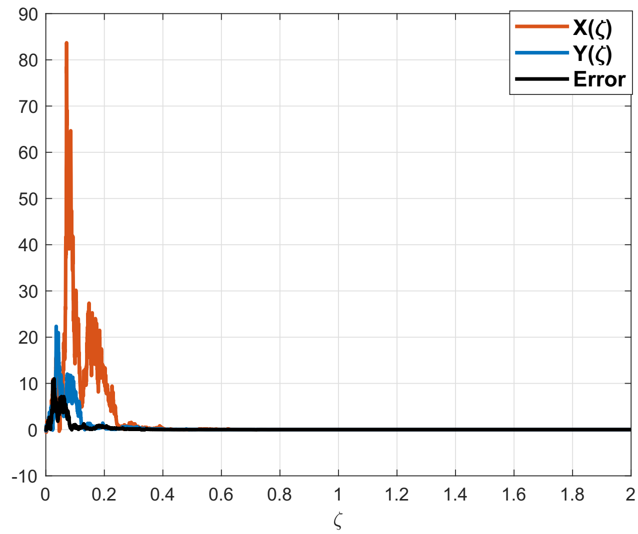

Figure 2: $\mathbf{X}(0) = 3$, $\hbar = \varepsilon^{1/2}/2$, $\mathbf{E}(\sup_{\zeta\in[0,8]}\|\hbar(\zeta)\|^2) = \mathbf{E}(\sup_{\zeta\in[0,8]}\|\varepsilon^{1/2}/2\|^2) = 0.007$.

Theorem 4.1, we discovered that equation (5.1) has only one solution, i.e., U-H stable.

Finally, we will perform a graphical illustration to ensure that the finding of (5.2) is U-H stable, as shown in Figure 1.

Example 5.2. Assume that U-H stability of the subsequent FSDEs having time delays

$${}^{c}\mathcal{D}_{0^+}^{\eta_1;\psi}\mathbf{X}(\zeta) = \frac{1}{\mathrm{n}}\mathbf{X}(\zeta) + \frac{\bar{\varepsilon}}{\sqrt{2}}\mathbf{X}(\zeta-\omega) + \left\{\frac{1}{\mathrm{n}^2}\mathbf{X}(\zeta) + \frac{\bar{\varepsilon}^2}{4}\sin\frac{1}{\mathbf{X}(\zeta-\omega)}\right\}\frac{\mathrm{d}\mathcal{W}(\zeta)}{\mathrm{d}\zeta}, \tag{5.3}$$

where $\eta_1 = 0.83$, $\zeta \in [0,2]$, $\mathrm{n} \geq 12$, $\mathcal{F}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)) = 1/\mathrm{n}\mathbf{X}(\zeta) + \bar{\varepsilon}\mathbf{X}(\zeta-\omega)$, and $\mathcal{G}(\zeta, \mathbf{X}(\zeta), \mathbf{X}(\zeta-\omega)) = 1/\mathrm{n}^2\mathbf{X}(\zeta) + \bar{\varepsilon}^2/4\sin 1/\mathbf{X}(\zeta-\omega)$ are the measurable continuous mappings, $\bar{\varepsilon}$ is an arbitrary number and $\bar{\varepsilon} \in (0, 1/4\sqrt{2})$.

Now, we have

$$\begin{aligned}
&\|\mathcal{F}(\zeta,\mathbf{X}(\zeta),\mathbf{X}(\zeta-\omega)) - \mathcal{F}(\zeta,\mathbf{Y}(\zeta),\mathbf{Y}(\zeta-\omega))\|^2 \\
&\quad + \|\mathcal{G}(\zeta,\mathbf{X}(\zeta),\mathbf{X}(\zeta-\omega)) - \mathcal{G}(\zeta,\mathbf{Y}(\zeta),\mathbf{Y}(\zeta-\omega))\|^2 \\
&= \left\|\frac{1}{\mathrm{n}}\mathbf{X}(\zeta) + \frac{\bar{\varepsilon}}{\sqrt{2}}\mathbf{X}(\zeta-\omega) - \frac{1}{\mathrm{n}}\mathbf{Y}(\zeta) - \frac{\bar{\varepsilon}}{\sqrt{2}}\mathbf{Y}(\zeta-\omega)\right\|^2 \\
&\quad + \left\|\frac{1}{\mathrm{n}^2}\mathbf{X}(\zeta) + \frac{\bar{\varepsilon}^2}{4}\sin\frac{1}{\mathbf{X}(\zeta-\omega)} - \frac{1}{\mathrm{n}^2}\mathbf{Y}(\zeta) - \frac{\bar{\varepsilon}^2}{4}\sin\frac{1}{\mathbf{Y}(\zeta-\omega)}\right\|^2 \\
&\leq \frac{\bar{\varepsilon}^4}{2} + \frac{4}{\mathrm{n}^2}\|\mathbf{X}(\zeta) - \mathbf{Y}(\zeta)\|^2 + \bar{\varepsilon}^2\|\mathbf{X}(\zeta-\omega) - \mathbf{Y}(\zeta-\omega)\|^2 \\
&= \Lambda(\zeta, \|\mathbf{X}(\zeta) - \mathbf{Y}(\zeta)\|^2, \|\mathbf{X}(\zeta-\omega) - \mathbf{Y}(\zeta-\omega)\|^2),
\end{aligned}$$

