Research Article

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Establishing breather and *N*-soliton solutions for conformable Klein–Gordon equation

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Abstract: This article develops and investigates the behavior of soliton solutions for the spatiotemporal conformable Klein–Gordon equation (CKGE), a well-known mathematical physics model that accounts for spinless pion and de-Broglie waves. To accomplish this task, we deploy an effective analytical method, namely, the modified extended direct algebraic method (mEDAM). This method first develops a nonlinear ordinary differential equation (NODE) through the use of a wave transformation. With the help of generalized Riccati NODE and balancing nonlinearity with the highest derivative term, it then assumes a finite series-form solution for the resulting NODE, from which four clusters of soliton solutions – generalized rational, trigonometric, exponential, and hyperbolic functions – are derived. Using contour and three-dimensional visuals, the behaviors of the soliton solutions – which are prominently described as dark kink, bright kink, breather, and other N-soliton waves – are examined and analyzed. These results have applications in solid-state physics, nonlinear optics, quantum field theory, and a more thorough knowledge of the dynamics of the CKGE.

Keywords: fractional partial differential equations, homogenous balancing principle, modified extended direct algebraic method, wave transformation, breather waves, generalized Riccati NODE

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1 Introduction

The prevalence of nonlinearity across the world emphasizes the necessity of creating nonlinear models, especially those that include fractional partial differential equations (FPDEs) and partial differential equations (PDEs) [1–4]. The wide range of applications in fluid dynamics, acoustics, image processing, vibration, biology, chemistry, physics, and control has attracted a lot of researchers to study nonlinear FPDEs [5–7]. Because of the great potential applications of these nonlinear FPDEs in various fields, researchers have invested a great deal of time and energy in finding analytical and numerical solutions [8–12]. Researchers have created a number of reliable and effective approaches to find and analyze exact solutions [13–17]. These methodologies encompass the first integral method [18], Laplace Adomian decomposition method [19,20], homotopy analysis method [21], natural transform decomposition method [22], modified simple equation method [23], (G'/G)-expansion method [24–26], extended direct algebraic method (mEDAM), [27,28] and numerous others.

Expanding on current methods to address nonlinear FPDEs (NFPDEs) is still a compelling and important area of research. Many NFPDEs, such as the impulsive fractional differential equations [29], the space-time fractional advection-dispersion equation [30–32], the fractional generalized Burgers' equation [33], the fractional heat and mass-transport equation [34], and others, have been studied and solved as a result of the efforts of numerous researchers. Motivated by the ongoing research on addressing NFPDEs, this study aims to address conformable Klein–Gordon equation (CKGE) with mEDAM. The fractional generalization of the Klein–Gordon equation (KGE) is known as the CKGE. It substitutes fractional order conformable derivatives for the integer ordered derivatives in KGE, a relativistic wave equation related to Schrödinger equation. In mathematical physics, particularly in relativistic quantum mechanics, the spinless pion and de-Broglie waves are well explained by the KGE, a well-known model that was first developed in 1926 by Klein and Gordon as a relativistic equation for the function of a wave of an individual particle with zero spins. Numerous scientific domains, including quantum field theory, solid-state physics, and nonlinear optics, are found in this model. The equation has shown tremendous interest in the fields of condensed matter physics, solitons in a collisionless plasma, nonlinear wave equations, and recurrence of initial states. In a mathematical model, this equation is used in many scientific fields, including quantum field theory, nonlinear optics, and solid-state physics. This model is stated as follows [31]:

$$D_t^{\delta}(D_t^{\delta}w(x,t)) - D_x^{\delta}(D_x^{\delta}w(x,t)) + \sigma w(x,t) - \rho w^2(x,t) = 0, \tag{1}$$

where $0 < \delta \leq 1$, $w(x,t)$ represents the quantum state of the particle and is a harmonic function, $\sigma = (mc)^2$, where m is the mass, c is the speed of the spinless particle, and ρ is the nonzero coefficient of nonlinear term. The last two terms describe an appropriate nonlinear function that shows the change in potential energy [35]. A series of mathematical equations, including the Landau-Ginzburg-Higgs, PHI-4 equations, Duffing, and sine-Gordon, can be obtained by choosing various representations of this nonlinear function. These equations may be employed to simulate a range of physical events, from wave propagation to quantum mechanical phenomena [36]. Renowned scientists such as Klein, Schrödinger, and Fock were crucial for discovering the connection among the general relativity theories with certain KGE's earlier renditions, in keeping with Kragh's account of the KGE's roots in the study by Kragh [37]. Some of the most significant equations is this one, which serves as the basis for the officially accepted nonlinear Schrödinger equation [38]. Galehouse [39] used the proper gauge changes to derive the KGE spatially. For the more generalized form of the KGE, Schechter also provides arguments in favor of the theory of scattering hypothesis [40]. The traditional method of generating a comparable equation in a finitely normed Hilbert space containing the first-order time is followed to create the analog theory of the KGE [41]. In this work, Weder also provided justification for the existence and coherence of the wave operators, the invariant principle, and the interconnected links. Strong findings, reminiscent of those given in the Schrödinger instance, can be obtained by adopting similar eigenfunction extensions to a specific field condition to broaden the spectral and dispersion theory of the KGE [42]. Tsukanov examines the mobility of a KGE in an alternative field [43].

The fractional derivative operators D_t^{δ} and D_t^{δ} in (1) are conformable fractional derivatives (CFDs), which are defined in the upcoming section. CFDs are a helpful tool for explaining the physical aspects of events that display noninteger behaviors, including memory-related events that also exhibit nonlocal as well as local activity. In mathematics, traditional derivatives stand for limited behavior, where a place's change in magnitude is only dependent on the environment around it. Nevertheless, a lot of real-world systems exhibit nonlocal behavior, in which the system's past or remote locations affect its present state at any particular moment. Fractional derivatives, or CFDs, are an extension of calculus used to explain these kinds of circumstances. Fractional interactions, for example, are properties of elastic components within the domain of substances that ascertain stresses or tension at a region by virtue of the behavior of the material during deformation and its current environment. This memory effect is accurately captured by recording sessions, which take into account the actions of the entire system throughout an uninterrupted spectrum of historical data. Furthermore, nonlocal structures are frequently observed in the diffusion processes that molecules go through. This causes problems with fractional diffusing because the molecules' dispersion over a nonlocal region determines the degree of intensity fluctuations at a certain site. To accurately simulate these kinds of occurrences, CFDs offer a mathematical tool that bridges the gap between both local and nonlocal behavior and takes into consideration memory-linked features that are essential to the system's operation.

Before this research, several researchers have already used various analytical and numerical methods to investigate this equation in integer and fractional orders. For instance, the homotopy perturbation method was employed by Golmankhaneh *et al.* [44] to study fractional (FKGE). Khan *et al.* in [25] utilized the (G'/G)-expansion method to construct families of travelling wave solutions for FKGE. Gepreel and Mohamed recently established the analytical approximate solution of the FKGE [45]. By using the homotopy analysis method, Jafari *et al.* [46] investigated the Cahn-Hilliard and KGE using the fractional sub-equation method. Moreover, Ran and Zhang used the compact difference method for the space FKGE in [47]. The modified extended tanh-method [48], the (w/g)-expansion [49], the Riccati expansion [50–52], and other techniques [53] are also used to look for exact and numerical solutions to the FKGE.

Nevertheless, the main goal of this study is to construct and examine new families of soliton solutions for CKGE using the enhanced mEDAM. Applications of these findings include quantum field theory, solid-state physics, nonlinear optics, and a deeper understanding of the dynamics of the CKGE. The proposed mEDAM is one of the most straightforward, important, and efficient algebraic procedures. Under the application of wave transformation and assumption of a series form solution, the strategic mEDAM transforms the CKGE into a system of nonlinear algebraic equations. Numerous soliton solutions in the form of generalized rational, trigonometric, exponential, and hyperbolic functions are produced when the resulting problem is solved using the Maple tool. From an academic perspective, soliton solutions for NFPDEs remain significant because they offer more depth and granularity than conventional solutions [54–57]. They are valuable in many technical and scientific fields due to their inherent stability and longevity. They provide effective information transfer and extensive concordance retention for nonlinear systems. To put it succinctly, the new findings of this study demonstrate the inventive character of our research by offering a unique and methodical discovery of numerous new soliton solutions. Soliton research is interested in studying nonlinear FPDEs that occur in quantum field theory, fluid dynamics, and optics. These soliton solutions offer a more profound understanding of the fundamental phenomena of CKGE in the associated scientific domains.

The rest of the article is organized as follows: Section 2 describes the CFD and proposed method's methodology, Section 3 presents soliton solutions for the targeted biological population models, while Section 4 presents graphs of some soliton solutions and a discussion of our findings. Finally, a brief conclusion has been provided.

2 Methodology and materials

2.1 The definition of CFD

Several fractional derivative operators such as the Caputo operator, Riemann-Liouville operator, Atangana-Baleanu operator, Caputo-Fabrizio operator, CFD operator, and many more have been introduced by different mathematicians in literature [58–68]. Among these fractional derivative operators, the CFD operator is preferred by academics due to its predominant applications over other derivative operators. For instance, employing Khalil's CFD and a matrix of Hessian, Lavín-Delgado *et al.* presented an innovative edge recognition technique that lowers noise and maintains image outlines regardless of dull contrasting situations [69]. Their method improved clarity of vision in edge identification and has potential uses in health care imaging for recognizing diseases, including cervical cancer along with medial cranial arterial ruptured arteries, hence enhancing clinical surveillance and the precision of diagnosis. One another advantage of CFD is that, unlike other derivative operators, it satisfies all derivative properties, particularly the chain rule, which is essential for our method for creating soliton solutions. By taking advantage of these beneficial properties of CFDs over alternative fractional derivative operators, explicit solutions for FPDEs can be derived. Notably, the soliton solutions of Eq. (1) cannot be obtained using alternative fractional derivative formulations because they violate the chain rule [70,71]. As a result, CFDs were introduced into Eq. (1). The study by Sarikaya *et al.* [72] defines the CFD operator of order α as follows:

$$D_{\psi}^{\alpha}w(\psi) = \lim_{\gamma\to 0}\frac{w(\gamma\psi^{1-\alpha}+\psi)-w(\psi)}{\gamma},\quad \alpha\in(0,1]. \tag{2}$$

In this investigation, the following properties of this derivative are utilized:

$$D_{\psi}^{\alpha}\psi^{r} = r\psi^{r-\alpha}, \tag{3}$$

$$D_{\psi}^{\alpha}(r_1\varpi(\psi)\pm r_2\pi(\psi)) = r_1D_{\psi}^{\alpha}(\varpi(\psi))\pm r_2D_{\psi}^{\alpha}(\pi(\psi)), \tag{4}$$

$$D_{\psi}^{\alpha}\chi[\tau(\psi)] = \chi_{\tau}'(\tau(\psi))D_{\psi}^{\alpha}\tau(\psi), \tag{5}$$

where $\varpi(\psi)$, $\pi(\psi)$, $\chi(\psi)$, and $\tau(\psi)$ are arbitrary differentiable functions, whereas r, r_1, and r_2 signify constants.

