Research Article

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Soliton, quasi-soliton, and their interaction solutions of a nonlinear (2 + 1)-dimensional ZK–mZK–BBM equation for gravity waves

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Abstract: The ZK–mZK–BBM equation plays a crucial role in actually depicting the gravity water waves with the long wave region. In this article, the bilinear forms of the (2 + 1)-dimensional ZK–mZK–BBM equation were derived using variable transformation. Then, the multiple soliton solutions of the ZK–mZK–BBM equation are obtained by bilinear forms and symbolic computation. Under complex conjugate transformations, quasi-soliton solutions and mixed solutions composed of one-soliton and one-quasi-soliton are derived from soliton solutions. These solutions are further studied graphically to observe the propagation characteristics of gravity water waves. The results enrich the research of gravity water wave in fluid mechanics.

Keywords: ZK–mZK–BBM equation, Hirota method, soliton solutions, quasi-soliton solutions

1 Introduction

Gravity water waves are known to play a crucial role in coastal, energy, and hydraulic engineering and attract much current interest [1–7]. A series of nonlinear partial differential equations are proposed to analyze the characteristics of gravity water waves [8–12]; a typical example is the ZK–mZK–BBM equation, which describes gravity water waves in a fluid [13].

The ZK–mZK–BBM equation is a nonlinear partial equation, which is a conjunction of the ZK and BBM equations, or the mZK and BBM equations, and it is of great significance to explore its solution for describing the motion law of water waves [13]. The ZK–mZK–BBM equation [13] has the following form:

$$u_t + \alpha u_x + \beta_1(u^2)_x + \beta_2(u^3)_x + \gamma(u_{xt} + u_{yy})_x = 0. \qquad (1)$$

When $\beta_2 = 0$, $\beta_1 \neq 0$, Eq. (1) is the ZK–BBM equation; when $\beta_1 = 0$, $\beta_2 \neq 0$, Eq. (1) is the mZK–BBM equation. Here, α, β_1, β_2, and γ are known coefficients, β_1 and β_2 are relative nonlinear coefficients, and γ is the dispersion coefficient.

There are many methods for solving nonlinear equations, such as the Hirota bilinear method [14–21], dressing method [22], Riemann–Hilbert method [23], steepest descent method [24], Lie symmetry analysis approach [25], Ansatz approach [26], auxiliary equation approach [27], new Kudryashov approach [28], sine-Gordon expansion approach [29], exp-function approach [30], new generalized ϕ 6-model expansion approach [31], new extended auxiliary equation approach [32], the Darboux transformations method [33–35], and so on [36–46].

The Hirota bilinear method is used to solve equations by utilizing bilinear operators, which are simple in form, easy to operate, and only related to the solved equation. Many partial differential equations are solved effectively by the Hirota bilinear method. Zhou *et al.* used the above method to solve the multiple-soliton and quasi-soliton solutions of the modified Korteweg–de Vries–Zakharov–Kuznetsov equation [47]. Yang *et al.* used the above method to obtain solitons and quasi-periodic behaviors in an inhomogeneous optical fiber [48]. Hong *et al.* used the above method to solve the multiple-soliton solution of the Hirota–Satsuma–Ito equation in shallow water [49–52].

In this article, we use the Hirota method to solve the soliton, quasi-soliton, and their interaction solutions of the ZK–mZK–BBM equation for gravity waves. The bilinear forms are deduced in Section 2. Multiple soliton solutions are presented in Section 3. Quasi-soliton solutions and mixed solutions are presented in Section 4. Section 5 concludes the article.

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2 Bilinear forms

Motivated by previous studies [53–57], we suppose

$$u = \sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left[\ln\frac{g}{f}\right]_x. \tag{2}$$

Eq. (1) becomes

$$\begin{aligned}
&\frac{D_t g\cdot f}{gf} + \alpha\frac{D_x g\cdot f}{gf} + \gamma\frac{D_t D_x^2 g\cdot f}{gf} + \gamma\frac{D_x D_y^2 g\cdot f}{gf}\\
&- \gamma\frac{D_x^2 g\cdot f}{gf}\frac{D_t g\cdot f}{gf} + \beta_1\sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left(\frac{D_x g\cdot f}{gf}\right)^2\\
&+ 2\gamma\frac{D_t g\cdot f}{gf}\left(\frac{D_x g\cdot f}{gf}\right)^2 - 2\gamma\frac{D_t D_x g\cdot f}{gf}\frac{D_x g\cdot f}{gf}\\
&- \gamma\frac{D_y^2 g\cdot f}{gf}\frac{D_x g\cdot f}{gf} - 2\gamma\frac{D_x D_y g\cdot f}{gf}\frac{D_y g\cdot f}{gf}\\
&- 2\alpha\gamma\left(\frac{D_x g\cdot f}{gf}\right)^3 + 2\gamma\left(\frac{D_y g\cdot f}{gf}\right)^2\frac{D_x g\cdot f}{gf} = 0,
\end{aligned} \tag{3}$$

where g and f are two real functions of x, y, and t, and D_x, D_y, and D_t are bilinear operators.

