Research Article

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Steady-state thermodynamic process in multilayered heterogeneous cylinder

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Abstract: The present study formulates and further examines a steady-state heat diffusion process in a generalized multilayered heterogeneous circular composite. Sufficient boundary and interfacial data are assumed at the endpoints of the circumferential length, and the interfaces, cutting across the respective perfectly welded cylinders. A well-known classical method for solving linear partial differential equations has been sought to derive a compacted solution for the diffusion process in governing heterogeneous cylinders. Certainly, among the significant novel findings of the current study is the acquisition of a generalized series solution for m-body multilayered heterogeneous circular composites, in addition to the portrayal of simple, yet an efficient method for solution; away from sophisticated numerical methods or integral transform methods that are not always invertible analytically. Moreover, three prototype situations of the structure have been profoundly examined, which are then found to satisfy all imposed structural assumptions. Moreover, the current examination finds relevance in the study and the analysis and design of multilayered bodies in engineering, material science, thermodynamics, and solid mechanics.

Keywords: steady-state process, heat diffusion equation, multilayered cylinder, material heterogeneity, method of separation of variables

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1 Introduction

The heat diffusion processes arise in a variety of thermodynamical-related processes and have been comprehensively analyzed in both the past and present literature [1–3]. Certainly, these processes occur in different forms and materials, including, in particular, their occurrence in single-layered and multilayered shapes [4,5]. Further, "diffusion processes through a multilayered material are of interest for a wide range of applications, including industrial, biological, electrical, and environmental areas," see Hickson *et al.* [6]. Moreover, there have been various studies with regard to the interaction of heat and wave processes in dissimilar elastic media, see the recent findings reported by Mubaraki *et al.* [7] and Althobaiti *et al.* [8] to mention a few. In light of the importance of the phenomena at hand, various methods have been devised to study the diffusional field relevance in several media. For instance, Akel *et al.* [9] employed the Laplace-typed transform to acquire a closed-form solution for diffusion problems in multilayer structures. Abro *et al.* [10] utilized the statistical method to solve the conductance and thermal resistance in conducting convection flow, while Al-Khaled and Momani [11] and Bokhari *et al.* [12] made use of the Adomian decomposition method (ADM) to treat wave-diffusion and nonlinear diffusion problems, respectively. Finally we make mention of the Wiener-Hopf method, which was used by Nuruddeen and Zaman [13] to solve a diffusion mixed problem in a cylindrical media, and on the other hand, a mixture of ADM with integral transform to tackle the class of fractional heat diffusion model [14]; see also the application of the method of separation of variables [15] in solving diverse mathematical physics problems. Besides, the study of heat transfer or diffusion in multilayered bodies has been comprehensively examined in the literature concerning its vast relevance in contemporary science and engineering applications. As the steady-state condition is attained upon reaching the equilibrium stage where time variation is immaterial, various researchers have profoundly scrutinized such a condition with the help of various analytical and approximation methods.

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Indeed, it is relevant to recall here that both Fourier's equation of heat transfer and that of the Cattaneo, which is popularly called the hyperbolic heat equation, have been used in the open literature to model a variety of heat conduction processes. Thus, for more on these studies as well as various analytical and approximation techniques, below is an up-to-date collection of various relevant methods in thermoelasticity and dynamical systems. To begin with, Eltayeb *et al.* [16] made use of the Laplace decomposition method, a famous semi-analytical method to obtain approximate closed-form solution for closed systems of Emden-Fowler equation. In addition, Islam *et al.* [17] equally deployed the same Laplace decomposition method to computationally examine the class of logistic differential equations. Yan *et al.* [18] proposed a novel series technique for diffusion equations gifted with fractional-order derivatives, while Ray and Bera [19] solved the same diffusion equations via the use of Adomian decomposition approach in the presence of fractional-order derivative in Caputo's sense. Accordingly, Bokhari *et al.* [12,20] deployed the same Adomian's approach to thermodynamic equations with local and nonlocal boundary conditions, respectively; besides, the consideration in the study by Bokhari *et al.* [12] was moved by nonlinearity and temperature-dependent material thermal property. Further, Nuruddeen and Garba [21] model the diffusion of heat amidst fractional derivative with nonlocal boundary data, while Anwar *et al.* [22] employed the double Laplace transform method on fractional heating models. In fact, double Laplace method is a sort of integral transform method, through the application of Laplace transform method twice to solve the partial differential equation under scrutiny. In addition, interested reader(s) can consult the recent work of Alotaibi and Althobaiti [23] and Althobaiti *et al.* [24] that examined some nonlinear evolution equations with the help of numerical and analytical methods, respectively, which are promising and portray agreeing solutions with the corresponding exact solutions.

Furthermore, with an emphasis on the diffusional heating transfers in multilayered, various research studies have examined the scenario extensively due to its vast application in contemporary thermodynamics. In this regard, we recall some of these studies: to start with, we make mention of the good work of Negi and Singh [25] that theoretically describes the heating process in a horizontally stratified spherical shell and slab presided over by temperature-depending heating sources with emerging application in the earth's crust. Noor and Burton [26] similarly examined the state-of-the-art of heating process in composite multilayer shells and plates; read also the study by Zhou [27] that analytically examined the transfer of heat in multilayered hollow cylindrical tubes containing stirred fluid with uniform heat sink. In addition, Yu *et al.* [28] made use of the Laplace transform method, to analytically derive the resulting diffusional fields in a multilayered conducting porous media that governs the transfer of chloride concentration fields in multiple layered systems. In the end, a comparative analysis was established between the derived analytical solution and the simulated corresponding numerical results. Further, Yuan *et al.* [29] analyzed the effect of the interfacial thermal resistance (ITR) on the diffusion of heat in multilayered materials; some application problems have been considered, where the effect of ITR was examined concerning the materials of a multiply coated substrate. Yang *et al.* [30] made use of integral (Laplace and Henkel) transform method, with a particular quadrature method to study the heat-emitting process in an axisymmetric cylindrical multilayer thermoelastic medium; moreover, an interesting real-life application was presented in geological science – see the work of Akbarzadeh and Pasini [31] that equally deployed the application of Laplace transform in the determination of the resulting temperature in phase-lag heat transfer associated with a multilayered media with sliding contact interfaces. Moreover, certain numerical procedures were equally used in the literature to determine the approximate fields for the transfer, In light of this, Kalis and Kangro [32] numerically deployed the finite difference method (FDM) and the finite element method to examine the transfer of two-dimensional (2D) heat in multilayer body, while Alaa *et al.* [33] coupled the analytical eigen-function method with the FDM to examine a 2D heating process in an inhomogeneous multilayered cylinder; read also the semi-analytical perspective of Ramadan [34] on dual-phase lag heating in perfectly conducting multilayered bodies, and the previous studies by Abbas *et al.* and Al Owidh *et al.* [35–37] for the recent findings and mathematical methods on the flow of thermodynamical and thermoelectric fluids in different settings.