evidently, Λ $(\zeta, \|\mathbf{X}(\zeta) - \mathbf{Y}(\zeta)\|^2, \|\mathbf{X}(\zeta-\omega) - \mathbf{Y}(\zeta-\omega)\|^2)$ is increasing, continuous and concave mapping and $\tilde{\mathcal{M}} = \max\{\sup_{\zeta\in[0,\Xi]}\bar{b}(\zeta), \sup_{\zeta\in[0,\Xi]}\bar{q}(\zeta)\} = \max(4/\mathrm{n}^2, \bar{\varepsilon}^2) = 1/25$, $6\mathcal{M}\Xi^{2\eta_1-0.5} = 25/50 < (4\eta_1-3)^{0.5}(\Gamma(\eta_1))^2 \approx 57/100$. For arbitrary constant $\bar{\varepsilon}$, it is noted that $\Lambda(\zeta,0,0) = 0$. For every $\varepsilon > 0$, there exists $\mathfrak{I} = 0.2\varepsilon > 0$ such that $(3(4\eta_1-3)^{0.5}\Xi^{2\eta_1} + 3(2\eta_1-1)\Xi^{2\eta_1-0.5}/(2\eta_1-1)(4\eta_1-3)^{0.5}(\Gamma(\eta_1))^2 \quad - 6(2\eta_1-1)\tilde{\mathcal{M}}\Xi^{2\eta_1-0.5})$. Thus, the $\sup_{\zeta\in[0,\Xi]}\bar{a}(\zeta) < (0.2\varepsilon)\varepsilon = \mathfrak{I}\varepsilon$. That accomplishes all of Theorem 4.2 settings. As a result, we can say that framework (5.3) is U-Hs on $[0,2]$.

Finally, we will perform a graphical illustration to ensure that the finding of (5.3) is U-Hs, as shown in Figure 2.

6 Conclusion

Carathéodory's approximation has helped approximate the fractional derivatives, ensuring that the resulting FSDEs are mathematically well posed. It has provided a framework for analyzing the stability and uniqueness of solutions, which is crucial for understanding the fluid's behavior under various conditions, especially in Lagrangian stochastic models of fluid particles.

Figuring out the descriptive characteristics of DEs is one of the most vital aspects of differential equation theory. Integral equations are valuable techniques for investigating such features. In this study, we first identified the well-posedness for generalized proportional FSDEs using various approximate techniques, and the restrictions implanted on the global E-U of the findings are coherent with the Caputo FSDEs and traditional SDEs. Then, we glanced at the continuity of findings with reverence for the fractional-order of such formulae, especially if the proportional index ψ and fractional-order η_1 were noticeable. When η_1 and ψ tend to 1, the solution of generalized proportional FSDEs reduces to the Caputo-type FSDE and the conventional SDEs solution. Besides that, we contemplate Carathéodory's approximate solution for GPF-SDEs as an extension task for SDEs. Moreover, various generalizations are employed to demonstrate the U-H stability of the GPF-SDEs with time delays. Ultimately, we reveal two examples to validate the envisaged method. Our forthcoming research will concentrate on investigating the U-H stability of multiple kinds of FDEs with weaker assumptions and the requirements discovered will be applicable to a broader spectrum of GPF-SDEs including an invariant manifold and an invariant measure. By ensuring the existence, uniqueness and stability of solutions, Carathéodory's approximation makes it feasible to apply these advanced mathematical tools to real-world fluid dynamics problems, leading to deeper insights and more accurate predictions for future research.

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References

[1] Podlubny I. Fractional differential equations. San Diego CA: Academic Press; 1999.

[2] Samko SG, Kilbas AA, Marichev OI. Fractional integrals and derivatives: theory and applications. Yverdon: Gordon and Breach; 1993.

[3] Kochubei AN. General fractional calculus, evolution equations, and renewal processes. Integr Equ Oper Theo. 2011;71:583–600.