2.2 The working procedure of mEDAM

The mEDAM is an efficient technique, which is utilized by many researchers to construct travelling wave and soliton solutions for nonlinear FPDEs [73]. The operational procedure of this proposed method is explained in this section [54]. Examine the FPDE of the structure:

$$P(w,\partial_t^{\alpha}w,\partial_{z_1}^{\beta}w,\partial_{z_2}^{\gamma}w,w\partial_{z_1}^{\beta}w,\ldots)=0,\quad 0<\alpha,\beta,\gamma\le 1, \tag{6}$$

where t, $z_1, z_2, z_3, \ldots, z_r$, and w are functions. The steps to solve Eq. (6) are as follows:

1) The variable transformation of the form $w(z_1, z_2, z_3, \ldots, z_r) = W(\psi)$, where ψ is defined in a number of ways, is first applied. This transformation converts (6) into a nonlinear ODE with the following form:

$$Q(W, W'W, W', \ldots) = 0, \tag{7}$$

where W in (7) has derivatives concerning ψ. To find the integration constant(s), (7) may be integrated one or more times.

2) Next, we assume that (7) has the following solution:

$$W(\psi) = \sum_{l=-\varsigma}^{\varsigma} d_l(Z(\psi))^l, \tag{8}$$

where $d_l(l=-\varsigma,\ldots,0,\ldots,\varsigma)$ are constants to be determined, and $Z(\psi)$ is the general solution of the following ODE:

$$Z'(\psi) = \ln(\Omega)(\eta + \psi Z(\psi) + \nu(Z(\psi))^2), \tag{9}$$

where $\Omega \neq 0, 1$ and the constants η, μ, and ν are used.

3) The positive integer ς presented in (8) is obtained by establishing the homogeneous balance between the

biggest nonlinear term and the highest order derivative in (7).

4) Next, we integrate (7) to obtain the equation that results from (8), and we put all of the terms of $Z(\psi)$ in the same order. Then, all of the coefficients of the ensuing polynomial are set to zero, yielding an algebraic system of equations for $d_l (l = -\varsigma, \ldots, 0, \ldots, \varsigma)$ and extra parameters.
5) We utilize MAPLE to solve this set of algebraic problems.
6) After figuring out the unknown values, they are added to (10) along with the $Z(\psi)$ (solution of Eq. (9)) to yield the analytical soliton solutions to (6). The following families of solutions can be generated by using the general solution of (9).

Family 1: When $M < 0$ and $\nu \neq 0$:

$$Z_1(\psi) = -\frac{\mu}{2\nu} + \frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{2\nu},$$

$$Z_2(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{2\nu},$$

$$Z_3(\psi) = -\frac{\mu}{2\nu} + \frac{\sqrt{-M}(\tan_\Omega(\sqrt{-M}\psi) \pm (\sec_\Omega(\sqrt{-M}\psi)))}{2\nu},$$

$$Z_4(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{-M}(\cot_\Omega(\sqrt{-M}\psi) \pm (\csc_\Omega(\sqrt{-M}\psi)))}{2\nu},$$

and

$$Z_5(\psi) = -\frac{\mu}{2\nu} + \frac{\sqrt{-M}\left(\tan_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right) - \cot_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)\right)}{4\nu}.$$

Family 2: When $M > 0$ and $\nu \neq 0$:

$$Z_6(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{2}\sqrt{M}\psi\right)}{2\nu},$$

$$Z_7(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{M}\coth_\Omega\left(\frac{1}{2}\sqrt{M}\psi\right)}{2\nu},$$

$$Z_8(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{M}(\tanh_\Omega(\sqrt{M}\psi) \pm i(\mathrm{sech}_\Omega(\sqrt{M}\psi)))}{2\nu},$$

$$Z_9(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{M}(\coth_\Omega(\sqrt{M}\psi) \pm (\mathrm{csch}_\Omega(\sqrt{M}\psi)))}{2\nu},$$

and

$$Z_{10}(\psi) = -\frac{\mu}{2\nu} - \frac{\sqrt{M}\left(\tanh_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right)\right)}{4\nu}.$$

Family 3: When $\eta\nu > 0$ and $\mu = 0$:

$$Z_{11}(\psi) = \sqrt{\frac{\eta}{\nu}}\tan_\Omega(\sqrt{\eta\nu}\psi),$$

$$Z_{12}(\psi) = -\sqrt{\frac{\eta}{\nu}}\cot_\Omega(\sqrt{\eta\nu}\psi),$$

$$Z_{13}(\psi) = \sqrt{\frac{\eta}{\nu}}(\tan_\Omega(2\sqrt{\eta\nu}\psi) \pm (\sec_\Omega(2\sqrt{\eta\nu}\psi))),$$

$$Z_{14}(\psi) = -\sqrt{\frac{\eta}{\nu}}(\cot_\Omega(2\sqrt{\eta\nu}\psi) \pm (\csc_\Omega(2\sqrt{\eta\nu}\psi))),$$

and

$$Z_{15}(\psi) = \frac{1}{2}\sqrt{\frac{\eta}{\nu}}\left(\tan_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\psi\right) - \cot_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\psi\right)\right).$$

Family 4: When $\eta\nu > 0$ and $\mu = 0$, then

$$Z_{16}(\psi) = -\sqrt{-\frac{\eta}{\nu}}\tanh_\Omega(\sqrt{-\eta\nu}\psi),$$

$$Z_{17}(\psi) = -\sqrt{-\frac{\eta}{\nu}}\coth_\Omega(\sqrt{-\eta\nu}\psi),$$

$$Z_{18}(\psi) = -\sqrt{-\frac{\eta}{\nu}}(\tanh_\Omega(2\sqrt{-\eta\nu}\psi) \pm (i\mathrm{sech}_\Omega(2\sqrt{-\eta\nu}\psi))),$$

$$Z_{19}(\psi) = -\sqrt{-\frac{\eta}{\nu}}(\coth_\Omega(2\sqrt{-\eta\nu}\psi) \pm (\mathrm{csch}_\Omega(2\sqrt{-\eta\nu}\psi))),$$

and

$$Z_{20}(\psi) = -\frac{1}{2}\sqrt{-\frac{\eta}{\nu}}\left(\tanh_\Omega\left(\frac{1}{2}\sqrt{-\eta\nu}\psi\right) + \coth_\Omega\left(\frac{1}{2}\sqrt{-\eta\nu}\psi\right)\right).$$

Family 5: When $\nu = \eta$ and $\mu = 0$:

$$Z_{21}(\psi) = \tan_\Omega(\eta\psi),$$

$$Z_{22}(\psi) = -\cot_\Omega(\eta\psi),$$

$$Z_{23}(\psi) = \tan_\Omega(2\eta\psi) \pm (\sec_\Omega(2\eta\psi)),$$

$$Z_{24}(\psi) = -\cot_\Omega(2\eta\psi) \pm (\csc_\Omega(2\eta\psi)),$$

and

$$Z_{25}(\psi) = \frac{1}{2}\tan_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\cot_\Omega\left(\frac{1}{2}\eta\psi\right),$$

Family 6: When $\nu = -\eta$ and $\mu = 0$:

$$Z_{26}(\psi) = -\tanh_\Omega(\eta\psi),$$

$$Z_{27}(\psi) = -\coth_\Omega(\eta\psi),$$

$$Z_{28}(\psi) = -\tanh_\Omega(\eta a\psi) \pm (i\mathrm{sech}_\Omega(2\eta\psi)),$$

$$Z_{29}(\psi) = -\coth_\Omega(2\eta\psi) \pm (\mathrm{csch}_\Omega(2\eta\psi)),$$

and

$$Z_{30}(\psi) = -\frac{1}{2}\tanh_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\coth_\Omega\left(\frac{1}{2}\eta\psi\right).$$

Family 7: When $M = 0$:

$$Z_{31}(\psi) = -2\frac{\eta(\mu\psi\ln(\Omega) + 2)}{\mu^2\psi\ln(\Omega)}.$$

Family 8: When $\mu = \tau$, $\eta = n\tau(n \neq 0)$ and $\nu = 0$:

$$Z_{32}(\psi) = \Omega^{\tau\psi} - n.$$

Family 9: When $\mu = \nu = 0$:

$$Z_{33}(\psi) = \eta\psi\ln(\Omega).$$

Family 10: When $\mu = \eta = 0$:

$$Z_{34}(\psi) = -\frac{1}{\nu\psi\ln(\Omega)}.$$

Family 11: When $\eta = 0$, $\mu \neq 0$ and $\nu \neq 0$:

$$Z_{35}(\psi) = -\frac{\mu}{\nu(\cosh_\Omega(\mu\psi) - \sinh_\Omega(\mu\psi) + 1)}$$

and

$$Z_{36}(\psi) = -\frac{\mu(\cosh_\Omega(\mu\psi) + \sinh_\Omega(\mu\psi))}{\nu(\cosh_\Omega(\mu\psi) + \sinh_\Omega(\mu\psi) + 1)}.$$

Family 12: When $\mu = \tau$, $\nu = n\tau(n \neq 0)$, and $\eta = 0$:

$$Z_{37}(\psi) = \frac{\Omega^{\tau\psi}}{1 - n\Omega^{\tau\psi}},$$

where $M = \mu^2 - 4\eta\nu$, while the generalized hyperbolic and trigonometric functions are described as follows:

$$\cos_\Omega(\psi) = \frac{\Omega^{i\psi} + \Omega^{-i\psi}}{2}, \quad \sin_\Omega(\psi) = \frac{\Omega^{i\psi} - \Omega^{-i\psi}}{2i},$$
$$\cot_\Omega(\psi) = \frac{\cos_\Omega(\psi)}{\sin_\Omega(\psi)}, \quad \tan_\Omega(\psi) = \frac{\sin_\Omega(\psi)}{\cos_\Omega(\psi)},$$
$$\csc_\Omega(\psi) = \frac{1}{\sin_\Omega(\psi)}, \quad \sec_\Omega(\psi) = \frac{1}{\cos_\Omega(\psi)}.$$