However, motivated by the findings of the submissions in previous studies [25–35] concerning the heat transfer process in diverse multilayered structural shapes, the present study thus formulates and analytically examines a steady-state heat diffusion process in a highly heterogeneous multilayered circular cylinder via the application of the method of separation of variables [15]. The method of separation of variables is surely among the oldest methods for the study of linear partial differential equations that is frequently used in modern times due to its efficiency. Besides, the choice of separation of variables is associated with its easy implementation process, yet efficient against many sophisticated numerical methods or integral transform methods where their inversion procedures are computationally expensive. Another unique feature of this research is the inventive utilization of the variable separable approach, which provides a comprehensive and efficient technique for solving linear differential equations in multilayered thermodynamics. Two advantages of the current work are the construction of a generalized compact system to address a wide range of multi-heat equations and the capability to precisely characterize multiple physical processes. In addition, the successful resolution of prototype difficulties demonstrates the practical relevance and versatility of the proposed approach. To sum up, this work seeks to address the limitations of existing methods for solving numerous models in multilayered settings; its novelty stems from the inventive method proposed, which may contribute to advances in the study of thermodynamics in modern applications. Further, sufficient boundary and interfacial conditions will be imposed at the end-points of the circumferential length and in between the respective bounded layers, sequentially; this assumption or rather the imposition of perfect contact conditions between the respect layers does away with imperfection conditions, which "invalidates the continuity condition of temperature at the interface, so that a special treatment is required" [29]. What is more, the study will deeply examine certain prototype cases of the structure graphically – by numerically simulating the resulting diffusional fields – to validate the formulated generalized model. Finally, the communication is organized as follows: Section 2 portrays the model formulation. Section 3 analytically tackles the formulated model. Sections 4 and 5 give the application and the numerical results and discussion, respectively; while Section 6 outlines the concluding remarks.

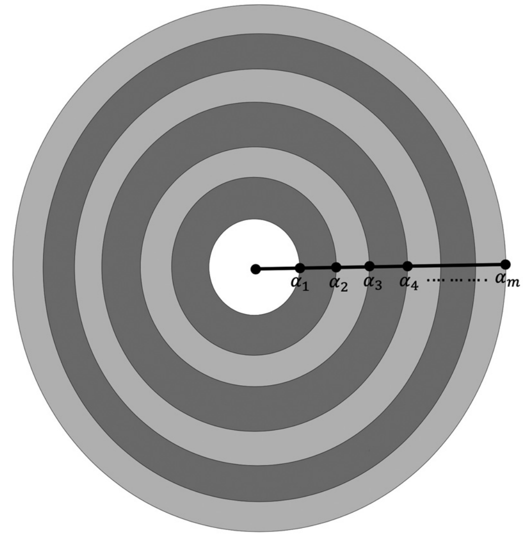

Figure 1: Multilayered heterogeneous circular cylinder.

2 Problem formulation

We make consideration to a multilayered heterogeneous circular cylinder, with constant thermal properties in all the layers and no heating source (Figure 1). Thus, the governing thermodynamic equations, featuring heat diffusion in the respective layers of the multilayered heterogeneous circular composite, are presided over by the following steady-state equations:

$$\frac{\partial^2 u_i}{\partial r^2}+\frac{1}{r}\frac{\partial u_i}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_i}{\partial \theta^2}=0, \quad i=1,2,\ldots,m, \tag{1}$$

where $u_i = u_i\ (r,\theta)$ are the diffusional fields in the respective layers of the multilayered heterogeneous cylinder for $i = 1, 2, \ldots, m$, with r as the radial variable, and θ as the azimuthal variable. Indeed, the explicit ranges for definitions of the respective circular cylinders are expressed as follows:

$$u_i \coloneqq \begin{cases} u_i(r,\theta) : r \in \begin{cases} 0 < r \le \alpha_1, & i = 1, \\ \alpha_{(i-1)} \le r \le \alpha_i, & i \ne 1, \end{cases} \quad -\pi < \theta < \pi, \\ \quad i = 1, 2, \ldots, m, \end{cases} \tag{2}$$

where α_1 is the radius of the inner most cylinder and α_i, for $i = 2, 3, \ldots, m$ are the thicknesses of the remaining cylinders.

Further, the imposition of appropriate endpoint boundary conditions is carried out on the structure. In fact, the related boundary conditions at $\theta = -\pi$ and $\theta = \pi$ are sequentially defined by periodic end conditions as follows [38]:

$$\begin{cases} u_i(r,-\pi) = u_i(r,\pi), \\ \dfrac{\partial u_i}{\partial \theta}(r,-\pi) = \dfrac{\partial u_i}{\partial \theta}(r,\pi), \end{cases} \quad i = 1, 2, \ldots, m, \tag{3}$$

which characterize equal boundary data on the two endpoints in the azimuthal direction, $\theta = -\pi$ and $\theta = \pi$ that are associated with both the diffusional fields $u_i(r,\theta)$ and heat fluxes $\frac{\partial u_i}{\partial \theta}(r,\theta)$ for $i = 1, 2, \ldots, m$. In addition, on the circumferential length, the body is further constrained to the boundary data at $r = 0$ and $r = \alpha_m$ of the following forms [38]:

$$\begin{cases} u_1(0,\theta) \ \text{is bounded}, \\ u_m(\alpha_m,\theta) = f_m(\theta), \end{cases} \tag{4}$$

where $f_m(\theta)$ is a θ-dependent diffusional field; virtually, a given nice function.

