Research Article

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Generalized model of thermoelasticity associated with fractional time-derivative operators and its applications to non-simple elastic materials

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Abstract: This study proposes a comprehensive heat conduction model that incorporates fractional time derivatives and two-phase lags to describe the behavior of non-simple thermoelastic materials accurately. Generalized fractional differential operators with non-singular kernels are introduced. This type of fractional derivative includes the Caputo–Fabrizio and the Atangana–Baleanu fractional derivatives. The model also consists of the two-temperature idea, which considers the effect of microstructure through a two-stage delay approach. Interactions of a thermoelastic nature caused by the rapid heating of an isotropic substance under the influence of an external body force were studied as a practical application of the new concept. There has been some discussion about the effect of the discrepancy index and fractional differential operators. Finally, the graphical representations obtained from the numerical simulations were used to explain the behavior of the studied physical fields. The generalized fractional heat transfer model is demonstrated to be capable of producing a temperature forecast that is in close agreement with experimental data. As a result, the proposed model may be useful for solving difficulties in heat transfer, anomalous transport, and other branches of engineering analysis.

Keywords: fractional thermoelasticity, two-temperature, phaselags, non-singular kernels

Nomenclature

λ, μ	Lamé's elastic parameters
α_t	thermal expansion parameter
C_s	specific heat
$\gamma = (3\lambda + 2\mu)\alpha_t$	thermal coupling parameter
T_0	reference temperature
$\theta = T - T_0$	temperature change
T	absolute temperature
u	displacement vector
$e = \mathrm{div}\,u$	dilatation
σ_{ij}	stresses
e_{ij}	strains
$\mathcal{H}$	heat flow vector
K	thermal conductivity
ρ	material density
Q	heat source
t	instant time
δ_{ij}	Kronecker's delta
ϕ	conductive temperature
τ_q	phase delay of heat flow
τ_ϕ	phase deley of conductive temperature
$D_t^{(\alpha)}$	fractional operator
τ_0	thermal relaxation time
α	fractional orders
η	The entropy

1 Introduction

Generalized thermoelasticity is a subfield of continuum mechanics that builds on the traditional theory of thermoelasticity to include more realistic and complicated responses from materials. In the field of conventional

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thermoelasticity, the prevailing assumption is that heat transmission occurs immediately. However, in the context of extended thermoelasticity, this assumption is loosened. The present theory considers a limited thermal wave velocity, a factor that holds significance in specific materials and circumstances [1]. The concept of generalized thermoelasticity has found use in various fields, including investigating heat propagation in semiconductors, examining thermal shock, and modeling and simulating biological tissues.

In contrast to conventional thermoelasticity, which assumes instantaneous heat transfer as described by the parabolic heat transfer equation, extensional thermoelasticity involves hyperbolic heat transfer models. These models often include thermal delays, measuring the amount of time required for temperature changes to propagate through a given material [2]. Lord and Shulman [3] proposed a comprehensive theory of thermoelasticity that incorporates a single relaxation period. Consequently, the governing system for the heat equation has transformed into a hyperbolic form. In their study, Green and Lindsay [4] put forward an alternative model that incorporates two distinct relaxation phases. In their research, Tzou [5,6] introduced a thermoelastic model incorporating the dual-phase-lag (DPL) approach to both the heat flow vector and the temperature gradient. In his work, Choudhuri [7] expanded the DPL model to derive the equation for heat conduction's three-phase lags (TPLs). This model allows for the estimation of TPL based on the heat flow component, temperature gradient, and thermal displacement, as opposed to relying on the classical Fourier transform law of heat transfer. The theories proposed by Green and Naghdi [8–10] encompass the fundamental principles of thermoelasticity, including both dissipative and non-dissipative systems. The aforementioned models introduce significant fundamental modifications to the constitutive equations, allowing the solution of a much wider range of heat flow problems, which can be classified as types I, II, and III. In recent times, several endeavors have been undertaken to alter the conventional heat transfer rule. Abouelregal *et al.* [11–15] made significant contributions by introducing generalized versions of heat conduction that incorporate higher-order time-derivative terms.

The two-temperature theory (2TT) approach is employed in examining thermal and heat transport processes in specific materials, necessitating the differentiation between two separate temperatures. This theory is especially relevant in materials where the movement of charge carriers, such as electrons, significantly affects the heat conduction process and where there may be a significant temperature difference between these carriers and the lattice or crystalline structure. Numerous researchers have investigated the linearized form of the two-temperature theory (2TT), focusing on two separate temperatures, namely, the conductive and thermodynamic temperatures [16–18]. The conductive temperature is a parameter linked to the heat conduction rate within a certain material. It refers to the local temperature at which heat conduction occurs. In the classical thermoelasticity, the temperature referred to is equivalent to the thermodynamic temperature, and heat conduction is believed to occur instantaneously [19]. Nevertheless, in the context of generalized thermoelasticity, the conductor may exhibit discrepancies with thermodynamics as a result of the limited velocity of heat conduction. This implies that the temperature associated with heat conduction, which is conductive, shows a delay in the thermodynamic temperature throughout the heat transfer process inside the material. The thermodynamic temperature refers to the typical temperature concept that is often employed in the field of thermodynamics. The parameter in question denotes the state of thermal equilibrium for a given substance and is the temperature variable utilized in equations of state and thermodynamic relationships. In the context of classical thermoelasticity, the thermodynamic temperature is employed to represent the temperature distribution inside the material [20]. Quintanilla [21] has engaged in a discussion over the existence, structural stability, and spatial behavior of the solution within the context of the 2TT model. Lesan [22] has proposed and proven the theorems of uniqueness and reciprocity in the context of the 2TT thermoelastic model. The two-temperature theory finds utility in several domains, such as investigating the dynamics of ultrafast laser interactions with materials, simulating semiconductor devices, and examining the response of materials under the influence of strong electromagnetic radiation. This facilitates a more precise characterization of temperature dynamics in such materials, wherein the electron temperature exhibits fast fluctuations in response to external influences.

Fractional derivatives are an extension of conventional derivatives, which are typically defined for integer orders to encompass non-integer orders. The concept of derivatives offers a means to quantify the rate of change of a function at a certain point or over a given interval, encompassing non-integer orders. The idea of fractional derivatives holds significant importance within the fractional calculus. Using integral operators commonly establishes fractional derivatives [23,24]. The Riemann–Liouville (RF) fractional and Caputo (C) fractional derivatives are widely recognized as the two most prevalent definitions in the field. The selection of a definition is contingent upon the particular problem under consideration. Fractional derivatives have been shown to have several applications in a wide range of disciplines, such as physics, engineering, economics, biology, and control theory. These models represent and examine phenomena characterized by intricate dynamics and memory effects, including, but not limited to, anomalous diffusion, viscoelasticity, and fractional Brownian motion [25,26].

