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Research Article

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Study of fractional telegraph equation *via* Shehu homotopy perturbation method

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Abstract: The iterative Shehu transform homotopy perturbation method (HPM) is used in the present research to address fractional telegraph equations in different dimensions, respectively. Considered equations particularly stand out in the field of material science and certain other significant fields. A graphic comparison of estimated and actual results is used to assess the validity and efficacy of the suggested technique. Graphs show a match of approximate to exact findings. Without any linearization or discretization, the iterative Shehu HPM offers a reliable and efficient way to deliver approximations and accurate outcomes that is also error-free. The development of numerical regimes based on discretization is difficult and expensive computationally. Additionally, discretization error is produced as a result of discretization in purely numerical regimes. The present regime has produced robust results and is time-efficient. Also, no discretization error was produced.

Keywords: fractional hyperbolic telegraph equations, Shehu transform, homotopy perturbation method, error analysis, numerical convergence

1 Introduction

Numerous areas of the physical sciences, including diffusion, control processes, elasticity, relaxation processes, and many others, use fractional calculus extensively [1–3]. The telegraph equation is employed in the reaction–diffusion process, as well as in signal observation for the transmission and propagation of electrical signals [4,5]. Many researchers have offered various types of solutions for fractional telegraph equations. Momani utilized Adomian decomposition method (ADM) [6]. Yildrim [7] put homotopy perturbation method (HPM) into the application. The variable separable technique was incorporated by Chen *et al.* [8]. Variation iteration method (VIM) was implemented by Sevimlican [9]. For the fractional hyperbolic telegraph (HT) equation, Khan *et al.* [10] adopted natural ADM, combining natural transform and ADM to produce a fusion. Fractional VIM was utilized by Jassim and Shahab [11] to solve the HT equation. The Shehu transform was used for the fractional HT equation in one dimension (1D), two dimension (2D), and three dimension (3D) by Kapoor *et al.* [12]. The concept of Laplace ADM in the Caputo sense was applied by Khan *et al.* [13] to cope with the fractional-order telegraph equation. The ECB-spline technique was utilized by Akram *et al.* [14] to solve the fractional telegraph problem. Regarding the pseudo HT equation, Modanli [15] provided compatibility between Caputo and Atangana–Baleanu fractional derivatives. The Legendre collocation method was applied to the fractional telegraph equation by Mishra *et al.* [16]. To obtain the precise analytical solution of fractional-order telegraph equations, Khan *et al.* [17] employed the triple Laplace transform.

1.1 Shehu transform

Integral transformation is necessary to effectively answer mathematical issues. Dealing with several partial differential equations (PDEs) is also made easier *via* an integral transform that has been proven to work. The integral transform is a straightforward method for managing numerous complex PDEs. Many studies on integral transformations have been conducted in recent years. Sumudu transformation, Elzaki transformation, natural transformation, Pourreza transformation, G transformation, Sawi transformation, Shehu transform, *etc.*, are a few examples of integral transformations. Some integral equations, ordinary differential equations, PDEs, and fractional PDEs (FPDEs) can be solved using transformations described in literature [18–23].

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Result 1

Iterative Shehu transform of fractional derivative in Caputo sense:

$$S[D_t^{\zeta}\theta] = \left(\frac{s}{v}\right)^{\zeta} S[\theta] - \sum_{r=0}^{\theta=1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^{r}(0). \qquad (1)$$

Result 2

The iterative Shehu transformation has been notified as follows:

$$S[Q(t)] = \int_0^{\infty} e^{\left(-\frac{st}{v}\right)} Q(t)\mathrm{d}t.$$

Laplace transform would be converted from iterative Shehu transform *via* considering $v = 1$.

Iterative Shehu transformation will lead to Yang transformation *via* considering $s = 1$.

Result 3

Considered $S[Q(t)] = J(s, v)$ and $S^{-1}[J(s, v)] = Q(t)$, then

$$Q(t) = S^{-1}[J(s, v)] = \frac{1}{2\pi i}\int_{-\beta+\infty}^{\beta+\infty} \frac{e^{-st}}{v} J(s, v)\mathrm{d}s.$$

where s and v are reported as Shehu transform variables.

Result 4

$$S[\sinh(at)] = \int_0^{\infty} \exp\left(-\frac{st}{v}\right)\sinh(at)\mathrm{d}t,$$

$$S[\sinh(at)] = \int_0^{\infty} \exp\left(-\frac{st}{v}\right)\frac{(e^{at} - e^{-at})}{2}\mathrm{d}t,$$

$$= \lim_{t\to\infty}\frac{v}{2(a^2v^2 - s^2)}\left[e^{\frac{t(av-s)}{v}}av + e^{-\frac{(av+s)t}{v}}av + e^{\frac{t(av-s)}{v}}s - e^{-\frac{(av+s)t}{v}}s - 2av\right]$$

$$= \frac{v^2}{(s^2 - v^2)}.$$

Result 5

$$S[\cosh(at)] = \int_0^{\infty} \exp\left(-\frac{st}{v}\right)\cosh(at)\mathrm{d}t,$$

$$S[\cosh(at)] = \int_0^{\infty} \exp\left(-\frac{st}{v}\right)\quad\frac{(e^{at} + e^{-at})}{2}\mathrm{d}t,$$

$$= \lim_{t\to\infty}\frac{v}{2(a^2v^2 - s^2)}\left[e^{\frac{t(av-s)}{v}}av - e^{-\frac{(av+s)t}{v}}av + e^{\frac{t(av-s)}{v}}s + e^{-\frac{(av+s)t}{v}}s - 2s\right]$$

$$= \frac{sv}{(s^2 - a^2v^2)}.$$

Result 6 Linearity property of Shehu transform.

$$S[c_1 f_1(t) + c_2 f_2(t)] = c_1 S[f_1(t)] + c_2 S[f_2(t)].$$

Proof

$$S[f(t)] = \int_0^{\infty}\exp\left(-\frac{st}{v}\right)f(t)\mathrm{d}t,$$

$$S[c_1 f_1(t) + c_2 f_2(t)] = \int_0^{\infty}\exp\left(-\frac{st}{v}\right)[c_1 f_1(t) + c_2 f_2(t)]\mathrm{d}t,$$

$$= c_1\int_0^{\infty}\exp\left(-\frac{st}{v}\right)f_1(t)\mathrm{d}t + c_2\int_0^{\infty}\exp\left(-\frac{st}{v}\right)f_2(t)\mathrm{d}t$$

$$= c_1 S[f_1(t)] + c_2 S[f_2(t)].$$

Proved.

Result 7 Shehu transform of single derivative of *f(t)*.

Proof

$$S[f(t)] = \int_0^{\infty}\exp\left(-\frac{st}{v}\right)f(t)\mathrm{d}t,$$

$$S[f'(t)] = \int_0^{\infty}\exp\left(-\frac{st}{v}\right)f'(t)\mathrm{d}t$$

$$= \left[e^{-\frac{st}{v}}f(t)\right]_0^{\infty} - \int_0^{\infty}\frac{\mathrm{d}}{\mathrm{d}t}e^{-\frac{st}{v}}f(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}}f(t)\right]_0^{\infty} - \int_0^{\infty}\left(-\frac{s}{v}\right)e^{-\frac{st}{v}}f(t)\mathrm{d}t$$

$$= \left[e^{-\frac{st}{v}}f(t)\right]_0^{\infty} + \left(\frac{s}{v}\right)\int_0^{\infty}e^{-\frac{st}{v}}f(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}}f(t)\right]_0^{\infty} + \left(\frac{s}{v}\right)S[f(t)]$$

$$= \lim_{t\to\infty}e^{-\frac{st}{v}}f(t) - f(0) + \left(\frac{s}{v}\right)S[f(t)],$$

$$S[f'(t)] = \left(\frac{s}{v}\right)S[f(t)] - f(0).$$

Proved.

Result 8 Shehu transform of double derivative of *f(t)*.

Proof

$$S[f(t)] = \int_0^\infty \exp\left(-\frac{st}{v}\right) f(t)\mathrm{d}t,$$

$$S[f''(t)] = \int_0^\infty \exp\left(-\frac{st}{v}\right) f''(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f'(t)\right]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} e^{-\frac{st}{v}} f'(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f'(t)\right]_0^\infty - \int_0^\infty \left(-\frac{s}{v}\right) e^{-\frac{st}{v}} f'(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f'(t)\right]_0^\infty + \left(\frac{s}{v}\right) \int_0^\infty e^{-\frac{st}{v}} f'(t)\mathrm{d}t,$$

$$= \lim_{t\to\infty} e^{-\frac{st}{v}} f'(t) - f'(0) + \left(\frac{s}{v}\right)\left[\frac{s}{v} S[f(t)] - f(0)\right],$$

$$S[f''(t)] = \frac{s^2}{v^2} S[f(t)] - \frac{s}{v} f(0) - f'(0).$$

Proved.

Result 9 Shehu transform of triple derivative of *f(t)*.

Proof

$$S[f(t)] = \int_0^\infty \exp\left(-\frac{st}{v}\right) f(t)\mathrm{d}t,$$

$$S[f'''(t)] = \int_0^\infty \exp\left(-\frac{st}{v}\right) f'''(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f''(t)\right]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t} e^{-\frac{st}{v}} f''(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f''(t)\right]_0^\infty - \int_0^\infty \left(-\frac{s}{v}\right) e^{-\frac{st}{v}} f''(t)\mathrm{d}t,$$

$$= \left[e^{-\frac{st}{v}} f''(t)\right]_0^\infty + \left(\frac{s}{v}\right) \int_0^\infty e^{-\frac{st}{v}} f''(t)\mathrm{d}t,$$

$$= \lim_{t\to\infty} e^{-\frac{st}{v}} f''(t) - f''(0) + \left(\frac{s}{v}\right)\left[\frac{s^2}{v^2} S[f(t)] - \frac{s}{v} f(0) - f'(0)\right],$$

$$S[f'''(t)] = \frac{s^3}{v^3} S[f(t)] - \frac{s^2}{v^2} f(0) - \frac{s}{v} f'(0) - f''0).$$

Proved.

Result 10 Shehu transform of nth derivative of *f(t)*.

Proof similarly,

$S[f^n(t)] = \frac{s^n}{v^n} S[f(t)] - \frac{s^{n-1}}{v^{n-1}} f(0) - \frac{s^{n-2}}{v^{n-2}} f'(0) - \ldots - f^{n-1}(0).$

Proved.

In Tables 1 and 2, charts are provided regarding Shehu transform and iterative Shehu transform.

1.2 HPM

He [18] first proposed the idea of HPM and then combined the conventional perturbation regime with the Homotopy regarding topology by producing a Homotopy (convex in nature). He then found the solution to the problem, which was then presented in series form and converged to the precise solution. He [24–26] used HPM to address non-linear issues in applied sciences, such as the Duffing equation and the Ear Drum equation, among others.

A Chinese scholar named He [27] launched HPM. He combined the conventional perturbation regime with topological Homotopy, which is used to resolve several crucial equations. Fundamentally, Homotopy is a concept borrowed from topology and differential geometry [28]. The French mathematician Poincare referred to the idea of Homotopy [29].

For HPM, the following equation is considered:

$$D(u) = 0. \tag{2}$$

Any convex homotopy deformation $H(u, p)$ in the case where D is regarded as a differential operator is as follows:

$$H(u, p) = (1 - p)F(u) + pD(u), \tag{3}$$

where u_0 is the known solution to $F(u)$, which is regarded as a basic operator.

In such a method, the equation's solution is given as a power series and embedding parameter *″p″* is originally employed.

$$U = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots + p^nu_n, \tag{4}$$

$$U = \lim_{p\to 1} U, \tag{5}$$

$$U = u_0 + u_1 + u_2 + u_3 + \ldots + u_n, \tag{6}$$

$$U = \lim_{p\to 1} U, \tag{7}$$

$$U = \sum_{n=0}^\infty u_n. \tag{8}$$

1.3 Time-fractional HT equation

Due to their relevance in numerous engineering and scientific sectors, FPDEs have recently become most crucial topic from the viewpoint of researchers and scientists. The fractional derivatives have a very high degree of flexibility, which results in an outstanding tool for expressing the varied past and inherited traits of the numerous prototypes. For the development of analytical and numeric outcomes of linear and non-linear FPDEs, large-scale research is conducted.

Fractional hyperbolic telegraph equation in 1D [10]. Some more references relevant to fractional calculus are provided in previous literature [30–37].

$$\theta_t^{\zeta} + \rho\,\theta(x,t) + \nu\theta_t = \theta_{xx} + \phi(x,t).$$

$\rho,\ \nu \rightarrow$ arbitrary constants. $\theta(x,t)$ is an unknown function.

For $\rho > 0,\ \nu = 0$, damp wave equation model would be retrieved.

For $\rho > 0,\ \nu > 0$, telegraph equation model would be retrieved.

In signal processing for transmission of electrical impulses and wave theory processes, the telegraph equation model is primarily and most frequently utilized. The biomedical sciences and aerospace have seen a number of these models implemented. Problems involving fractional derivatives are attracting the attention of researchers. Fractional-order PDEs are modelled specifically by linear PDEs of integer order. For outcomes of integer-order techniques, fractional-order schemes converge. Some latest references are provided in previous literature [38–48].

Fractional HT equation in 2D [10].

$$D_t^{2\zeta}\theta + 2\zeta D_t^{\zeta} + \beta^2\theta = \theta_{xx} + \theta_{yy} + g(x,y,t).$$

I.C.: $\theta(x,y,0) = f_1(x,y)$ and $\theta_t(x,y,0) = f_2(x,y)$.

Fractional HT equation in 3D [10].

$$D_t^{2\zeta}\theta + 2\zeta D_t^{\zeta}\theta + \beta^2\theta = \theta_{xx} + \theta_{yy} + \theta_{zz} + g(x,y,z,t).$$

I.C.: $\theta(x,y,z,0) = f_1(x,y,z)$ and $\theta_t(x,y,z,0) = f_2(x,y,z)$.

1.3.1 Motivation of the study

There are many literary solutions to fractional telegraph equation in 1D, 2D, or 3D , but there are very few approaches that address the fractional telegraph equation in all three dimensions. The development of a method that can demonstrate the reliability of the estimated analytical solutions to the aforementioned equations in one, two, and three dimensions is therefore the main goal. In the present study, an iterative approach to tackle fractional HT equation in 1D, 2D, and 3D is presented. The present approach is simple to use and does not require intricate numerical discretization programming. It is difficult to create numerical programs for FPDEs, hence creating such iterative techniques is necessary to obtain approximate analytical solutions. Numerous transforms are offered in literature; however, from the perspective of calculation, some transforms are simple to use and others are not. Among all the available integral transforms, Shehu transform HPM is regarded as one of the simplest to use. Due to the significance of these equations, this research concentrates on their solution, maintaining the study's uniqueness. Additionally, the article includes assessments of convergence and error.