In the same vein, the respective layers of the governing multilayered heterogeneous circular cylinder are presumed to be perfectly welded. That is, perfect interfacial conditions hold among the related diffusional fields u_i, and the diffusional fluxes $\kappa_i \frac{\partial u_i}{\partial r}$, at $r = \alpha_i$ as follows [34]:

$$\begin{cases} u_i(a_i,\theta) = u_{i+1}(a_i,\theta), \\ \kappa_i \dfrac{\partial u_i}{\partial r}(a_i,\theta) = \kappa_{i+1}\dfrac{\partial u_{i+1}}{\partial r}(a_i,\theta), \end{cases} \quad i = 1, 2, \ldots, m, \tag{5}$$

where κ_i for $i = 1, 2, \ldots, m$, are the diffusivity constants in the respective cylindrical layers.

Above and beyond, such types of perfect continuity conditions at the respective interfaces, as imposed in (5), are typical to thermodynamic and elasticity problems in multilayered and composite bodies, refer to [2,3,7,8,39] for more on perfect continuity conditions in multilayered bodies, while [40,41] examined the parallel imperfect interfacial conditions.

3 Problem solution

To solve the governing problem, an analytical approach through the application of the method of separation of variables [15] is employed. Thus, the solution of Eq. (1) is considered to admit the following solution pattern:

$$u_i(r,\theta) = R_i(r)Q(\theta), \quad i = 1, 2, \ldots, m, \tag{6}$$

where $R_i(r)$ are the respective radial solutions, while $Q(\theta)$ is the corresponding azimuthal solution. Further, upon substituting the latter equation into the governing equation in Eq. (1), one obtains

$$\frac{R_i''}{R_i} + \frac{1}{r}\frac{R_i'}{R_i} + \frac{1}{r^2}\frac{Q''}{Q} = 0, \tag{7}$$

or equally,

$$-\left(r^2\frac{R_i''}{R_i} + r\frac{R_i'}{R_i}\right) = \frac{Q''}{Q} = -\lambda, \tag{8}$$

which subsequently separated to the following boundary-value problem (BVP):

$$Q'' + \lambda Q = 0, \quad Q(-\pi) = Q(\pi), \quad Q'(-\pi) = Q'(\pi), \tag{9}$$

and the following associated differential equation:

$$r^2R_i'' + rR_i' - \lambda R_i = 0, \quad i = 1, 2, \ldots, m. \tag{10}$$

More so, the azimuthal solution $Q(\theta)$ is then obtained by solving the BVP expressed in Eq. (9) as follows:

$$Q(\theta) = C_{1n}\cos(n\theta) + C_{2n}\sin(n\theta), \tag{11}$$

where C_{1n} and C_{2n} are constants to be determined later on, while the eigenvalues λ_n are found to be

$$\lambda_n = n^2, \quad n = 0, 1, 2, \ldots$$

In addition, the radial solutions $R_i(r)$ are further obtained by solving Eq. (10), which are Euler equations that admit the following solution forms:

$$R_i(r) = \begin{cases} {}^1C_3 r^n, & 0 < r \le a_1, \quad i = 1, \\ {}^iC_3 r^n + {}^iC_4 r^{-n}, & a_{(i-1)} \le r \le a_i, \quad i \ne 1, \end{cases} \tag{12}$$
$$i = 1, 2, \ldots, m,$$

where ${}^iC_3$ and ${}^iC_4$ for $i = 1, 2, \ldots, m$, are constants to be determined. In fact, ${}^iC_4$ vanishes at the inner most cylinder, that is, when $i = 1$ due to the boundedness condition at $r = 0$ that was prescribed in Eq. $(4)_1$.

Hence, the overall solution without the involvement of the interfacial and boundary conditions, which was earlier expressed in Eq. (6) is now rewritten using Eqs. (11) and (12) as follows:

$$u_i(r,\theta) = \begin{cases} \sum\limits_{n=0}^{\infty}\{{}^1a_n r^n\cos(n\theta) + {}^1b_n r^n\sin(n\theta)\}, & i = 1, \\ \sum\limits_{n=0}^{\infty}\{({}^ia_n r^n + {}^i\bar{a}_n r^{-n})\cos(n\theta) & i \ne 1, \\ \quad + ({}^ib_n r^n + {}^i\bar{b}_n r^{-n})\sin(n\theta)\}, & \end{cases} \tag{13}$$

where

$$\begin{array}{lll} C_{1n}{}^1C_3 = {}^1a_n, & C_{2n}{}^1C_3 = {}^1b_n, & C_{1n}{}^iC_3 = {}^ia_n, \\ C_{1n}{}^iC_4 = {}^i\bar{a}_n, & C_{2n}{}^iC_3 = {}^ib_n, & C_{2n}{}^iC_4 = {}^i\bar{b}_n, \end{array} \tag{14}$$

all for $i = 1, 2, \ldots, m$, are new constants to be determined little later. What is more, one obtains the resulting diffusional/heat fluxes $q_i(r,\theta)$ in the respective layers form Eq. (13) as follows:

$$q_i(r,\theta) = -\begin{cases} \sum\limits_{n=0}^{\infty}\kappa_1 n\{{}^1a_n r^{n-1}\cos(n\theta) & i = 1, \\ \quad + {}^1b_n r^{n-1}\sin(n\theta)\}, & \\ \sum\limits_{n=0}^{\infty}\kappa_i n\{({}^ia_n r^{n-1} - {}^i\bar{a}_n r^{-n-1})\cos(n\theta) & i \ne 1, \\ \quad + ({}^ib_n r^{n-1} - {}^i\bar{b}_n r^{-n-1})\sin(n\theta)\}, & \end{cases} \tag{15}$$

where κ_i for $i = 1, 2, \ldots, m$, are the diffusivity constants in the respective layers, while the diffusional/heat flux $q(r,\theta)$ is defined along the radial direction as $q(r,\theta) = -\kappa\frac{\partial u}{\partial r}(r,\theta)$.