Similarly,

$$\cosh_\Omega(\psi) = \frac{\Omega^{\psi} + \Omega^{-\psi}}{2}, \quad \sinh_\Omega(\psi) = \frac{\Omega^{\psi} - \Omega^{-\psi}}{2},$$
$$\coth_\Omega(\psi) = \frac{\cosh_\Omega(\psi)}{\sinh_\Omega(\psi)}, \quad \tanh_\Omega(\psi) = \frac{\sinh_\Omega(\psi)}{\cosh_\Omega(\psi)},$$
$$\mathrm{csch}_\Omega(\psi) = \frac{1}{\sinh_\Omega(\psi)}, \quad \mathrm{sech}_\Omega(\psi) = \frac{1}{\cosh_\Omega(\psi)}.$$

3 Execution of the mEDAM

In the present section, we use the suggested method mEDAM for constructing soliton solutions for CKGE stated in (1). We offer the subsequent transformation so that this approach may be expanded to solve (1):

$$w(x,t) = W(\psi); \quad \psi = \beta\frac{x^\delta}{\delta} - \alpha\frac{t^\delta}{\delta}, \tag{10}$$

where α and β stands for nonzero constants. When this transformation is applied, Eq. (1) converts to the following ordinary equation:

$$(\alpha^2 - \beta^2)W'' + \sigma W - \rho W^2 = 0. \tag{11}$$

Establishing the homogenous balance principle between W'' and W^2, we obtain $\varsigma = 2$. By substituting $\varsigma = 2$ in (8), we obtain the following series form solution for (11):

$$W(\psi) = \sum_{l=-2}^{2} d_l(Z(\psi))^l. \tag{12}$$

With the help of (9), we insert (12) into (11) which create a polynomial in $Z(\psi)$ by collecting the terms with same power of $Z(\psi)$. By equating the coefficients of the polynomial to zero, we obtain a system of nonlinear algebraic equations. We use Maple to solve the system and reach at the following four distinct cases of solutions:

Case 1:

$$\begin{gathered}\alpha = \frac{\sqrt{\sigma - \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{-M}}, \quad \beta = \beta, d_{-2} = -6\frac{\eta^2\sigma}{M\rho}, \\ d_{-1} = -6\frac{\eta\mu\sigma}{M\rho}, \quad d_0 = -6\frac{\sigma\eta\nu}{M\rho}, \quad d_1 = 0, \quad d_2 = 0.\end{gathered} \tag{13}$$

Case 2:

$$\begin{gathered}\alpha = \frac{\sqrt{\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}}, \quad \beta = \beta, \\ d_{-2} = 6\frac{\eta^2\sigma}{M\rho}, d_{-1} = 6\frac{\eta\mu\sigma}{M\rho}, \quad d_0 = \frac{\sigma(2\nu\eta + \mu^2)}{M\rho}, \\ d_1 = 0, \quad d_2 = 0.\end{gathered} \tag{14}$$

Case 3:

$$\begin{gathered}\alpha = \frac{\sqrt{-\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}}, \quad \beta = \beta, \\ d_{-2} = 0, \quad d_{-1} = 0, \quad d_0 = -6\frac{\sigma\eta\nu}{M\rho}, \quad d_1 = -6\frac{\mu\nu\sigma}{M\rho}, \\ d_2 = -6\frac{\nu^2\sigma}{M\rho}.\end{gathered} \tag{15}$$

Case 4:

$$\begin{gathered}\alpha = \frac{\sqrt{\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}}, \quad \beta = \beta, \\ d_{-2} = 0, \quad d_{-1} = 0, \quad d_0 = \frac{\sigma(2\nu\eta + \mu^2)}{M\rho}, \\ d_1 = 6\frac{\mu\nu\sigma}{M\rho}, \quad d_2 = 6\frac{\nu^2\sigma}{M\rho}.\end{gathered} \tag{16}$$

Assuming case 1, we obtain the following families of soliton solutions for (1):

Family 1.1: When $M < 0$ and $\nu \neq 0$:

$$w_{1,1}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{\mu}{2\nu} + \frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{2\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{\mu}{2\nu} + \frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{2\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{17}$$

$$w_{1,2}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{18}$$

$$w_{1,3}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_{\Omega}(\sqrt{-M}\psi) \pm (\sec_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_{\Omega}(\sqrt{-M}\psi) \pm (\sec_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{19}$$

$$w_{1,4}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\cot_{\Omega}(\sqrt{-M}\psi) \pm (\csc_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\cot_{\Omega}(\sqrt{-M}\psi) \pm (\csc_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{20}$$

and

$$w_{1,5}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{4}\frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right) - \cot_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{4}\frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right) - \cot_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)} - 6\frac{\sigma\nu\eta}{M\rho}. \tag{21}$$

Family 1.2: When $M > 0$, and $\nu \neq 0$:

$$w_{1,6}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{22}$$

$$w_{1,7}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\coth_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\coth_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{23}$$

$$w_{1,8}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{M}\tanh_{\Omega}(\sqrt{-M}\psi) \pm (\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{Z}\tanh_{\Omega}(\sqrt{-M}\psi) \pm (\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{24}$$

$$w_{1,9}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{M}\coth_{\Omega}(\sqrt{-M}\psi) \pm (\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tanh_{\Omega}(\sqrt{-M}\psi) \pm (\mathrm{csch}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{25}$$

and

$$w_{1,10}(x,t) = \frac{-6\eta^2\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)^2} - \frac{6\eta\mu\sigma}{M\rho\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)} - 6\frac{\sigma\eta\nu}{M\rho}. \tag{26}$$

Family 1.3: When $\eta\nu > 0$ and $\mu = 0$:

$$w_{1,11}(x,t) = -6\frac{\sigma\eta\nu}{(\tan_\Omega(\sqrt{\eta\nu}\,\psi))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho} \tag{27}$$

$$w_{1,12}(x,t) = -6\frac{\sigma\eta\nu}{(\cot_\Omega(\sqrt{\eta\nu}\,\psi))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{28}$$

$$w_{1,13}(x,t) = -6\frac{\sigma\eta\nu}{(\tan_\Omega(2\sqrt{\eta\nu}\,\psi) \pm (\sec_\Omega(2\sqrt{\eta\nu}\,\psi)))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{29}$$

$$w_{1,14}(x,t) = -6\frac{\sigma\eta\nu^2}{(\cot_\Omega(2\sqrt{\eta\nu}\,\psi) \pm (\csc_\Omega(2\sqrt{\eta\nu}\,\psi)))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{30}$$

and

$$w_{1,15}(x,t) = -24\frac{\sigma\eta\nu}{\left(\tan_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\,\psi\right) - \cot_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\,\psi\right)\right)^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}. \tag{31}$$

Family 1.4: When $\eta\nu < 0$ and $\mu = 0$;

$$w_{1,16}(x,t) = 6\frac{\sigma\eta\nu}{(\tanh_\Omega(\sqrt{-\eta\nu}\,\psi))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{32}$$

$$w_{1,17}(x,t) = 6\frac{\sigma\eta\nu}{(\coth_\Omega(\sqrt{-\nu\eta}\,\psi))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{33}$$

$$w_{1,18}(x,t) = 6\frac{\sigma\eta\nu}{(\tanh_\Omega(2\sqrt{-\eta\nu}\,\psi) \pm (i\,\mathrm{sech}_\Omega(2\sqrt{-\eta\nu}\,\psi)))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{34}$$

$$w_{1,19}(x,t) = 6\frac{\sigma\eta\nu}{(\tanh_\Omega(2\sqrt{-\eta\nu}\,\psi) \pm (i\,\mathrm{sech}_\Omega(2\sqrt{-\eta\nu}\,\psi)))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}, \tag{35}$$

and

$$w_{1,20}(x,t) = 6\frac{\sigma\eta\nu}{(\coth_\Omega(2\sqrt{-\eta\nu}\,\psi) \pm (i\,\mathrm{csch}_\Omega(2\sqrt{-\eta\nu}\,\psi)))^2 M\rho} - 6\frac{\sigma\eta\nu}{M\rho}. \tag{36}$$

Family 1.5: When $\nu = \eta$ and $\mu = 0$;

$$w_{1,21}(x,t) = -6\frac{\eta^2\sigma}{M\rho(\tan_\Omega(\eta\psi))^2} - 6\frac{\eta^2\sigma}{M\rho}, \tag{37}$$

$$w_{1,22}(x,t) = -6\frac{\eta^2\sigma}{M\rho(\cot_\Omega(\eta\psi))^2} - 6\frac{\eta^2\sigma}{M\rho}, \tag{38}$$

$$w_{1,23}(x,t) = -6\frac{\eta^2\sigma}{M\rho(\tan_\Omega(2\eta\psi) \pm (\sec_\Omega(2\eta\psi)))^2} - 6\frac{\eta^2\sigma}{M\rho}, \tag{39}$$

$$w_{1,24}(x,t) = -6\frac{\eta^2\sigma}{M\rho(-\cot_\Omega(2\eta\psi) \pm (\csc_\Omega(2\eta\psi)))^2} - 6\frac{\eta^2\sigma}{M\rho}, \tag{40}$$

and

$$w_{1,25}(x,t) = -6\frac{\eta^2\sigma}{M\rho\left(\frac{1}{2}\tan_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\cot_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2} - 6\frac{\eta^2\sigma}{M\rho}. \tag{41}$$

Family 1.6: When $\nu = -\eta$ and $\mu = 0$:

$$w_{1,26}(x,t) = -6\frac{\eta^2\sigma}{M\rho(\tanh_\Omega(\eta\psi))^2} + 6\frac{\eta^2\sigma}{M\rho} \tag{42}$$

$$w_{1,27}(x,t) = -6\frac{\eta^2\sigma}{M\rho(\coth_\Omega(\eta\psi))^2} + 6\frac{\eta^2\sigma}{M\rho}, \tag{43}$$

$$w_{1,28}(x,t) = -6\frac{\eta^2\sigma}{M\rho(-\tanh_\Omega(2\eta\psi) \pm (i\,\mathrm{sech}_\Omega(2\eta\psi)))^2} + 6\frac{\eta^2\sigma}{M\rho}, \tag{44}$$

$$w_{1,29}(x,t) = -6\frac{\eta^2\sigma}{M\rho(-\coth_\Omega(2\eta\psi) \pm (i\,\mathrm{cech}(2\eta\psi)))^2} + 6\frac{\eta^2\sigma}{M\rho}, \tag{45}$$

and

$$w_{1,30}(x,t) = -6\frac{\eta^2\sigma}{M\rho\left(-\frac{1}{2}\tanh_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\coth_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2} + 6\frac{\eta^2\sigma}{M\rho}. \tag{46}$$

Family 1.7: When $\mu = \tau, \eta = n\tau (n \neq 0)$ and $\nu = 0$:

$$w_{1,31}(x,t) = -6\frac{n\sigma\Omega^{\tau\psi}}{\rho(\Omega^{\tau\psi} - n)^2}, \tag{47}$$

where $\psi = \beta\frac{x^\delta}{\delta} - \left(\frac{\sqrt{\sigma - \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{-M}}\right)\frac{t^\delta}{\delta}$.