1.3.2 Research questions

1Q. What is the novelty of this work?

Reply – In the present study, a numerical convergence aspect is discussed in detail regarding the proposed regime. Moreover, graphical and tabular compatibility of results are also validated.

2Q. Why only Shehu transform is used in this study?

Reply – Some integral transforms are simple to use from a calculation standpoint, whereas others are more difficult. The Shehu transform is one of the simplest integral transforms, hence it is being used.

3Q. Why is any discretization-based technique not implemented?

Reply – The development of numerical regimes based on discretization is difficult and expensive computationally. Additionally, discretization error is produced as a result of discretization.

1.3.3 Outline of the research work

The present study is framed into diversified sections to create a better notion regarding the work done. In Section 2, general formulas are developed for fractional Telegraph equation in 1D. In Section 3, five examples are evaluated regarding fractional HT equations in different dimensions. In Section 4, graphical and tabular analyses of the work are provided along with numerical convergence. Section 5 is related to the concluding remarks.

2 Implementation of the regime

Implementation upon 1D fractional Telegraph equation.

$$D_t^{\zeta}u(x,t) + L[u(x,t)] + N[u(x,t)] = q(x,t).$$

Applying Shehu transform

$$S[D_t^{\zeta}u(x,t)] = -S[L[u(x,t)] - N[u(x,t)] + q(x,t)],$$

$$\left(\frac{s}{v}\right)^{\zeta} S[u(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1} = -S[Lu(x,t) + Nu(x,t)] + S[q(x,t)],$$

$$\left(\frac{s}{v}\right)^{\zeta} S[u(x,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1} - S[Lu(x,t) + Nu(x,t)] + S[q(x,t)],$$

$$S[u(x,t)] = \left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1} - \left(\frac{v}{s}\right)^{\zeta}[S[Lu(x,t) + Nu(x,t)] + S[q(x,t)]],$$

$$u(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\right] - S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}[S[Lu(x,t) + Nu(x,t)] + S[q(x,t)]]\right].$$

Applying HPM

$$\sum_{n=0}^{\infty} p^n u_n = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\right] - pS^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left[S\left[L\left(\sum_{n=0}^{\infty} p^n u_n\right) + \sum_{n=0}^{\infty} p^n H_n(u)\right] + S[q(x,t)]\right]\right],$$

where $N[u(x,t)] = \sum_{n=0}^{\infty} p^n H_n(u)$.

Comparing p^0

$$u_0(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1} - S[q(x,t)]\right],$$

Comparing p^1

$$u_1(x,t) = -S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}[S[L(u_0) + H_0(u)]]\right],$$

Comparing p^2

$$u_2(x,t) = -S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}[S[L(u_1) + H_1(u)]]\right],$$

Comparing p^3

$$u_3(x,t) = -S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}[S[L(u_2) + H_2(u)]]\right],$$

and so on.

In Figure 1, steps regarding the solution of the proposed regime are provided.

Uniqueness and convergence theorems

Theorem 1. *Let X be a Banach space and let $\theta_m(\Omega_1, \alpha_1)$ and $\theta_n(\Omega_1, \alpha_1)$ be in X. Suppose $\gamma \in (0, 1)$, then the series solution $\{\theta_m(\Omega_1, \alpha_1)\}_{m=0}^{\infty}$ which is defined converges to the lower bound solution whenever $\theta_m(\Omega_1, \alpha_1) \le \gamma\theta_{m-1}(\Omega_1, \alpha_1)$, $\forall m > N$, that is for any given $\varepsilon > 0$, there exists a positive number N, such that $||\theta_{m+n}(\Omega_1, \alpha_1)|| \le \epsilon$, $\forall m, n > N$.*

Proof provided.

$$\begin{aligned}
M_0(\Omega_1, \alpha_1) &= \theta_0(\Omega_1, \alpha_1),\\
M_1(\Omega_1, \alpha_1) &= \theta_0(\Omega_1, \alpha_1) + \theta_1(\Omega_1, \alpha_1),\\
M_2(\Omega_1, \alpha_1) &= \theta_0(\Omega_1, \alpha_1) + \theta_1(\Omega_1, \alpha_1) + \theta_2(\Omega_1, \alpha_1),\\
M_3(\Omega_1, \alpha_1) &= \theta_0(\Omega_1, \alpha_1) + \theta_1(\Omega_1, \alpha_1) + \theta_2(\Omega_1, \alpha_1) + \theta_3(\Omega_1, \alpha_1),\\
&\dots\\
M_m(\Omega_1, \alpha_1) &= \theta_0(\Omega_1, \alpha_1) + \theta_1(\Omega_1, \alpha_1) + \theta_2(\Omega_1, \alpha_1) + \theta_3(\Omega_1, \alpha_1) + \dots + \theta_m(\Omega_1, \alpha_1).
\end{aligned}$$

The aim is to prove that $M_m(\Omega_1, \alpha_1)$ is a Cauchy sequence in the Banach space.

It is provided that for $\gamma \in (0,1)$,

$$\begin{aligned}
\| M_{m+1}(\Omega_1, \alpha_1) - M_m(x, \alpha_1)\| = \| \theta_{m+1}(\Omega_1, \alpha_1)\| &\le \gamma \| \theta_m(\Omega_1, \alpha_1)\|\\
&\le \gamma^2 \| \theta_{m-1}(\Omega_1, \alpha_1)\|\\
&\le \gamma^3 \| \theta_{m-2}(\Omega_1, \alpha_1)\|\\
&\dots\\
&\le \gamma^{m+1}\|\theta_0(\Omega_1, \alpha_1)\|
\end{aligned}$$

Let us find

$$\begin{aligned}
&\| M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\|\\
&= \| M_m(\Omega_1, \alpha_1) - M_{m-1}(\Omega_1, \alpha_1) + M_{m-1}(\Omega_1, \alpha_1) - M_{m-2}(\Omega_1, \alpha_1)\\
&\quad + M_{m-2}(\Omega_1, \alpha_1) - M_{m-3}(\Omega_1, \alpha_1) + \dots + M_{n+1}(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\|
\end{aligned}$$

$$\begin{aligned}
&\| M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\|\\
&\le ||M_m(\Omega_1, \alpha_1) - M_{m-1}(\Omega_1, \alpha_1)|| + \| M_{m-1}(\Omega_1, \alpha_1) - M_{m-2}(\Omega_1, \alpha_1)\|\\
&\quad + \| M_{m-2}(\Omega_1, \alpha_1) - M_{m-3}(\Omega_1, \alpha_1)\| + \dots + \| M_{n+1}(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\|
\end{aligned}$$

$$\begin{aligned}
&\| M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\|\\
&= \gamma^m \| \theta_0(\Omega_1, \alpha_1)\| + \gamma^{m-1} \| \theta_0(\Omega_1, \alpha_1)\| + \gamma^{m-2} \| \theta_0(\Omega_1, \alpha_1)\|\\
&\quad + \gamma^{m-3} \| \theta_0(\Omega_1, \alpha_1)\| + \dots + \gamma^{n+1} \| \theta_0(\Omega_1, \alpha_1)\|
\end{aligned}$$

$$\| M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\| \le \frac{(1-\gamma^{m-n})}{(1-\gamma)}\gamma^{n+1} \| \theta_0(\Omega_1, \alpha_1)\|.$$

Considered $\epsilon = \frac{1-\gamma}{(1-\gamma^{m-n})\gamma^{n+1} \| \theta_0(\Omega_1, \alpha_1) \|}$,

$$||M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)|| < \epsilon,$$

$$\lim_{m,n\to\infty} \| M_m(\Omega_1, \alpha_1) - M_n(\Omega_1, \alpha_1)\| = 0.$$

$\Rightarrow \{M_m\}_{m=0}^{\infty}$ is a Cauchy sequence.

Theorem 2. Let $\sum_{i=0}^{j}\theta_i(\Omega_1, \alpha_1)$ *be finite and* $\theta_i(\Omega_1, \alpha_1)$ *be its approximate solution. Suppose* $\gamma > 0$, *such that* $||\theta_{i+1}(\Omega_1, \alpha_1)|| \leq \gamma \, || \theta_i(\Omega_1, \alpha_1)||$, $\gamma \in (0,1)$, $\forall i$, *then the max. absolute error for the lower bound solution is*

$$|| \underline{\theta}(\Omega_1, \alpha_1) - \sum_{i=0}^{j}\theta_i(\Omega_1, \alpha_1)|| \leq \frac{\gamma^{j+1}}{1-\gamma} \, || \theta_0(\Omega_1, \alpha_1)||.$$

Proof Let $\sum_{i=0}^{j}\underline{\theta}_i(\Omega_1, \alpha_1) < \infty$, then

$$|| \underline{\theta}(\Omega_1, \alpha_1) - \sum_{i=0}^{j}\theta_i(\Omega_1, \alpha_1)||$$
$$= || \sum_{i=j+1}^{\infty} \theta_i(\Omega_1, \alpha_1)||$$
$$\leq \sum_{i=j+1}^{\infty} || \theta_i(\Omega_1, \alpha_1)||$$
$$\leq \sum_{i=j+1}^{\infty} \gamma^i \, || \theta_0(\Omega_1, \alpha_1)||$$
$$\leq || \theta_0(\Omega_1, \alpha_1)||[\gamma^{j+1} + \gamma^{j+2} + \gamma^{j+3} + \ldots]$$
$$\leq \frac{|| \theta_0(\Omega_1, \alpha_1)|| \, \gamma^{j+1}}{1-\gamma}.$$

3 Examples and discussion

Example 1

Considered fractional 1D HT equation is as follows [12]:

$$D_t^{\zeta} \, \theta = \theta - 2\, \theta_t - \theta_{xx}. \tag{9}$$

I.C.: $\theta(x, 0) = e^x$ and $\theta_t(x, 0) = -2e^x$, $0 < \zeta \leq 2$.

Applying Shehu transform upon Eq. (9)

$$S[D_t^{\zeta}\theta] = S[\theta - 2\theta_t - \theta_{xx}],$$

$$\Rightarrow \left(\frac{s}{v}\right)^{\zeta} S[\theta(x, t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0) = S[\theta - 2\theta_t - \theta_{xx}],$$

$$\Rightarrow \left(\frac{s}{v}\right)^{\zeta} S[\theta(x, t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0) = S[\theta - 2\theta_t - \theta_{xx}],$$

$$\Rightarrow S[\theta(x, t)] = \left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0) + \left(\frac{v}{s}\right)^{\zeta} S[\theta - 2\theta_t - \theta_{xx}],$$

$$\Rightarrow \theta(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta - 2\theta_t - \theta_{xx}]\right].$$

Applying HPM

$$\Rightarrow \sum_{n=0}^{\infty} p^n\theta_n(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0)\right] + pS^{-1}\left[\left(\frac{v}{s}\right)^{\alpha} S\left[\left(\sum_{r=0}^{\theta-1} p^n\theta_n(x, t)\right) - 2\left(\sum_{r=0}^{\theta-1} p^n\theta_n(x, t)\right)_t - \left(\sum_{r=0}^{\theta-1} p^n\theta_n(x, t)\right)_{xx}\right]\right].$$

Comparing p^0:

$$\Rightarrow \theta_0(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\sum_{n=0}^{\infty}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x, 0)\right].$$

Considered $\theta = 1$:

$$\Rightarrow \theta_0(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta(0)\right],$$

$$\Rightarrow \theta_0(x, t) = \theta(0),$$

$$\Rightarrow \theta_0(x, t) = \theta(x, 0) + t\theta_t(x, 0) \Rightarrow \theta_0(x, t) = (1 - 2t)e^x.$$

Comparing p^1:

$$\Rightarrow \theta_1(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[(\theta_0) - 2(\theta_0)_t - (\theta_0)_{xx}]\right],$$

where

$$\theta_0 = (\theta_0)_x = (\theta_0)_{xx},$$

$$\Rightarrow \theta_1(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[-2(\theta_0)_t]\right] \Rightarrow \theta_1(x, t) = 4e^x S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(\frac{v}{s}\right)\right],$$

$$\Rightarrow \theta_1(x, t) = 4e^x S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta+1}\right] \Rightarrow \theta_1(x, t) = 4e^x\frac{t^{\zeta}}{\Gamma(\alpha+1)}.$$

Comparing p^2

$$\Rightarrow \theta_2(x, t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[(\theta_1) - 2(\theta_1)_t - (\theta_1)_{xx}]\right], \text{ where}$$
$$\theta_1 = (\theta_1)_x = (\theta_1)_{xx},$$

$$\Rightarrow \theta_2(x, t) = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[(\theta_1)_t]\right],$$

$$\Rightarrow \theta_2(x, t) = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S\left[4e^x\zeta\frac{t^{\zeta-1}}{\Gamma(\alpha+1)}\right]\right],$$

$$\Rightarrow \theta_2(x, t) = -8e^x\frac{\zeta}{\Gamma(\zeta+1)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[t^{\zeta-1}]\right],$$

$$\Rightarrow \theta_2(x, t) = -8e^x\frac{\zeta}{\Gamma(\zeta+1)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(\Gamma\zeta\left(\frac{v}{s}\right)^{\zeta}\right)\right],$$

$$\Rightarrow\theta_2(x,t) = -8e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\right]$$

$$\Rightarrow\ \theta_2(x,t) = -8e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{t^{2\zeta-1}}{\Gamma(2\zeta)}.$$

Comparing p^3

$$\Rightarrow\theta_3(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\alpha}S[(u_2) - 2(u_2)_t - (u_2)_{xx}]\right],$$

where

$$\theta_2 = (\theta_2)_x = (\theta_2)_{xx},$$

$$\Rightarrow\theta_3(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[-2(\theta_2)_t]\right],$$

$$\Rightarrow\theta_3(x,t) = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S\left[-8e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}t^{2\zeta-2}\right]\right],$$

$$\Rightarrow\ \theta_3(x,t) = 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[t^{2\zeta-2}]\right],$$

$$\Rightarrow\ \theta_3(x,t) = 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\Gamma(2\zeta-1)\left(\frac{v}{s}\right)^{2\zeta-1}\right],$$

$$\Rightarrow\theta_3(x,t) = 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\Gamma(2\zeta-1)\left(\frac{v}{s}\right)^{3\zeta-1}\right],$$

$$\Rightarrow\theta_3(x,t) = 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^{3\zeta-1}\right],$$

$$\Rightarrow\theta_3(x,t) = 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{t^{3\zeta-2}}{\Gamma(3\zeta-1)}.$$