Certainly, one might unambiguously express the aforementioned compacted solution in each layer of the multilayer body in a more comprehending form as follows:

$$u_1(r,\theta)=\sum_{n=0}^{\infty}\{{}^1a_nr^n\cos(n\theta)+{}^1b_nr^n\sin(n\theta)\},$$
$$u_2(r,\theta)=\sum_{n=0}^{\infty}\{({}^2a_nr^n+{}^2\bar{a}_nr^{-n})\cos(n\theta)+({}^2b_nr^n+{}^2\bar{b}_nr^{-n})\sin(n\theta)\},$$
$$u_3(r,\theta)=\sum_{n=0}^{\infty}\{({}^3a_nr^n+{}^3\bar{a}_nr^{-n})\cos(n\theta)+({}^3\bar{b}_nr^n+{}^3\bar{b}_nr^{-n})\sin(n\theta)\},$$
$$u_4(r,\theta)=\sum_{n=0}^{\infty}\{({}^4a_nr^n+{}^4\bar{a}_nr^{-n})\cos(n\theta)+({}^4b_nr^n+{}^4\bar{b}_nr^{-n})\sin(n\theta)\},\quad(16)$$
$$u_5(r,\theta)=\sum_{n=0}^{\infty}\{({}^5a_nr^n+{}^5\bar{a}_nr^{-n})\cos(n\theta)+({}^5b_nr^n+{}^5\bar{b}_nr^{-n})\sin(n\theta)\},$$
$$\vdots$$
$$u_m(r,\theta)=\sum_{n=0}^{\infty}\{({}^ma_nr^n+{}^m\bar{a}_nr^{-n})\cos(n\theta)+({}^mb_nr^n+{}^m\bar{b}_nr^{-n})\sin(n\theta)\}.$$

Moreover, to determine the explicit solution of the governing model, we now employ the prescribed boundary condition in Eq. $(4)_2$ and the interfacial conditions given in Eq. (5). Thus, we begin by substituting the interfacial conditions at $r=\alpha_i$ for $i=1,2,\dots,m$ into the acquired solution in Eq. (13) to obtain the following system of algebraic equations:

at $r=\alpha_1$, one obtains

$$\begin{aligned}{}^1a_n&={}^2a_n+{}^2\bar{a}_n\alpha_1^{-2n},\\ {}^1b_n&={}^2b_n+{}^2\bar{b}_n\alpha_1^{-2n},\\ \kappa_1{}^1a_n&=\kappa_2({}^2a_n-{}^2\bar{a}_n\alpha_1^{-2n}),\\ \kappa_1{}^1b_n&=\kappa_2({}^2b_n-{}^2\bar{b}_n\alpha_1^{-2n}),\end{aligned}\quad(17)$$

while at $r=\alpha_j$, $j=2,3,\dots,(m-1)$, one obtains

$$\begin{aligned}{}^ja_n+{}^j\bar{a}_n\alpha_j^{-2n}&={}^{j+1}a_n+{}^{j+1}\bar{a}_n\alpha_j^{-2n},\\ {}^jb_n+{}^j\bar{b}_n\alpha_j^{-2n}&={}^{j+1}b_n+{}^{j+1}\bar{b}_n\alpha_j^{-2n},\\ \kappa_j({}^ja_n-{}^j\bar{a}_n\alpha_j^{-2n})&=\kappa_{j+1}({}^{j+1}a_n-{}^{j+1}\bar{a}_n\alpha_j^{-2n}),\\ \kappa_j({}^jb_n-{}^j\bar{b}_n\alpha_j^{-2n})&=\kappa_{j+1}({}^{j+1}b_n-{}^{j+1}\bar{b}_n\alpha_j^{-2n}),\end{aligned}\quad(18)$$

for $j=2,3,\dots,(m-1)$.

In addition, on utilizing the second boundary condition expressed in Eq. $(4)_2$ when $r=\alpha_m$, one virtually obtains

$$f_m(\theta)=\sum_{n=0}^{\infty}\{({}^ma_n\alpha_m^n+{}^m\bar{a}_n\alpha_m^{-n})\cos(n\theta)+({}^mb_n\alpha_m^n+{}^m\bar{b}_n\alpha_m^{-n})\sin(n\theta)\},\quad(19)$$

which upon employing the application of Fourier's series [15] then yields the explict expressions for the involving coefficients as follows:

$$\begin{aligned}{}^ma_n\alpha_m^n+{}^m\bar{a}_n\alpha_m^{-n}&=\frac{1}{\pi}\int_{-\pi}^{\pi}f_m(\theta)\cos(n\theta)\mathrm{d}\theta,\\ {}^bb_n\alpha_m^n+{}^m\bar{b}_n\alpha_m^{-n}&=\frac{1}{\pi}\int_{-\pi}^{\pi}f_m(\theta)\sin(n\theta)\mathrm{d}\theta.\end{aligned}\quad(20)$$

Finally, the overall general solution is explicitly determined while solving the $(m+4)\times(m+4)$ coupled algebraic system of equations in Eqs. (17), (18), and (20) for ${}^ia_n$, ${}^ib_n$,${}^s\bar{a}_n$, and ${}^s\bar{b}_n$ when $i=1,2,\dots,m$ and $s=2,3,\dots,m$. In particular, these resulting system of algebraic equations are solved using symbolic computation via the help of *Mathematica* software in this study.

Remarkably, when the examining multilayered heterogeneous cylinder is considered to be a hollow one, that is, the inner most cylinder is vacuum layer, where fluid can be flowed in; then, the acquired solution in Eq. (13) take the following general form:

$$u_i(r,\theta)=\sum_{n=0}^{\infty}\{({}^ia_nr^n+{}^i\bar{a}_nr^{-n})\cos(n\theta)+({}^ib_nr^n+{}^i\bar{b}_nr^{-n})\sin(n\theta)\},\quad(21)$$

where $\alpha_i\le r\le\alpha_{(i+1)}$, with ${}^ia_n$, ${}^i\bar{a}_n$, ${}^ib_n$, and $\bar{b}_n$ are constants to be determined, for $i=1,2,\dots,(n+1)$. In this regard, various impositions can be made in the innermost layer, as well as, on the outermost layer, including the prescription of internal and external heating sources, such as the typical laser sources [7] among others. In addition, the two outer faces of the reduced multilayered hollow cylinder can be prescribed with different boundary conditions, including insulation conditions, imposition of specific heat fluxes, convective boundary conditions, and mixed boundary conditions among others – by mixing the aforementioned boundary conditions on the two boundaries of the structure.