Fractional derivatives involving non-singular kernels can be considered an extension of classical derivatives. In classical calculus, derivatives characterize the rate of change of a function at a certain location, typically denoted by integer-order derivatives. The definition of fractional derivatives expands the notion of differentiation to include non-integer orders, with a notable feature being the utilization of non-singular (or non-local) kernels. Researchers have recognized the significance of discovering novel fractional derivatives utilizing distinct unique or non-singular kernels to address the requirement for effectively simulating real-world phenomena across several disciplines, including hydrodynamics, viscoelasticity, physics, biological sciences, and mechanical engineering [27]. Non-singular kernels sometimes incorporate transcendental functions such as gamma, exponential, or more intricate functions such as Mittag–Leffler functions. The selection of these kernels is based on ensuring that the fractional derivatives exhibit well-defined mathematical features and hold practical importance in many applications.

Caputo and Fabrizio [28] proposed a solution to deal with the single kernel problem that occurs in Liouville–Caputo and RL fractional derivative definitions of fractional-order derivatives, among others. The solution is provided through the use of an exponential function. Nevertheless, this particular operator had some challenges, one of which pertains to its non-local nature. Furthermore, it should be noted that the integral corresponding to the fractional-order derivative is not a fractional integral. Atangana and Baleanu [29] successfully addressed these challenges. Two fractional derivatives, namely, the Liouville–Caputo and RL derivatives, were presented. These derivatives are defined using the extended Mittag–Leffler function. A comprehensive formulation encompassing all previously established fractional derivative operators featuring non-singular kernels was recently introduced in the study by Hattaf [30]. The definitions discussed earlier have been included in various physical and engineering models [31–52].

In the fields of material science and continuum mechanics, the idea of fractional thermoelasticity with phase delays and two temperatures is relatively new. The new aspect of this method is that it can help us better understand the thermal processes occurring within materials, even ones with complex and non-local behavior. Non-local and memory-dependent impacts, which standard models of thermoelasticity do not capture, can be modeled using this framework thanks to the introduction of fractional derivatives. To accurately characterize the dynamic behavior of materials, it is necessary to account for the time lag between the application of a thermal stimulus and the resulting mechanical response, and this can be done by integrating phase delays. The accuracy of temperature forecasts and thermal analysis can be improved by considering two temperatures, a technique known as dual-phase-lag modeling. This allows for a more accurate description of the thermal interactions between distinct components or phases inside a material. The evaluation of the suggested model involves the utilization of fractional derivatives, incorporating both singular and non-singular kernels. This proposed approach also allows extracting some previous cases from the thermoelasticity model, whether with one temperature or two temperatures, both when fractional differentiation is present and when it is not.

The current work utilizes the newly developed model to investigate a thermoelasticity problem involving two temperatures. Specifically, the thermodynamic response of a one-dimensional elastic medium with a surface free from traction under the influence of thermal shock and a decreasing external force is examined. Discussions were conducted to determine the effect of temperature difference coefficients, phase delay coefficients, and fractional factors on the propagation behavior of thermal and mechanical waves within the material. The numerical results in this study were compared with those in previous literature and were determined to be comparable and consistent. In addition, the results and observations showed a high degree of agreement between the analytical solutions and their numerical counterparts.

2 Derivation of the fractional thermoelasticity model

Fourier's law holds significant importance within heat conduction, serving as a foundational concept for comprehending the mechanisms by which heat is transmitted across solid and liquid mediums. This statement elucidates the correlation between heat flux $\mathcal{H}$, which represents the rate of heat transfer per unit area, and thermal gradient $\nabla\theta$, which denotes the spatial change of temperature within a material. The mathematical expression of Fourier's law can be represented as:

$$\mathcal{H}(\boldsymbol{r}, t) = -K\nabla\theta(\boldsymbol{r}, t). \tag{1}$$

The laws of energy conservation are essential notions within the fields of physics and engineering. These concepts articulate the fundamental law that energy is conserved, meaning it cannot be generated or destroyed. Instead, energy can only transform or be moved between other systems. The aforementioned concepts are encompassed under the laws of thermodynamics, which are widely recognized as fundamental and firmly established principles in physics. The equations that follow may be obtained using the increment in entropy denoted by η:

$$\mathrm{div}(\mathcal{H}) + Q = -\rho\dot{\eta}T_0, \tag{2}$$

$$\eta = \frac{\rho C_s}{\rho T_0}\theta + \frac{\gamma}{\rho}\mathrm{div}(u). \tag{3}$$

We may obtain the energy equation using Eqs. (2) and (3) as follows:

$$\rho C_e\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial}{\partial t}\mathrm{div}(u) = -\mathrm{div}(\mathcal{H}) + Q. \tag{4}$$

The following has changed the conventional form of Fourier's law (1), according to Quintanilla [21]:

$$\mathcal{H}(\boldsymbol{r}, t) = -K\nabla\phi(\boldsymbol{r}, t). \tag{5}$$

Chen and Gurtin [17] and Chen *et al.* [18,19] developed the theory of heat transfer in flexible structures. This theory depends on two different temperatures: the conductive temperature, ϕ, and the thermodynamical temperature, θ. The equation that describes the relationship between the two temperatures, θ and ϕ, may be written as [17–19]:

$$\theta = \phi - \beta\nabla^2\phi, \tag{6}$$

where β is the characteristic parameter to distinguish between temperatures.

Tzou [5,6] included the influence of microstructural influences on the delayed temporal response in the macroscopic formulation. This was achieved by considering the delayed rise in lattice temperature resulting from phonon–electron interactions at the macroscopic scale. A macroscopic lagging, or delayed, reaction between the temperature gradient and the heat flow vector seems like it could happen because of interactions that happen over time.

In this case, the usual Fourier law is replaced by a rough approximation that uses a changed version of the law. This approximation has two separate phase lags for the heat flow vector (τ_q) and the conductive temperature gradient (τ_ϕ):

$$\mathcal{H}(\boldsymbol{r}, t + \tau_q) = -K\nabla\phi(\boldsymbol{r}, t + \tau_\phi). \tag{7}$$

In order to obtain a close approximation of the modified heat transfer rule, the Taylor series expansion of Eq. (7) is utilized at a specific position $\boldsymbol{r}$ and time t, using only the first-order variables in τ_q and τ_ϕ:

$$\left(1 + \tau_q\frac{\partial}{\partial t}\right)\mathcal{H} = -K\left(1 + \tau_\phi\frac{\partial}{\partial t}\right)\nabla\phi. \tag{8}$$

According to Quintanilla [21], if $2\tau_\theta > \tau_q$ holds, then the system is (even exponentially) stable, but if $2\tau_\theta < \tau_q$ holds, then it is not (exponentially) stable.

The fractional heat equation is a partial differential equation (PDE) that extends the traditional heat equation by incorporating fractional derivatives. The given sentence explains how the temperature of a substance changes over time when non-local or fractional-order diffusion phenomena are considered. The fractional heat equation represents the heat conduction in materials with intricate characteristics and behavior.