Assuming case 2, we obtain the following families of soliton solutions for (1):

Family 2.1: When $M < 0 \quad \nu \neq 0$:

$$w_{2,1}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{48}$$

$$w_{2,2}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{49}$$

$$w_{2,3}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi)\pm(\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi)\pm(\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{50}$$

$$w_{2,4}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi)\pm(\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi)\pm(\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{51}$$

and

$$w_{2,5}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{4}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)-\cot_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{4}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)-\cot_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}. \tag{52}$$

Family 2.2: When $M > 0 \quad \nu \neq 0$:

$$w_{2,6}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{53}$$

$$w_{2,7}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\coth_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\coth_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{54}$$

$$w_{2,8}(x,t)=\frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{M}\tanh_\Omega(\sqrt{-M}\psi)\pm(\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-2}+\frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{Z}\tanh_\Omega(\sqrt{-M}\psi)\pm(\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-1}+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{55}$$

$$w_{2,9}(x,t) = \frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{M}\coth_\Omega(\sqrt{-M}\psi) \pm (\text{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-2} + \frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\frac{\sqrt{-M}\tanh_\Omega(\sqrt{-M}\psi) \pm (\text{csch}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^{-1} + \frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \tag{56}$$

and

$$w_{2,10}(x,t) = \frac{6\eta^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)^{-2} + \frac{6\eta\mu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)^{-1} + \frac{\sigma(\mu^2+2\nu\eta)}{M\rho}. \tag{57}$$

Family 2.3: When $\eta\nu > 0$ and $\mu = 0$:

$$w_{2,11}(x,t) = \frac{6\eta\nu\sigma}{(\tan_\Omega(\sqrt{\nu\eta}\psi))^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}, \tag{58}$$

$$w_{2,12}(x,t) = \frac{6\eta\nu\sigma}{(\cot_\Omega(\sqrt{\nu\eta}\psi))^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}, \tag{59}$$

$$w_{2,13}(x,t) = \frac{6\eta\nu\sigma}{M\rho(\tan_\Omega(2\sqrt{\nu\eta}\psi) \pm (\sec_\Omega(2\sqrt{\nu\eta}\psi)))^2} + \frac{2\sigma\nu\eta}{M\rho}, \tag{60}$$

$$w_{2,14}(x,t) = \frac{6\eta\nu\sigma}{M\rho(-\cot_\Omega(2\sqrt{\nu\eta}\psi) \pm (\csc_\Omega(2\sqrt{\nu\eta}\psi)))^2} + \frac{2\sigma\nu\eta}{M\rho}, \tag{61}$$

and

$$w_{2,15}(x,t) = \frac{6\eta\nu\sigma}{M\rho\left(\frac{1}{2}\tan_\Omega\left(\frac{1}{2}\sqrt{\nu\eta}\psi\right) - \cot_\Omega\left(\frac{1}{2}\sqrt{\nu\eta}\psi\right)\right)^2} + \frac{2\sigma\nu\eta}{M\rho}. \tag{62}$$

Family 2.4: When $\eta\nu < 0$ and $\mu = 0$:

$$w_{2,16}(x,t) = \frac{6\eta\nu\sigma}{(\tanh_\Omega(\sqrt{-\nu\eta}\psi))^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}, \tag{63}$$

$$w_{2,17}(x,t) = \frac{6\eta\nu\sigma}{(\coth_\Omega(\sqrt{-\nu\eta}\psi))^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}, \tag{64}$$

$$w_{2,18}(x,t) = \frac{6\eta\nu\sigma}{M\rho(\tanh_\Omega(2\sqrt{-\nu\eta}\psi) \pm (i\text{sech}_\Omega(2\sqrt{-\nu\eta}\psi)))^2} + \frac{2\sigma\nu\eta}{M\rho}, \tag{65}$$

$$w_{2,19}(x,t) = \frac{6\eta\nu\sigma}{(\coth_\Omega(2\sqrt{-\nu\eta}\psi) \pm (\text{csch}_\Omega(2\sqrt{-\nu\eta}\psi)))^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}, \tag{66}$$

and

$$w_{2,20}(x,t) = \frac{24\eta\nu\sigma}{\left(\tanh_\Omega\left(\frac{1}{2}\sqrt{-\nu\eta}\psi\right) \pm \left(\coth_\Omega\left(\frac{1}{2}\sqrt{-\nu\eta}\psi\right)\right)\right)^2 M\rho} + \frac{2\sigma\nu\eta}{M\rho}. \tag{67}$$

Family 2.5: When $\nu = \eta$ and $\mu = 0$:

$$w_{2,21}(x,t) = 6\frac{\eta^2\sigma}{M\rho(\tan_\Omega(\eta\psi))^2} + \frac{2\sigma\eta^2}{M\rho}, \tag{68}$$

$$w_{2,22}(x,t) = 6\frac{\eta^2\sigma}{M\rho(\cot_\Omega(\eta\psi))^2} + \frac{2\sigma\eta^2}{M\rho}, \tag{69}$$

$$w_{2,23}(x,t) = 6\frac{\eta^2\sigma}{M\rho(\tan_\Omega(2\eta\psi) \pm (\sec_\Omega(2\eta\psi)))^2} + \frac{2\sigma\eta^2}{M\rho}, \tag{70}$$

$$w_{2,24}(x,t) = 6\frac{\eta^2\sigma}{M\rho(-\cot_\Omega(2\eta\psi) \pm (\csc_\Omega(2\eta\psi)))^2} + \frac{2\sigma\eta^2}{M\rho}, \tag{71}$$

and

$$w_{2,25}(x,t) = 6\frac{\eta^2\sigma}{M\rho\left(\frac{1}{2}\tan_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\cot_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2} + \frac{2\sigma\eta^2}{M\rho}. \tag{72}$$

Family 2.6: When $\nu = -\eta$ and $\mu = 0$:

$$w_{2,26}(x,t) = 6\frac{\eta^2\sigma}{M\rho(\tanh_\Omega(\eta\psi))^2} - 2\frac{\eta^2\sigma}{M\rho}, \tag{73}$$

$$w_{2,27}(x,t) = 6\frac{\eta^2\sigma}{M\rho(\coth_\Omega(\eta\psi))^2} - 2\frac{\eta^2\sigma}{M\rho}, \tag{74}$$

$$w_{2,28}(x,t) = 6\frac{\eta^2\sigma}{M\rho(-\tanh_\Omega(2\eta\psi) \pm (i\operatorname{sech}_\Omega(2\eta\psi)))^2} - 2\frac{\eta^2\sigma}{M\rho}, \tag{75}$$

$$w_{2,29}(x,t) = 6\frac{\eta^2\sigma}{M\rho(-\coth_\Omega(2\eta\psi) \pm (i\operatorname{cech}(2\eta\psi)))^2} - 2\frac{\eta^2\sigma}{M\rho}, \tag{76}$$

and

$$w_{2,30}(x,t) = 6\frac{\eta^2\sigma}{M\rho\left(-\frac{1}{2}\tanh_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\coth_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2} - 2\frac{\eta^2\sigma}{M\rho}. \tag{77}$$

Family 2.7: When $\mu = \tau$, $\eta = n\tau (n \neq 0)$ and $\nu = 0$:

$$w_{2,31}(x,t) = \frac{\sigma(n^2 + 4\Omega^{\tau\psi}n + \Omega^{2\tau\psi})}{\rho(\Omega^{\tau\psi} - n)^2}, \tag{78}$$

where $\psi = \beta\frac{x^\delta}{\delta} - (\frac{\sqrt{\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}})\frac{t^\delta}{\delta}$.

Assuming case 3, we obtain the following families of soliton solutions for (1):

Family 3.1: When $M < 0 \quad \nu \neq 0$:

$$w_{3,1}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho} \times \left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2, \tag{79}$$

$$w_{3,2}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho} \times \left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2, \tag{80}$$

$$w_{3,3}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi) \pm (\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi) \pm (\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2, \tag{81}$$

$$w_{3,4}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi) \pm (\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi) \pm (\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2, \tag{82}$$

and

$$w_{3,5}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{4} \times \frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right) - \cot_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right) - \frac{\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{4} \times \frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right) - \cot_\Omega\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)^2. \tag{83}$$

Family 3.2: When $M > 0 \quad \nu \neq 0$:

$$w_{3,6}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho} \times \left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2, \tag{84}$$

$$w_{3,7}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho} \times \left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\coth_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{M}\coth_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2, \tag{85}$$

$$w_{3,8}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho} \times \left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{M}\tanh_\Omega(\sqrt{-M}\psi) \pm (\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{M}\tanh_\Omega(\sqrt{-M}\psi) \pm (\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2, \tag{86}$$

$$w_{3,9}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{M}\coth_\Omega(\sqrt{-M}\psi) \pm (\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2} \times \frac{\sqrt{M}\coth_\Omega(\sqrt{-M}\psi) \pm (\mathrm{sech}_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2, \tag{87}$$

and

$$w_{3,10}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4} \times \frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right) - \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{4} \times \frac{\sqrt{M}\tanh_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right) - \coth_\Omega\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)^2. \tag{88}$$