4 Application

The present section makes consideration of three prototypical multilayered cylinders as an application of the governing generalized multilayered heterogeneous cylinder that was successfully examined in the aforementioned section. More precisely, we will determine the diffusional fields in the following prototypes: a single-layered cylinder, a two-layered heterogeneous cylinder, and a three-layered heterogeneous cylinder. Indeed, all the related unknown coefficients – in each of the three cases – will be explicitly determined by solving the posed system of algebraic equations (see Figure 2 for the schema of the prototype multilayered cylinders of concern).

4.1 Single-layered cylinder

Explicit solution for the resulting diffusional field with regard to a single-layered cylinder for $i = 1$ is obtained from Eq. (13) as follows:

$$u_1(r,\theta) = \sum_{n=0}^{\infty}\{{}^1a_n r^n\cos(n\theta) + {}^1b_n r^n\sin(n\theta)\}, \tag{22}$$

where the coefficients ${}^1a_n$ and ${}^1b_n$ are obtained from Eq. (20) as follows:

$$\begin{aligned} {}^1a_n &= \frac{1}{\pi\alpha_1^n}\int_{-\pi}^{\pi} f_1(\theta)\cos(n\theta)\mathrm{d}\theta,\\ {}^1b_n &= \frac{1}{\pi\alpha_1^n}\int_{-\pi}^{\pi} f_1(\theta)\sin(n\theta)\mathrm{d}\theta. \end{aligned} \tag{23}$$

4.2 Two-layered heterogeneous cylinder

Equally, the explicit expressions for the resulting diffusional fields with regard to a two-layered cylinder for $i = 2$ are obtained in the respective layers from Eq. (13) as follows:

$$\begin{aligned} u_1(r,\theta) &= \sum_{n=0}^{\infty}\{{}^1a_n r^n\cos(n\theta) + {}^1b_n r^n\sin(n\theta)\},\\ u_2(r,\theta) &= \sum_{n=0}^{\infty}\{({}^2a_n r^n + {}^2\bar{a}_n r^{-n})\cos(n\theta)\\ &\quad + ({}^2b_n r^n + {}^2\bar{b}_n r^{-n})\sin(n\theta)\}, \end{aligned} \tag{24}$$

where the involving coefficients ${}^1a_n$, ${}^1b_n$, ${}^2a_n$, ${}^2\bar{a}_n$, ${}^2b_n$, and ${}^2\bar{b}_n$ are determined from Eqs. (17), (18), and (20) as follows:

$$\begin{aligned} {}^1a_n &= 2\kappa_2\alpha_2^n v_1(n), & {}^1b_n &= 2\kappa_2\alpha_2^n v_2(n),\\ {}^2a_n &= (\kappa_1+\kappa_2)\alpha_2^n v_1(n), & {}^2b_n &= (\kappa_1+\kappa_2)\alpha_2^n v_2(n),\\ {}^2\bar{a}_n &= (\kappa_2-\kappa_1)\alpha_1^{2n}\alpha_2^n v_1(n), & {}^2\bar{b}_n &= (\kappa_2-\kappa_1)\alpha_1^{2n}\alpha_2^n v_2(n), \end{aligned}$$

where $v_1(n)$ and $v_2(n)$ appearing earlier is expressed as follows:

$$\begin{aligned} v_1(n) &= \frac{1}{\pi\zeta}\int_{-\pi}^{\pi} f_2(\theta)\cos(n\theta)\mathrm{d}\theta,\\ v_2(n) &= \frac{1}{\pi\zeta}\int_{-\pi}^{\pi} f_2(\theta)\sin(n\theta)\mathrm{d}\theta, \end{aligned} \tag{25}$$

with ζ appearing above equally expressed as follows:

$$\zeta = (\kappa_2-\kappa_1)\alpha_1^{2n} + (\kappa_1+\kappa_2)\alpha_2^{2n}. \tag{26}$$

4.3 Three-layered heterogeneous cylinder

As proceed, the explicit expressions for the resulting diffusional fields in this regard for $i = 3$ are obtained from Eq. (13) as follows:

$$\begin{aligned} u_1(r,\theta) &= \sum_{n=0}^{\infty}\{{}^1a_n r^n\cos(n\theta) + {}^1b_n r^n\sin(n\theta)\},\\ u_2(r,\theta) &= \sum_{n=0}^{\infty}\{({}^2a_n r^n + {}^2\bar{a}_n r^{-n})\cos(n\theta)\\ &\quad + ({}^2b_n r^n + {}^2\bar{b}_n r^{-n})\sin(n\theta)\},\\ u_3(r,\theta) &= \sum_{n=0}^{\infty}\{({}^3a_n r^n + {}^3\bar{a}_n r^{-n})\cos(n\theta)\\ &\quad + ({}^3\bar{b}_n r^n + {}^3\bar{b}_n r^{-n})\sin(n\theta)\}, \end{aligned} \tag{27}$$

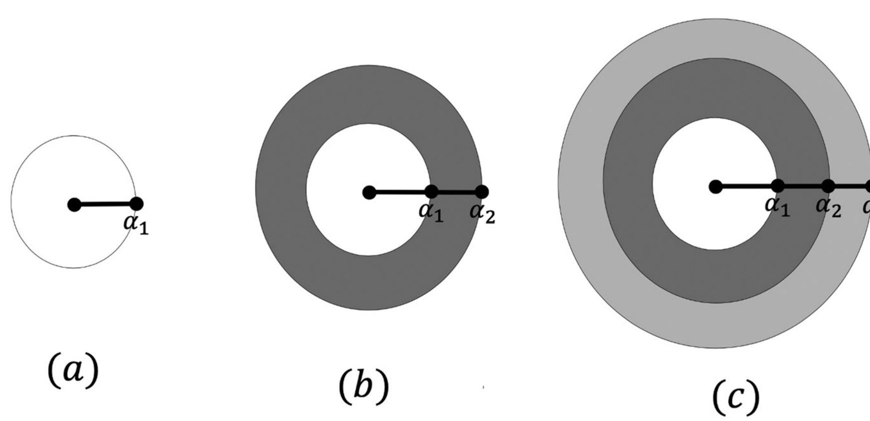

Figure 2: Prototype schema for: (a) a single-layered cylinder, (b) a two-layered heterogeneous cylinder, and (c) a three-layered heterogeneous cylinder.