The fractional heat equation is a mathematical model that describes the phenomenon of heat conduction, in which the local temperature gradient and previous temperature behavior in the surrounding area both impact the heat transfer rate at a given location. This phenomenon has special significance in materials characterized by fractal or multifractal features, as well as materials that display anomalous diffusion or sub-diffusion.

The established formulation for the C time-fractional derivative is expressed as follows:

$${}^{C}D_t^{(\alpha)}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\mathfrak{s})^{\alpha}}\frac{\mathrm{d}f(\mathfrak{s})}{\mathrm{d}\mathfrak{s}}\mathrm{d}\mathfrak{s},\ 0 < \alpha < 1. \tag{9}$$

The representation of the fractional derivative of fractional order α, as proposed by Caputo and Fabrizio [28], may be expressed as follows:

$${}^{CF}D_t^{(\alpha)}f(t) = \frac{1}{1-\alpha}\int_0^t \exp\left(-\frac{\alpha(t-\mathfrak{s})}{1-\alpha}\right)\frac{\mathrm{d}f(\mathfrak{s})}{\mathrm{d}\mathfrak{s}}\mathrm{d}\mathfrak{s}, \quad 0 < \alpha < 1. \tag{10}$$

The investigation into the development of fractional differentiation in latex has also brought to light the notion that the derivative introduced by Caputo and Fabrizio can be considered a filter rather than a true fractional derivative. This conclusion is based on the observation that the employed kernel is localized and may not adequately capture the intricate dynamics of the heat flow process. The idea of elasticity can be characterized by employing a non-local framework, exemplified by the extended Mittag–Leffler function.

The mathematical expression for the Atangana–Baleanu (AB) fractional time derivative is given by the following formula [29]:

$${}^{AB}D_t^{(\alpha)}f(t) = \frac{1}{1-\alpha}\int_0^t \mathrm{E}_\alpha\left(-\frac{\alpha}{1-\alpha}(t-\mathfrak{s})^{\alpha}\right)\frac{\mathrm{d}f(\mathfrak{s})}{\mathrm{d}\mathfrak{s}}\mathrm{d}\mathfrak{s}, \quad 0 < \alpha < 1. \tag{11}$$

The function $E_\alpha(z)$ represents a Mittag–Leffler function, which may be mathematically represented as

$$E_\alpha(z) = \sum_{r=0}^{\infty} \frac{z^r}{\Gamma(1 + r\alpha)}, \quad z \in \mathbb{C}, 0 < \alpha < 1. \tag{12}$$

By substituting a derivative of fractional order in place of the time derivative in Eq. (8), we may convert the Fourier law (8) into an improved formula of fractional order as

$$(1 + \tau_q^\alpha D_t^{(\alpha)})\mathcal{H} = -K(1 + \tau_\phi^\alpha D_t^{(\alpha)})\nabla\phi. \tag{13}$$

By performing the divergence operation on both sides of Eq. (4), the resulting equation may be obtained:

$$(1 + \tau_q^\alpha D_t^{(\alpha)})\mathrm{div}(\mathcal{H}) = -K(1 + \tau_\phi^\alpha D_t^{(\alpha)})\nabla^2\phi. \tag{14}$$

By combining Eq. (9) with the energy Eq. (4), we obtain the fractional two-temperature heat transfer equation:

$$K(1 + \tau_\phi^\alpha D_t^{(\alpha)})\nabla^2\phi = (1 + \tau_q^\alpha D_t^{(\alpha)})(\rho C_s\dot{\theta} + \gamma T_0\dot{e} - \rho Q). \tag{15}$$

The fractional heat equation has utility in several domains, such as the representation of heat propagation in intricate substances, including fractal media, porous materials, biological tissues, and polymers. Also, understanding heat transport in systems that display phenomena outside the scope of conventional diffusion models has considerable importance.

Furthermore, the stress–strain–temperature, the relationships describing the material properties, and the relation between strain and displacement for thermoelastic isotropic materials at a constant ambient temperature T_0 are given by

$$\sigma_{kl} = \lambda\mathrm{div}(u)\delta_{kl} + \mu(u_{l,k} + u_{k,l}) - \gamma\theta\delta_{kl}, \quad 2e_{kl} = u_{l,k} + u_{k,l}, \tag{16}$$

$$\mu u_{k,ll} + (\lambda + \mu)u_{l,kl} - \gamma\theta_{,k} + \mathcal{F}_k = \rho\ddot{u}_k. \tag{17}$$

In this article, we define the equilibrium point and give stability concepts and accompanying conclusions for fractional-order systems. Now, let us look at a generic kind of fractional differential equation that uses the AB fractional time derivative

$$^{AB}D_t^{(\alpha)}f(t) = g(t, f(t)), \quad g(0) = g_0, \tag{18}$$

where $0 < \alpha < 1$, and $g \in \mathbb{R}^n$.

The equilibrium point P is defined as a point of stability for the AB fractional-order system (18) if and only if the function $g(t, P) = 0$. Also, the local asymptotic stability of the equilibrium points of the fractional-order system (18) is determined by the satisfaction of the condition $|\arg(\lambda_i)| = \alpha\pi/2$ for all eigenvalues of the Jacobian matrix of the system, which are assessed at said equilibrium sites [53].

3 Special cases

Three previous models of thermal elasticity with one temperature can be derived without the discrepancy coefficient and fractional derivatives ($\beta = 0$, $\alpha = 1$). These models include the classical thermoelasticity (CTE) model ($\tau_q = \tau_\phi = 0$), the Lord and Shulman (LS) model with a single delay coefficient ($\tau_\phi = 0$), and the dual-phase lag model (DPL). Furthermore, in the event of the discrepancy factor being absent ($\beta = 0$) and considering the inclusion of fractional differential derivatives ($0 < \alpha < 1$), two distinct fractional models of thermoelasticity may be derived at a given temperature. These models are fractional Lord and Shulman (FLS) and fractional dual-phase lag (FDPL). Finally, when considering the incorporation of the model of thermoelasticity at two temperatures ($\beta > 0$) and the utilization of fractional operators with orders ($0 < \alpha < 1$), two further models for fractional thermoelasticity emerge. These models are two temperature fractional Lord and Shulman (2TFLS) and two temperature fractional dual-phase lag (2TFDPL).

