

Research Article

Gamal M. Ismail*, Alwaleed Kamel, and Abdulaziz Alsarrani

Studying nonlinear vibration analysis of nanoelectro-mechanical resonators *via* analytical computational method

<https://doi.org/10.1515/phys-2024-0011>
received December 09, 2023; accepted March 13, 2024

Abstract: Periodic behaviour analysis of nano/microelectro-mechanical systems (N/MEMS) is an important area due to its numerous prospective applications in micro instruments. The intriguing and unique qualities of these systems, notably their tiny size, batch manufacturing, low power consumption, and great dependability have piqued the attention of academics and enterprises in using these structures to manufacture various microdevices. This article presents the parameter expansion method (PEM) to obtain the approximate solutions of N/MEMS. The present approach, as well as its speed and simplicity in providing analytical solutions that converge quickly to the exact numerical ones, distinguishes this study. The PEM has the benefit of immediately providing analytical solutions to nonlinear differential equations while avoiding costly calculations. Furthermore, in terms of establishing numerous terms of semi-analytic solutions, this approach is very faster and superior to other established analytical techniques in the literature.

Keywords: nanoelectro-mechanical resonators, nonlinear vibration, analytical method, cantilever beam, numerical solution

1 Introduction

The scientific phenomena frequently take the form of nonlinear types. Nonlinear differential equations (NDEs) are used in various fields to express numerous problems related to mathematics, biology, engineering, and physics.

In particular, most of the branches can be modelled by using derivatives in NDEs [1–4]. The nano/microelectro-mechanical system (N/MEMS) has sparked global interest, and its applications have transformed technologies in a wide range of sophisticated industries, including wearable sensors, 5G communication technology, aircraft, and energy harvesting. When the applied force is sufficiently strong, however, the pull-in instability occurs, and dependable functioning is prohibited. MEMS are of significant attention and have been the topic of various studies.

MEMS are widely used in a variety of disciplines, including medicine, industry, robotics, aerospace, automotive, and many others [5–8]. MEMS are also widely employed in many engineering domains, including optical and bio-medical engineering, and are widely used in many applications, such as micro-pumps, accelerometers, micro-switches, and so on [8]. MEMS is a technology that integrates computers with small mechanical devices contained in semiconductor chips such as gears and actuators. These systems can detect, control, and trigger mechanical processes on the microscale, and they can work alone or in groups to produce effects on the macroscale. MEMS devices, which need minimal mechanical components and low-voltage levels for actuation, are rapidly expanding in current technology [9–11]. Numerous researches works in the literature deal with nonlinear differential equations in diverse sectors of science and engineering, therefore emphasizing the relevance of mathematical calculations was critical. Many nonlinear differential equations can be statistically studied, but only a handful can be directly solved. Numerous approximation strategies have been employed in the literature to determine the interaction between frequency and amplitude. The perturbation technique, which was commonly used to obtain approximate analytic solutions to nonlinear differential equations, was the most versatile tool in evaluating nonlinear engineering issues. The determination of periodic solutions of nonlinear oscillators is an important subject in physical nonlinear research. Traditional analytical approaches were utilized to address nonlinear differential equations, which are often employed to solve nonlinear oscillators with modest parameters. There are several powerful

* **Corresponding author: Gamal M. Ismail**, Department of Mathematics, Faculty of Science, Islamic University of Madinah, 42351, Madinah, Saudi Arabia, e-mail: gismail@iu.edu.sa

Alwaleed Kamel, Abdulaziz Alsarrani: Department of Mathematics, Faculty of Science, Islamic University of Madinah, 42351, Madinah, Saudi Arabia

analytical and approximate methods have been studied to solve a large number of NDEs with various assumptions such as variational iteration method [12,13], modified multi-stage decomposition method [14], homotopy perturbation method [15–19], He's frequency formulation [20,21], generalized residual power series method [22], harmonic balance method [23,24], Exp-function method [25], Laplace variational iteration method [26], reduced differential transform method [27], variational iteration method Padé technique [28], energy balance method (EBM) [29,30], global residue harmonic balance method (GRHBM) [31,32], Gamma function method [33,34], global error minimization method [35], Levenberg–Marquardt backpropagation neural networks [36], Hamiltonian approach [37], parameter expanding method (PEM) [38–42], and finite element method [43–45]. Recently, refs. [46,47] proposed a variational principle for a fractal N/MEMS, where the new variational theory opens a new direction in fractal in these systems, it provides an excellent physical knowledge of the iteration approach, and variational numerical techniques always have a conservation scheme with a fast rate of convergence. Many analytical methods compatible with nonlinear fractional differential equations can be constructed, and the results simulated using other existing numerical and analytical techniques in the literature. The current work provides considerable insight and critical analysis of the dynamic features of nano-scaled structures, as well as the essential guidelines for assessing the dynamic response of nano-components in advanced MEMS/NEMS. The advantage of the PEM is that it provides direct analytical solutions to nonlinear differential equations without requiring costly computations. Furthermore, this technique is faster and superior to other analytical approaches in the literature in developing a variety of analytic solutions.