Comparing p^4

$$\Rightarrow\theta_4(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[(\theta_3) - 2(\theta_3)_t - (\theta_3)_{xx}]\right],$$

where

$$\theta_3 = (\theta_3)_x = (\theta_3)_{xx},$$

$$\Rightarrow\theta_4(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[-2(\theta_3)_t]\right],$$

$$\Rightarrow\theta_4(x,t) = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S\left[16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}t^{3\zeta-3}\right]\right],$$

$$\Rightarrow\theta_4(x,t) = -32e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}S^{-1}\left[\Gamma(3\zeta-2)\left(\frac{v}{s}\right)^{4\zeta-3}\right],$$

$$\Rightarrow\ \theta_4(x,t) = -32e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)}S^{-1}\left[\left(\frac{v}{s}\right)^{4\zeta-3}\right],$$

$$\Rightarrow\theta_4(x,t) = -32e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{t^{4\zeta-2}}{\Gamma(4\zeta-3)}.$$

Comparing p^5

$$\Rightarrow\ \theta_5(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[(\theta_4) - 2(\theta_4)_t - (\theta_4)_{xx}]\right],$$

where

$$\theta_4 = (\theta_4)_x = (\theta_4)_{xx},$$

$$\Rightarrow\theta_5(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[-2(\theta_4)_t]\right],$$

$$\Rightarrow\theta_5(x,t) = 64e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)} \times \frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{(4\zeta-3)}{\Gamma(4\zeta-3)} S^{-1}\left[\Gamma(4\zeta-2)\left(\frac{v}{s}\right)^{5\zeta-2}\right],$$

$$\Rightarrow\theta_5(x,t) = 64e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)} \times \frac{(4\zeta-3)}{\Gamma(4\zeta-3)}\frac{\Gamma(4\zeta-2)}{\Gamma(5\zeta-2)}t^{5\zeta-3}.$$

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \ldots$$

$$\Rightarrow\ \theta = (1-2t)e^x + 4e^x\frac{t^{\zeta}}{\Gamma(\zeta+1)} - 8e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{t^{2\zeta-1}}{\Gamma(2\zeta)} + 16e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{t^{3\zeta-2}}{\Gamma(3\zeta-1)} - 32e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)} \times \frac{t^{4\zeta-2}}{\Gamma(4\zeta-3)} + 64e^x\frac{\zeta\Gamma\zeta}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)\Gamma(3\zeta-2)}{\Gamma(3\zeta-1)} \times \frac{(4\zeta-3)}{\Gamma(4\zeta-3)}\frac{\Gamma(4\zeta-2)}{\Gamma(5\zeta-2)}t^{5\zeta-3}.$$

Considered $\zeta = 2$: $\Rightarrow \theta(x,t) = \exp[x-2t]$.

Example 2

Considered fractional 1D HT equation as follows [12]:

$$D_t^{\zeta}\theta = \theta - \theta_t - \theta_{xx}. \tag{10}$$

I.C.: $\theta(x,0) = e^x$ and $\theta_t(x,0) = -e^x$; $0 < \zeta \le 2$.
Applying Shehu transform upon Eq. (10)

$$\Rightarrow S[D_t^\zeta \theta] = S[\theta - \theta_t - \theta_{xx}],$$

$$\Rightarrow \left(\frac{s}{v}\right)^\zeta S[\theta(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0) = S[\theta - \theta_t - \theta_{xx}],$$

$$\Rightarrow \left(\frac{s}{v}\right)^\zeta S[\theta(x,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0) + S[\theta - \theta_t - \theta_{xx}],$$

$$\Rightarrow S[\theta(x,t)] = \left(\frac{v}{s}\right)^\zeta \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0) + \left(\frac{v}{s}\right)^\zeta S[\theta - \theta_t - \theta_{xx}],$$

$$\Rightarrow S[\theta(x,t)] = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[\theta - \theta_t - \theta_{xx}]\right],$$

$$\Rightarrow \theta(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[\theta - \theta_t - \theta_{xx}]\right].$$

Applying HPM

$$\Rightarrow \sum_{n=0}^{\infty} p^n\theta_n(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \sum_{n=0}^{\infty}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S\left[\left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right) - \left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right)_t - \left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right)_{xx}\right]\right].$$

Comparing p^0

$$\Rightarrow \theta_0(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta^r(x,0)\right].$$

Considered $\theta = 1$

$$\Rightarrow \theta_0(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \left(\frac{s}{v}\right)^{\zeta-1}\theta(x,0)\right],$$

$$\Rightarrow \theta_0(x,t) = \theta(x,0)\, S^{-1}\left[\frac{v}{s}\right] \Rightarrow \theta_0(x,t) = \theta(0),$$

$$\Rightarrow \theta_0(x,t) = \theta(x,0) + t\,\theta_t(x,0) \Rightarrow \theta_0(x,t) = (1-t)e^x.$$

Comparing p^1

$$\Rightarrow \theta_1 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[(\theta_0) - (\theta_0)_t - (\theta_0)_{xx}]\right],$$

$$\Rightarrow \theta_1 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[e^x]\right] \Rightarrow \theta_1 = e^x S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta+1}\right] \Rightarrow \theta_1 = e^x\frac{t^\zeta}{\Gamma(\alpha+1)}.$$

Comparing p^2

$$\Rightarrow \theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[(\theta_1) - (\theta_1)_t - (\theta_1)_{xx}]\right],$$

$$\Rightarrow \theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[-(\theta_1)_t]\right],$$

$$\Rightarrow \theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S\left[-\left(e^x\frac{\alpha\, t^{\zeta-1}}{\Gamma(\zeta+1)}\right)\right]\right],$$

$$\Rightarrow \theta_2 = -e^x\frac{\zeta}{\Gamma(\zeta+1)}S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[t^{\zeta-1}]\right],$$

$$\Rightarrow \theta_2 = -e^x\frac{\zeta}{\Gamma(\zeta+1)}S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \Gamma(\alpha)\left(\frac{v}{s}\right)^\zeta\right],$$

$$\Rightarrow \theta_2 = -e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{t^{2\zeta-1}}{\Gamma(2\zeta)}.$$

Comparing p^3

$$\Rightarrow \theta_3 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[(\theta_2) - (\theta_2)_t - (\theta_2)_{xx}]\right],$$

$$\Rightarrow \theta_3 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S\left[-\left(-e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)t^{2\zeta-2}}{\Gamma(2\zeta)}\right)\right]\right],$$

$$\Rightarrow \theta_3 = e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[t^{2\zeta-2}]\right],$$

$$\Rightarrow \theta_3 = e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^\zeta \Gamma(2\zeta-1)\left(\frac{v}{s}\right)^{2\zeta-1}\right],$$

$$\Rightarrow \theta_3 = e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}S^{-1}\left[\left(\frac{v}{s}\right)^{3\zeta-1}\right],$$

$$\Rightarrow \theta_3 = e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{t^{3\zeta-2}}{\Gamma(3\zeta-1)}.$$

Comparing p^4

$$\Rightarrow \theta_4 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S[(\theta_3) - (\theta_3)_t - (\theta_3)_{xx}]\right],$$

$$\Rightarrow \theta_4 = S^{-1}\left[\left(\frac{v}{s}\right)^\zeta S\left[-\left(e^x\frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{t^{3\zeta-2}}{\Gamma(3\zeta-1)}\right)\right]\right],$$

$$\Rightarrow \theta_4 = -e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[t^{3\zeta-3}]\right],$$

$$\Rightarrow \theta_4 = -e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\Gamma(3\zeta-2)\left(\frac{v}{s}\right)^{3\zeta-2}\right],$$

$$\Rightarrow \theta_4 = -e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}S^{-1}\left[\Gamma(3\zeta-2)\left(\frac{v}{s}\right)^{4\zeta-2}\right],$$

$$\Rightarrow \theta_4 = -e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{\Gamma(3\zeta-2)}{\Gamma(4\zeta-2)}\,t^{4\zeta-3}.$$

Comparing p^5

$$\Rightarrow \theta_5 = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[(\theta_4)-(\theta_4)_t-(\theta_4)_{xx}]\right],$$

$$\Rightarrow \theta_5 = S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[-(\theta_4)_t]\right],$$

$$\Rightarrow \theta_5 = e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{\Gamma(3\zeta-2)}{\Gamma(4\zeta-2)}(4\zeta-3)\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[(t^{4\zeta-4})]\right],$$

$$\Rightarrow \theta_5 = e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{\Gamma(3\zeta-2)}{\Gamma(4\zeta-2)}(4\zeta-3)\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\Gamma(4\zeta-3)\left(\frac{v}{s}\right)^{4\zeta-3}\right],$$

$$\Rightarrow \theta_5 = e^x \frac{\zeta\,\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{\Gamma(3\zeta-2)}{\Gamma(4\zeta-2)} \times (4\zeta-3)\,\Gamma(4\zeta-3) S^{-1}\left[\left(\frac{v}{s}\right)^{5\zeta-3}\right],$$

$$\Rightarrow \theta_5 = e^x \frac{\zeta\Gamma(\zeta)}{\Gamma(\zeta+1)}\frac{(2\zeta-1)\Gamma(2\zeta-1)}{\Gamma(2\zeta)}\frac{(3\zeta-2)}{\Gamma(3\zeta-1)}\frac{\Gamma(3\zeta-2)}{\Gamma(4\zeta-2)} \times \frac{(4\zeta-3)\Gamma(4\zeta-3)}{\Gamma(5\zeta-3)}\,t^{5\zeta-4}.$$

Considered $\zeta = 1$

$$\Rightarrow \theta(x,t) = \exp[x-t].$$

Example 3

Considered fractional 1D HT equation as follows [12]:

$$D_t^{2\zeta}\,\theta + 2\,D_t^{\zeta}\,\theta + \theta = \theta_{xx}. \tag{11}$$

I.C.: $\theta(x,0) = e^x$ and $\theta_t(x,0) = -2\,e^x$.

Applying Shehu transform upon Eq. (11)

$$\Rightarrow S[D_t^{2\zeta}\,\theta + 2\,D_t^{\zeta}\,\theta + \theta] = S[\theta_{xx}],$$

$$\Rightarrow S[D_t^{2\zeta}\,\theta] = S[\theta_{xx} - \theta - 2\,D_t^{\zeta}\,\theta\,],$$

$$\Rightarrow \left(\frac{s}{v}\right)^{2\zeta} S[\theta(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0) = S[\theta_{xx} - \theta - 2\,D_t^{\zeta}\theta],$$

$$\Rightarrow \left(\frac{s}{v}\right)^{2\zeta} S[\theta(x,t)] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0) + S[\theta_{xx} - \theta - 2D_t^{\zeta}\theta],$$

$$\Rightarrow S[\theta(x,t)] = \left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0) + \left(\frac{v}{s}\right)^{2\zeta} S[\theta_{xx} - \theta - 2\,D_t^{\zeta}\,\theta\,],$$

$$\Rightarrow \theta(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[\theta_{xx} - \theta - 2\,D_t^{\zeta}\,\theta]\right],$$

$$\Rightarrow \theta(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[\theta_{xx} - \theta]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,\theta]\right].$$

Applying HPM

$$\Rightarrow \sum_{n=0}^{\infty} p^n\theta_n(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S\left[\left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right)_{xx} - \left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right)\right]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S\left[D_t^{\zeta}\left(\sum_{n=0}^{\infty} p^n\theta_n(x,t)\right)\right]\right].$$

Comparing p^0

$$\Rightarrow \theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left(\frac{s}{v}\right)^{2\zeta-1}\theta(0)\right],$$

$$\Rightarrow \theta_0 = \theta(0) S^{-1}\left[\frac{v}{s}\right] \Rightarrow \theta_0 = \theta(0),$$

$$\Rightarrow \theta_0 = \theta(x,0) + t\,\theta_t(x,0) \Rightarrow \theta_0 = (1-2t)e^x.$$

Comparing p^1

$$\Rightarrow \theta_1 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_0)_{xx} - (\theta_0)]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_0)]\right],$$

where $\theta_0 = (\theta_0)_{xx}$,

$$\Rightarrow\theta_1 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_0)]\right],$$

$$\Rightarrow\theta_1 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_0] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_0^r(x,0)\right\}\right],$$

Considered $\theta = 1$

$$\Rightarrow\theta_1 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_0(0)\right\}\right],$$

$$\Rightarrow\theta_1 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_0(0)\right\}\right],$$

Where $S[\theta_0] = S[(1-2t)e^x] = e^x\left[\left(\frac{v}{s}\right) - 2\left(\frac{v}{s}\right)^2\right]$,

$$\Rightarrow\theta_1 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta}\left(e^x\left[\left(\frac{v}{s}\right) - 2\left(\frac{v}{s}\right)^2\right]\right) - \left(\frac{s}{v}\right)^{\zeta-1} e^x\right\}\right],$$

$$\Rightarrow\theta_1 = -2e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta}\left(\left[\left(\frac{v}{s}\right) - 2\left(\frac{v}{s}\right)^2\right]\right) - \left(\frac{s}{v}\right)^{\zeta-1}\right\}\right],$$

$$\Rightarrow\theta_1 = 4e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{v}{s}\right)^{2-\zeta}\right\}\right],$$

$$\Rightarrow\theta_1 = 4e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{2+\zeta}\right] \Rightarrow \theta_1 = 4e^x\frac{t^{\zeta}}{\Gamma(\zeta+2)}.$$

Comparing p^2

$$\Rightarrow\theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_1)_{xx} - (\theta_1)]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_1)]\right],$$

where

$$\theta_1 = (\theta_1)_{xx},$$

$$\Rightarrow\theta_2 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_1)]\right],$$

where

$$S[D_t^{\zeta}\theta_1] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_1(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_1^r(0),$$

$$\Rightarrow\theta_2 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_1(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_1^r(0)\right\}\right],$$

$$\Rightarrow\theta_2 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_1] - \left(\frac{v}{s}\right)^{\zeta+1}\theta_1(0)\right],$$

$$\Rightarrow\theta_2 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(4\,e^x\left(\frac{v}{s}\right)^{\zeta+2}\right)\right] \Rightarrow \theta_2 = -8e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta+2}\right],$$

$$\Rightarrow\theta_2 = -8e^x\frac{t^{2\zeta+1}}{\Gamma(2\,\zeta+2)}.$$

Comparing p^3

$$\Rightarrow\theta_3 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_2)_{xx} - (\theta_2)]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\alpha}\,(\theta_2)]\right],$$

where

$$S[D_t^{\zeta}\,(\theta_2)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_2] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_2(0),$$

$$\Rightarrow\theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_2)]\right],$$

$$\Rightarrow\theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_2] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_2(0)\right\}\right],$$

$$\Rightarrow\theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_2] - \left(\frac{v}{s}\right)^{\zeta+1}\theta_2(0)\right],$$

$$\Rightarrow\theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_2]\right] \Rightarrow \theta_3 = 16\,e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(\frac{v}{s}\right)^{2\zeta+2}\right],$$

$$\Rightarrow\theta_3 = 16\,e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{3\zeta+2}\right] \Rightarrow \theta_3 = 16\,e^x\frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}.$$