4 Problem formulation

In order to assess the precision of the suggested model, its utilization in the context of a thermoelastic problem is being examined. This study will discuss the thermoelastic behavior of a half-space with infinite area ($x \geq 0$), where the x-axis is directed inward into the medium and is perpendicular to the surface $x = 0$ (Figure 1). The medium under investigation is initially at rest without pressure or strain and is at an initial temperature of T_0. A laser pulse of uniform intensity is directed into the medium at a level defined by the surface $x = 0$. In this study, a one-

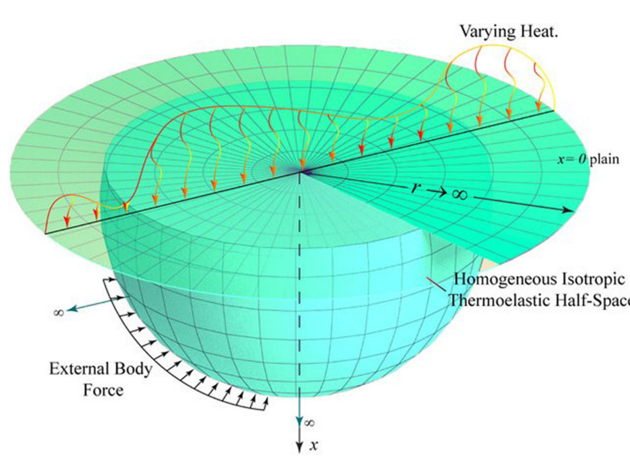

Figure 1: Representation diagram of an infinite half-space.

dimensional problem can be conceptualized as a dynamic problem within a half-space. Thus, all areas under investigation depend only on time and space variables, x and t.

The expression for the displacement components can be described as follows:

$$u_1 = u(x,t), \quad u_2 = 0, u_3 = 0. \tag{19}$$

Also, the expression for the only non-zero strain is provided by

$$e = \frac{\partial u(x,t)}{\partial x}. \tag{20}$$

In mechanics and physics, external body forces are those that come from outside of a body as opposed to those that come from internal tensions or interactions within the body. The forces in question may possess diverse components contingent upon the particular problem and the underlying physical phenomena at play. Forces can be considered to be exponentially fading with time in several different physical systems. In the present study, the selection of the components of the external body force can be determined as

$$\mathcal{F}_1 = e^{-\ell x}, \mathcal{F}_2 = 0, \mathcal{F}_3 = 0, \tag{21}$$

where the parameter $\ell > 0$ is responsible for determining the rate of decay.

Eqs. (6), (15), (16), and (18) can therefore be reduced to the following forms:

$$\theta = \phi - \beta\frac{\partial^2\phi}{\partial x^2}, \tag{22}$$

$$\sigma_{xx} = \sigma = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\left(\phi - \beta\frac{\partial^2\phi}{\partial x^2}\right), \tag{23}$$

$$(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial\theta}{\partial x} + \rho e^{-\ell x} = \rho\frac{\partial^2 u}{\partial t^2}, \tag{24}$$

$$K(1 + \tau_\phi^\alpha D_t^{(\alpha)})\frac{\partial^2\phi}{\partial x^2} = (1 + \tau_q^\alpha D_t^{(\alpha)})\left(\rho C_s\frac{\partial\theta}{\partial t} + \gamma T_0\frac{\partial^2 u}{\partial t\partial x}\right). \tag{25}$$

In physical and mathematical modeling, dimensionless quantities are essential for simplifying and comparing systems or circumstances. The subsequent dimensionless amounts are introduced to streamline the resolution of the problem at hand:

$$\begin{aligned} &x' = \chi c_0 x,\ u' = \chi c_0 u,\ \sigma' = \frac{1}{\rho c_0^2}\sigma, \\ &\{\tau_q', \tau_\phi'\} = c_0^2\chi\{\tau_q, \tau_\phi\}, \\ &t' = c_0^2\chi t,\ \beta' = \chi^2 c_0^2\beta, \quad \{\theta', \phi'\} = \frac{\gamma}{\rho c_0^2}\{\theta, \phi\}, \\ &\mathcal{F}' = \frac{\rho}{\chi\rho c_0^3}\mathcal{F}, \end{aligned} \tag{26}$$

where $c_0^2 = \frac{\lambda + 2\mu}{\rho}$ and $\chi = \frac{\rho C_s}{K}$.

Eqs. (22)–(25) can be rewritten in non-dimensional forms by utilizing Eq. (26) while disregarding the primes

$$\theta = \phi - \beta\frac{\partial^2\phi}{\partial x^2}, \tag{27}$$

$$\sigma = \frac{\partial u}{\partial x} - \phi + \beta\frac{\partial^2\phi}{\partial x^2}, \tag{28}$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \frac{\partial\theta}{\partial x} - e^{-\ell x}, \tag{29}$$

$$(1 + \tau_\phi^\alpha D_t^{(\alpha)})\frac{\partial^2\phi}{\partial x^2} = (1 + \tau_q^\alpha D_t^{(\alpha)})\left(\frac{\partial\theta}{\partial t} + \varepsilon\frac{\partial^2 u}{\partial t\partial x}\right), \tag{30}$$

where $\varepsilon = \frac{\gamma^2 T_0}{\rho^2 c_0^2 C_s}$.

The initial conditions, along with the conditions of regularity, are provided by

$$\begin{aligned} &\theta = \phi = u = \sigma = 0, \text{ at } \quad t = 0, \\ &\frac{\partial\theta}{\partial t} = \frac{\partial\phi}{\partial t} = \frac{\partial u}{\partial t} = 0, \text{ at } \quad t = 0, \\ &\theta = \phi = u = \sigma = 0, \text{ at } \quad x \to \infty. \end{aligned} \tag{31}$$

Since the upper part of the medium is not subjected to traction and is loaded with heat, the following boundary conditions are applicable:

$$\sigma(x,t) = 0, \text{ at } \quad x = 0, \tag{32}$$

$$\phi(x,t) = h(t), \text{ at } \quad x = 0, \tag{33}$$

where

$$h(t) = \begin{cases} 0, & t \le 0, \\ \frac{\phi_0}{t_0}t, & 0 < t \le t_0, \\ \phi_0, & t > t_0, \end{cases} \tag{34}$$

where ϕ_0 and t_0 are the constants.