The goal of this study is to find an analytical approximation solution to the free vibration equation that arises in MEMS. To show the efficiency and accuracy of the current technique, the acquired findings are compared with the EBM [30], GRHBM [32], and numerical solutions. The current technique produces more dependable findings for the current situation. Finally, it is demonstrated that the PEM has significant promise and that it might be applied to other substantially nonlinear situations.

2 Basic concept of the PEM

Consider the following differential equation:

$$\ddot{u} + au + \beta f(u, t) = 0, \quad (1)$$

where a and β are the coefficients of u and the nonlinear function $f(u, t)$ and will be defined in Eqs. (3) and (4).

According to PEM [48,49], the solution and coefficients of Eq. (1) are expanded in a similar way:

$$u(t) = u_0(t) + pu_1(t) + p^2u_2(t) + \dots, \quad (2)$$

$$\alpha = \omega^2 + p\omega_1 + p^2\omega_2 + \dots, \quad (3)$$

$$\beta = pa_1 + p^2a_2 + \dots \quad (4)$$

Inserting Eqs. (2)–(4) into Eq. (1) and then equating the terms in powers of p , we obtain

$$p^0: \quad \ddot{u}_0 + \omega^2u_0 = 0, \quad (5)$$

$$p^1: \quad \ddot{u}_1 + \omega^2u_1 + \omega_1u_0 + a_1f(u_0, t) = 0. \quad (6)$$

The solutions of Eq. (5) is

$$u_0 = A \cos(\omega t). \quad (7)$$

By substituting Eq. (7) into Eq. (6), we obtain

$$p^1: \quad \ddot{u}_1 + \omega^2u_1 + \omega_1A \cos(\omega t) + a_1f(A \cos(\omega t), t) = 0. \quad (8)$$

We utilize the Fourier expansion series to achieve the secular term as follows:

$$(A \cos(\omega t), t) = \sum_{k=0}^{\infty} b_{2k+1} \cos((2k+1)\omega t). \quad (9)$$

Inserting (9) into (8) yields:

$$p^1: \quad \ddot{u}_1 + \omega^2u_1 + (\omega_1A + a_1b_1) \cos(\omega t) = 0. \quad (10)$$

We examine the following to avoid using secular terms:

$$\omega_1A + a_1b_1 = 0. \quad (11)$$

Putting $p = 1$ in Eqs. (3) and (4), we have:

$$\omega_1 = \alpha - \omega^2, \quad (12)$$

$$a_1 = \beta. \quad (13)$$

By inserting Eqs. (12) and (13) into Eq. (11), we shall attain the oscillator's first-order approximation frequency (1). It is worth noting that, according to Eqs. (4) and (13), we can discover that $a_i \neq 0$ for all $i = 1, 2, 3, 4, \dots$

3 Mathematical model

In this section, we show a severely nonlinear issue with enormous motion amplitudes and physical parameters by considering a nano resonator made up of a moving electrode and two stationary substrates. The mobile electrode might be thought of as a clamped-clamped or cantilever beam with a rectangular cross section that is implanted

between two fixed substrates [30,32]. Figure 1 depicts a schematic for a clamped-clamped resonator.

$$(a_2 - 4a_4u^2 + 6a_6u^4 - 4a_8u^6 + a_{10}u^8)\ddot{u} + K_1u + K_2u^3 + K_3u^5 + K_4u^7 + K_5u^9 + K_6u^{11} = 0, \quad (14)$$

$$u(0) = A, \quad \dot{u}(0) = 0. \quad (15)$$

The coefficients of $K_1 - K_6$, $a_0 - a_{10}$, $b_1 - b_5$, and $c_1 - c_5$ are listed in the Appendix.