Comparing p^4

$$\Rightarrow\theta_4 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_3)_{xx} - (\theta_3)]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_3)]\right],$$

where

$$S[D_t^{\zeta}(\theta_3)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_3] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_3(0),$$

$$\Rightarrow\theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_3] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_3(0)\right\}\right],$$

$$\Rightarrow\theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_3]\right\}\right],$$

$$\Rightarrow\theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_3]\right] \Rightarrow \theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S\left[16\,e^x\frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}\right]\right],$$

$$\Rightarrow\theta_4 = -32\,e^x\,S^{-1}\left[\left(\frac{v}{s}\right)^{4\zeta+2}\right] \Rightarrow \theta_4 = -32\,e^x\frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}.$$

Comparing p^5

$$\Rightarrow\theta_5 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_4)_{xx} - (\theta_4)]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\,(\theta_4)]\right],$$

where

$$S[D_t^{\zeta}(\theta_4)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_4] - \left(\frac{s}{v}\right)^{\zeta-1} \theta_4(0),$$

$$\Rightarrow \theta_5 = -2\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_4)]\right],$$

$$\Rightarrow \theta_5 = -2\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_4] - \left(\frac{s}{v}\right)^{\zeta-1} \theta_4(0)\right\}\right],$$

$$\Rightarrow \theta_5 = -2\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S\left[-32\, e^x \frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}\right]\right]$$

$$\Rightarrow \theta_5 = 64\, e^x\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left(\frac{v}{s}\right)^{4\zeta+2}\right],$$

$$\Rightarrow \theta_5 = 64\, e^x\, S^{-1}\left[\left(\frac{v}{s}\right)^{5\zeta+2}\right] \Rightarrow \theta_5 = 64\, e^x \frac{t^{5\zeta+1}}{\Gamma(5\,\zeta+2)}.$$

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \ldots$$

$$\Rightarrow \theta = (1-2t)e^x + 4e^x \frac{t^{\zeta}}{\Gamma(\zeta+2)} - 8e^x \frac{t^{2\zeta+1}}{\Gamma(2\,\zeta+2)} + 16\, e^x \frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)} - 32\, e^x \frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)} + 64\, e^x \frac{t^{5\zeta+1}}{\Gamma(5\,\zeta+2)} - \ldots$$

Considered $\zeta = 1$

$$\Rightarrow \theta(x,t) = \exp[x - t].$$

Example 4.

Considered fractional 2D HT equation as follows [12]:

$$D_t^{2\zeta}\,\theta + 3\, D_t^{\zeta}\theta + 2\theta = \theta_{xx} + \theta_{yy},\ 0 < \alpha \le 1. \qquad (12)$$

I.C.: $\theta(x,y,0) = \exp(x+y)$ and $\theta_t(x,y,0) = -3\exp(x+y)$.

Applying Shehu transform upon Eq. (12)

$$\Rightarrow S[D_t^{2\zeta}\,\theta + 3\, D_t^{\zeta}\theta + 2\theta] = S[\theta_{xx} + \theta_{yy}],$$

$$\Rightarrow S[D_t^{2\zeta}\,\theta] = S[\theta_{xx} + \theta_{yy} - 2\theta] - 3S[D_t^{\zeta}\theta],$$

$$\Rightarrow \left(\frac{s}{v}\right)^{2\zeta} S[\theta(x,t)] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1} \theta^r(x,0) = S[\theta_{xx} + \theta_{yy} - 2\theta] - 3\, S[D_t^{\zeta}\theta],$$

$$\Rightarrow S[\theta(x,t)] = \left(\frac{v}{s}\right)^{2\zeta} \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1} \theta^r(x,0) + \left(\frac{v}{s}\right)^{2\zeta} S[\theta_{xx} + \theta_{yy} - 2\theta] - 3\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\theta],$$

$$\Rightarrow \theta(x,t) = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1} \theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[\theta_{xx} + \theta_{yy} - 2\theta]\right] - 3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\theta]\right].$$

Applying HPM

$$\Rightarrow \sum_{n=0}^{\infty} p^n \theta_n = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1} \theta^r(x,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S\left[\left(\sum_{n=0}^{\infty} p^n \theta_n\right)_{xx} + \left(\sum_{n=0}^{\infty} p^n \theta_n\right)_{yy} - 2\left(\sum_{n=0}^{\infty} p^n \theta_n\right)\right]\right] - 3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S\left[D_t^{\zeta}\left(\sum_{n=0}^{\infty} p^n \theta_n\right)\right]\right].$$

Comparing p^0

$$\Rightarrow \theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1} \theta^r(x,0)\right].$$

Considered $\theta = 1$

$$\Rightarrow \theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left(\frac{s}{v}\right)^{2\zeta-1} \theta(0)\right],$$

$$\Rightarrow \theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)\theta(0)\right] \Rightarrow \theta_0 = u(0)\, S^{-1}\left[\left(\frac{v}{s}\right)\right],$$

$$\Rightarrow \theta_0 = \theta(0),$$

$$\Rightarrow \theta_0 = \theta(x,y,0) + t\,\theta_t(x,y,0) \Rightarrow \theta_0 = (1-3t)\exp(x+y).$$

Comparing p^1

$$\Rightarrow \theta_1 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_0)_{xx} + (\theta_0)_{yy} - 2(\theta_0)]\right] - 3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_0)]\right],$$

$$\Rightarrow \theta_1 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_0)]\right],$$

where

$$S[D_t^{\zeta}\,\theta_0] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1} \theta_0(x,y,0),$$

$$S[D_t^{\zeta}\,\theta_0] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1} \exp(x+y),$$

$$\Rightarrow\theta_1 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[u_0] - \left(\frac{s}{v}\right)^{\zeta-1}\exp(x+y)\right\}\right],$$

$$\Rightarrow\theta_1 = -3S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_0] - \left(\frac{v}{s}\right)^{\zeta-1}\exp(x+y)\right],$$

where

$$S[\theta_0] = S[(1-3t)\exp(x+y)],$$

$$S[\theta_0] = \exp(x+y)\left[\frac{v}{s} - 3\left(\frac{v}{s}\right)^2\right],$$

$$\Rightarrow\theta_1 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{\exp(x+y)\left[\frac{v}{s} - 3\left(\frac{v}{s}\right)^2\right]\right\} - \left(\frac{v}{s}\right)^{\zeta-1}\exp(x+y)\right],$$

$$\Rightarrow\theta_1 = -3\exp(x+y)\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta+1} - 3\left(\frac{v}{s}\right)^{\zeta+2} - \left(\frac{v}{s}\right)^{\zeta+1}\right],$$

$$\Rightarrow\theta_1 = -3\exp(x+y)\,S^{-1}\left[-3\left(\frac{v}{s}\right)^{\zeta+2}\right],$$

$$\Rightarrow\theta_1 = 9\exp(x+y)\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta+2}\right] \Rightarrow \theta_1 = 9\exp(x+y)\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}.$$

Comparing p^2

$$\Rightarrow\theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_1)_{xx} + (\theta_1)_{yy} - 2(\theta_1)]\right] - 3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_1)]\right],$$

$$\Rightarrow\theta_2 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_1)]\right],$$

where

$$S[D_t^{\zeta}\theta_1] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_1] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_1(0),$$

$$S[D_t^{\zeta}\theta_1] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_1],$$

$$\Rightarrow\theta_2 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_1]\right\}\right] \Rightarrow \theta_2 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_1]\right],$$

where

$$S[\theta_1] = S\left[9\exp(x+y)\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}\right],$$

$$S[\theta_1] = 9\exp(x+y)\left(\frac{v}{s}\right)^{\zeta+2},$$

$$\Rightarrow\theta_2 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{9\exp(x+y)\left(\frac{v}{s}\right)^{\zeta+2}\right\}\right],$$

$$\Rightarrow\theta_2 = -27\exp(x+y)\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta+2}\right] \Rightarrow \theta_2 = -27\exp(x+y)\frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)}.$$

Comparing p^3

$$\Rightarrow\theta_3 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_2)_{xx} + (\theta_2)_{yy} - 2(\theta_2)]\right] - 3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_2)]\right],$$

$$\Rightarrow\theta_3 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_2)]\right],$$

where

$$S[D_t^{\zeta}\theta_2] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_2] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_2(0),$$

$$S[D_t^{\zeta}\theta_2] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_2],$$

$$\Rightarrow\theta_3 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_2]\right\}\right] \Rightarrow \theta_3 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_2]\right],$$

where

$$S[\theta_2] = S\left[-27\exp(x+y)\frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)}\right],$$

$$S[\theta_2] = -27\exp(x+y)\left(\frac{v}{s}\right)^{2\zeta+2},$$

$$\Rightarrow\theta_3 = -3\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{-27\exp(x+y)\left(\frac{v}{s}\right)^{2\zeta+2}\right\}\right],$$

$$\Rightarrow\theta_3 = 81\exp(x+y)\,S^{-1}\left[\left(\frac{v}{s}\right)^{3\zeta+2}\right] \Rightarrow \theta_3 = 81\exp(x+y)\frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}.$$

Comparing p^4

$$\Rightarrow\theta_4 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_3)_{xx} + (\theta_3)_{yy} - 2(\theta_3)]\right] - 3\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_3)]\right],$$

$$\Rightarrow \theta_4 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_3)]\right],$$

where

$$S[D_t^{\zeta}\theta_3] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_3] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_3^{r}(0),$$

$$S[D_t^{\zeta}\theta_3] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_3],$$

$$\Rightarrow \theta_4 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_3]\right\}\right] \Rightarrow \theta_4 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_3]\right],$$

where

$$S[\theta_3] = S\left[81 \exp(x+y)\, \frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}\right],$$

$$S[\theta_3] = 81 \exp(x+y)\left(\frac{v}{s}\right)^{3\zeta+2},$$

$$\Rightarrow \theta_4 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{81 \exp(x+y)\left(\frac{v}{s}\right)^{3\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_4 = -243 \exp(x+y)\, S^{-1}\left[\left(\frac{v}{s}\right)^{4\zeta+2}\right]$$

$$\Rightarrow \theta_4 = -243 \exp(x+y)\frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}.$$

Comparing p^5

$$\Rightarrow \theta_5 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_4)_{xx} + (\theta_4)_{yy} - 2(\theta_4)]\right]$$

$$- 3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_4)]\right],$$

$$\Rightarrow \theta_5 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_4)]\right],$$

where

$$S[D_t^{\zeta}\theta_4] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_4] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_4^{r}(0),$$

$$S[D_t^{\zeta}\theta_4] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_4],$$

$$\Rightarrow \theta_5 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_4]\right\}\right] \Rightarrow \theta_5 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_4]\right],$$

where

$$S[\theta_4] = S\left[-243 \exp(x+y)\frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}\right],$$

$$S[\theta_4] = -243 \exp(x+y)\left(\frac{v}{s}\right)^{4\zeta+2},$$

$$\Rightarrow \theta_5 = -3\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{-243 \exp(x+y)\left(\frac{v}{s}\right)^{4\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_5 = 729 \exp(x+y)\, S^{-1}\left[\left(\frac{v}{s}\right)^{5\zeta+2}\right]$$

$$\Rightarrow \theta_5 = 729 \exp(x+y)\, \frac{t^{5\zeta+1}}{\Gamma(5\zeta+2)}.$$

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \ldots$$

$$\Rightarrow \theta = (1-3t)\exp(x+y) + 9\exp(x+y)\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}$$

$$- 27\exp(x+y)\frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)} + 81\exp(x+y)\,\frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}$$

$$- 243\exp(x+y)\frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)} + 729\exp(x+y)\,\frac{t^{5\zeta+1}}{\Gamma(5\zeta+2)} - \ldots$$

Considered $\zeta = 1$

$$\Rightarrow \theta = \exp[x + y - 3t].$$

Example 5.

Considered 3D time-fractional HT equation as follows [12]:

$$D_t^{2\zeta}\theta + 2\, D_t^{\zeta}\theta + 3\theta = \theta_{xx} + \theta_{yy} + \theta_{zz},\ \ 0 < \zeta \le 1. \tag{13}$$

I.C.: $\theta(x, y, z, 0) = \sinh x \, \sinh y \, \sinh z$ and $\theta_t(x, y, z, 0) = -\sinh x \, \sinh y \, \sinh z$.