5 Methodology used to solve the problem

The Laplace transform is a useful mathematical tool for analyzing and solving differential equations in various scientific and engineering disciplines, especially for time-varying linear systems. This method allows complex systems to be solved by transferring them from the space–time domain to the space domain only. The integral representation for the Laplace transforms of a function $g(x,t)$, designated as $\bar{g}(x,s)$, is expressed as follows:

$$\bar{g}(x,s)=\int_0^\infty g(x,t)e^{-st}\mathrm{d}t. \tag{35}$$

Transforming Eqs. (23)–(26), we obtain

$$\bar{\theta}=\bar{\phi}-\beta\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}x^2}, \tag{36}$$

$$\bar{\sigma}=\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}-\bar{\phi}+\beta\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}x^2}, \tag{37}$$

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}x^2}-s^2\bar{u}=\frac{\mathrm{d}\theta}{\mathrm{d}x}-\frac{1}{s}e^{-\ell x}, \tag{38}$$

$$\ell_\phi\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}x^2}=\ell_q\left(\bar{\phi}-\beta\frac{\mathrm{d}^2\bar{\phi}}{\mathrm{d}x^2}+\varepsilon\frac{\mathrm{d}\bar{u}}{\mathrm{d}x}\right), \tag{39}$$

where

$$\ell_q=1+\tau_q^\alpha\frac{s^\alpha}{\alpha+s^\alpha(1-\alpha)},\qquad \ell_\phi=1+\tau_\phi^\alpha\frac{s^\alpha}{\alpha+s^\alpha(1-\alpha)}. \tag{40}$$

Eqs. (38) and (39) can be rewritten as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\beta\frac{\mathrm{d}^4}{\mathrm{d}x^4}\right)\bar{\phi}=\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-s^2\right)\bar{e}-\alpha_1e^{-\ell x}, \tag{41}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}-\alpha_2\right)\bar{\phi}=\alpha_3\bar{e}, \tag{42}$$

where

$$\alpha_1=\frac{\ell}{s},\qquad \alpha_2=\frac{\ell_q}{\beta\ell_q+\ell_\phi},\qquad \alpha_3=\frac{\varepsilon\ell_q}{\beta\ell_q+\ell_\phi}. \tag{43}$$

By eliminating the variable $\bar{e}$ from Eqs. (41) and (42), the following DF may be derived:

$$\left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}-\ell\frac{\mathrm{d}^2}{\mathrm{d}x^2}+m\right)\bar{\phi}=\alpha_4e^{-\ell x}, \tag{44}$$

where

$$\ell=\frac{s^2+\alpha_2+\alpha_3}{1+\beta\alpha_3},\qquad m=\frac{s^2\alpha_2}{1+\beta\alpha_3},\qquad \alpha_4=\frac{\alpha_1\alpha_3}{1+\beta\alpha_3}. \tag{45}$$

The solution of Eq. (44) that is guaranteed to remain finite as x approaches infinity is expressed as follows:

$$\bar{\phi}(x,s)=\mathcal{M}_1\mathrm{e}^{-k_1x}+\mathcal{M}_2\mathrm{e}^{-k_2x}+\mathcal{M}_3e^{-\ell x}, \tag{46}$$

where $\mathcal{M}_3=\alpha_4/(\ell^4-\ell\ell^2+m)$ and $\mathcal{M}_1$ and $\mathcal{M}_2$ are the integration coefficients.

Moreover, the coefficients k_1 and k_1 represent the solutions to the equation

$$k^4-\ell k^2+m=0. \tag{47}$$

By substituting the value of (46) into Eqs. (36) and (42), we can obtain the following expressions:

$$\bar{e}(x,s)=\sum_{i=1}^{2}\frac{(k_i^2-\alpha_2)}{\alpha_3}\mathcal{M}_i\mathrm{e}^{-k_ix}+\frac{(\ell^2-\alpha_2)}{\alpha_3}\mathcal{M}_3e^{\ell x}, \tag{48}$$

$$\bar{\theta}(x,s)=\sum_{i=1}^{2}(1-\beta k_i^2)\mathcal{M}_i\mathrm{e}^{-k_ix}+(1-\beta\ell^2)\mathcal{M}_3e^{-\ell x}. \tag{49}$$

By substituting Eq. (48) into Eq. (20), the resulting expression yields the converted displacement:

$$\bar{u}(x,s)=-\sum_{i=1}^{2}\frac{(k_i^2-\alpha_2)}{k_i\alpha_3}\mathcal{M}_i\mathrm{e}^{-k_ix}-\frac{(\ell^2-\alpha_2)}{\ell\alpha_3}\mathcal{M}_3e^{\ell x}. \tag{50}$$

Furthermore, by substituting Eqs. (46) and (48) into Eq. (37), we obtain

$$\begin{aligned}\bar{\sigma}(x,s)=&\sum_{i=1}^{2}\left[\frac{(k_i^2-\alpha_2)}{\alpha_3}-(1-\beta k_i^2)\right]\mathcal{M}_i\mathrm{e}^{-k_ix}\\&+\left[\frac{(\ell^2-\alpha_2)}{\alpha_3}-(1-\beta\ell^2)\right]\mathcal{M}_3e^{-\ell x}.\end{aligned} \tag{51}$$

In Eqs. (32) and (33), the Laplace transform is employed to obtain the modified boundary conditions as

$$\begin{aligned}&\bar{\sigma}(x,s)=0\quad\text{at}\quad x=0,\\&\bar{\phi}(x,s)=\frac{\phi_0(1-e^{-st_0})}{s^2t_0}=\bar{\mathcal{R}}(s)\quad\text{at}\quad x=0.\end{aligned} \tag{52}$$

By substituting the expressions for $\bar{\phi}$ and $\bar{\sigma}$ provided in Eqs. (41) and (46) into the boundary conditions (47), the integral coefficients $\mathcal{M}_1$ and $\mathcal{M}_2$ can be determined.

6 Numerical methods for Laplace transform inversions

The inverse Laplace transform is employed to obtain the initial function from its corresponding Laplace transform. Numerical methods for Laplace transform inversions are used to approximate the inverse Laplace transform of a given function. There exist multiple techniques for quantitatively comparing the inverse Laplace transform. When conducting numerical inversions of Laplace transforms, it is crucial to consider the selection of numerical methods, the desired level of precision, and the processing resources at hand. The choice of a suitable methodology is contingent upon the particular situation and the characteristics exhibited by the function that has undergone the Laplace transformation.

The Bromwich integral is a widely used technique employed in the numerical inversion of Laplace transforms. The process encompasses the utilization of contour integration inside the complicated plane. The inverse Laplace transform can be approximated by picking a suitable contour and employing numerical integration algorithms. Further information regarding these strategies can be located in the work of Honig and Hirdes [54].

7 Numerical results

The present section aims to present a realistic illustration to substantiate the precision of the existing fractional thermoelastic model incorporating phase lags and two temperatures. In order to facilitate numerical analyses, we have adopted the copper material coefficients as the values to be utilized [55]:

$$C_E = 383.1\ \mathrm{J\,kg^{-1}\,K^{-1}},\quad T_0 = 293\ \mathrm{K},\quad \alpha_t = 1.78\times10^{-5}\ \mathrm{K^{-1}},$$
$$K = 386\ \mathrm{Wm^{-1}\,K^{-1}},\quad t = 0.12\ \mathrm{s},\quad \lambda = 7.76\times10^{10}\ \mathrm{Nm^{-2}},$$
$$\mu = 3.86\times10^{10}\ \mathrm{Nm^{-2}},\quad \varphi_0 = 1,\quad \rho = 8{,}954\ \mathrm{kgm^{-3}}.$$

We provide a selection of illustrations to illustrate and enhance our understanding of observable physical events. To analyze the effect of phase lag and the coefficient of variation of the two temperatures on measurable thermophysical quantities in the range of $0 \le x \le 5$, the corresponding graphs are plotted. In addition, the results obtained can be arranged in a tabular manner to facilitate the use of these results by other researchers, enabling them to conduct comparative analyses and verify the accuracy of their results. Based on the aforementioned physical coefficients, the field quantities are calculated using MATHEMATICA programming and visually shown through graphical means. The simulations were conducted for three different scenarios.