To normalize Eq. (14), the following nondimensional parameters are used

$$\left. \begin{aligned} w &= \frac{\hat{w}}{g}, \quad x = \frac{X}{l}, \quad \tau = t \sqrt{\frac{\hat{E}I}{\rho phl^4}}, \quad \alpha = 6 \left(\frac{g_0}{h} \right), \\ \alpha_{vdw} &= \frac{abl^4}{6\pi g_0^4 \hat{E}I}, \quad \alpha_{Ca} = \frac{\pi^2 h c b l^4}{240 g_0^5 \hat{E}I} \\ \delta &= \frac{\mu b h \lambda^2}{\hat{E}I}, \quad \gamma = 0.65 \frac{g}{b}, \quad \beta = \frac{\varepsilon_0 b V^2 L^4}{2g_0^3 \hat{E}I}, \quad N = \frac{\hat{N} l^2}{\hat{E}I}. \end{aligned} \right\} \quad (16)$$

According to Fu *et al.* [50], assuming $w(\xi, \tau) = \phi(\xi)u(\tau)$ where $\phi(\xi)$ is the first eigen mode of the clamped-clamped beam, which can be written as $\phi(\xi) = 16\xi^2(1 - \xi^2)$, and $u(\tau)$ constricts the time-dependent part of the solution.

Rewrite Eq. (14) in the following form:

$$\ddot{u} + \left(\frac{K_1}{a_2} \right) u + 1 \left(-\frac{4a_4}{a_2} u^2 \ddot{u} + \frac{6a_6}{a_2} u^4 \ddot{u} - \frac{4a_8}{a_2} u^6 \ddot{u} + \frac{a_{10}}{a_2} u^8 \ddot{u} + \frac{K_1}{a_2} u + \frac{K_2}{a_2} u^3 + \frac{K_3}{a_2} u^5 + \frac{K_4}{a_2} u^7 + \frac{K_5}{a_2} u^9 + \frac{K_6}{a_2} u^{11} \right) = 0. \quad (17)$$

Assume that the solution is a power series in p :

$$u = u_0 + pu_1 + p^2u_2 + \dots, \quad (18)$$

let

$$K_1/a_2 = \omega^2 + p\omega_1 + p^2\omega_2 + \dots, \quad (19)$$

$$1 = pb_1 + p^2b_2 + \dots \quad (20)$$

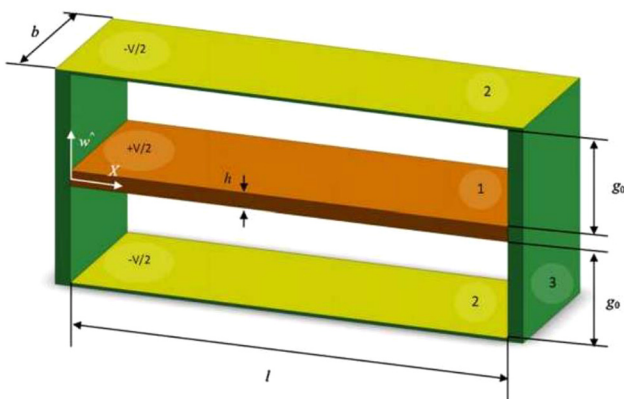


Figure 1: Electrostatically actuated microbeam geometry.

Substituting Eqs. (18)–(20) into Eq. (17) and the coefficient of p should be zero, then a set of linear differential equations was created.

$$\ddot{u}_0 + \omega^2 u_0 = 0, \quad (21)$$

$$\ddot{u}_1 + \omega^2 u_1 + \frac{b_1 K_2}{a_2} u_0^3 + \frac{b_1 K_3}{a_2} u_0^5 + \frac{b_1 K_4}{a_2} u_0^7 + \frac{b_1 K_5}{a_2} u_0^9 + \frac{b_1 K_6}{a_2} u_0^{11} - \frac{4a_4 b_1}{a_2} u_0^2 \ddot{u}_0 + \frac{6a_6 b_1}{a_2} u_0^4 \ddot{u}_0 - \frac{4a_8 b_1}{a_2} u_0^6 \ddot{u}_0 + \frac{a_{10} b_1}{a_2} u_0^8 \ddot{u}_0 = 0. \quad (22)$$