[4] Luchko Y. General fractional integrals and derivatives with the Sonine kernels. Mathematics. 2021;9:594.

[5] Khan H, Chen W, Khan A, Khan TS, Al-Madlal QM. Hyers-Ulam stability and existence criteria for coupled fractional differential equations involving p-Laplacian operator. Adv Differ Equ. 2018;2018:1–6.

[6] Khan H, Tunç C, Khan A. Stability results and existence theorems for nonlinear delay-fractional differential equations with φ_p^*-operator. J Appl Anal Comput. 2020;10(2):584–97.

[7] Chu YM, Rashid S, Alzahrani T, Alhulayyil H, Alsagri H, Rehman S. Complex adaptive learning cortical neural network systems for solving time-fractional difference equations with bursting and mixed-mode oscillation behaviors. Sci Rep. 2024;13(1):22447.

[8] Chu YM, Rashid S, QUA Asif, Abdalbagi M. On configuring new choatic behaviors for a variable fractional-order memristor-based circuit in terms of Mittag-Leffler kernel. Res Phy. 2023;53(1–4):106939.

[9] Khan H, Alzabut J, Shah A, He Z-Y, Etemad S, Rezapour S, Zada A. On fractal-fractional waterborne disease model: A study on theoretical and numerical aspects of solutions via simulations. Fractals. 2023;31:2340055.

[10] Li Y, Chen YQ, Podlubny I. Mittag-Leffler stability of fractional order nonlinear dynamic systems. Automatica. 2009;45:2965–9.

[11] Abed-Elhameed TM, Aboelenen T. Mittag-Leffler stability, control, and synchronization for chaotic generalized fractional-order systems. Adv Cont Discr Mod. 2022;2022:50.

[12] Begum R, Tunç O, Khan H, Gulzar H, Khan A. A fractional order Zika virus model with Mittag-Leffler kernel. Chaos Solit Fract. 2021;146:110898.

[13] Chu YM, Alzahrani T, Rashid S, Alhulayyil H, Rashidah W, Rehman S. Complex dynamical analysis of fractional differences Willamowski-Rössler chemical reaction model in time-scale analysis, Res Phy. 2023;54(6):107023.

[14] Chu YM, Alzahrani T, Rashid S, Rashidah W, Rehman S, Alkhatib M. An advanced approach for the electrical responses of discrete fractional-order biophysical neural network models and their dynamical responses. Sci Rep. 2023;13(1):18180.

[15] Tajadodi H, Khan A, Gómez-Aguilar JF, Khan H. Optimal control problems with Atangana-Baleanu fractional derivative. Optimal Cont Appl Meth. 2021;42(1):96–109.

[16] Khan H, Abdeljawad T, Gómez-Aguilar JF, Tajadodi H, Khan A. Fractional order Volterra integro-differential equation with Mittag-Leffler kernel. Fractals. 2021;29(6):2150154.

[17] Shah K, Abdeljawad T, Study of a mathematical model of COVID-19 outbreak using some advanced analysis. Waves Rand Comp Media. 2022;1–18.

[18] Shah K, Abdeljawad T, Alrabaiah H. On coupled system of drug therapy via piecewise equations. Fractals. 2022:30;2240206.

[19] Jarad F, Abdeljawad T, Alzabut J, Generalized fractional derivatives generated by a class of local proportional derivatives. Eur Phys J Spec Top. 2017;226(16–18):3457–71.

[20] Sk N, Mondal B, Thirthar AA, Alqudah MA, Abdeljawad T. Bistability and tristability in a deterministic prey-predator model: Transitions and emergent patterns in its stochastic counterpart. Chaos Solit Fract. 2023;176:114073.

[21] Gafel HS, Rashid S, Elagan SK. Novel codynamics of the HIV-1/HTLV-I model involving humoral immune response and cellular outbreak: A new approach to probability density functions and fractional operators. AIMS Math. 2023;8(12):28246–79.

[22] Rashid S, Hamidi SZ, Akram S, Aon M, Elagan SK, Alsubei SMT. Theoretical and mathematical codynamics of nonlinear tuberculosis and COVID-19 model pertaining to fractional calculus and probabilistic approach. Sci Rep. 2024;14(1):8827.

[23] Rashid S, Jarad F. Novel investigation of stochastic fractional differential equations measles model via the white noise and global derivative operator depending on Mittag-Leffler kernel. Comp Model Engin Sci. 2024;139(3):2289–327.