Applying Shehu transform upon Eq. (13)

$$\Rightarrow S[D_t^{2\zeta}\theta + 2\, D_t^{\zeta}\theta + 3\theta] = S[\theta_{xx} + \theta_{yy} + \theta_{zz}],$$

$$\Rightarrow S[D_t^{2\zeta}u] = S[u_{xx} + u_{yy} + u_{zz} - 3u] - 2S[D_t^{\zeta}u],$$

$$\Rightarrow\left(\frac{s}{v}\right)^{2\zeta} S[\theta] - \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^{r}(x,y,0) = S[\theta_{xx} + \theta_{yy} + \theta_{zz} - 3\theta]$$

$$- 2S[D_t^{\zeta}\theta],$$

$$\Rightarrow\left(\frac{s}{v}\right)^{2\zeta} S[\theta] = \sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^{r}(x,y,0) + S[\theta_{xx} + \theta_{yy} + \theta_{zz} - 3\theta]$$

$$- 2S[D_t^{\zeta}\theta],$$

$$\Rightarrow S[\theta] = \left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^{r}(x,y,0) + \left(\frac{v}{s}\right)^{2\zeta}$$

$$S[\theta_{xx} + \theta_{yy} + \theta_{zz} - 3\theta] - 2\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}\theta],$$

$$\Rightarrow \theta = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,y,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[\theta_{xx}+\theta_{yy}+\theta_{zz}-3\theta]\right] - 2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[D_t^{\zeta}\theta]\right]. \tag{14}$$

Applying HPM upon Eq. (14)

$$\Rightarrow \sum_{n=0}^{\infty}p^n\theta_n = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(x,y,0)\right] + S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S\left[\left(\sum_{n=0}^{\infty}p^n\theta_n\right)_{xx}+\left(\sum_{n=0}^{\infty}p^n\theta_n\right)_{yy}+\left(\sum_{n=0}^{\infty}p^n\theta_n\right)_{zz} - 3\sum_{n=0}^{\infty}p^n\theta_n\right]\right] - 2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S\left[D_t^{\zeta}\left(\sum_{n=0}^{\infty}p^n\theta_n\right)\right]\right].$$

Comparing p^0

$$\Rightarrow\theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\sum_{r=0}^{\theta-1}\left(\frac{s}{v}\right)^{2\zeta-r-1}\theta^r(0)\right].$$

Considered $\theta = 1$

$$\Rightarrow\theta_0 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left(\frac{s}{v}\right)^{2\zeta-1}\theta(0)\right] \Rightarrow \theta_0 = \theta(0)\,S^{-1}\left[\frac{v}{s}\right] \Rightarrow \theta_0 = \theta(0),$$

$$\Rightarrow\theta_0 = \theta(x,y,z,0) + t\,\theta_t(x,y,z,0) \Rightarrow \theta_0 = (1-t)\sinh x\,\sinh y\,\sinh z.$$

Comparing p^1

$$\Rightarrow\theta_1 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[(\theta_0)_{xx}+(\theta_0)_{yy}+(\theta_0)_{zz}-3\theta_0]\right] - 2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[D_t^{\zeta}(\theta_0)]\right],$$

$$\Rightarrow\theta_1 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[D_t^{\zeta}(\theta_0)]\right],$$

where

$$S[D_t^{\zeta}\,\theta_0] = \left(\frac{s}{v}\right)^{\zeta}S[\theta_0] - \sum_{n=0}^{\theta-1}\left(\frac{s}{v}\right)^{\zeta-r-1}\theta_0^r(0),$$

Considered $\theta = 1$

$$S[D_t^{\zeta}\theta_0] = \left(\frac{s}{v}\right)^{\zeta}S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1}\sinh x\,\sinh y\,\sinh z,$$

$$\Rightarrow\theta_1 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta}S[\theta_0] - \left(\frac{s}{v}\right)^{\zeta-1}\sinh x\,\sinh y\,\sinh z\right\}\right],$$

$$\Rightarrow\theta_1 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[u_0] - \left(\frac{v}{s}\right)^{\zeta+1}\sinh x\,\sinh y\,\sinh z\right],$$

where

$$S[\theta_0] = S[(1-t)\sinh x\,\sinh y\,\sinh z],$$

$$S[\theta_0] = \sinh x\,\sinh y\,\sinh z\left[\frac{v}{s} - \left(\frac{v}{s}\right)^2\right],$$

$$\Rightarrow\theta_1 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{\sinh x\,\sinh y\,\sinh z\left[\frac{v}{s}-\left(\frac{v}{s}\right)^2\right]\right\} - \left(\frac{v}{s}\right)^{\zeta+1}\sinh x\,\sinh y\,\sinh z\right],$$

$$\Rightarrow\theta_1 = -2\sinh x\,\sinh y\,\sinh z\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{\left[\frac{v}{s}-\left(\frac{v}{s}\right)^2\right]\right\} - \left(\frac{v}{s}\right)^{\zeta+1}\right],$$

$$\Rightarrow\theta_1 = -2\sinh x\,\sinh y\,\sinh z\,S^{-1}\left[-\left(\frac{v}{s}\right)^{\zeta+2}\right] \Rightarrow \theta_1 = 2\sinh x\,\sinh y\,\sinh z\,\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}.$$

Comparing p^2

$$\Rightarrow\theta_2 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[(\theta_1)_{xx}+(\theta_1)_{yy}+(\theta_1)_{zz}-3\theta_1]\right] - 2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[D_t^{\zeta}(\theta_1)]\right],$$

$$\Rightarrow\theta_2 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}S[D_t^{\zeta}(\theta_1)]\right],$$

where

$$S[D_t^{\zeta}\,\theta_1] = \left(\frac{s}{v}\right)^{\zeta}S[\theta_1] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_1(0),$$

$$S[D_t^{\zeta}\,\theta_1] = \left(\frac{s}{v}\right)^{\zeta}S[\theta_1],$$

$$\Rightarrow\theta_2 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta}S[\theta_1]\right\}\right] \Rightarrow \theta_2 = -2S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}S[\theta_1]\right],$$

where

$$S[\theta_1] = S\left[2\sinh x\,\sinh y\,\sinh z\,\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}\right],$$

$$S[\theta_1] = 2\sinh x\,\sinh y\,\sinh z\,S\left[\frac{t^{\zeta+1}}{\Gamma(\zeta+2)}\right],$$

$$S[\theta_1] = 2\sinh x\,\sinh y\,\sinh z\left(\frac{v}{s}\right)^{\zeta+2},$$

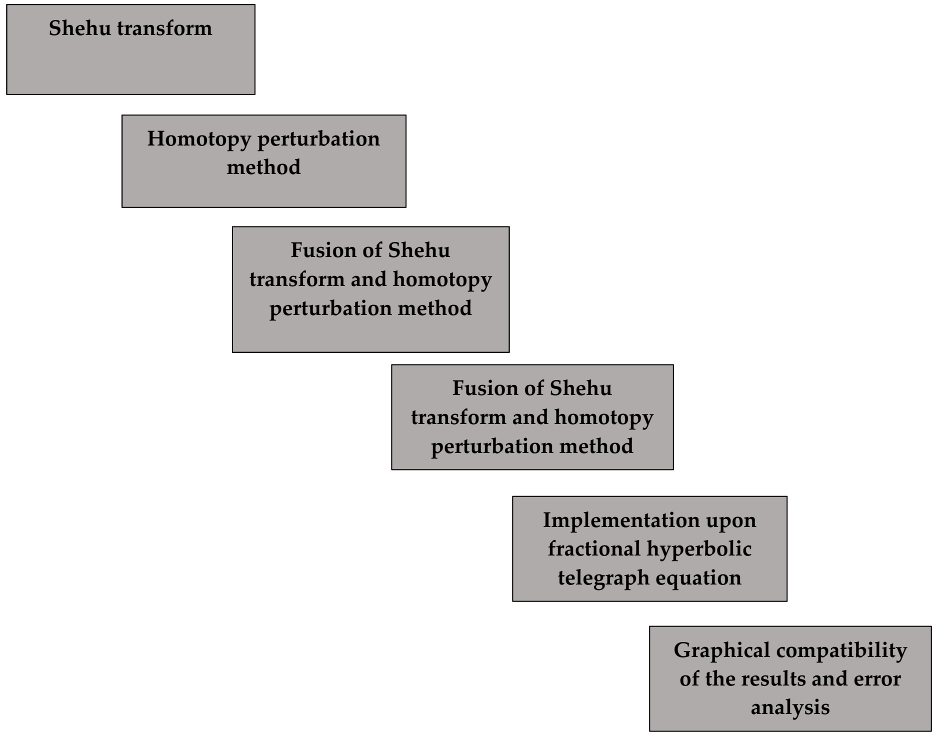

Figure 1: Steps regarding the solution of the proposed regime.

$$\Rightarrow \theta_2 = -2\, S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{2 \sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_2 = -4 \sinh x \sinh y \sinh z\, S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta+2}\right]$$

$$\Rightarrow \theta_2 = -4 \sinh x \sinh y \sinh z \frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)}.$$

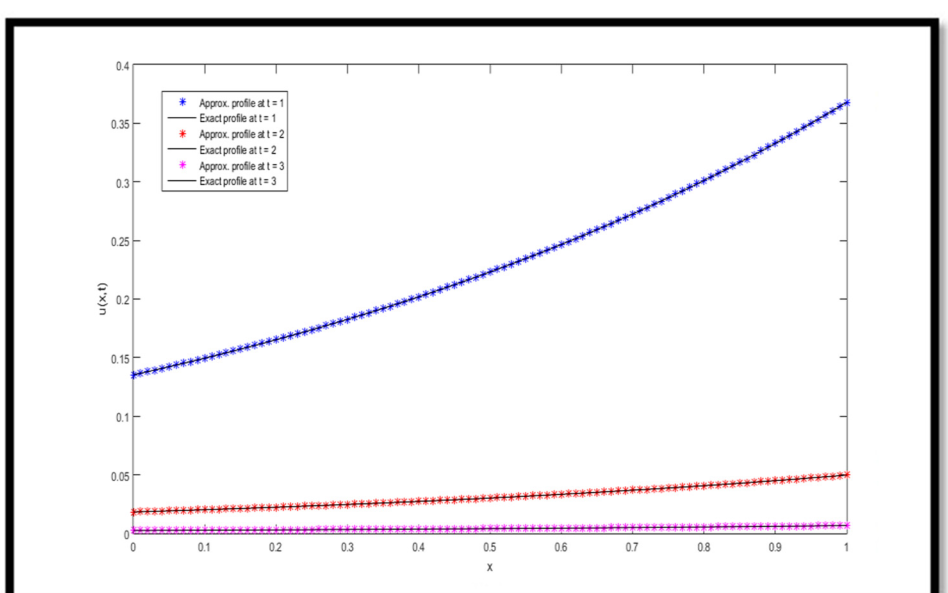

Figure 2: Comparison of approximate and exact outcomes at t = 1, 2, and 3 for Example 1.

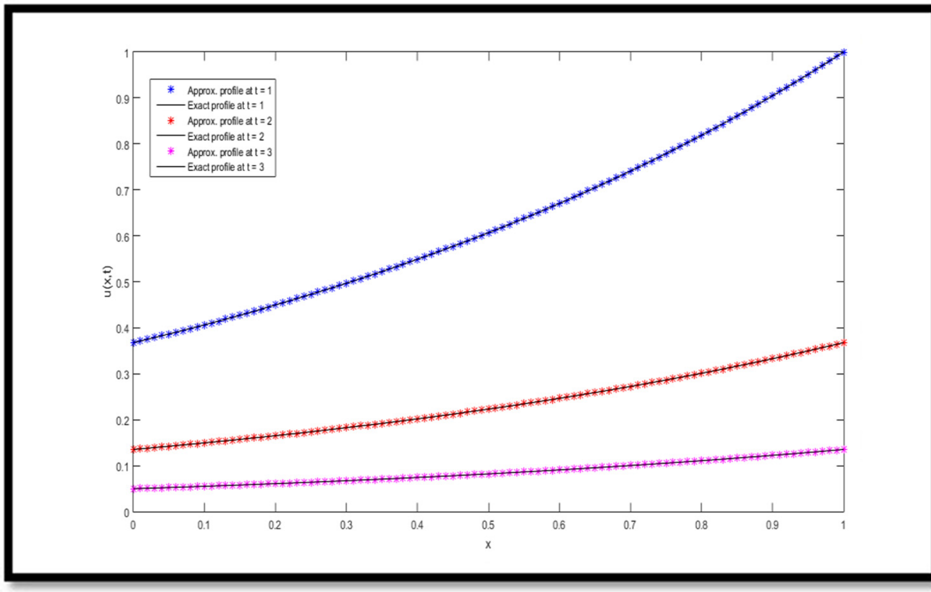

Figure 3: Comparison of approximate and exact outcomes at t = 1, 2, and 3 for Example 2.

Comparing p^3

$$\Rightarrow \theta_3 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_2)_{xx} + (\theta_2)_{yy} + (\theta_2)_{zz} - 3\theta_2]\right] - 2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_2)]\right],$$

$$\Rightarrow \theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_2)]\right],$$

where

$$S[D_t^{\zeta}(\theta_2)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_2] - \left(\frac{s}{v}\right)^{\zeta-1} \theta_2(0),$$

$$S[D_t^{\zeta}(\theta_2)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_2],$$

$$\Rightarrow \theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_2]\right\}\right] \Rightarrow \theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_2]\right],$$

where

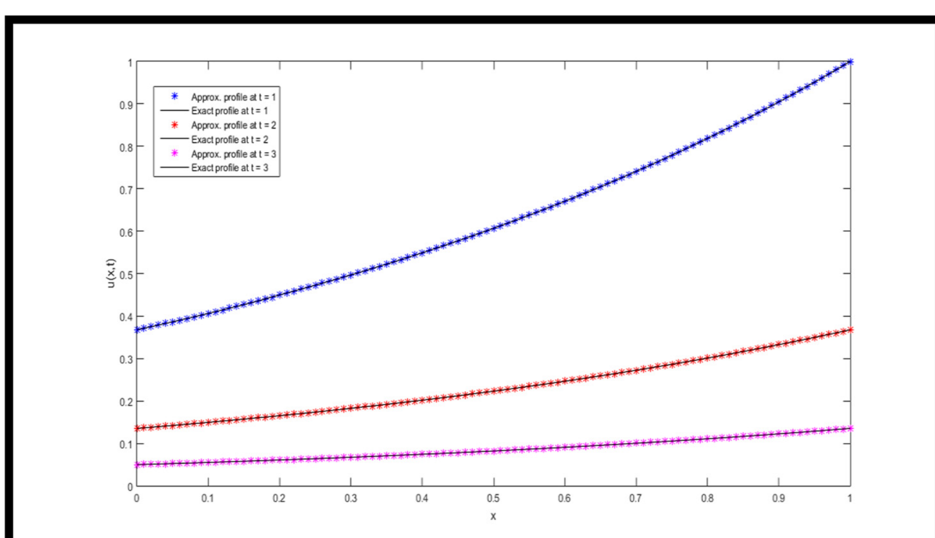

Figure 4: Comparison of approximate and exact outcomes at t = 1, 2, and 3 for Example 3.