We achieved the following conclusions by solving Eqs. (21) and (22) using the initial conditions stated in Eq. (15), and we obtain

$$u_0 = A \cos(\omega t), \quad (23)$$

$$\begin{aligned} u_1 &= \frac{1}{8\omega^2} \left(\frac{165A^{11}b_1K_6}{512a_2} + \frac{21A^9b_1K_5}{64a_2} - \frac{21a_{10}A^9b_1\omega^2}{64a_2} + \frac{21A^7b_1K_4}{64a_2} \right. \\ &\quad + \frac{21a_8A^7b_1\omega^2}{16a_2} + \frac{5A^5b_1K_3}{16a_2} - \frac{15a_6A^5b_1\omega^2}{8a_2} + \frac{A^3b_1K_2}{4a_2} \\ &\quad \left. + \frac{a_4A^3b_1\omega^2}{a_2} \right) \times \cos(3t\omega) + \frac{1}{24\omega^2} \left(\frac{165A^{11}b_1K_6}{1024a_2} + \frac{9A^9b_1K_5}{64a_2} \right. \\ &\quad - \frac{9a_{10}A^9b_1\omega^2}{64a_2} + \frac{7A^7b_1K_4}{64a_2} + \frac{7a_8A^7b_1\omega^2}{16a_2} + \frac{A^5b_1K_3}{16a_2} \\ &\quad \left. - \frac{3a_6A^5b_1\omega^2}{8a_2} \right) \times \cos(5t\omega) + \frac{1}{48\omega^2} \left(\frac{55A^{11}b_1K_6}{1024a_2} + \frac{9A^9b_1K_5}{256a_2} \right. \\ &\quad - \frac{9a_{10}A^9b_1\omega^2}{256a_2} + \frac{A^7b_1K_4}{64a_2} + \frac{a_8A^7b_1\omega^2}{16a_2} \left. \right) \times \cos(7t\omega) \\ &\quad + \frac{1}{80\omega^2} \left(\frac{11A^{11}b_1K_6}{1024a_2} + \frac{A^9b_1K_5}{256a_2} - \frac{a_{10}A^9b_1\omega^2}{256a_2} \right) \cos(9t\omega) \\ &\quad + \frac{1}{120\omega^2} \left(\frac{A^{11}b_1K_6}{1024a_2} \right) \cos(11t\omega). \end{aligned} \quad (24)$$

with the nonlinear frequency–amplitude relationship:

$$\omega_1 = (\Delta_1 + \Delta_2)/\Delta_3, \quad (25)$$

where

$$\begin{aligned} \Delta_1 &= -A^2(231a_2A^8K_6 - 252a_{10}A^6K_1 + 252a_2A^6K_5 + 1120a_8A^4K_1 \\ &\quad + 280a_2A^4K_4), \\ \Delta_2 &= -A^2(-1,920a_6A^2K_1 + 320a_2A^2K_3 + 1,536a_4K_1 \\ &\quad + 384a_2K_2), \\ \Delta_3 &= 4a_2(63a_{10}A^8 - 280a_8A^6 + 480a_6A^4 - 384a_4A^2 + 128a_2). \end{aligned}$$

Finally, we obtain the first-order approximation provided by putting Eqs. (23) and (24) into Eq. (18):

4 Results and discussion

To examine the accuracy of the parameter-expansion method, we plot the analytical approximate solutions with the numerical solutions and those in the literature, for example, the EBM [30] and the GRHBM [32]. The calculations are plotted for the