Family 3.3: When $\eta\nu > 0$ and $\mu = 0$:

$$w_{3,11}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho}(\tan_\Omega(\sqrt{\eta\nu}\psi))^2 \tag{89}$$

$$w_{3,12}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho}(\cot_\Omega(\sqrt{\eta\nu}\psi))^2 \tag{90}$$

$$w_{3,13}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho} \times (\tan_\Omega(2\sqrt{\eta\nu}\psi) \pm (\sec_\Omega(2\sqrt{\eta\nu}\psi)))^2, \tag{91}$$

$$w_{3,14}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho} \times (\cot_\Omega(2\sqrt{\eta\nu}\psi) \pm (\csc_\Omega(2\sqrt{\eta\nu}\psi)))^2, \tag{92}$$

and

$$w_{3,15}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{3\nu\sigma\eta}{2M\rho} \times \left(\tan_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\psi\right) - \cot_\Omega\left(\frac{1}{2}\sqrt{\eta\nu}\psi\right)\right)^2 \tag{93}$$

Family 3.4: When $\eta\nu < 0$ and $\mu = 0$:

$$w_{3,16}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho}(\tanh_\Omega(\sqrt{-\eta\nu}\psi))^2, \tag{94}$$

$$w_{3,17}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho}(\coth_\Omega(\sqrt{-\eta\nu}\psi))^2, \tag{95}$$

$$w_{3,18}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho} \times (\tanh_\Omega(2\sqrt{-\eta\nu}\psi) \pm (i\,\mathrm{sech}_\Omega(2\sqrt{-\eta\nu}\psi)))^2, \tag{96}$$

$$w_{3,19}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{6\nu\sigma\eta}{M\rho} \times (\coth_\Omega(2\sqrt{-\eta\nu}\psi) \pm (\mathrm{csch}_\Omega(2\sqrt{-\eta\nu}\psi)))^2, \tag{97}$$

and

$$w_{3,20}(x,t) = -6\frac{\sigma\eta\nu}{M\rho} - \frac{3\nu\sigma\eta}{2M\rho} \times \left(\tanh_\Omega\left(\frac{1}{2}\sqrt{-\eta\nu}\psi\right) + \coth_\Omega\left(\frac{1}{2}\sqrt{-\eta\nu}\psi\right)\right)^2. \tag{98}$$

Family 3.5: When $\nu = \eta$ and $\mu = 0$:

$$w_{3,21}(x,t) = -6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(\tan_\Omega(\eta\psi))^2}{M\rho}, \tag{99}$$

$$w_{3,22}(x,t) = -6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(\cot_\Omega(\eta\psi))^2}{M\rho}, \tag{100}$$

$$w_{3,23}(x,t) = -6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(\tan_\Omega(2\eta\psi) \pm (\sec_\Omega(2\eta\psi)))^2}{M\rho}, \tag{101}$$

$$w_{3,24}(x,t) = -6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(-\cot_\Omega(2\eta\psi) \pm (\csc_\Omega(2\eta\psi)))^2}{M\rho}, \tag{102}$$

and

$$w_{3,25}(x,t) = -6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2\left(\frac{1}{2}\tan_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\cot_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2}{M\rho}. \tag{103}$$

Family 3.6: When $\nu = -\eta$ and $\mu = 0$:

$$w_{3,26}(x,t) = 6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(\tanh_\Omega(\eta\psi))^2}{M\rho}, \tag{104}$$

$$w_{3,27}(x,t) = 6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(\coth_\Omega(\eta\psi))^2}{M\rho}, \tag{105}$$

$$w_{3,28}(x,t) = 6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(-\tanh_\Omega(2\eta\psi) \pm (i\operatorname{sech}_\Omega(2\eta\psi)))^2}{M\rho}, \tag{106}$$

$$w_{3,29}(x,t) = 6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2(-\coth_\Omega(2\eta\psi) \pm (i\operatorname{icech}(2\eta\psi)))^2}{M\rho}, \tag{107}$$

and

$$w_{3,30}(x,t) = 6\frac{\sigma\eta^2}{M\rho} - 6\frac{\sigma\eta^2\left(-\frac{1}{2}\tanh_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\coth_\Omega\left(\frac{1}{2}\eta\psi\right)\right)^2}{M\rho}. \tag{108}$$

Family 3.7: When $\mu = \tau$, $c = n\tau (n \neq 0)$, and $\eta = 0$:

$$w_{3,31}(x,t) = -6\frac{\tau^2 n\sigma\Omega^{\tau\psi}}{M\rho(-1 + n\Omega^{\tau\psi})^2}, \tag{109}$$

where $\psi = \beta\frac{x^\delta}{\delta} - \left(\frac{\sqrt{-\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}}\right)\frac{t^\delta}{\delta}$.

Assuming case 4, we obtain the following families of soliton solutions for (1):

Family 4.1: When $M < 0 \quad \nu \neq 0$:

$$w_{4,1}(x,t) = \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) + \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2 + \frac{\sigma(\mu^2 + 2\nu\eta)}{M\rho}, \tag{110}$$

$$w_{4,2}(x,t) = \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right) + \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} - \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2 + \frac{\sigma(\mu^2 + 2\nu\eta)}{M\rho}, \tag{111}$$

$$w_{4,3}(x,t) = \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi) \pm (\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right) + \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\tan_\Omega(\sqrt{-M}\psi) \pm (\sec_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2 + \frac{\sigma(\mu^2 + 2\nu\eta)}{M\rho}, \tag{112}$$

$$w_{4,4}(x,t) = \frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi) \pm (\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right) + \frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu} + \frac{1}{2}\frac{\sqrt{-M}\cot_\Omega(\sqrt{-M}\psi) \pm (\csc_\Omega(\sqrt{-M}\psi))}{\nu}\right)^2 + \frac{\sigma(\mu^2 + 2\nu\eta)}{M\rho}, \tag{113}$$

and

$$w_{4,5}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{4}\,\frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)-\cot_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+1/4\,\frac{\sqrt{-M}\tan_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)-\cot_{\Omega}\left(\frac{1}{4}\sqrt{-M}\psi\right)}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}. \quad (114)$$

Family 4.2: When $M>0 \quad \nu\neq 0$:

$$w_{4,6}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \quad (115)$$

$$w_{4,7}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\coth_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{2}\frac{\sqrt{M}\coth_{\Omega}\left(\frac{1}{2}\sqrt{-M}\psi\right)}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \quad (116)$$

$$w_{4,8}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\times\frac{\sqrt{M}\tanh_{\Omega}(\sqrt{-M}\psi)\pm(\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\times\frac{\sqrt{M}\tanh_{\Omega}(\sqrt{-M}\psi)\pm(\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \quad (117)$$

$$w_{4,9}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\times\frac{\sqrt{M}\coth_{\Omega}(\sqrt{-M}\psi)\pm(\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}+\frac{1}{2}\times\frac{\sqrt{M}\coth_{\Omega}(\sqrt{-M}\psi)\pm(\mathrm{sech}_{\Omega}(\sqrt{-M}\psi))}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}, \quad (118)$$

and

$$w_{4,10}(x,t)=\frac{6\mu\nu\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{4}\times\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)-\coth_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)+\frac{6\nu^2\sigma}{M\rho}\left(-\frac{1}{2}\frac{\mu}{\nu}-\frac{1}{4}\times\frac{\sqrt{M}\tanh_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)-\coth_{\Omega}\left(\frac{1}{4}\sqrt{M}\psi\right)}{\nu}\right)^2+\frac{\sigma(\mu^2+2\nu\eta)}{M\rho}. \quad (119)$$

Family 4.3: When $\eta\nu>0$ and $\mu=0$:

$$w_{4,11}(x,t)=\frac{-3\sigma}{2}(\tan_{\Omega}(\sqrt{\nu\eta}\,\psi))^2-\frac{1}{2}\frac{\sigma}{\rho}, \quad (120)$$

$$w_{4,12}(x,t)=\frac{-3\sigma}{2}(\cot_{\Omega}(\sqrt{\nu\eta}\,\psi))^2-\frac{1}{2}\frac{\sigma}{\rho}, \quad (121)$$

$$w_{4,13}(x,t)=\frac{-3\sigma}{2}(\tan_{\Omega}(2\sqrt{\nu\eta}\,\psi)\pm(\sec_{\Omega}(2\sqrt{\nu\eta}\,\psi)))^2-\frac{1}{2}\frac{\sigma}{\rho}, \quad (122)$$

$$w_{4,14}(x,t)=\frac{-3\sigma}{2}(\cot_{\Omega}(2\sqrt{\nu\eta}\,\psi)\pm(\csc_{\Omega}(2\sqrt{\nu\eta}\,\psi)))^2-\frac{1}{2}\frac{\sigma}{\rho}, \quad (123)$$

and

$$w_{4,15}(x,t)=\frac{-3\sigma}{2}\left(\tan_{\Omega}\left(\frac{1}{2}\sqrt{\nu\eta}\,\psi\right)-\cot_{\Omega}\left(\frac{1}{2}\sqrt{\nu\eta}\,\psi\right)\right)^2-\frac{1}{2}\frac{\sigma}{\rho}. \quad (124)$$

Family 4.4: When $\eta\nu < 0$ and $\mu = 0$:

$$w_{4,16}(x,t) = \frac{-3\sigma}{2}(\tanh_\Omega(\sqrt{-\nu\eta}\,\psi))^2 - \frac{1}{2}\frac{\sigma}{\rho}, \tag{125}$$

$$w_{4,17}(x,t) = \frac{-3\sigma}{2}((\tanh_\Omega(2\sqrt{-\nu\eta}\,\psi) \pm (i\operatorname{sech}_\Omega(2\sqrt{-\nu\eta}\,\psi))))^2 - \frac{1}{2}\frac{\sigma}{\rho}, \tag{126}$$

$$w_{4,18}(x,t) = \frac{-3\sigma}{2}(\tanh_\Omega(2\sqrt{-\nu\eta}\,\psi) \pm (i\operatorname{sech}_\Omega(2\sqrt{-\nu\eta}\,\psi)))^2 - \frac{1}{2}\frac{\sigma}{\rho}, \tag{127}$$

$$w_{4,19}(x,t) = \frac{-3\sigma}{2}((\coth_\Omega(2\sqrt{-\nu\eta}\,\psi) \pm (\operatorname{csch}_\Omega(2\sqrt{-\nu\eta}\,\psi))))^2 - \frac{1}{2}\frac{\sigma}{\rho}, \tag{128}$$

and

$$w_{4,20}(x,t) = \frac{-3\sigma}{2}\left(\tanh_\Omega\left(\frac{1}{2}\sqrt{-\nu\eta}\,\psi\right) + \coth_\Omega\left(\frac{1}{2}\sqrt{-\nu\eta}\,\psi\right)\right)^2 - \frac{1}{2}\frac{\sigma}{\rho}. \tag{129}$$