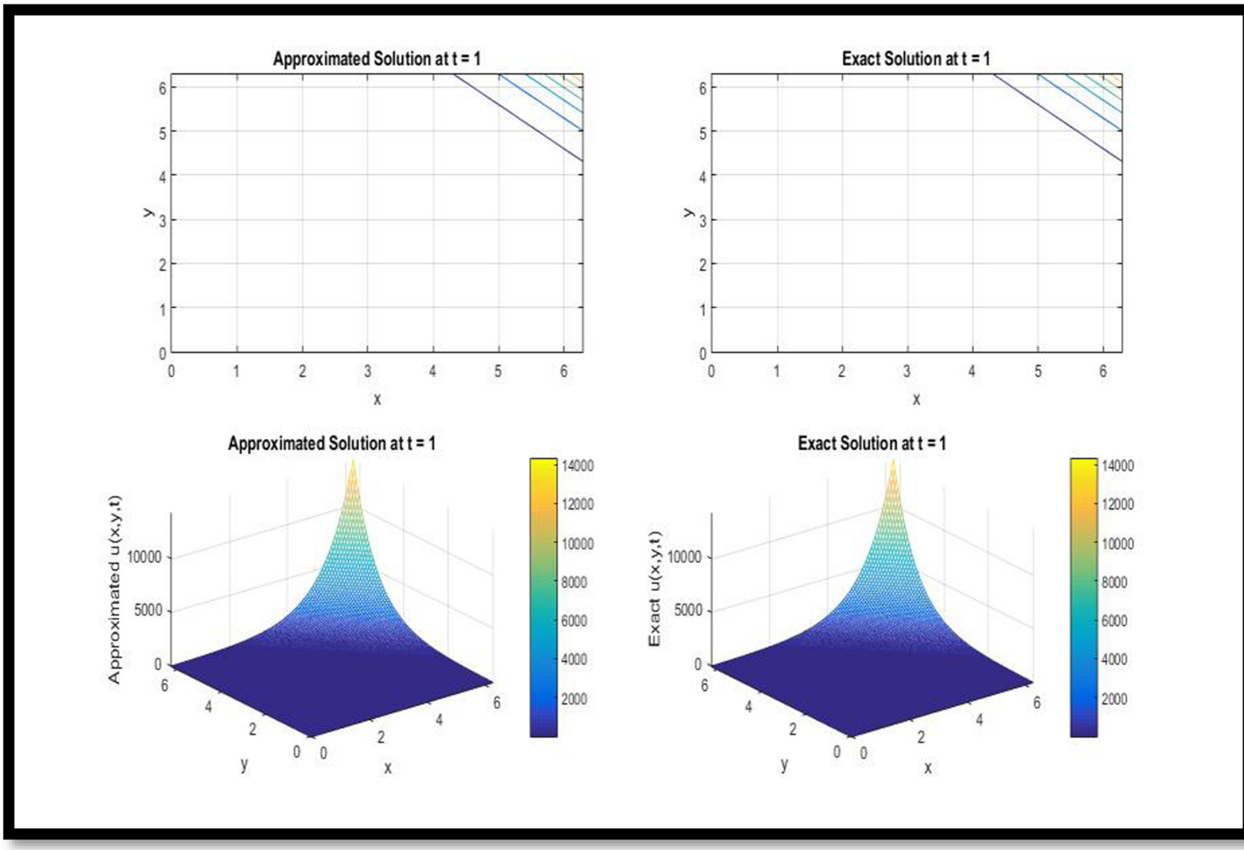

Figure 5: Comparison of approximate and exact outcomes at $t = 1$ for Example 4.

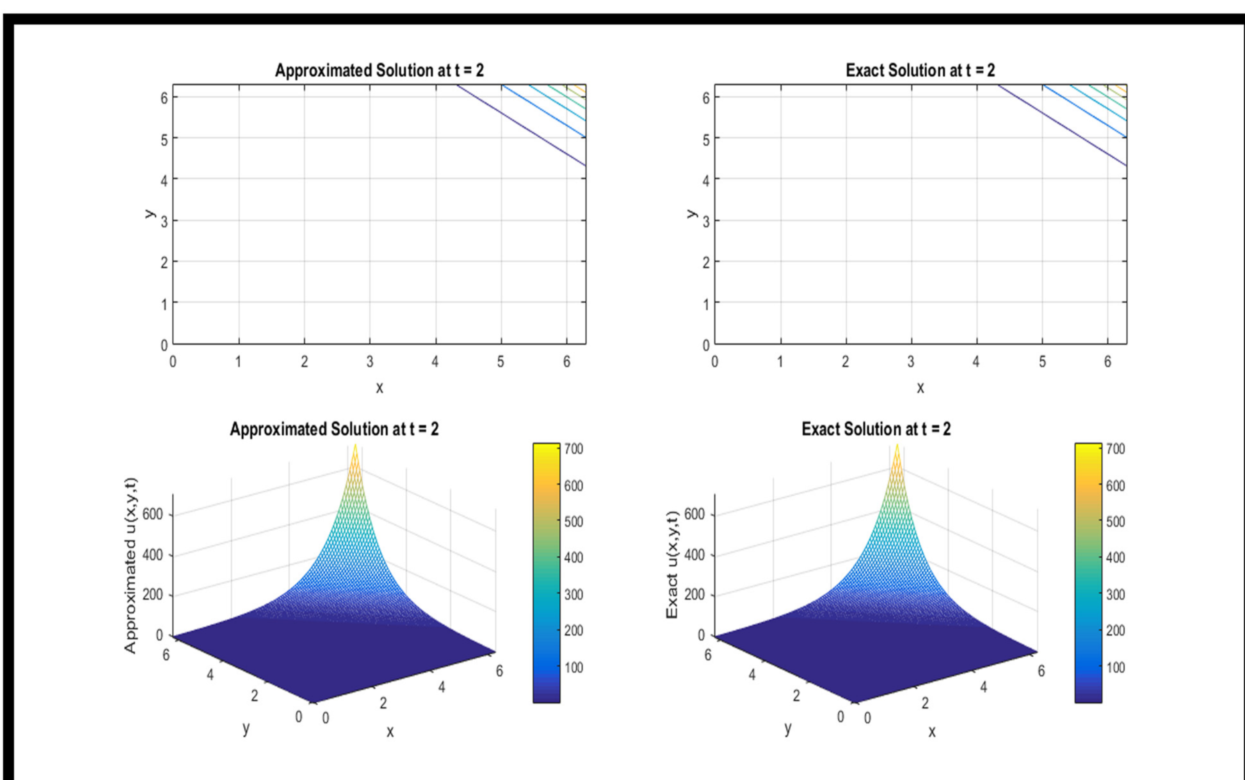

Figure 6: Comparison of approximate and exact outcomes at $t = 2$ for Example 4.

$$S[\theta_2] = S\left[-4\sinh x \sinh y \sinh z \frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)}\right],$$

$$S[\theta_2] = -4\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{2\zeta+2},$$

$$\Rightarrow \theta_3 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{-4\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{2\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_3 = 8\sinh x \sinh y \sinh z\, S^{-1}\left[\left(\frac{v}{s}\right)^{3\zeta+2}\right]$$

$$\Rightarrow \theta_3 = 8\sinh x \sinh y \sinh z \frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}.$$

Comparing p^4

$$\Rightarrow \theta_4 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_3)_{xx} + (\theta_3)_{yy} + (\theta_3)_{zz} - 3\theta_3]\right]$$

$$-\,2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_3)]\right],$$

$$\Rightarrow \theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_3)]\right],$$

where

$$S[D_t^{\zeta}(\theta_3)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_3] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_3(0),$$

$$S[D_t^{\zeta}(\theta_3)] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_3],$$

$$\Rightarrow \theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_3]\right\}\right] \Rightarrow \theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_3]\right],$$

where

$$S[\theta_3] = S\left[8\sinh x \sinh y \sinh z \frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)}\right],$$

$$S[\theta_3] = 8\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{3\zeta+2},$$

$$\Rightarrow \theta_4 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{8\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{3\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_4 = -16\sinh x \sinh y \sinh z\, S^{-1}\left[\left(\frac{v}{s}\right)^{4\zeta+2}\right]$$

$$\Rightarrow \theta_4 = -16\sinh x \sinh y \sinh z \frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}.$$

Comparing p^5

$$\Rightarrow \theta_5 = S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[(\theta_4)_{xx} + (\theta_4)_{yy} + (\theta_4)_{zz} - 3\theta_4]\right]$$

$$-\,2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_4)]\right],$$

$$\Rightarrow \theta_5 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta} S[D_t^{\zeta}(\theta_4)]\right],$$

where

$$S[D_t^{\zeta}\theta_4] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_4] - \left(\frac{s}{v}\right)^{\zeta-1}\theta_4(0),$$

$$S[D_t^{\zeta}\theta_4] = \left(\frac{s}{v}\right)^{\zeta} S[\theta_4],$$

$$\Rightarrow \theta_5 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{2\zeta}\left\{\left(\frac{s}{v}\right)^{\zeta} S[\theta_4]\right\}\right] \Rightarrow \theta_5 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta} S[\theta_4]\right],$$

where

$$S[\theta_4] = S\left[-16\sinh x \sinh y \sinh z \frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)}\right],$$

$$S[\theta_4] = -16\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{4\zeta+2},$$

$$\Rightarrow \theta_5 = -2\,S^{-1}\left[\left(\frac{v}{s}\right)^{\zeta}\left\{-16\sinh x \sinh y \sinh z \left(\frac{v}{s}\right)^{4\zeta+2}\right\}\right],$$

$$\Rightarrow \theta_5 = 32\sinh x \sinh y \sinh z\, S^{-1}\left[\left(\frac{v}{s}\right)^{5\zeta+2}\right]$$

$$\Rightarrow \theta_5 = 32\sinh x \sinh y \sinh z \frac{t^{5\zeta+1}}{\Gamma(5\zeta+2)}.$$

$$\theta = \theta_0 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 + \ldots$$

$$\begin{aligned}\Rightarrow \theta &= (1-t)\sinh x \sinh y \sinh z \\ &+ 2\sinh x \sinh y \sinh z \frac{t^{\zeta+1}}{\Gamma(\zeta+2)} \\ &- 4\sinh x \sinh y \sinh z \frac{t^{2\zeta+1}}{\Gamma(2\zeta+2)} \\ &+ 8\sinh x \sinh y \sinh z \frac{t^{3\zeta+1}}{\Gamma(3\zeta+2)} \\ &- 16\sinh x \sinh y \sinh z \frac{t^{4\zeta+1}}{\Gamma(4\zeta+2)} \\ &+ 32\sinh x \sinh y \sinh z \frac{t^{5\zeta+1}}{\Gamma(5\zeta+2)} - \ldots\end{aligned}$$

Considered

$$\zeta = 1: \Rightarrow \theta = \sinh x \sinh y \sinh z\left[1 - t + t^2 - \frac{4t^3}{3!} + \ldots\right].$$

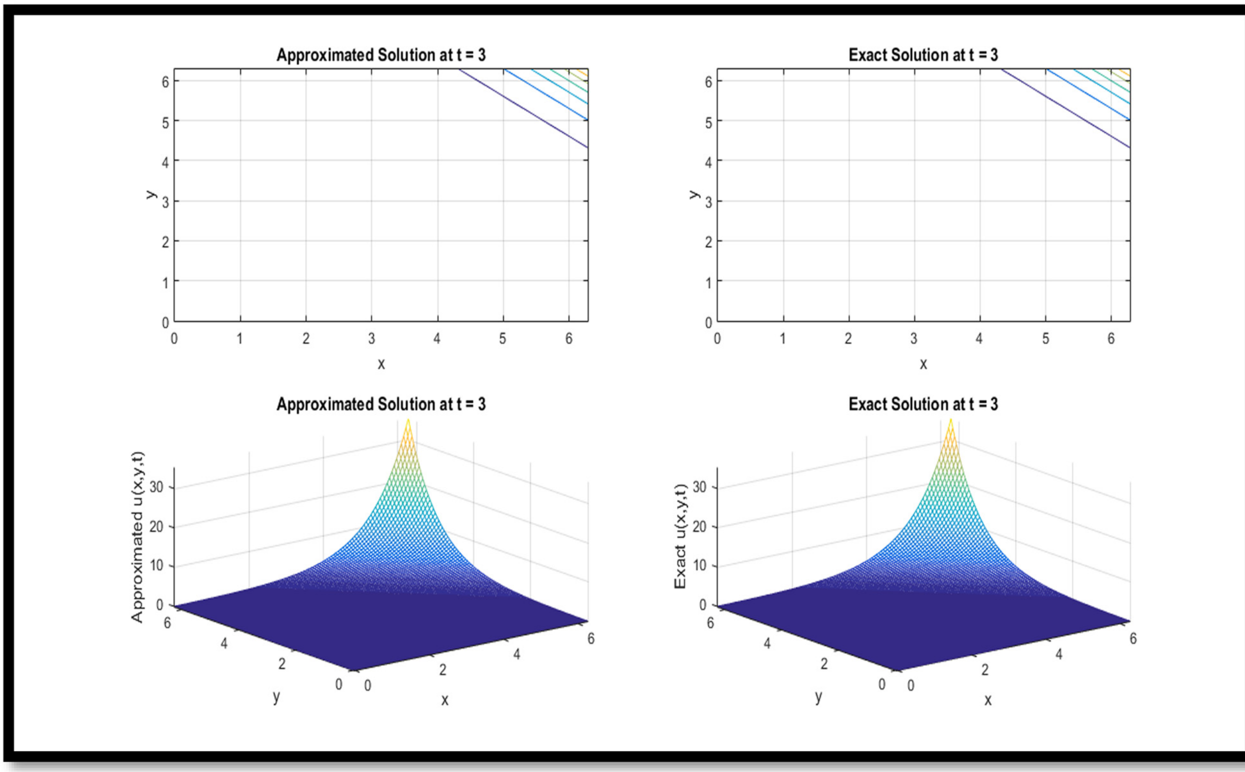

Figure 7: Comparison of approximate and exact outcomes at t = 3 for Example 4.

4 Graphs and error analysis

In the present section, graphical and tabular analyses of the present regime is proposed for several examples. Along with it, the error analysis *via* L_∞ error norm is provided. Compatibility of approx. and exact profiles is also notified.

In Figure 2, comparison of approximate and exact outcomes is notified at t = 1, 2, and 3 for Example 1. In Figure 3, comparison of approximate and exact outcomes is mentioned at t = 1, 2, and 3 for Example 2. In Figure 4, comparison of approximate and exact outcomes is notified at t = 1, 2, and 3 for Example 3. In Figure 5, comparison of approximate and exact outcomes is notified at t = 1 regarding Example 4. In Figure 6, comparison of approximate and exact outcomes is mentioned at t = 2 regarding Example 4. In Figure 7, comparison of approximate and exact outcomes is provided at t = 3 regarding Example 4. Using Figures 1–3, the compatibility of the outcomes is matched at t = 1, 2, and 3 for Example 1–3, respectively. Compatibility of outcomes is claimed at t = 1, 2, and 3, respectively, for Example 4. In Table 3, L_∞ error is calculated at t = 1, 2, and 3 for N = 21, 31, and 41 for Example 1. In Table 4, approximate and exact outcomes are matched at t = 1 and 2, respectively, for Example 1. In Table 5, L_∞ error is evaluated at t = 1, 2, and 3 for N = 11, 21, and 31, respectively, for Example 2. In Table 6, approximate and exact outcomes are compared at t = 1 and 2 for Example 2. Through Table 7, an approximate and exact solutions are provided at t = 1 and 2 regarding Example 3. In Table 8, L_∞ error is evaluated at t = 1.1, 1.2, and 1.3 for Example 4. In Table 9, approximate and exact outcomes are matched at t = 1.1 and 1.2, respectively, for Example 4.