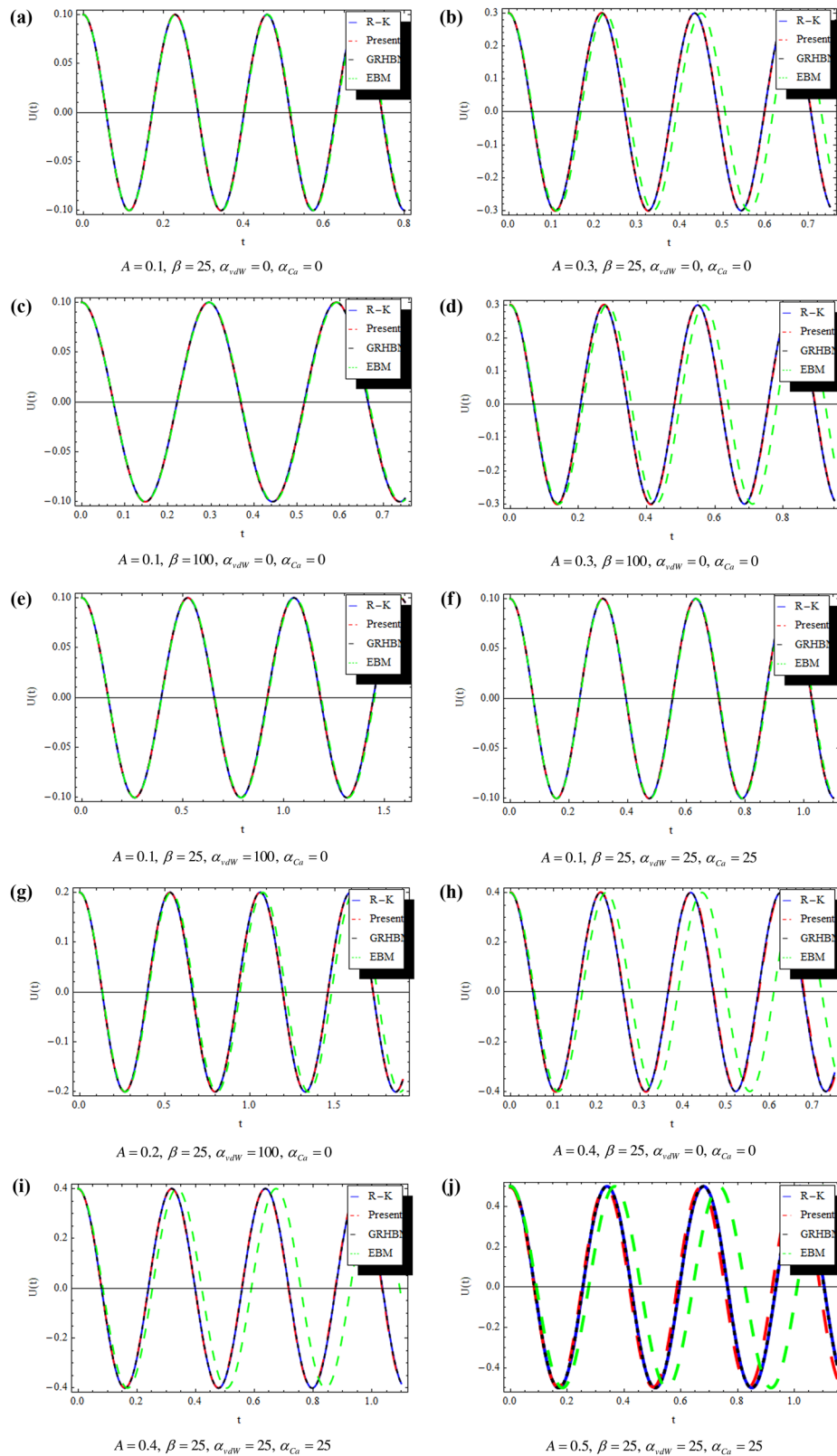


Figure 2: Comparison of the present analytical solution, EBM [30], GRHBM [32], and the numerical solution. (a) $A = 0.1, \beta = 25, \alpha_{vdW} = 0, \alpha_{Ca} = 0$, (b) $A = 0.3, \beta = 25, \alpha_{vdW} = 0, \alpha_{Ca} = 0$, (c) $A = 0.1, \beta = 100, \alpha_{vdW} = 0, \alpha_{Ca} = 0$, (d) $A = 0.3, \beta = 100, \alpha_{vdW} = 0, \alpha_{Ca} = 0$, (e) $A = 0.1, \beta = 25, \alpha_{vdW} = 100, \alpha_{Ca} = 0$, (f) $A = 0.1, \beta = 25, \alpha_{vdW} = 25, \alpha_{Ca} = 25$, (g) $A = 0.2, \beta = 25, \alpha_{vdW} = 100, \alpha_{Ca} = 0$, (h) $A = 0.4, \beta = 25, \alpha_{vdW} = 0, \alpha_{Ca} = 0$, (i) $A = 0.4, \beta = 25, \alpha_{vdW} = 25, \alpha_{Ca} = 25$, (j) $A = 0.5, \beta = 25, \alpha_{vdW} = 25, \alpha_{Ca} = 25$.

values of Van der Waals parameter α_{vdW} by choosing the given values of parameters $\delta = 0$ and $\gamma = 0$ in the first four figures while $\delta = 0.65$ and $\gamma = 0.65$ at the last six figures, respectively, and for different values of α_{vdW} , α_{Ca} , β , and A , where the values $N = 10$ and $\alpha = 24$ in the all cases for a clamped-clamped beam. The comparison results are shown in Figure 2. Figure 2 shows that the numerical findings from the fourth-order Runge–Kutta technique correspond quite well with the analytical approximation, demonstrating the high precision of the solutions utilizing the suggested technique. The analytical solutions show that the second term in series expansions is enough to obtain a very accurate solution to the current model. Figure 2 shows a comparison of different beginning frequencies of micro-beams estimated using different approaches, revealing that the PEM results correspond quite well with the numerical and experimental results reported in the literature. Also, the PEM performance implies that it might be used for more sophisticated nonlinear differential equations. We may therefore infer that the current technique is a more powerful computational methodology for analysing nonlinear problems than other known methods currently in use.