[24] Bouftouh O, Kabbaj S, Abdeljawad T, Khan A. Quasi controlled K metric spaces over C^* algebras with an application to stochastic integral equations. Comput Model Eng Sci. 2023;135:2649–63.
[25] Kilbas A, Srivastava H, Trujillo J. Theory and applications of fractional differential equations. North-Holland Mathematics Studies. Vol. 207. Amsterdam, Netherlands: Elsevier; 2006.
[26] Zhou Y, Wang JR, Zhang L. Basic: theory of fractional differential equations. Singapore: World Scientific; 2014.
[27] Miller KS, Ross B. An introduction to the fractional calculus and fractional differential equations. Hoboken, NJ, USA: Wiley-Interscience Publication; 1993.
[28] Cong ND, Doan TS, Siegmund S, Tuan T. On stable manifolds for planar fractional differential equations. Appl Math Comput. 2014;226:157–68.
[29] Xu W, Xu W, Zhang S. The averaging principle for stochastic differential equations with Caputo fractional derivative. Appl Math Lett. 2019;93:79–84.
[30] Wang Y, Xu J, Kloeden P. Asymptotic behavior of stochastic lattice systems with a Caputo fractional time derivative. Nonlinear Anal Theory Meth Appl. 2016;135:205–22.
[31] Doan T, Kloeden P, Huong P, Tuan H. Asymptotic separation between solutions of Caputo fractional stochastic differential equations. Stoch Anal Appl. 2018;36:654–64.
[32] Doan T, Huong P, Kloeden P. Euler-Maruyama scheme for Caputo stochastic fractional differential equations. J Comput Appl Math. 2020;112989:1–26.
[33] Mao X, Stochastic differential equations and their applications. Chichester: Horwood; 2008.
[34] Wang W, Cheng S, Guo Z, Yan X. A note on the continuity for Caputo fractional stochastic differential equations. Chaos. 2020;30:073106.
[35] Guo Z, Hu J, Wang W. Carathéodoryas approximation for a type of Caputo fractional stochastic differential equations. Adv Diff Equ. 2020;2020:636.
[36] Ulam S. A collection of mathematical problem. New York, NY, USA: Interscience; 1960.
[37] Hyers DH, On the stability of the linear functional equation. Proc Nat Acad Sci. 1941;27:222–4.
[38] Wang J, Li X. A uniform method to Ulam-Hyers stability for some linear fractional equations, Mediterranean J Math. 2016;13:625–35.
[39] Zada A, Alzabut J, Waheed H, Ioan-Lucian P. Ulam-Hyers stability of impulsive integrodifferential equations with Riemann-Liouville boundary conditions. Adv Diff Equ. 2020;2020;64.
[40] Luo D, Abdeljawad T, Luo Z. Ulam-Hyers stability results for a novel nonlinear Nabla Caputo fractional variable-order difference system. Turkish J Math. 2021;45:456–70.
[41] Wang J, Zhang Y. Ulam-Hyers-Mittag-Leffler stability of fractional-order delay differential equations. Optimization. 2014;63:1181–90.
[42] Zada A, Fatima S, Ali Z, Xu J, Cui Y. Stability results for a coupled system of impulsive fractional differential equations. Mathematics. 2019;7:927.
[43] Luo DF, Shah K, Luo ZG. On the novel Ulam-Hyers stability for a class of nonlinear ψ Hilfer fractional differential equation with time-varying delays. Mediterranean J Math. 2019;16:112.
[44] Sathiyaraj T, Wang JR, Balasubramaniam P. Ulams stability of Hilfer fractional stochastic differential systems. European Phy J Plus. 2019;134:605.
[45] Abdeljawad T. On conformable fractional calculus. J Comput Appl Math. 2015;279:57–66.
[46] Khalil R, Al Horani M, Yousef A, Sababheh M. A new definition of fractional derivative. J Comput Appl Math. 2014;264:65–70.
[47] Anderson DR, Ulness DJ. Newly defined conformable derivatives. Adv Dyn Sys Appl. 2015;10(2):109–37.
[48] Anderson DR. Second-order self-adjoint differential equations using a proportional-derivative controller. Commun Appl Nonlin Anal. 2017;24:17–48.
[49] Alzabut J, Abdeljawad T, Jarad F, Sudsutad W. A Gronwall inequality via the generalized proportional fractional derivative with applications. J Ineq Appl. 2019;2019:101.
[50] Balasundaram K, Raja R, Zhu Q, Chandrasekaran S, Zhou H. New global asymptotic stability of discrete-time recurrent neural networks with multiple time-varying delays in the leakage term and impulsive effects. Neurocomputing. 2016;214:420–9.
[51] Senthilraj S, Raja R, Jiang F, Zhu Q, Samidurai R. New delay-interval-dependent stability analysis of neutral type BAM neural networks with successive time delay components. Neurocomputing. 2016;171:1265–80.