Case I: The effect of the temperature discrepancy

Thermoelasticity with dual temperatures represents a non-classical framework within the field of thermomechanics, specifically addressing the behavior of elastic solids. The main difference between this theory and the traditional model is its dependence on temperature. This analysis will examine the impact of temperature discrepancies on the thermo-physical variables under study. In this particular scenario, the utilization of a FDPL thermoelastic model is employed, which incorporates the AB fractional derivative operator. It should be noted that the value of $\beta = 0$ represents the one-temperature fractional DPL model (1TFDPL), whereas a value of $\beta > 0$ represents the 2TFDPL. Figures 2–5 exhibit the variances of thermodynamic and conductive temperatures, θ and ϕ, as well as displacement u and normal stress σ, respectively, for various values of distances x. In addition, for the purpose of computation, we have assumed that $\tau_\phi = 0.02$, $\tau_q = 0.05$, $t_0 = 0.1$, and $\alpha = 0.85$.

The observation of limited transmission of wave speeds can be deduced from the analysis of tables and figures. Furthermore, it can be observed from the presented tables and figures that the various models exhibit notable variations in values close to the surface borders. These discrepancies gradually diminish as the distance from the boundaries increases. This trend can be attributed to the influence of thermal shock exerted on the stress-free boundary. The conductive temperature pattern exhibits a peak value at the surface $x = 0$, indicating thermal loading, and subsequently diminishes progressively until reaching zero. It is observed that the thermal stress σ becomes negligible at the surface $x = 0$, consistent with the limited mechanical condition described in Eq. (32) for the discussed problem.

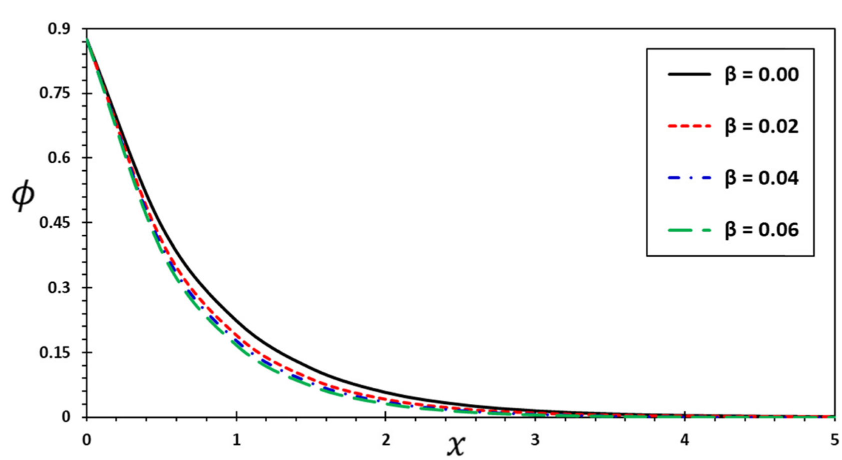

Figure 2: Influence of discrepancy index β on the variation of conductive temperature ϕ.

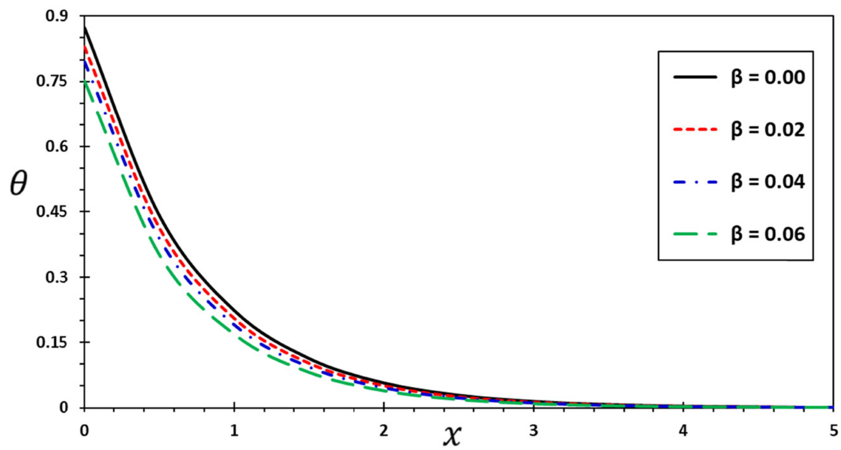

Figure 3: Influence of discrepancy index β on the variation of thermodynamic temperature θ.

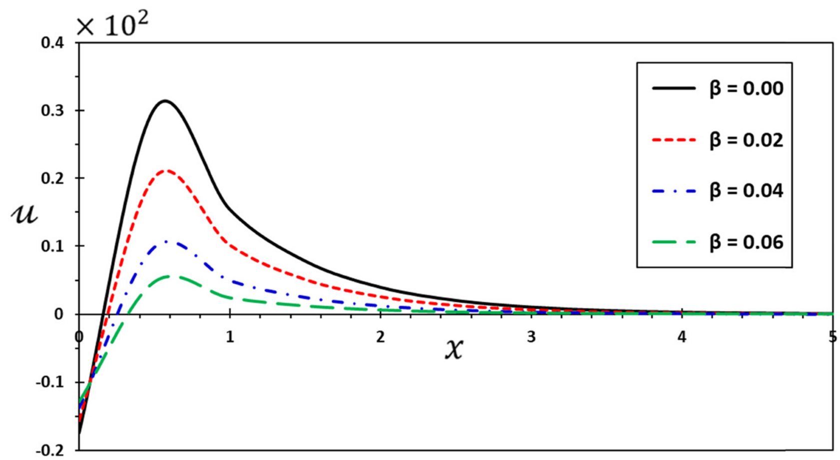

Figure 4: Influence of discrepancy index β on the variation of displacement u.

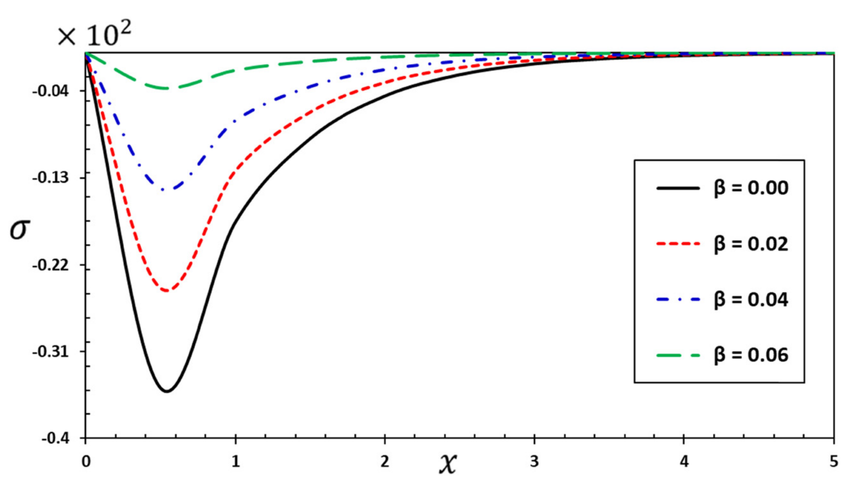

Figure 5: Influence of discrepancy index β on the variation of stress σ.