5 Conclusion

In this article, the PEM was employed to obtain the approximate solution of a nanoelectro mechanical resonator system. The frequency–amplitude relationships are achieved in the closed forms. The high accuracy of the PEM is presented by comparing the current solutions with the EBM [30], GRHBM [32], and the numerical solutions. The results show that the PEM is very useful in analysing nonlinear oscillations. Finally, we can conclude that the PEM can produce the approximate analytic solution and the corresponding quite accurate frequency with the first-order approximation. The suggested approach in this study is superior to the traditional perturbation method since it does not rely on small parameter assumptions. The new solution's correctness is validated by comparing the acquired findings to previously published results in the literature, as well as the numerical solution. Excellent agreement was found between the present and numerical solution, while better results have been obtained as compared to other techniques available in the literature.

Acknowledgments: The researchers wish to extend their sincere gratitude to the deanship of scientific research at the Islamic University of Madinah for the support provided to the post-publishing program.

Funding information: The deanship of scientific research at the Islamic University of Madinah.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

References

- [1] He JH. The simplest approach to nonlinear oscillators. *Results Phys.* 2019;15:102546.
- [2] He JH, Hou WF, Qie N, Gepreel KA, Shirazi AH, Sedighi HM. Hamiltonian-based frequency-amplitude formulation for nonlinear oscillators. *Facta Universitatis Ser: Mech Eng.* 2021;19:199–208.
- [3] He JH, Skrzypacz PS, Zhang Y, Pang J. Approximate periodic solutions to microelectromechanical system oscillator subject to magnetostatic excitation. *Math Methods Appl Sci.* 2020. doi: 10.1002/mma.7018.
- [4] Nadeem M, He JH. He-Laplace variational iteration method for solving the nonlinear equations arising in chemical kinetics and population dynamics. *J Math Chem.* 2021;59:1234–45.
- [5] Leissa AW. *Ibration of plates*. Washington, DC: Acoustical Society of America; 1993.
- [6] Chopra AK. *Dynamic of structures, theory and application to earthquake engineering*. New Jersey: Prentice-Hall; 1995.
- [7] Nayfeh AH, Mook DT. *Nonlinear oscillations*. New York: John Wiley & Sons; 1979.
- [8] Senturia SD. *Mircrosystem design*. Norwell, MA, USA: Kluwer Academic Publishers; 2001.
- [9] Sasayama T, Suzuki S, Tsuchitani S, Koide A, Suzuki M, Nakazawa T, et al. Highly reliable silicon micromachined physical sensors in mass production. *Sens Actuators A: Phys.* 1996;54:714–7.
- [10] Bao MH, Yang H, Yin H, Shen SQ. Effects of electrostatic forces generated by the driving signal on capacitive sensing devices. *Sens Actuators A: Phys.* 2000;84:213.
- [11] Lee S, Ramadoss R, Buck M, Bright VM, Gupta KC, Lee YC. Reliability testing of flexible printed circuit-based RF MEMS capacitive switches. *Microelectron Reliab.* 2004;44:245.
- [12] He JH. Variational iteration method-a kind of non-linear analytical technique: some examples. *Int J Non-Linear Mech.* 1999;34:699–708.
- [13] Tang W, Anjum N, He JH. Variational iteration method for the nanobeams-based N/MEMS system. *MethodsX.* 2023;11:102465.
- [14] Az-Zo'bi EA, Al-Khaled K, Darweesh A. Numeric-analytic solutions for nonlinear oscillators *via* the modified multi-stage decomposition method. *Mathematics.* 2019;7:550.
- [15] He JH. A coupling method of a homotopy technique and a perturbation technique for non-linear problems. *Int J Non-Linear Mech.* 2000;35:37–43.
- [16] He CH, El-Dib YQ. A heuristic review on the homotopy perturbation method for non-conservative oscillators. *J Low Freq Noise Vib Act Control.* 2022;41:572–603.
- [17] He JH, Amer TS, Elnaggar S, Galal AH. Periodic property and instability of a rotating pendulum system. *Axioms.* 2021;10:191.
- [18] He JH, El-Dib YO. The enhanced homotopy perturbation method for axial vibration of strings. *Facta Universitatis Ser: Mech Eng.* 2021;19:735–50. doi: 10.22190/FUME210125033H.