where the involving coefficients ${}^{1}a_n$, ${}^{1}b_n$, ${}^{2}a_n$, ${}^{2}\bar{a}_n$, ${}^{2}b_n$, ${}^{2}\bar{b}_n$, ${}^{3}a_n$, ${}^{3}\bar{a}_n$, ${}^{3}\bar{b}_n$, and ${}^{3}\bar{b}_n$ are determined from Eqs. (17), (18), and (20) as follows:

$$
\begin{aligned}
&{}^{1}a_n = -4\kappa_2\kappa_3\alpha_2^{2n}\alpha_3^{n}w_1(n), && {}^{1}b_n = -4\kappa_2\kappa_3\alpha_2^{2n}\alpha_3^{n}w_2(n),\\
&{}^{2}a_n = -2\kappa_3(\kappa_1+\kappa_2)\alpha_2^{2n}\alpha_3^{n}w_1(n), && {}^{2}b_n = -2\kappa_3(\kappa_1+\kappa_2)\alpha_2^{2n}\alpha_3^{n}w_2(n),\\
&{}^{2}\bar{a}_n = 2\kappa_3(\kappa_1-\kappa_2)\alpha_1^{2n}\alpha_2^{2n}\alpha_3^{n}w_1(n), && {}^{2}\bar{b}_n = 2\kappa_3(\kappa_1-\kappa_2)\alpha_1^{2n}\alpha_2^{2n}\alpha_3^{n}w_2(n),\\
&{}^{3}a_n = g_1(n)w_1(n), && {}^{3}b_n = g_1(n)w_2(n),\\
&{}^{3}\bar{a}_n = g_2(n)w_1(n), && {}^{3}\bar{b}_n = g_2(n)w_2(n),
\end{aligned}
$$

where $g_1(n)$, and $g_2(n)$, appearing in the later equation, are expressed as follows:

$$
\begin{aligned}
g_1(n) &= -\alpha_3^{n}((\kappa_1-\kappa_2)(\kappa_2-\kappa_3)\alpha_1^{2n} + (\kappa_1+\kappa_2)(\kappa_2+\kappa_3)\alpha_2^{2n}),\\
g_2(n) &= \alpha_2^{2n}\alpha_3^{n}((\kappa_1-\kappa_2)(\kappa_2+\kappa_3)\alpha_1^{2n} + (\kappa_1+\kappa_2)(\kappa_2-\kappa_3)\alpha_2^{2n}),
\end{aligned} \tag{28}
$$

while $w_1(n)$ and $w_2(n)$ take the following expressions:

$$
\begin{aligned}
w_1(n) &= \frac{1}{\pi\chi}\int_{-\pi}^{\pi} f_3(\theta)\cos(n\theta)\mathrm{d}\theta,\\
w_2(n) &= \frac{1}{\pi\chi}\int_{-\pi}^{\pi} f_3(\theta)\sin(n\theta)\mathrm{d}\theta,
\end{aligned} \tag{29}
$$

with χ appearing above equally expressed as follows:

$$
\begin{aligned}
\chi = (\kappa_1-\kappa_2)\alpha_1^{2n}((\kappa_2+\kappa_3)\alpha_2^{2n} + (\kappa_3-\kappa_2)\alpha_3^{2n})\\
+ (\kappa_1+\kappa_2)\alpha_2^{2n}((\kappa_2-\kappa_3)\alpha_2^{2n} - (\kappa_2+\kappa_3)\alpha_3^{2n}).
\end{aligned} \tag{30}
$$

and three-dimensional (3D) illustrations are used in depicting the obtained diffusional Âăsolutions while fixing the infinite summation $\sum_{n=0}^{\infty}(.)$ as $\sum_{n=0}^{150}(.)$, with the prescribed field function $f_i(\theta)$ considered to be a decaying one as follows:

$$
f_i(\theta) = e^{-\alpha_i\theta}, \quad \text{for } i = 1, 2, 3. \tag{31}
$$

Comparatively, the acquired general solution for the transfer of heat in a steady-state multilayered heterogeneous circular composite is very complicated to be established numerically, as rightly asserted by Haberman [38], where the single-layered was examined – this assertion has equally been affirmed by deploying several scheme by the study. In fact, Haberman [38] made use of the effective Green's function method to acquire the solution of single-layered problem, which admits the following closed-integral solution form:

$$
u_1(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2 - r^2}{a^2 - r^2 - g(r,\theta,\tau)}f_1(\tau)\mathrm{d}\tau, \tag{32}
$$

where

$$
g(r,\theta,\tau) = 2ar\cos(\theta-\tau).
$$

The latter integral, realized in Eq. (32), is the known as the Poisson's formula. Moreover, the method of Green's function is equally not that straightforward method concerning the determination of perfect Green's function of the given problem. Therefore, with the consideration of $f_i(\theta)$ in the single layered to be $f_1(\theta) = e^{-\alpha_1\theta}$, a closed-form solution of the integral is realized with the help of *Mathematica* software to be

$$
u_1(r,\theta) = -\frac{e^{-2\pi\alpha_1}(e^{2\pi\alpha_1}-1)\left({}^{2}F_1\left(1, i\alpha_1; 1+i\alpha_1; \frac{\alpha_1 e^{-i\theta}}{r}\right) - {}^{2}F_1\left(1, i\alpha_1; 1+i\alpha_1; \frac{e^{-i\theta}r}{\alpha_1}\right)\right)}{2\pi\alpha_1}, \tag{33}
$$

which is, indeed, a complex-valued function amidst the presence of hypergeometric function ${}^{2}F_1(\cdot,\cdot;\ \cdot;\ \cdot)$, where $i = \sqrt{-1}$. Hence, the adopted separation of variable method turns out to be the best approach for solving such a generalized m-body model, taking into account all the prescribed interfacial and boundary data. Besides, one may read previous articles [25–35] for several sophisticated approaches, in addition to [31,42,43], which deployed the finite element method, numerical Laplace method, and approximated analytical procedure for the solution of thermal conduction-based model in multilayered bodies.