Family 4.5: When $\nu = \eta$ and $\mu = 0$:

$$w_{4,21}(x,t) = \frac{-3\sigma(\tan_\Omega(\eta\psi))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{130}$$

$$w_{4,22}(x,t) = \frac{-3\sigma(\cot_\Omega(\eta\psi))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{131}$$

$$w_{4,23}(x,t) = \frac{-3\sigma((\tan_\Omega(2\eta\psi) \pm (\sec_\Omega(2\eta\psi))))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{132}$$

$$w_{4,24}(x,t) = \frac{-3\sigma((-\cot_\Omega(2\eta\psi) \pm (\csc_\Omega(2\eta\psi))))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{133}$$

and

$$w_{4,25}(x,t) = \frac{-3\sigma\left(\left(\frac{1}{2}\tan_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\cot_\Omega\left(\frac{1}{2}\eta\psi\right)\right)\right)^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}. \tag{134}$$

Family 4.6: When $\nu = -\eta$ and $\mu = 0$:

$$w_{4,26}(x,t) = \frac{-3\sigma(\tanh_\Omega(\eta\psi))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{135}$$

$$w_{4,27}(x,t) = \frac{-3\sigma(\coth_\Omega(\eta\psi))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{136}$$

$$w_{4,28}(x,t) = \frac{-3\sigma((-\tanh_\Omega(2\eta\psi) \pm (i\operatorname{sech}_\Omega(2\eta\psi))))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{137}$$

$$w_{4,29}(x,t) = \frac{-3\sigma((-\coth_\Omega(2\eta\psi) \pm (\operatorname{cech}(2\eta\psi))))^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}, \tag{138}$$

and

$$w_{4,30}(x,t) = \frac{-3\sigma\left(\left(-\frac{1}{2}\tanh_\Omega\left(\frac{1}{2}\eta\psi\right) - \frac{1}{2}\coth_\Omega\left(\frac{1}{2}\eta\psi\right)\right)\right)^2}{2\rho} - \frac{1}{2}\frac{\sigma}{\rho}. \tag{139}$$

Family 4.7: When $\eta = 0$, $\mu \neq 0$ and $\nu \neq 0$:

$$w_{4,31}(x,t) = 2\frac{\sigma((\cosh_\Omega(\mu\psi))^2 - \cosh_\Omega(\mu\psi)\sinh_\Omega(\mu\psi) - 2\cosh_\Omega(\mu\psi) + 2\sinh_\Omega(\mu\psi))}{\rho(\cosh_\Omega(\mu\psi) - \sinh_\Omega(\mu\psi) + 1)^2}, \tag{140}$$

and

$$w_{4,32}(x,t) = 2\frac{\sigma((\cosh_\Omega(\mu\psi))^2 + \cosh_\Omega(\mu\psi)\sinh_\Omega(\mu\psi) - 2\cosh_\Omega(\mu\psi) - 2\sinh_\Omega(\mu\psi))}{\rho(\cosh_\Omega(\mu\psi) + \sinh_\Omega(\mu\psi) + 1)^2}. \tag{141}$$

Family 4.8: When $\mu = \tau$, $\nu = n\tau (n \neq 0)$ and $\eta = 0$:

$$w_{4,33}(x,t) = \frac{\sigma(1 + 4n\Omega^{\tau\psi} + n^2\Omega^{2\tau\psi})}{\rho(-1 + n\Omega^{\tau\psi})^2}, \tag{142}$$

where $\psi = \beta\frac{x^\delta}{\delta} - \left(\frac{\sqrt{\sigma + \beta^2(\ln(\Omega))^2 M}}{\ln(\Omega)\sqrt{M}}\right)\frac{t^\delta}{\delta}$.

4 Discussion and graphs

In this section, we graphically depict a number of wave patterns that were seen in the system that is being studied. By using the mEDAM, we were able to identify and display wave patterns in contour and 3D graphs. Predominantly, breather, dark kink, periodic, bright kinks, and other N-soliton structures are found in the soliton solutions constructed for CKGE. They continue travelling at the same pace and in the identical direction. They clarify the behavior of scalar fields in relativistic quantum physics when applied to the CKGE. Similar to vacuum state transitions, kink solitons, stable borders traversing scalar field zones, signify abrupt phase or amplitude shifts. Understanding

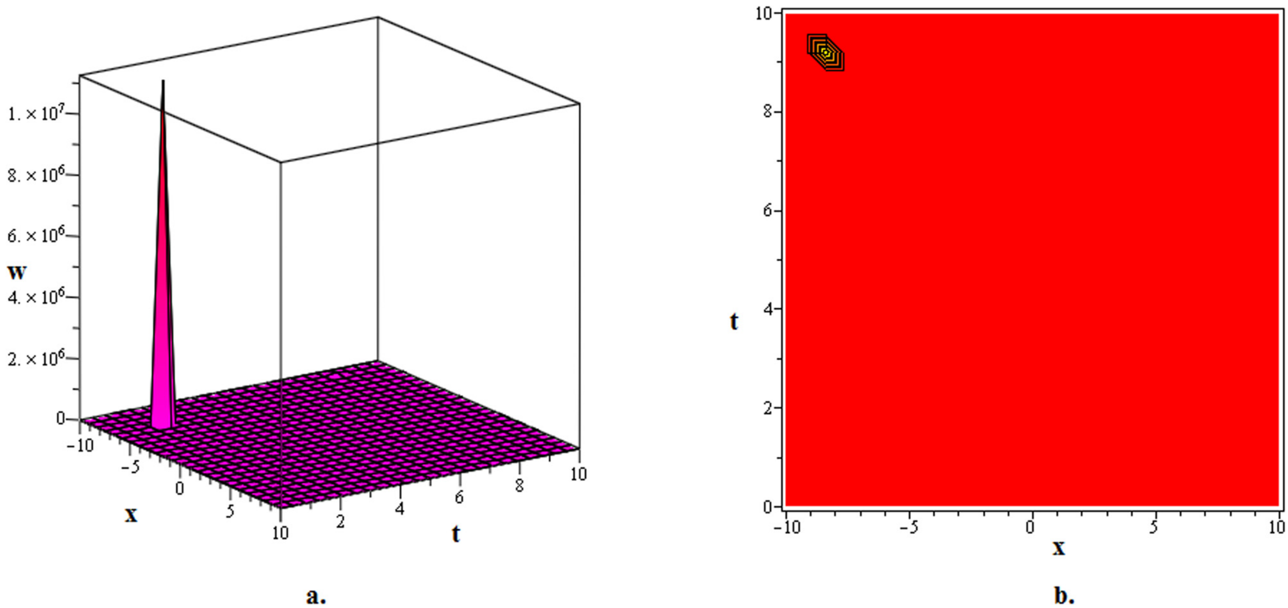

Figure 1: The three-dimensional and contour graphics of the singular bright kink soliton solution $w_{1,1}$ stated in (17) are visualized for $\mu = 1$; $\nu = 1$; $\eta = 2$; $\delta = 1$; $\beta = 1$; $\sigma = 2$; $\Omega = e$; $\rho = 6$.

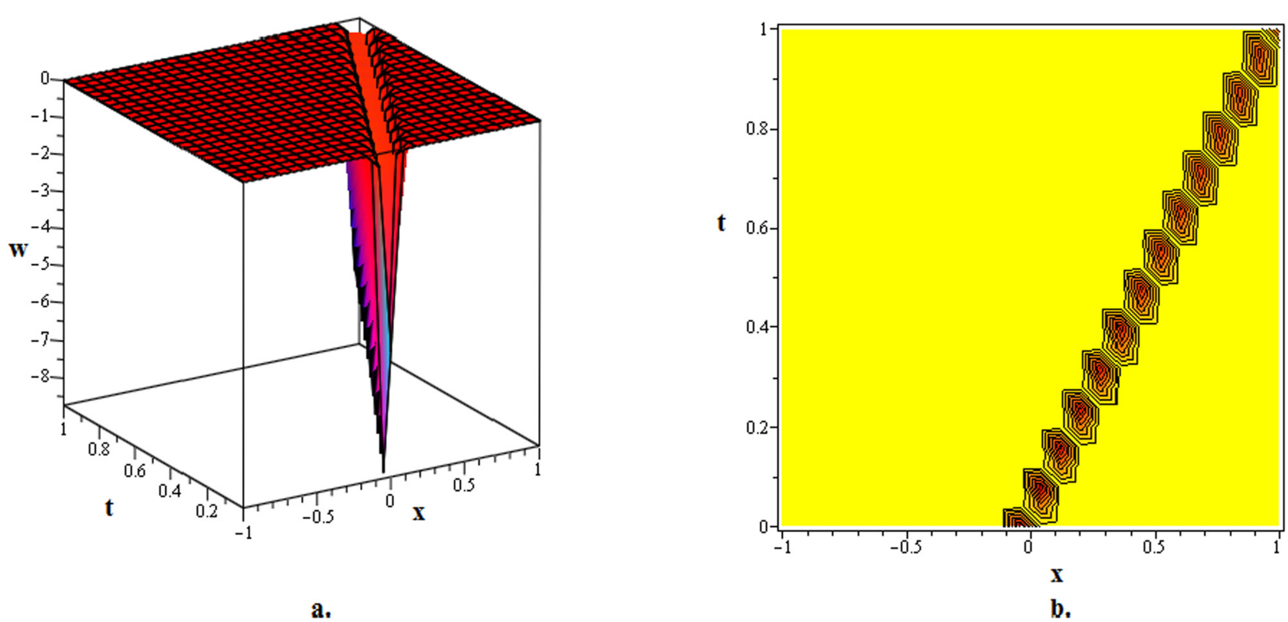

Figure 2: The three-dimensional and contour graphics of the dark soliton solution $w_{1,7}$ stated in (23) are visualized for $\mu = 10$; $\nu = 8$; $\eta = 2$; $\delta = 1$; $\beta = 10$; $\sigma = 20$; $\Omega = e$; $\rho = 10$.

the basic characteristics of relativistic quantum mechanical phenomena and how they relate to particles and vacuum features is possible through studying solitons in the CKGE. Breather solitons, which are localized disturbances going through energy state transitions, exhibit periodic amplitude oscillations. Stable, recurrent structures with periodic solitons include standing waves. In the field description, dark kinks show localized depressive issues that could be indicative of strong soliton-like qualities. Bright kinks, on the other hand, show peaks or humps in the field's persona, which could be signs of temporary or meta-stable excitations. Kink solitons, stable boundaries