- [19] Anjum N, He JH, Ain QT, Tian D. Li-He's modified homotopy perturbation method for doubly-clamped electrically actuated microbeams- based microelectromechanical system. *Facta Universitatis Ser: Mech Eng.* 2021;19(4):601–12. doi: 10.22190/FUME210112025A.
- [20] Zhang Y, Tian D, Pang J. A fast estimation of the frequency property of the microelectromechanical system oscillator. *J Low Freq Noise Vib Act Control.* 2022;41:160–6.
- [21] Yang YQ. A mathematical control for the pseudo-pull-in stability arising in a micro-electromechanical system. *J Low Freq Noise Vib Act Control.* 2023;42:927–34.
- [22] Az-Zo'bi EA. A reliable analytic study for higher-dimensional telegraph equation. *J Math Comput Sci.* 2018;18:423–9.
- [23] Wu B, Sun W, Lim C. An analytical approximate technique for a class of strongly non-linear oscillators. *Int J Non-Linear Mech.* 2006;41:766–74.
- [24] Farea NM, Zayed M, Ismail GM. Accurate analytical solution of the circular sector oscillation by the modified harmonic balance method. *J Low Freq Noise Vib Act Control.* 2022;41:1446–53.
- [25] He JH, Abdou M. New periodic solutions for nonlinear evolution equations using Exp-function method. *Chaos Solitons Fractals.* 2007;34:1421–9.
- [26] Khan MN, Haider JA, Wang Z, Lone SA, Almutlak SA, Elseesy IE. Application of Laplace-based variational iteration method to analyze generalized nonlinear oscillations in physical systems. *Mod Phys Lett B.* 2023;37:2350169.
- [27] Az-Zo'bi EA, Al Dawoud K, Marashdeh M. Numeric-analytic solutions of mixed-type systems of balance laws. *Appl Math Comput.* 2015;265:133–43.
- [28] Lu J, Ma L. The VIM-Padé technique for strongly nonlinear oscillators with cubic and harmonic restoring force. *J Low Freq Noise Vib Active Control.* 2019;38:1272–8.
- [29] Mehdipour I, Ganji DD, Mozaffari M. Application of the energy balance method to nonlinear vibrating equations. *Curr Appl Phys.* 2020;10:104–12.
- [30] Ghalambaz M, Ghalambaz M, Edalatifar M. Nonlinear oscillation of nanoelectro-mechanical resonators using energy balance method: considering the size effect and the van der Waals force. *Appl Nanosci.* 2016;6:309–17.
- [31] Ju P. Global residue harmonic balance method for Helmholtz-Duffing oscillator. *App Math Model.* 2015;39:2172–9.
- [32] Ismail GM, Abul-Ez M, Farea NM, Saad N. Analytical approximations to nonlinear oscillation of nanoelectro-mechanical resonators. *Eur Phys J Plus.* 2019;134:47.
- [33] He JH. Special functions for solving nonlinear differential equations. *Int J Appl Comput Math.* 2021;7:84.
- [34] Ismail GM, El-Moshneb MM, Zayed M. Analytical technique for solving strongly nonlinear oscillator differential equations. *Alex Eng J.* 2023;74:547–57.
- [35] Ismail GM, El-Moshneb MM, Zayed M. A modified global error minimization method for solving nonlinear Duffing-harmonic oscillators. *AIMS Math.* 2023;8:484500.
- [36] Ruttanaprommarin N, Sabir Z, Núñez RAS, Az-Zo'bi EA, Weera W, Botmart T, et al. A stochastic framework for solving the prey-predator delay differential model of Holling Type-III, *Computers. Mater Continua.* 2023;74:5915–30.
- [37] Ismail GM, Cveticanin L. Higher order Hamiltonian approach for solving doubly clamped beam type N/MEMS subjected to the van der Waals attraction. *Chin J Phys.* 2021;72:6977.
- [38] Koroglu C, Ozis T. Applications of parameter-expanding method to nonlinear oscillators in which the restoring force is inversely proportional to the dependent variable or in form of rational function of dependent variable. *Comput Model Eng Sci.* 2011;75:223–34.
- [39] Hieu DV, Thoa NTK, Duy LQ. Analysis of nonlinear oscillator arising in the microelectromechanical system by using the parameter expansion and equivalent linearization methods. *Int J Eng Tech.* 2018;7:597–604.
- [40] Kaya MO, Demirbag SA. Application of parameter expansion method to the generalized nonlinear discontinuity equation. *Chaos, Solitons Fractals.* 2009;42:1967–73.
- [41] Wang SQ, He JH. Nonlinear oscillator with discontinuity by parameter-expansion method. *Chaos, Solitons Fractals.* 2008;35:688–91.
- [42] Darvishi MT, Karami A, Shin BC. Application of He's parameter-expansion method for oscillators with smooth odd nonlinearities. *Phys Lett A.* 2008;372:5381–4.
- [43] Abbas I. Finite element analysis of the thermoelastic interactions in an unbounded body with a cavity. *Forsch Ingenieurwes.* 2007;71:215–22.
- [44] Zenkour AM, Abbas I. Nonlinear transient thermal stress analysis of temperature-dependent hollow cylinders using a finite element model. *Int J Struct Stab Dyn.* 2014;7:1450025.
- [45] Marin M, Hobiny A, Abbas I. The effects of fractional time derivatives in porothermoelastic materials using finite element method. *Mathematics.* 2021;9:1606.
- [46] Faghidian SA, Tounsi A. Dynamic characteristics of mixture unified gradient elastic nanobeams. *Facta Universitatis Ser: Mech Eng.* 2022;20:539–52.
- [47] He CH. A variational principle for a fractal nano/microelectromechanical (N/MEMS) system. *Int J Numer Methods Heat Fluid Flow.* 2023;33:351–9.
- [48] He JH. An elementary introduction to recently developed asymptotic methods and nanomechanics in textile engineering. *Int J Mod Phys B.* 2008;22:3487–578.
- [49] He JH. Bookkeeping parameter in perturbation methods. *Int J Nonlinear Sci Numer Simul.* 2001;2:257–64.
- [50] Fu Y, Zhang J, Wan L. Application of the energy balance method to a nonlinear oscillator arising in the microelectromechanical system (MEMS). *Curr Appl Phys.* 2022;11:482.