Figure 2 shows a constant decreasing trend of the field ϕ with increasing depth into the medium. The figure also shows that the amplitude of the function ϕ in the case of the 1TFDPL model at $\beta = 0$ exceeds the amplitude of the 2TFDPL model with two temperatures. In addition, the decay rate of the ϕ curve associated with the 1TFDPL model is higher compared to the histogram representing the 2FDPL model. Furthermore, it has been observed that the parameter "β," which represents the two-temperature index, exerts a substantial impact on the fields under investigation. Therefore, based on the findings, it is crucial to distinguish between thermodynamic and conductive temperatures.

Figure 3 shows the effect of the two temperature difference indexes on the thermodynamic temperature change θ. Figure 3 shows the relationship between θ and the spatial coordinate x in both the 1FDPL and 2FDPL models, considering two different values of $\beta = 0$ (representing a single-type temperature) and $\beta > 0$ (representing a two-type temperature). The observed actions in this figure exhibit similarities to those depicted in Figure 2. As one moves deeper into the medium, the dynamic and conductive temperatures decrease from their maximum values at the surface. These results are consistent with the physical properties of thermoelasticity and confirm the validity of the generalized models [56]. Based on the proposed novel framework, it is imperative to establish a revised categorization scheme for materials based on their two temperature parameters. These parameters serve as a novel metric for evaluating the thermal conductivity of materials [57].

Figure 4 depicts the graphical representations of the displacement u as a function of x for the 1TFDPL and 2TFDPL systems. The displacement field exhibits both expansion and compression deformations. This phenomenon arises due to the occurrence of thermal shock, leading to the expansion of the half-space areas next to the boundary edge in an unconstrained direction. First, this results in negative displacement, which gradually transitions from negative to positive. Eventually, the displacement reaches a steady state and becomes zero at the heat wavefront [58]. The smallest displacement values are obtained using the 2TFDPL model with ($\beta = 0$), whereas the biggest displacement values are obtained using the 1TFDPL theory (Figure 4).

Figure 4 illustrates the fluctuation of the stress σ with respect to the variable x. The normal stress σ exhibits a high cost across all values of t. The stress differences have similar characteristics for both models, while disparities become more pronounced at higher temporal values. The stress σ first assumes a zero value due to the boundary condition, then experiences a rapid increase, and ultimately approaches zero. It is worth mentioning that the radial stress increases as the parameter β grows, regardless of the values of x. The provided graphic illustrates that the stress experienced at the boundary is compressive.

Case II: Comparison of fractional differentiation operators

Various mathematical strategies are employed to solve non-integer-order models of PDEs in mathematical analysis. The mathematical theory of fractional calculus deals with nonlocal integration and differentiation. Fractional calculus has found extensive application in mathematical modeling and several physical phenomena. Numerous scholars have recently addressed thermoelastic difficulties in heat transmission under diverse mechanical and thermal circumstances by utilizing fractional C derivatives and Riemann–Liouville (RL) fractional derivatives. The AB time-fractional derivative is a recent and adapted version of the Caputo–Fabrizio (CF) technique. It possesses desirable characteristics such as non-singularity, non-locality of the kernel, prominent memory influences, and transmission effects.

This study introduces a thermal conductivity model incorporating fractional differential operators ($D_t^{(\alpha)}$) and two distinct temperatures. In the present section, we will conduct a comparative analysis of fractional derivatives AB, CF, and C, focusing on the potential benefits associated with fractional derivative AB in relation to fractional derivatives CF and C. The impact of certain fractional differentiation operators ($D_t^{(\alpha)}$) and the fractional parameter α on system fields such as temperature and displacement was analyzed graphically and tabularly, respectively. The study also included a comparison of thermoelastic models incorporating fractional derivatives and those incorporating integer derivatives ($\alpha = 1$). Furthermore, for the purpose of computational analysis, it has been postulated that the values of τ_ϕ, τ_q, t_0, and β are 0.02, 0.05, 0.1, and 0.001, respectively.

- The new FDPL thermoelastic model appears to improve the temperature field, as shown in Table 1. This improvement can be attributed to the incorporation of fractional operators, which effectively decrease thermal diffusion within the medium. This observation aligns with both experimental findings and established physical facts.
- It was found that the two-temperature thermoelastic C fractional model with phase delays (2TFDPL-C) has much higher thermodynamic and conductive temperatures as the fractional parameter is raised. This is in contrast to the CF fractional model (2TFDPL-CF) and the AB fractional model (2TFDPL-AB).
- It is imperative to acknowledge that the transmission of thermal and mechanical waves within an elastic medium is contingent upon the specific fractional-order derivative operator utilized. Hence, it can be inferred that the

Table 1: Dynamic temperature distribution under the influence of different fractional differential operators

x	2TDPL	2TFDPL-C		2TFDPL-CF		2TFDPL-AB	
	$\alpha = 1$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$
0.0	0.826009	0.792505	0.7609570	0.731233	0.703209	0.676771	0.651815
0.5	0.423119	0.405957	0.3897960	0.374570	0.360215	0.346673	0.333889
1.0	0.215114	0.206389	0.1981730	0.190432	0.183134	0.176249	0.169750
1.5	0.109364	0.104928	0.1007510	0.096816	0.093105	0.089605	0.086301
2.0	0.055601	0.053346	0.0512222	0.049222	0.047335	0.045556	0.043876
2.5	0.028268	0.027122	0.0260421	0.025025	0.024066	0.023161	0.022307
3.0	0.014373	0.013790	0.0132407	0.012724	0.012236	0.011776	0.011342
3.5	0.007308	0.007012	0.00673259	0.006470	0.006222	0.005988	0.005767
4.0	0.003716	0.003566	0.00342388	0.003290	0.003164	0.003045	0.002933
4.5	0.001890	0.001814	0.00174174	0.001674	0.001610	0.001549	0.001492
5.0	0.000962	0.000923	0.000886547	0.000852	0.000819	0.000789	0.000760