4.1 Observation regarding numerical convergence of the proposed regime

Through Table 3, numerical convergence of the proposed regime is provided. It is notified that on increasing the value

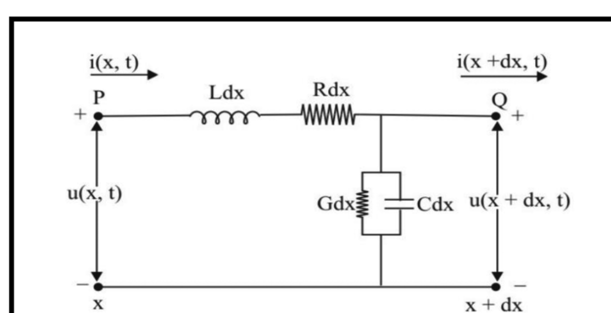

Figure 8: Telegraph transmission line with leakage.

Table 1: Iterative Shehu transform of diversified functions

S. No.	$f(t)$	$S[f(t)]$
1	1	$\frac{v}{s}$
2	t	$\frac{v^2}{s^2}$
3	t^2	$\frac{2v^3}{s^3}$
4	t^3	$\frac{6v^4}{s^4}$
6	$\exp(at)$	$\frac{v}{s-av}$
7	$\sin(at)$	$\frac{av^2}{s^2+a^2v^2}$
8	$\cos(at)$	$\frac{sv}{s^2+a^2v^2}$
9	$\sinh(at)$	$\frac{av^2}{s^2-a^2v^2}$
10	$\cosh(at)$	$\frac{sv}{s^2-a^2v^2}$
11	$\frac{\exp(bt)\sin(at)}{a}$	$\frac{v^2}{(a^2+b^2)v^2-2bvs+s^2}$
12	$\exp(bt)\ \cos(at)$	$\frac{v(s-bv)}{(a^2+b^2)v^2-2b\,sv+s^2}$
13	$\frac{\exp(bt)\sinh(at)}{a}$	$\frac{2v^2}{((bv-s)^2-a^2v^2)}$
14	$\exp(bt)\ \cosh(at)$	$\frac{2v(bv-s)}{(a^2v^2-(bv-s)^2)}$
15	$\frac{\exp(bt)-\exp(at)}{(b-a)}, a \neq b$	$\frac{v^2}{(av-s)(bv-s)}$
16	$\frac{b\exp(bt)-a\exp(at)}{(b-a)}, a \neq b$	$\frac{sv}{(av-s)(bv-s)}$
17	$\frac{(\sin(at)-at\cos(at))}{2a^3}$	$\frac{v^4}{(a^2v^2+s^2)^2}$
18	$\frac{t\sin(at)}{2a}$	$\frac{sv^3}{(a^2v^2+s^2)^2}$
19	$\frac{\sin(at)+at\cos(at)}{2a}$	$\frac{v^2s^2}{(a^2v^2+s^2)^2}$
20	$\cos(at)-\frac{1}{2}at\sin(at)$	$\frac{v(2s-av)}{2(a^2v^2+s^2)}$
21	$t\cos(at)$	$\frac{v^2(s^2-a^2v^2)}{(a^2v^2+s^2)^2}$
23	$\frac{t\sinh(at)}{2a}$	$\frac{v^3s}{(av-s)^2(av+s)^2}$
26	$t\cosh(at)$	$\frac{v^2s^2}{(av-s)^2(av+s)^2}$
27	$\frac{(3-a^2t^2)\sin(at)-3at\cos(at)}{8a^5}$	$\frac{v^6}{(a^2v^2+s^2)^3}$
28	$\frac{t\sin(at)-at^2\cos(at)}{8a^3}$	$\frac{v^5s}{(a^2v^2+s^2)^3}$
29	$\frac{(1+a^2t^2)\sin(at)-at\cos(at)}{8a^3}$	$\frac{v^4s^2}{(a^2v^2+s^2)^3}$
30	$\frac{3t\sin(at)+at^2\cos(at)}{8a}$	$\frac{v^3s^3}{(a^2v^2+s^2)^3}$
31	$\frac{t^2\sin(at)}{2a}$	$\frac{(3s^2-v^2a^2)v^4}{(a^2v^2+s^2)^3}$

of N at different time levels, the L_∞ error got reduced and thereafter became stable regarding Example 1. Through Table 3, the reduced error is obtained on an increase in the number of grid points at different values of time levels. The method got converged up to 10^{-16} regarding Example 2. Through Table 8, the reduced L_∞ error is notified at different values of "t" on an increment in the number of grid points regarding Example 4.

Table 2: Inverse Shehu transform of diversified functions [19]

	$f(s,v)$	$Q(t) = S^{-1}[f(s,v)]$
1	$\frac{v}{s}$	1
2	$\frac{v^2}{s^2}$	t
3	$\left(\frac{v}{s}\right)^{m+1}$	$\frac{t^m}{\angle m}$
4	$\left(\frac{v}{s}\right)^{m+1}$	$\frac{t^m}{\Gamma(m+1)}$
5	$\frac{v}{s-av}$	e^{at}
6	$\frac{mv^2}{s^2+m^2v^2}$	$\sin(mt)$
7	$\frac{sv^2}{s^2+m^2v^2}$	$\cos(mt)$
8	$\frac{mv^2}{s^2-m^2v^2}$	$\sinh(mt)$
9	$\frac{sv^2}{s^2-m^2v^2}$	$\cosh(mt)$

4.2 Application of the proposed regime

4.2.1 Telegraph transmission line with leakage

Any system that transmits telegraph signals over a long distance – usually by a wire or cable – is referred to as a telegraph transmission line with leakage. In this context, "leakage" refers to the weakening of the signal down the transmission line, which can be caused by a number of different things, including capacitance, inductance, and resistance.

Table 3: Analysis of errors for Example 1

N	L_∞ error		
	$t = 1$	$t = 2$	$t = 3$
21	-3.1042×10^{-13}	-6.0062×10^{-7}	-2.7791×10^{-3}
31	-4.4409×10^{-16}	-9.3814×10^{-15}	-1.3578×10^{-9}
41	-4.4409×10^{-16}	-4.2744×10^{-15}	3.0923×10^{-14}
	↓ **Converging up to** 10^{16}	↓ **Converging up to** 10^{15}	↓ **Converging up to** 10^{14}

Table 4: Approximate and exact outcomes match for Example 1

x	Approx.	Exact	Approx.	Exact
	$t = 1$		$t = 2$	
1.00×10^{-1}	1.50×10^{-1}	1.50×10^{-1}	2.02×10^{-2}	2.02×10^{-2}
1.50×10^{-1}	1.57×10^{-1}	1.57×10^{-1}	2.13×10^{-2}	2.13×10^{-2}
2.00×10^{-1}	1.65×10^{-1}	1.65×10^{-1}	2.24×10^{-2}	2.24×10^{-2}

Table 5: Analysis of errors for Example 2

N	L_∞ error		
	$t=1$	$t=2$	$t=3$
11	-1.3916×10^{-7}	-2.6433×10^{-4}	-2.1305×10^{-2}
21	8.8818×10^{-16}	-3.1042×10^{-13}	-1.4863×10^{-9}
31	-4.4409×10^{-16}	-4.4409×10^{-16}	-1.6653×10^{-16}
	↓	↓	↓
	Converging up to 10^{16}	**Converging up to** 10^{16}	**Converging up to** 10^{16}

Table 6: Approximate and exact outcomes match for Example 2

x	Approx.	Exact	Approx.	Exact
	$t=1$		$t=2$	
1.00×10^{-1}	4.07×10^{-1}	4.07×10^{-1}	1.50×10^{-1}	1.50×10^{-1}
1.50×10^{-1}	4.27×10^{-1}	4.27×10^{-1}	1.57×10^{-1}	1.57×10^{-1}
2.00×10^{-1}	4.49×10^{-1}	4.49×10^{-1}	1.65×10^{-1}	1.65×10^{-1}

Some main elements and traits of a leaky telegraph transmission line:

Transmitter: A telegraph transmitter is located at one end of the wire. This device produces electrical signals that must be transferred in order to represent telegraph codes, such as Morse code.

Table 7: Approximate and exact outcomes match for Example 3

x	Approx.	Exact	Approx.	Exact
	$t=1$		$t=2$	
1.00×10^{-1}	0.4066	0.4066	0.1496	0.1496
1.50×10^{-1}	0.4274	0.4274	0.1572	0.1572
2.00×10^{-1}	0.4493	0.4493	0.1653	0.1653

Transmission line: An electrical signal travels from the transmitter to the receiver *via* a physical conductor, typically a wire or cable. When it comes to telegraphy, this connection can link offices or telegraph stations over great distances.

Leakage: Losses in electrical impulses along the transmission line result in leakage. This may be caused by elements such as the line's inductance, capacitance between the conductors, and wire resistance. The signal gradually weakens as a result of these losses.

Receiver: A telegraph receiver at the other end of the wire deciphers the weaker signals and transforms them back into legible telegraph code. Despite any communication losses, the recipient bears the responsibility of deriving the provided data.

Plotting the transmission line: A straightforward line representing the physical link between the transmitter and the receiver would be used to depict the telegraph transmission line visually. In keeping with this, you might

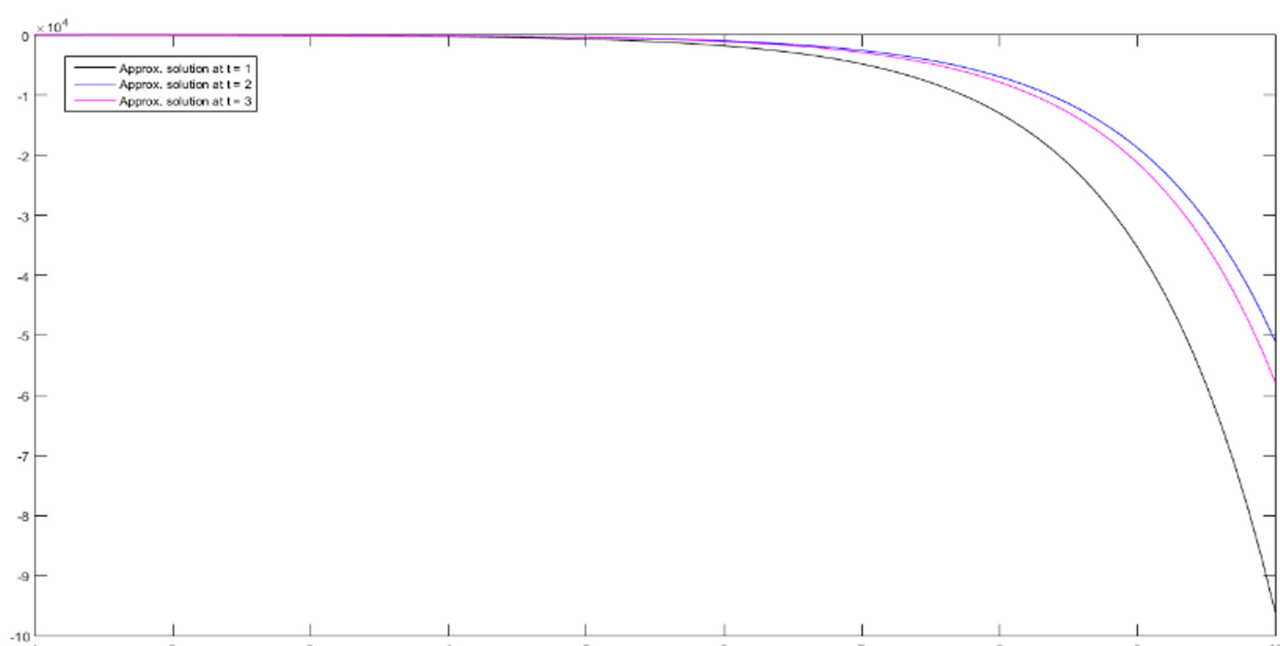

Figure 9: Approximated solution up to fourth term for $\alpha=0.25$ at $t=1$, 2, and 3 regarding Example 1.

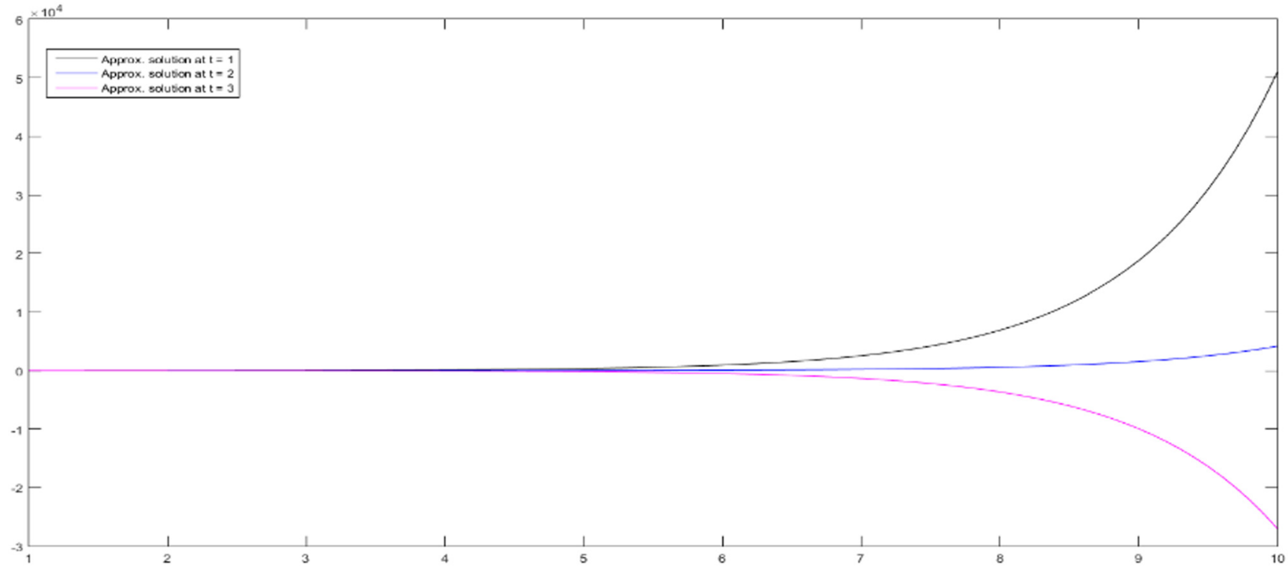

Figure 10: Approximated solution up to fourth term for $\alpha = 0.45$ at $t = 1$, 2, and 3 regarding Example 1.

incorporate components that depict leakage, like arrows that show the signal intensity decreasing at different spots.