5 Results and discussion

This section discusses the obtained results in the above section numerically, by graphically portraying the resulting diffusional fields in the three prototype heterogeneous cylinders under consideration. Certainly, these prototypes include a single-layered cylinder, a two-layered heterogeneous cylinder, and a three-layered heterogeneous cylinder. In addition, perfect interfacial conditions were imposed in their respective interfaces, while the outer boundary face was assumed to be at a fixed decaying temperature. Such consideration of perfect interfacial condition is industrially practical as an imperfection on boundary causes a lot of drawbacks in thermodynamical systems. More so, the two-dimensional (2D)

Thus, back to the acquired solution in Section 4, without much delay, we depict in Figures 3 and 4 the graphical view of the diffusional field for a single-layered cylinder

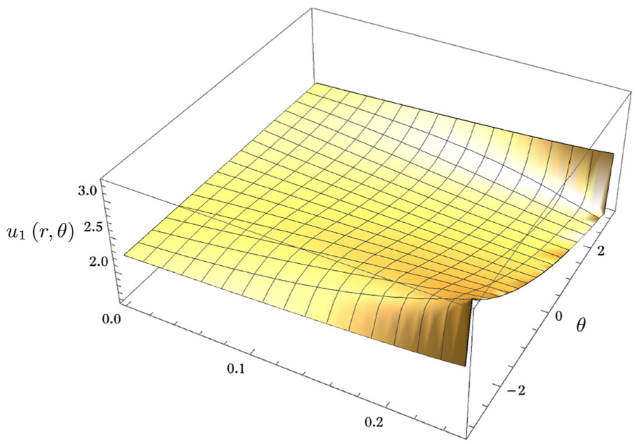

Figure 3: Graphical view of the diffusional field for a single-layered cylinder Eq. (22) portrayed using a 3D plot when $0 \le r \le 0.25$, $-\pi \le \theta \le \pi$.

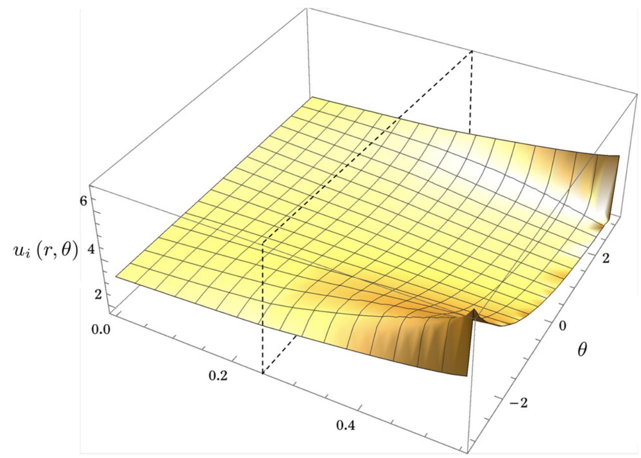

Figure 5: Graphical view of the diffusional fields for a two-layered cylinder Eq. (24) portrayed using a 3D plot when $0 \le r \le 0.25 \cup 0.25 \le r \le 0.55$, $-\pi \le \theta \le \pi$.

determined in Eq. (22) when $\alpha_1 = 0.25$ and $f_1(\theta) = e^{-\alpha_1\theta}$. Certainly, from Figure 3, the 3D view is shown when $0 \le r \le 0.25$, and $-\pi \le \theta \le \pi$; while in Figure 4, a 2D plot is shown when $0 \le r \le 0.25$, and $\theta = \pi$. Notably, the field is noted to increase along the radius of the cylinder. Further, Figures 5 and 6 give the graphical view of the diffusional fields for a two-layered cylinder Eq. (24) when $\alpha_1 = 0.25$, $\alpha_2 = 0.55$, $\kappa_1 = 0.50$, $\kappa_2 = 0.25$, and $f_2(\theta) = e^{-\alpha_2\theta}$. In fact, from Figure 5, the 3D view is shown when $0 \le r \le 0.25 \cup 0.25 \le r \le 0.55$, and $-\pi \le \theta \le \pi$; while in Figure 6, a 2D plot is shown when $0 \le r \le 0.25 \cup 0.25 \le r \le 0.55$, and $\theta = \pi$. Remarkably, the continuity conditions are noted to perfectly hold between the inner and the outer layers of the two-layered heterogamous cylinder; moreover, the respective fields increase along the thickness of the cylinder with an increase in the radius.

Finally, Figures 7 and 8 depict the graphical view of the diffusional fields for a three-layered cylinder as determined in Eq. (27) when $\alpha_1 = 0.25$, $\alpha_2 = 0.55$, $\alpha_3 = 0.80$, $\kappa_1 = 0.50$, $\kappa_2 = 0.25$, $\kappa_3 = 0.50$, and $f_3(\theta) = e^{-\alpha_3\theta}$. In fact,

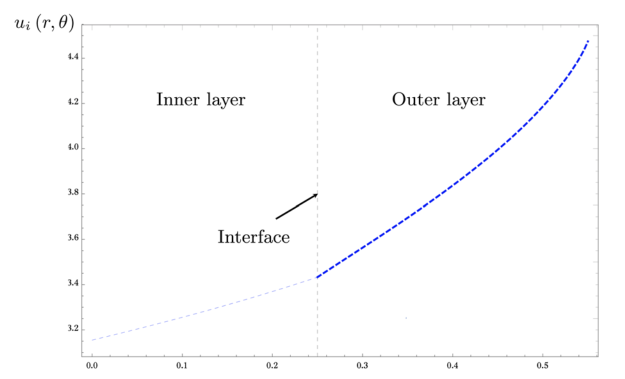

Figure 6: Graphical view of the diffusional fields for a two-layered cylinder Eq. (24) portrayed using a 2D plot when $0 \le r \le 0.25 \cup 0.25 \le r \le 0.55$, $\theta = \pi$.