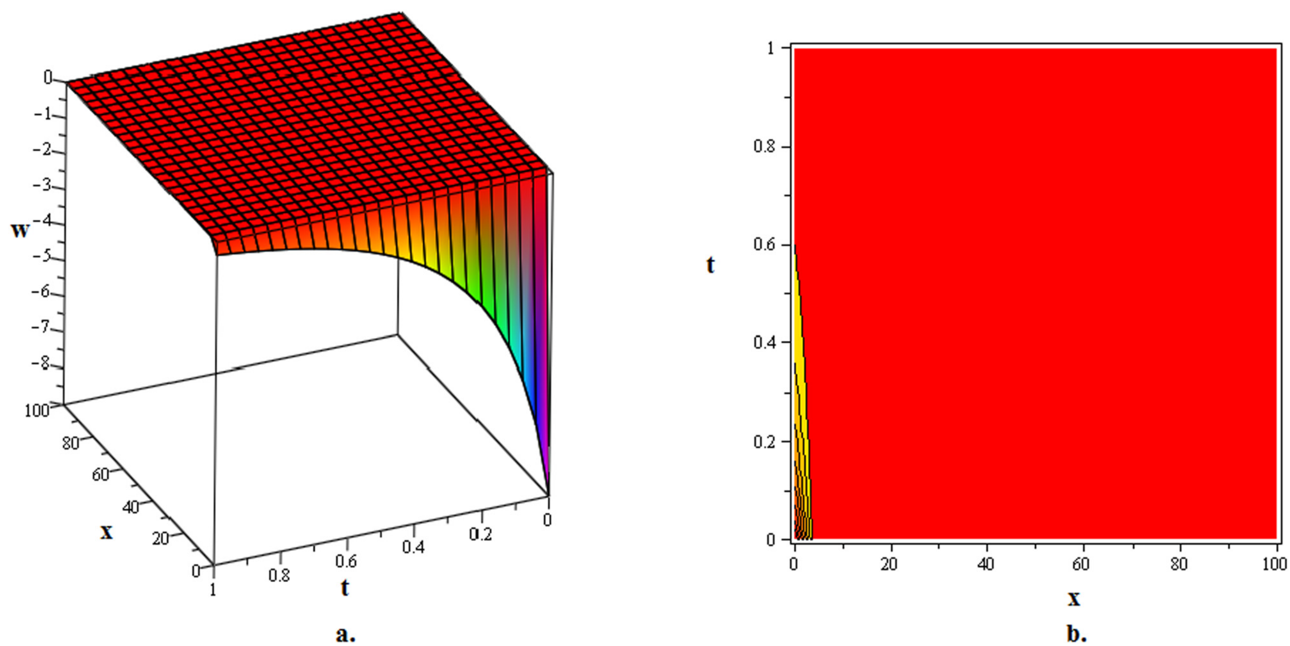

Figure 3: The three-dimensional and contour graphics of the dark soliton solution $w_{1,31}$ stated in (47) are visualized for $\mu = 2$; $\nu = 0$; $\eta = 6$; $\delta = 0.9$; $\beta = 1$; $\sigma = 20$; $\Omega = 3$; $\rho = 10$; $\tau = 2$; $n = 3$.

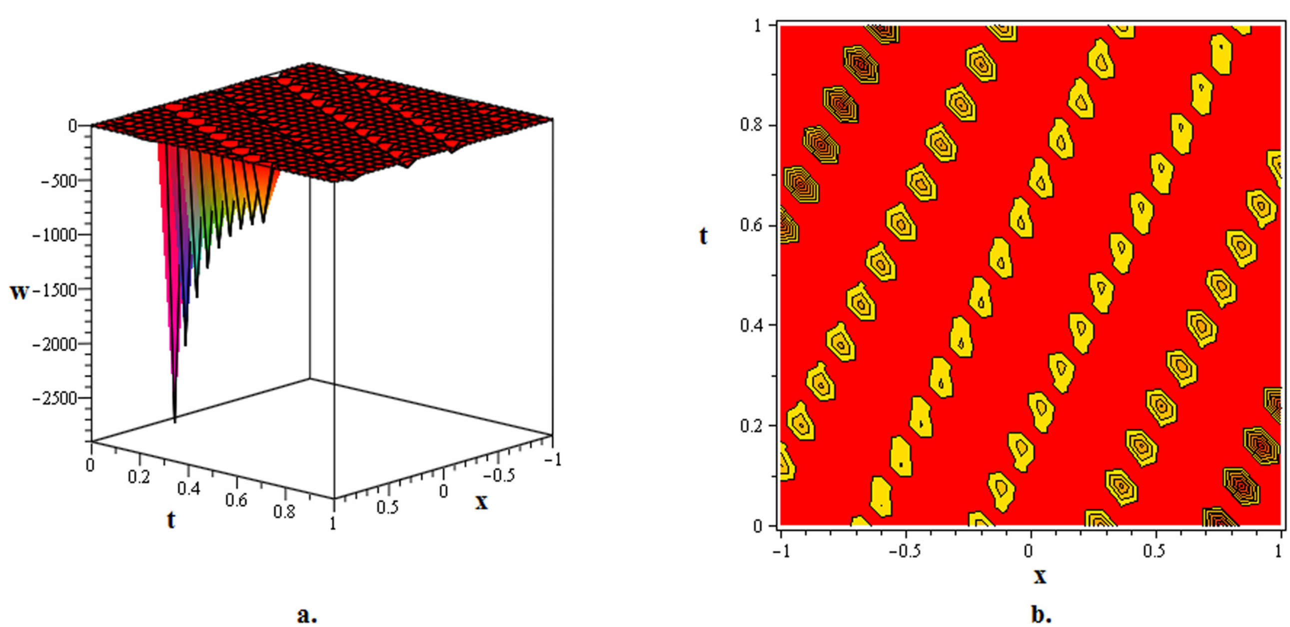

Figure 4: The three-dimensional and contour graphics of the breather dark soliton solution interacts with breather soliton solution $w_{2,5}$ stated in (52) are visualized for $\mu = 1$; $\nu = 1$; $\eta = 2$; $\delta = 1$; $\beta = 5$; $\sigma = 1$; $\Omega = e$; $\rho = 6$.

across scalar field zones, represent abrupt phase or amplitude changes, much like vacuum state transitions do. Multi-soliton solutions, or *N*-solitons, explain the interaction between localized disturbances that result in bounded states, fusion, and dispersion. Studying solitons in the CKGE allows us to comprehend the fundamental properties of relativistic quantum mechanical systems and their relationship in particles and vacuum features. In a nutshell, the novelty of the current study lies in the provision of the systematic and unique unearthing of the new plethora of soliton solutions that showcase the novel and innovative nature of our investigation. Investigations into nonlinear

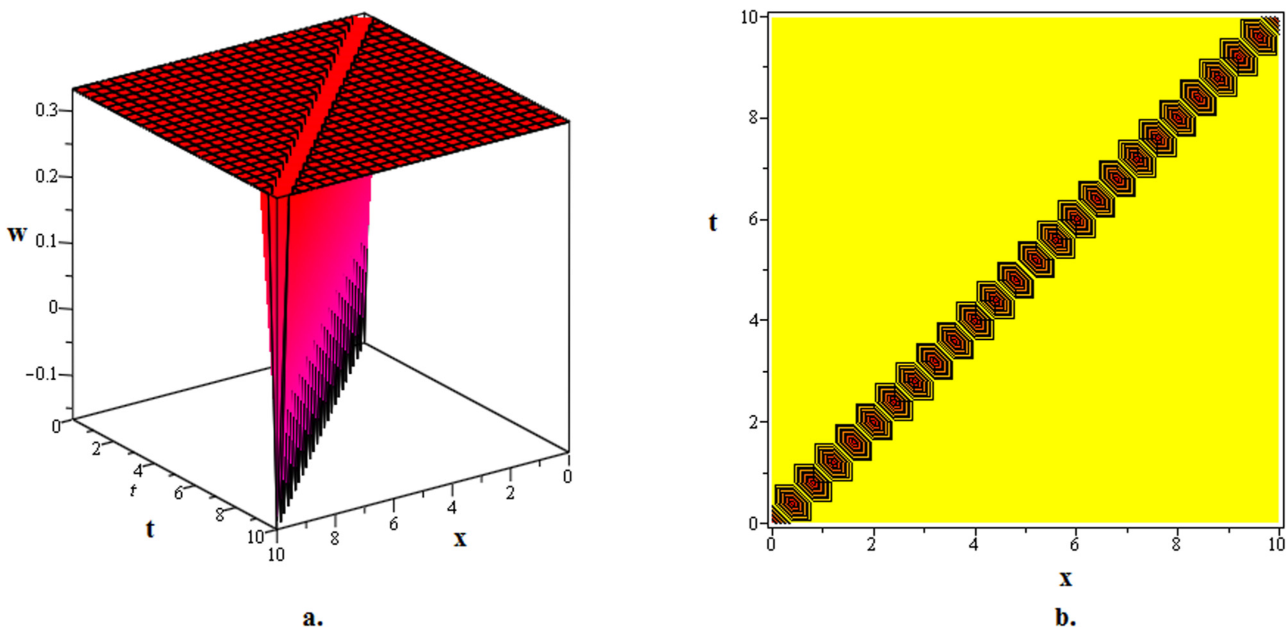

Figure 5: The three-dimensional and contour graphics of the breather dark soliton solution $w_{2,17}$ stated in (64) are visualized for $\mu = 0$; $\nu = 2$; $\eta = 8$; $\delta = 0.9$; $\beta = 30$; $\sigma = 2$; $\Omega = e$; $\rho = 6$.