Table 2: Conductive temperature distribution under the influence of different fractional differential operators

x	2TDPL	2TFDPL-C		2TFDPL-CF		2TFDPL-AB	
	$\alpha = 1$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$
0.0	0.874464	0.838995	0.8055960	0.774128	0.744460	0.716472	0.690051
0.5	0.443852	0.425848	0.4088960	0.392924	0.377866	0.363660	0.350249
1.0	0.225652	0.216499	0.2078810	0.199761	0.192105	0.184883	0.178065
1.5	0.114721	0.110068	0.1056860	0.101558	0.097666	0.093994	0.090528
2.0	0.058325	0.055959	0.0537314	0.051633	0.049654	0.047787	0.046025
2.5	0.029653	0.028450	0.0273178	0.026251	0.025245	0.024296	0.023400
3.0	0.015076	0.014465	0.0138893	0.013347	0.012835	0.012353	0.011898
3.5	0.007666	0.007355	0.00706229	0.006787	0.006527	0.006281	0.006050
4.0	0.003898	0.003740	0.00359149	0.003451	0.003319	0.003194	0.003077
4.5	0.001983	0.001903	0.00182696	0.001756	0.001688	0.001625	0.001565
5.0	0.001009	0.000968	0.00092987	0.000894	0.000859	0.000827	0.000797

Table 3: Displacement distribution under the influence of different fractional differential operators

x	2TDPL	2TFDPL-C		2TFDPL-CF		2TFDPL-AB	
	$\alpha = 1$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$
0.0	−0.12601	−0.12505	−0.1095700	−0.091230	−0.08032	−0.06768	−0.05182
0.5	0.379483	0.364091	0.3495980	0.335943	0.323068	0.310923	0.299458
1.0	0.193224	0.185387	0.1780080	0.171055	0.164500	0.158316	0.152478
1.5	0.098241	0.094256	0.0905047	0.086970	0.083637	0.080494	0.077526
2.0	0.049951	0.047926	0.0460186	0.044222	0.042527	0.040929	0.039420
2.5	0.025402	0.024372	0.0234020	0.022488	0.021627	0.020815	0.020048
3.0	0.012920	0.012397	0.0119039	0.011439	0.011002	0.010588	0.010199
3.5	0.006575	0.006309	0.0060583	0.005822	0.005600	0.005390	0.005191
4.0	0.003349	0.003214	0.00308639	0.002966	0.002853	0.002746	0.002646
4.5	0.001709	0.001640	0.00157544	0.001514	0.001457	0.001403	0.001351
5.0	0.000875	0.000840	0.000807237	0.000776	0.000747	0.000719	0.000693

fractional arrangement considers the variability of the elastic medium, as each operator and fractional order signifies a distinct characteristic of elasticity.
- The aforementioned theoretical findings possess potential utility in addressing some practical issues. Based on the theoretical findings and empirical evidence, selecting a fractional mathematical model that exhibits congruence between the experimental findings and theoretical outcomes is possible.
- Tables 1–4 showcase the empirical results, which suggest that the power-law kernel exhibits a resilient memory effect in long-term historical data. On the other hand, non-singular kernels may be more suitable for describing relaxation or diffusion processes characterized by exceptionally strong memory.
- It is demonstrated that the decay rate of the model with a stretched exponential kernel, which is of the CF type, is much quicker than that of the model with a power-law kernel. This indicates that a CF-type derivative model with a stretched exponential kernel, as opposed to one with a power-law kernel, may be used to describe a wider variety of relaxation events.
- It is important to acknowledge that a mathematical model employing the AB-type derivative with a stretched Mittag–Leffler function kernel exhibits a slower diffusive motion compared to a model utilizing an exponential kernel.
- Furthermore, it is highly anticipated that the novel fractional derivatives of AB and CF will significantly contribute to investigating the macroscopic characteristics of particular materials associated with nonlocal exchanges. These exchanges predominantly govern the properties of these materials.

8 Conclusions

The physical aspects of relaxation and thermoelastic heat transfer models are studied in this work. It focuses on the newly introduced temporal derivative that uses a non-power function kernel. The use of analytical analysis shows that the C-type derivative idea, when combined with a power function kernel, is not good at describing the non-exponential changes that are often seen in asymmetrical heat conduction. Because of this, a new version of the thermoelasticity theory has been made that includes two temperatures, two separate phase delays, and fractional-order differential operators that are not in a single position. The revised definition's primary benefit is replacing the singular power-law kernel with a Mittag–Leffler function and a non-singular exponential kernel. This replacement facilitates using the revised definition in theoretical analysis, numerical calculations, practical scenarios, and real-world applications.

The results show that the field variables, such as temperature, displacement, and thermal stress, are affected by the two-temperature parameter (β) and fractional-order differential operators, as well as the position (x) and time (t). The impact of fractional differential operators on various field quantities is particularly important in academic research. All the investigated field quantities are considerably impacted by fractional differential operators. The fractional coefficient characterizes the thermal conductivity of a thermoelastic material, necessitating the development of new categories of materials. Moreover, there is a great expectation that the unique fractional derivatives proposed by AB and CF will make a substantial contribution to the study of the macroscopic properties of specific materials that involve nonlocal interactions.

Table 4: Thermal stress distribution under the influence of different fractional differential operators

x	2TDPL	2TFDPL-C		2TFDPL-CF		2TFDPL-AB	
	$\alpha = 1$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.75$
0.0	0	0	0	0	0	0	0
0.5	−0.4218	−0.40469	−0.388578	−0.37340	−0.35909	−0.34559	−0.33285
1.0	−0.21474	−0.20603	−0.197825	−0.19010	−0.18281	−0.17594	−0.16945
1.5	−0.10918	−0.10475	−0.100580	−0.09665	−0.09295	−0.08945	−0.08616
2.0	−0.05551	−0.05326	−0.0511408	−0.04914	−0.04726	−0.04548	−0.04381
2.5	−0.02823	−0.02708	−0.0260063	−0.02499	−0.02403	−0.02313	−0.02228
3.0	−0.01436	−0.01378	−0.0132280	−0.01271	−0.01223	−0.01177	−0.01133
3.5	−0.00731	−0.00701	−0.00673156	−0.00647	−0.00622	−0.00599	−0.00577
4.0	−0.00372	−0.00357	−0.00342877	−0.00330	−0.00317	−0.00305	−0.00294
4.5	−0.0019	−0.00182	−0.00174961	−0.00168	−0.00162	−0.00156	−0.00150
5.0	−0.00097	−0.00093	−0.000895892	−0.00086	−0.00083	−0.00080	−0.00077

These interactions primarily regulate the characteristics of these materials.

The AB and CF fractional operators offer reliable mathematical frameworks for comprehensively capturing nonlocal interactions inside materials. These operators empower researchers to construct advanced models that account for the impact of remote events on the macroscopic features of the material. By integrating these fractional operators into the governing equations, scholars are able to adequately consider the long-range correlations and spatial dependencies that are inherent in materials with nonlocal interactions. This enables a more precise depiction of their macroscopic behavior.

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