Consider an infinitesimal piece of the telegraph cable wire as an electrical circuit (Figure 9), and consider that the cable has the perfect insulation, so that the capacitor and leakage to the floor are present. C is the capacitance to the ground; x is the distance from the end of cable; $u(x, t)$ is the voltage; G is the inductance; $i(x, t)$ is the current; L is the inductance of the cable. Figure 8 is provided regarding the notion of Telegraph transmission line with leakage.

4.2.2 Fractional derivative model equations are as follows [10]:

$$c^2 D_x^{\delta} i = D_t^{\beta} i + (\theta + \phi) D_t^{\alpha} i + \theta \phi i,$$

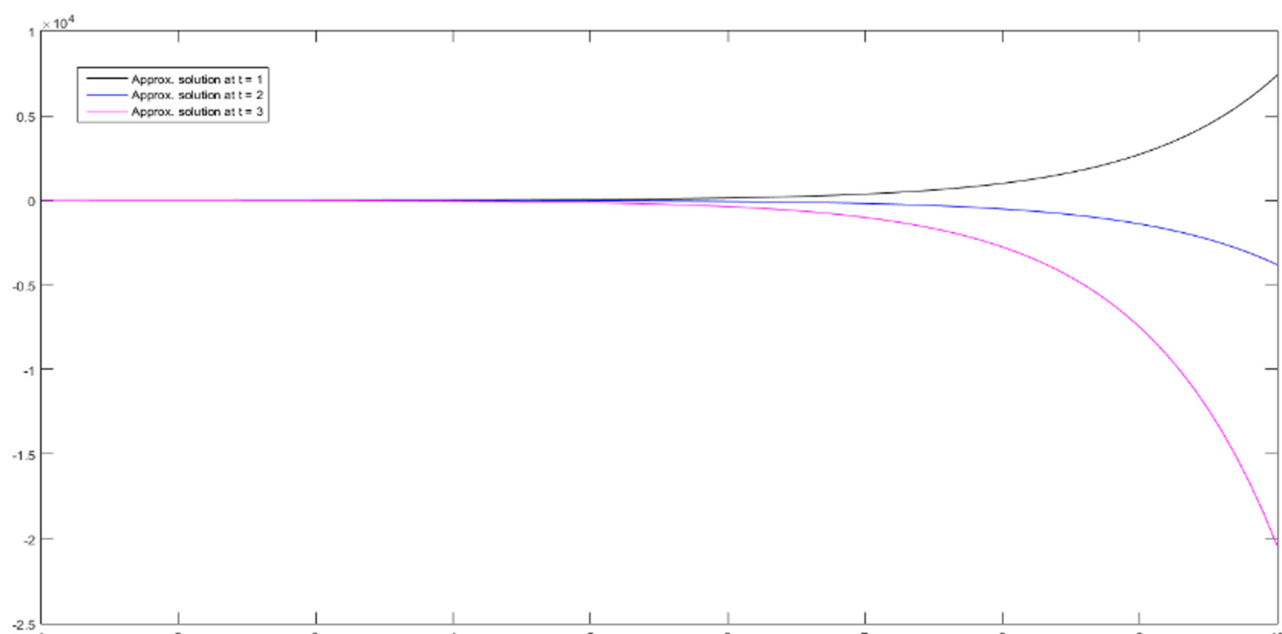

Figure 11: Approximated solution up to fourth term for $\alpha = 0.25$ at $t = 1$, 2, and 3 regarding Example 2.

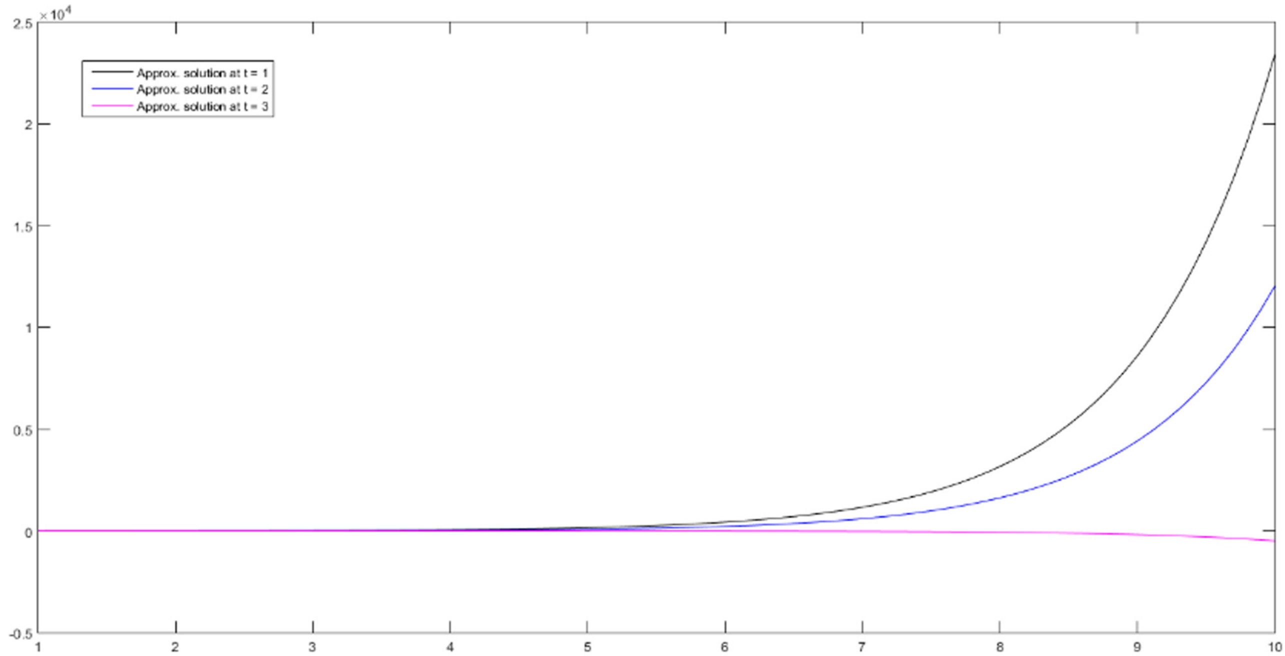

Figure 12: Approximated solution up to fourth term for $\alpha = 0.75$ at $t = 1$, 2, and 3 regarding Example 2.

and

$$c^2 D_x^\delta u = D_t^\beta u + (\theta + \phi) D_t^\alpha u + \theta\phi u.$$

and where $0 < \alpha \le 1,\ 1 < \delta, \beta \le 2$.

4.2.3 Study of effect of changed values of "α" on the numerical solution

The effect of different values of fractional order derivative is claimed *via* Figures 9–14. For Example 1, $\alpha = 0.25$ and 0.45 is taken into account. For Examples 2 and 3, $\alpha = 0.25$ and 0.75 is considered. These figures will be useful to

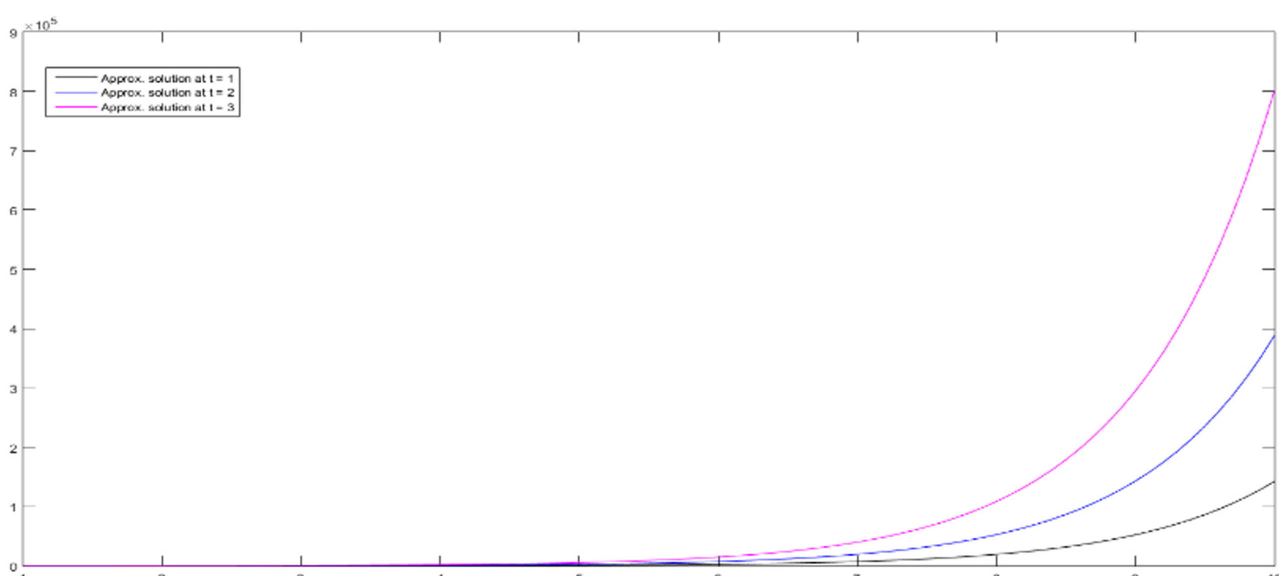

Figure 13: Approximated solution up to fourth term for $\alpha = 0.25$ at $t = 1$, 2, and 3 regarding Example 3.

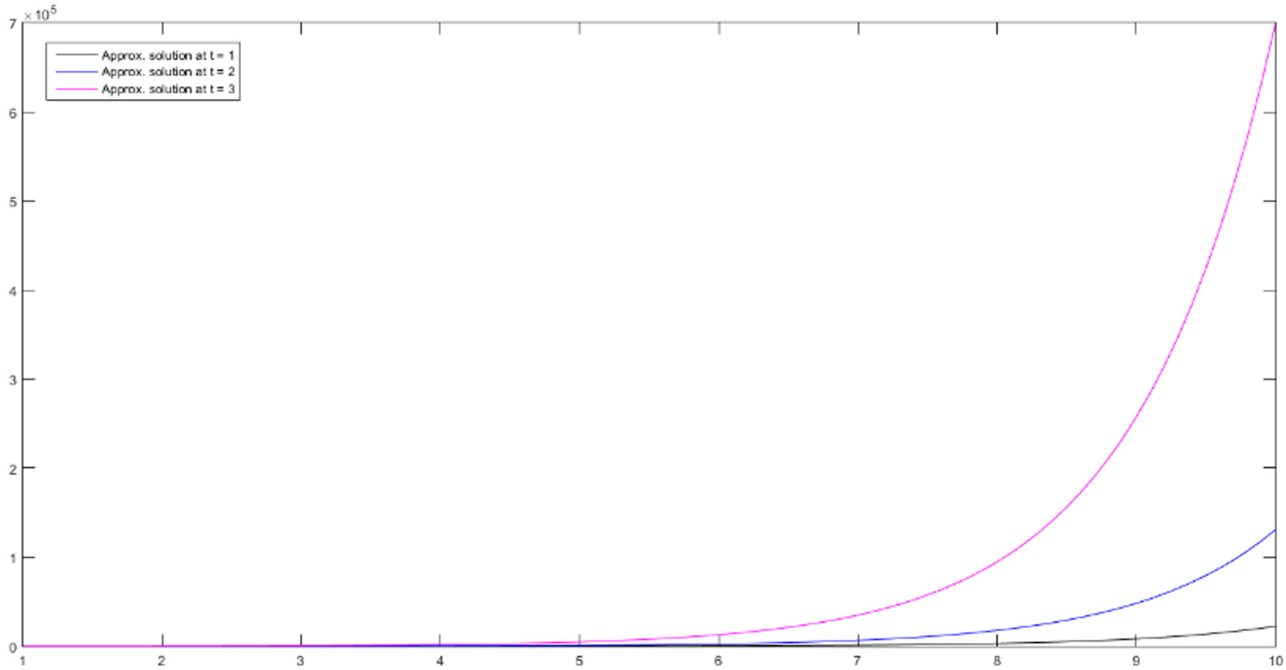

Figure 14: Approximated solution up to fourth term for $\alpha = 0.75$ at $t = 1$, 2, and 3 regarding Example 3.

Table 8: Error analysis for Example 4

N	L_∞ error		
	$t = 1.1$	$t = 1.2$	$t = 1.3$
20	2.3873×10^{-3}	1.3436×10^{-2}	6.5796×10^{-2}
30	2.5102×10^{-10}	4.3929×10^{-10}	1.1569×10^{-9}
40	1.7462×10^{-10}	1.9918×10^{-10}	6.2482×10^{-10}
	↓	↓	↓
	Converging up to 10^{10}	**Converging up to** 10^{10}	**Converging up to** 10^{10}

understand the changed values of fractional derivative and approximated solution.

Table 9: Approximate and exact outcomes match regarding Example 4

(x, y)	Approx.	Exact	Approx.	Exact
	$t = 1.1$		$t = 1.2$	
(3.14×10^{-1}, 3.14 $\times 10^{-1}$)	6.91×10^{-2}	0.069136	5.12×10^{-2}	0.051217
(6.28×10^{-1}, 6.28 $\times 10^{-1}$)	1.30×10^{-1}	0.129592	9.60×10^{-2}	0.096004
(9.42×10^{-1}, 9.42 $\times 10^{-1}$)	2.43×10^{-1}	0.242915	1.80×10^{-1}	0.179956

(x, y) are points in the spatial domain. t is the time level.

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5 Concluding remarks

This article presents the integration of the Shehu transform with the homotopy perturbation method. The series and accurate approximation of the fractional HT equation in various dimensions are the subjects of this research. There are five instances of fractional HT equations in 1D, 2D, and 3D that are covered. Tabular and graphic discussions are also included. It is evident from the graphs that the approximate and accurate profiles are compatible with one another. Tables are another way that the result's numerical convergence is indicated. Numerical convergence is clearly verified by the fact that L_∞ error decreased as the values of the various grid points increased. In the literature, the suggested regime will undoubtedly be useful in solving a variety of fractional differential equations, fuzzy differential equations, higher-order fractional differential equations, and partial-integro differential equations.

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