Assuming that

$$D_y g\cdot f = \sqrt{\alpha}D_x g\cdot f, \tag{4}$$

Eq. (3) becomes

$$\begin{aligned}
&\frac{D_t g\cdot f}{gf} + \alpha\frac{D_x g\cdot f}{gf} + \gamma\frac{D_t D_x^2 g\cdot f}{gf} + \gamma\frac{D_x D_y^2 g\cdot f}{gf}\\
&+ \sqrt{\frac{-\alpha\gamma\beta_1^2}{2\beta_2}}\frac{D_x^2 g\cdot f}{gf} - \left(2\gamma\frac{D_t D_x g\cdot f}{gf} + \gamma\frac{D_y^2 g\cdot f}{gf}\right.\\
&\left.+ 2\sqrt{\alpha}\gamma\frac{D_x D_y g\cdot f}{gf}\right)\frac{D_x g\cdot f}{gf} = 0,
\end{aligned} \tag{5}$$

and the bilinear forms of Eq. (1) can be derived as

$$\left(D_t + \alpha D_x + \gamma D_t D_x^2 + \gamma D_x D_y^2 + \sqrt{\frac{-\alpha\gamma\beta_1^2}{2\beta_2}}D_x^2\right)g\cdot f = 0, \tag{6}$$

$$(2D_t D_x + D_y^2 + 2\sqrt{\alpha}D_x D_y)g\cdot f = 0, \tag{7}$$

$$D_y g\cdot f = \sqrt{\alpha}D_x g\cdot f. \tag{8}$$

According to Eq. (2), we can obtain

$$\sqrt{\frac{-2\alpha\gamma}{\beta_2}} = \begin{cases}\sqrt{\dfrac{-2\alpha\gamma}{\beta_2}}, & \alpha\gamma\beta_2 < 0\\[2ex] \sqrt{\dfrac{2\alpha\gamma}{\beta_2}}\,i, & \alpha\gamma\beta_2 > 0,\end{cases} \tag{9}$$

where $i = \sqrt{-1}$.

3 Soliton solutions

3.1 *N*-soliton solutions

Considering the N-soliton solution of Eq. (1), we suppose that

$$g = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3 + \cdots \varepsilon^N g_N, \tag{10}$$

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 + \cdots + \varepsilon^N f_N, \tag{11}$$

where g_i's and f_i's $(i = 1, 2, 3,\cdots)$ are real functions. Substituting Eqs. (10) and (11) into Eqs. (6)–(8) and eliminating the coefficients of all powers of ε, we obtain a series of equations. By solving these equations, the N-soliton solution can be expressed as

$$u = \sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left[\ln\left(\frac{G_N}{F_N}\right)\right]_x, \tag{12}$$

where

$$G_N = \begin{cases}\displaystyle\sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j\theta_j + \sum_{j<l}^N(\mu_j\mu_l A_{jl})\right], & \alpha\gamma\beta_2 < 0\\[3ex] \displaystyle\sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j\left(\theta_j + \frac{i\pi}{2}\right) + \sum_{j<l}^N(\mu_j\mu_l A_{jl})\right], & \alpha\gamma\beta_2 > 0,\end{cases}$$

$$F_N = \begin{cases}\displaystyle\sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j(\theta_j + i\pi) + \sum_{j<l}^N(\mu_j\mu_l A_{jl})\right], & \alpha\gamma\beta_2 < 0\\[3ex] \displaystyle\sum_{\mu=0,1}\exp\left[\sum_{j=1}^N \mu_j\left(\theta_j - \frac{i\pi}{2}\right) + \sum_{j<l}^N(\mu_j\mu_l A_{jl})\right], & \alpha\gamma\beta_2 > 0,\end{cases}$$

while $\sum_{\mu=0,1}$ is the sum of all the permutations of $\{\mu_1, \mu_2,\cdots, \mu_N\} = 0, 1$, and $\theta_j = k_j x + h_j y + \omega_j t$, $h_j = \sqrt{\alpha}k_j$, $\omega_j = -\alpha k_j$, $1 \le j \le N$, $A_{jl} = \frac{(k_j - k_l)^2}{(k_j + k_l)^2}$, $1 \le j < l \le N$, k_j's being the constants.

3.2 One-soliton solutions

By taking $N = 1$ in Eqs. (10)–(12), we have

$$g = 1 + \varepsilon g_1, \tag{13}$$

$$f = 1 + \varepsilon f_1, \tag{14}$$

with $g_1 = e^{\theta_1}$, $f_1 = -e^{\theta_1}$.