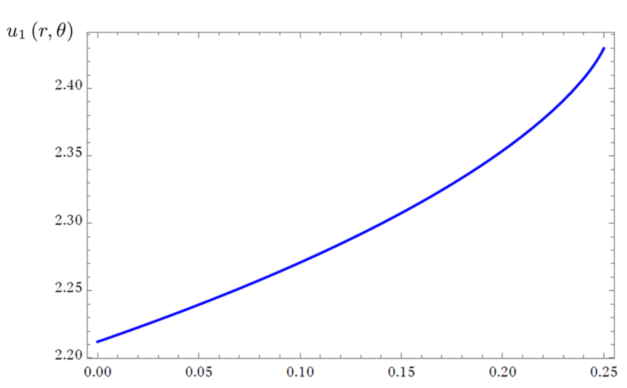

Figure 4: Graphical view of the diffusional field for a single-layered cylinder Eq. (22) portrayed using a 2D plot when $0 \le r \le 0.25$, $\theta = \pi$.

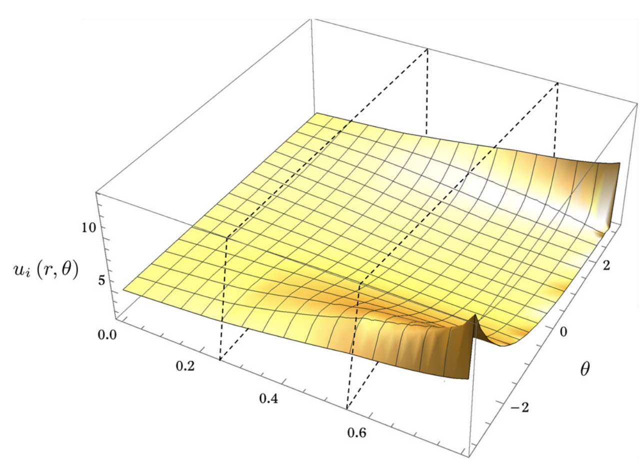

Figure 7: Graphical view of the diffusional fields for a three-layered cylinder Eq. (27) portrayed using a 3D plot when $0 \le r \le 0.25 \cup 0.25 \le r \le 0.55 \cup 0.55 \le r \le 0.80$, $-\pi \le \theta \le \pi$.

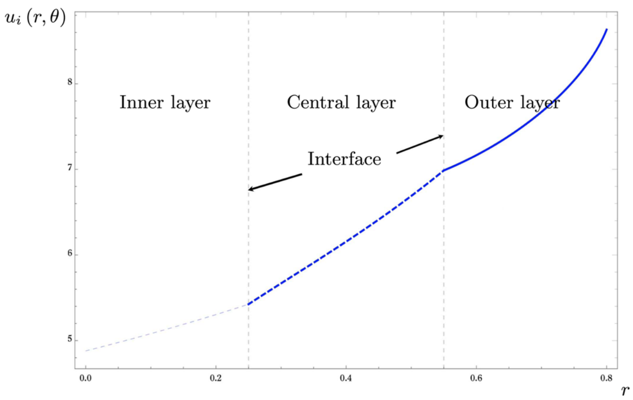

Figure 8: Graphical view of the diffusional fields for a three-layered cylinder Eq. (27) portrayed using a 2D plot when $0 \leq r \leq 0.25 \cup 0.25 \leq r \leq 0.55 \cup 0.55 \leq r \leq 0.80$, $\theta = \pi$.

from Figure 7, the 3D view is shown when $0 \leq r \leq 0.25 \cup 0.25 \leq r \leq 0.55 \cup 0.55 \leq r \leq 0.80$, and $^{-}\pi \leq \theta \leq \pi$; while in Figure 8, a 2D plot is shown when $0 \leq r \leq 0.25 \cup 0.25 \leq r \leq 0.55 \cup 0.55 \leq r \leq 0.80$, and $\theta = \pi$. Amazingly, the continuity conditions rightly hold across all the two interfaces of the multilayered cylinder; that is, the interfacial conditions between the inner cylinder and the middle cylinder, and that of the middle cylinder and that of the outer cylinder. Furthermore, the respective fields increase along the thickness of the multilayered heterogeneous cylinder. Moreover, the satisfaction of the interfacial conditions on the respective interfaces is alone an affirmation of the genuineness of the established study. In addition, the same tread is expected to hold when more layers are incorporated for strongly heterogeneous cylindrical tubes. Comparatively, most of the researchers in [25–35] tackled specific cases of heat transfer in multilayered media, unlike the current study that examined a generalized n-layer heterogeneous cylindrical tube. In addition, the study uses the method by the name separation of variables method, which is an old method, yet very powerful for the analysis of linear partial differential equations; this is indeed against the deployment of mostly integral transformation methods – see [28,30,31] – that end in numerical inversion procedures to be able to invert the solution back to the original domain; see also the application of certain computational techniques in [32–34] that result in approximate solutions. All-in-all, the present study made use of a simple analytical method to examine a perfectly conducting n-body heterogeneous tube, leading to the acquisition of the resulting diffusional fields and their corresponding fluxes, which gives a complete thermodynamical perspective of the problem at hand.

6 Concluding remarks

The present communication formulated and analytically examined a steady-state heat diffusion process in a multilayered heterogeneous cylinder. Indeed, such structural configurations arise in many real-world applications, most especially, with the current advancement in contemporary science and engineering applications. Moreover, sufficient boundary and interfacial conditions were imposed at the endpoints of the circumferential length and in between the respective bounded layers, respectively. The promising method of separation of variables was adopted as the focal approach for the solution, which then revealed a compacted solution for the governing model for $i = 1, 2, \ldots, m$. In addition, three prototypical situations consisting of a single-layered cylinder, a two-layered heterogeneous cylinder, and a three-layered heterogeneous cylinder were practically examined in deep. Amazingly, as expected, the graphically portrayed related diffusional fields in the respective prototype cases were found to fully satisfy all imposed structural assumptions. Besides, the present study could be extended to the unsteady-state scenario, in addition to its possible extension to higher-order dimensions. In addition, the present study generalized several other considerations by various researchers by considering m-body multilayered heterogeneous circular composites, against many findings in the literature that examined either two- or three-layered composites. In addition, the beseeched method is equally of paramount importance, considering its easy implementation, yet efficient as against many sophisticated numerical methods or integral transform methods where their inversion procedures are computationally expensive. Finally, this study can be used in the design and analysis of multilayered media in engineering, material science, thermodynamics, and solid mechanics among others.

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