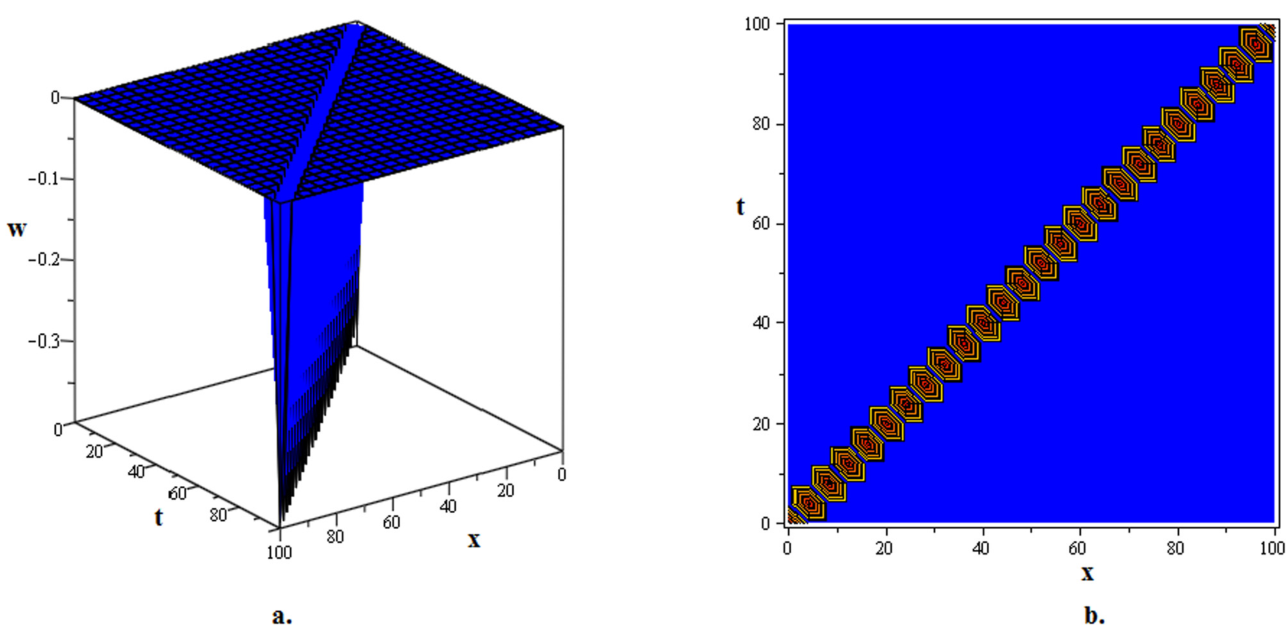

Figure 6: The three-dimensional and contour graphics of the breather dark soliton solution $w_{3,31}$ stated in (109) are visualized for $\mu = 5$; $\nu = 25$; $\eta = 0$; $\delta = 0.5$; $\beta = 50$; $\sigma = 1$; $\Omega = e$; $\rho = 5$; $\tau = 5$; $n = 5$.

FPDEs that arise in quantum field theory, fluid dynamics, and optics are of importance to solitons. These soliton solutions present a deeper insight into the CKGE underlying phenomena in the related scientific fields (Figures 1–8).

5 Conclusion

We have used the upgraded mEDAM to comprehensively study the transmission of solitons in the CKGE, an

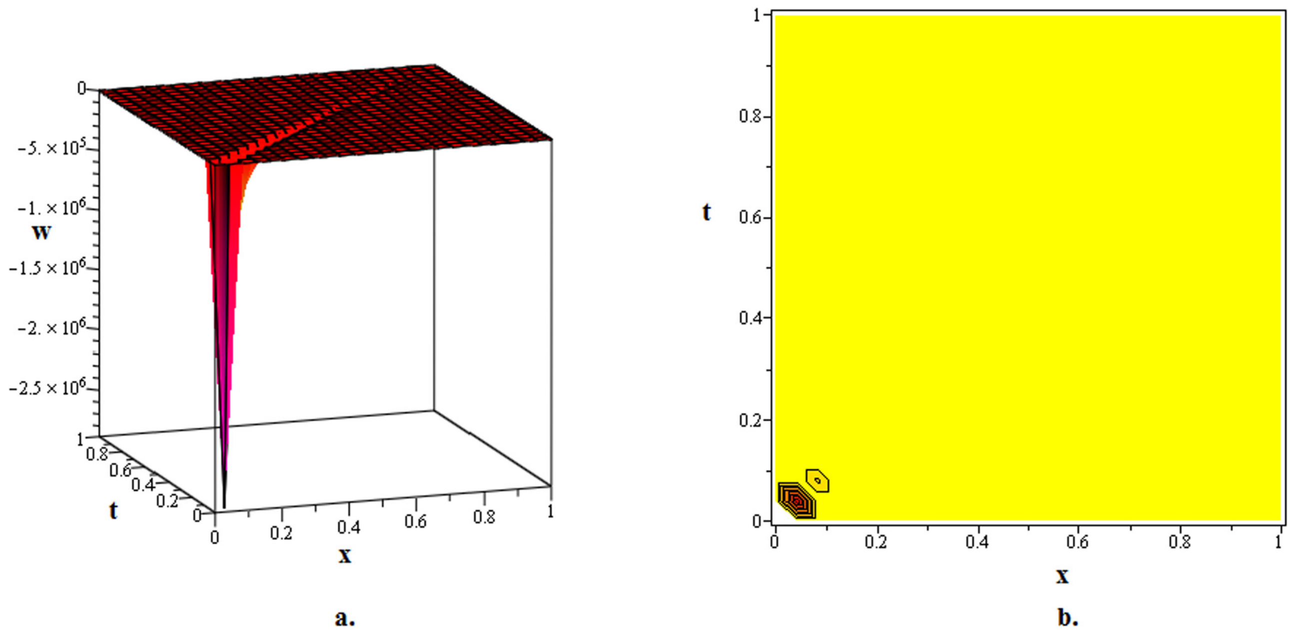

Figure 7: The three-dimensional and contour graphics of the dark soliton solution $w_{4,25}$ stated in (134) are visualized for $\mu = 0$; $\nu = 7$; $\eta = 7$; $\delta = 1$; $\beta = 4$; $\sigma = 3$; $\Omega = e$; $\rho = 5$.

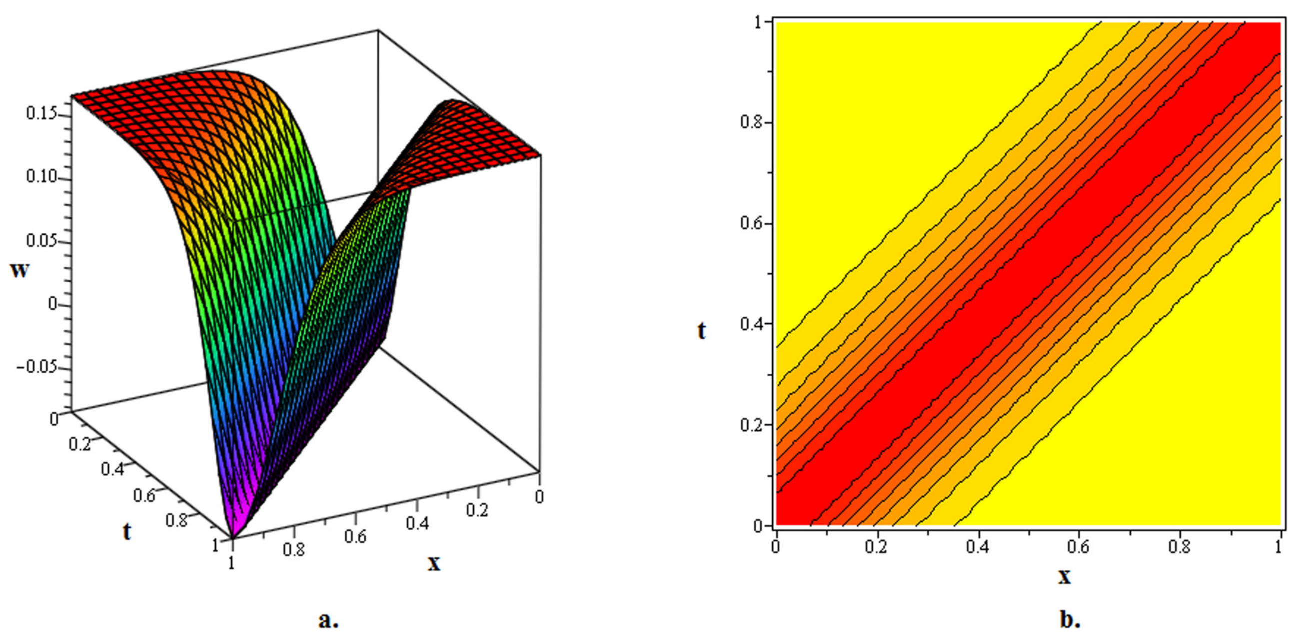

Figure 8: The three-dimensional and contour graphics of the bell-shaped or dark kink soliton solution $w_{4,31}$ stated in (140) are visualized for $\mu = 2$; $\nu = 5$; $\eta = 0$; $\delta = 1$; $\beta = 5$; $\sigma = 1$; $\Omega = e$; $\rho = 6$.

established model in the fields of solid-state physics, quantum field theory, and nonlinear optics. Several dark kink, bright kink, breather, and other N-soliton solutions, including generalized hyperbolic, trigonometric, exponential, and rational functions, have been found by translating the suggested model into NODEs and assuming closed-form solutions. Crucial insights into the dynamics of propagating soliton processes can be gained from contour and three-dimensional graphs, which graphically represent the propagation behavior of certain soliton solutions. These graphs directly relate to domains related to the models under consideration. The unusual outcomes of applying the mEDAM to the model demonstrate its originality, as they have not been examined in academic literature previously. These findings improve our knowledge of nonlinear dynamics and temporal evolution processes, which have important ramifications for our comprehension of related physical phenomena. The effectiveness and consistency of the methods used in this work also show how widely relevant they are to nonlinear issues in many different scientific fields. Even though the mEDAM has made a substantial contribution to our knowledge of soliton dynamics and how they impact the targeted model, it is crucial to recognize the technique's limits, especially in cases where the largest derivative and the nonlinear term are not uniformly balanced. Despite this drawback, the study makes clear how many concerns about nonlinear behaviors and soliton dynamics remain unresolved and provides fresh avenues for future research in the area. To put it briefly, what makes our study unusual is that we have discovered a variety of new soliton solutions in a methodical and distinctive way, demonstrating the creative and inventive nature of our research. Solitons are interested in studying nonlinear FPDEs that occur in fluid dynamics, optics, and quantum field theory. In the associated scientific domains, these soliton solutions offer a more profound understanding of the CKGE underlying phenomena. The authors also aim to make the mEDAM efficient in addressing nonlinear FPDEs with variable coefficients and stochastic behaviors for the construction and analysis of soliton solutions.

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