Appendix

The coefficients of $K_1 - K_6$, $a_0 - a_{10}$, $b_1 - b_5$ and $c_1 - c_5$ in Eq. (14) are as follows:

$$K_1 = (1 + \delta)b_1 - \chi_{\text{typ}}Nc_1 - 8\alpha_{\text{Ca}}a_2 - 6\alpha_{\text{vdw}}a_2 - 4\beta a_2 - 2\gamma\beta a_2,$$

$$K_2 = -4(1 + \delta)b_2 + 4\chi_{\text{typ}}Nc_2 - \chi_{\text{typ}}\alpha_{\text{Ca}}a_0 - 8\alpha_{\text{Ca}}a_4 + 4\alpha_{\text{vdw}}a_4 + 8\beta a_4 + 6\gamma\beta a_4,$$

$$K_3 = 6(1 + \delta)b_3 - 6\chi_{\text{typ}}Nc_3 + 4\chi_{\text{typ}}\alpha_{\text{Ca}}a_0 + 2\alpha_{\text{vdw}}a_6 - 4\beta a_6 - 6\gamma\beta a_6,$$

$$K_4 = -4(1 + \delta)b_4 + 4\chi_{\text{typ}}Nc_4 - 6\chi_{\text{typ}}\alpha_{\text{Ca}}a_0 + 2\gamma\beta a_8,$$

$$K_5 = (1 + \delta)b_5 - \chi_{\text{typ}}Nc_5 + 4\chi_{\text{typ}}\alpha_{\text{Ca}}a_0,$$

$$K_6 = -\chi_{\text{typ}}\alpha_{\text{Ca}}a_0.$$

$$a_0 = \int_0^1 \phi'^2 dx, \quad a_1 = \int_0^1 \phi dx, \quad a_2 = \int_0^1 \phi^2 dx, \quad a_3 = \int_0^1 \phi^3 dx,$$

$$a_4 = \int_0^1 \phi^4 dx, \quad a_5 = \int_0^1 \phi^5 dx, \quad a_6 = \int_0^1 \phi^6 dx, \quad a_7 = \int_0^1 \phi^7 dx,$$

$$a_8 = \int_0^1 \phi^8 dx, \quad a_9 = \int_0^1 \phi^9 dx, \quad a_{10} = \int_0^1 \phi^{10} dx.$$

$$b_1 = \int_0^1 \phi \phi^{(iv)} dx, \quad b_2 = \int_0^1 \phi^3 \phi^{(iv)} dx, \quad b_3 = \int_0^1 \phi^5 \phi^{(iv)} dx,$$

$$b_4 = \int_0^1 \phi^7 \phi^{(iv)} dx, \quad b_5 = \int_0^1 \phi^9 \phi^{(iv)} dx,$$

$$c_1 = \int_0^1 \phi \phi' dx, \quad c_2 = \int_0^1 \phi^3 \phi' dx, \quad c_3 = \int_0^1 \phi^5 \phi' dx,$$

$$c_4 = \int_0^1 \phi^7 \phi' dx, \quad c_5 = \int_0^1 \phi^9 \phi' dx.$$