Then, based on Eq. (9) and Eqs. (12)–(14), the one-soliton solution can be given as

$$u_1 = \sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left[\ln\left(\frac{g}{f}\right)\right]_x = \begin{cases} \pm\sqrt{\frac{-2\alpha\gamma}{\beta_2}}\frac{k_1}{\sinh(\theta_1)}, & \varepsilon = \pm 1,\ \alpha\gamma\beta_2 < 0 \\ \mp\sqrt{\frac{2\alpha\gamma}{\beta_2}}\frac{k_1}{\cosh(\theta_1)}, & \varepsilon = \pm i,\ \alpha\gamma\beta_2 > 0, \end{cases} \tag{15}$$

where θ_1 is given by Eq. (12), and k_1 is constant. The figures of one-soliton can be obtained by selecting the appropriate parameters, $\alpha = 1$, $k_1 = 0.1$, $\gamma = 0.3$, $\beta_2 = -12$, $\varepsilon = 1$, and $t = 0$, as shown in Figure 1(a). Compared with Figure 1(a), when β_2 and γ, respectively, increase, it can be found that the amplitudes of one-soliton increase in Figure 1(b) and (c), but the shapes and widths of one-soliton remain unchanged.

3.3 Two-soliton solutions

By taking $N = 2$ in Eqs. (10)–(12), we have

$$g = 1 + \varepsilon g_1 + \varepsilon^2 g_2, \tag{16}$$

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2, \tag{17}$$

with $g_1 = e^{\theta_1} + e^{\theta_2}$, $g_2 = A_{12}e^{\theta_1+\theta_2}$, $f_1 = -g_1 = -(e^{\theta_1} + e^{\theta_2})$, $f_2 = g_2 = A_{12}e^{\theta_1+\theta_2}$.

Then, based on Eqs. (9), (12), (16), and (17), the two-soliton solution can be expressed as

$$u_2 = \sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left[\ln\left(\frac{g}{f}\right)\right]_x = \begin{cases} \pm\sqrt{\frac{-2\alpha\gamma}{\beta_2}}\frac{2[(1+g_2)g_{1,x} - g_{2,x}g_1]}{(1+g_2)^2 - g_1^2}, \\ \varepsilon = \pm 1,\ \alpha\gamma\beta_2 < 0 \\ \mp\sqrt{\frac{2\alpha\gamma}{\beta_2}}\frac{2[(1-g_2)g_{1,x} + g_{2,x}g_1]}{(1-g_2)^2 + g_1^2}, \quad \varepsilon = \pm i, \\ \alpha\gamma\beta_2 > 0, \end{cases} \tag{18}$$

whereas θ_1, θ_2, and A_{12} are given by Eq. (12). The effect of parameters β_2 and γ in a two-soliton solution is the same as in a one-soliton; when β_2 and γ, respectively, increase, the amplitudes of two-soliton increase, but the shapes and widths of two-soliton remain unchanged. The figure of two-soliton is shown in Figure 2(b); it can be found that the two-soliton is parallel.

3.4 Three-soliton solutions

By taking $N = 3$ in Eqs. (10)–(12), we have

$$g = 1 + \varepsilon g_1 + \varepsilon^2 g_2 + \varepsilon^3 g_3, \tag{19}$$

$$f = 1 + \varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3, \tag{20}$$

with

$$g_1 = e^{\theta_1} + e^{\theta_2} + e^{\theta_3},\quad g_2 = A_{12}e^{\theta_1+\theta_2} + A_{13}e^{\theta_1+\theta_3} + A_{23}e^{\theta_2+\theta_3},$$
$$g_3 = A_{12}A_{13}A_{23}e^{\theta_1+\theta_2+\theta_3},\quad f_1 = -g_1 = -(e^{\theta_1} + e^{\theta_2} + e^{\theta_3}),$$
$$f_2 = g_2 = A_{12}e^{\theta_1+\theta_2} + A_{13}e^{\theta_1+\theta_3} + A_{23}e^{\theta_2+\theta_3},$$
$$f_3 = -g_3 = -A_{12}A_{13}A_{23}e^{\theta_1+\theta_2+\theta_3}$$

Then, based on Eqs. (9), (12), (19), and (20), the three-soliton solution can be expressed as

$$u_3 = \sqrt{\frac{-2\alpha\gamma}{\beta_2}}\left[\ln\left(\frac{g}{f}\right)\right]_x = \begin{cases} \pm\sqrt{\frac{-2\alpha\gamma}{\beta_2}}\frac{2[(1+g_2)(g_{1,x}+g_{3,x}) - g_{2,x}(g_1+g_3)]}{(1+g_2)^2 - (g_1+g_3)^2}, \\ \varepsilon = \pm 1,\ \alpha\gamma\beta_2 < 0 \\ \mp\sqrt{\frac{2\alpha\gamma}{\beta_2}}\frac{2[(1-g_2)(g_{1,x}-g_{3,x}) - (-g_{2,x})(g_1-g_3)]}{(1-g_2)^2 + (g_1-g_3)^2} \\ ,\ \varepsilon = \pm i,\ \alpha\gamma\beta_2 > 0, \end{cases} \tag{21}$$

while θ_j and $A_{jl}(j, l = 1, 2, 3)$ are given by Eq. (12). The effect of parameters β_2 and γ in a three-soliton solution is the same as in one-soliton; when β_2 and γ, respectively, increase, the

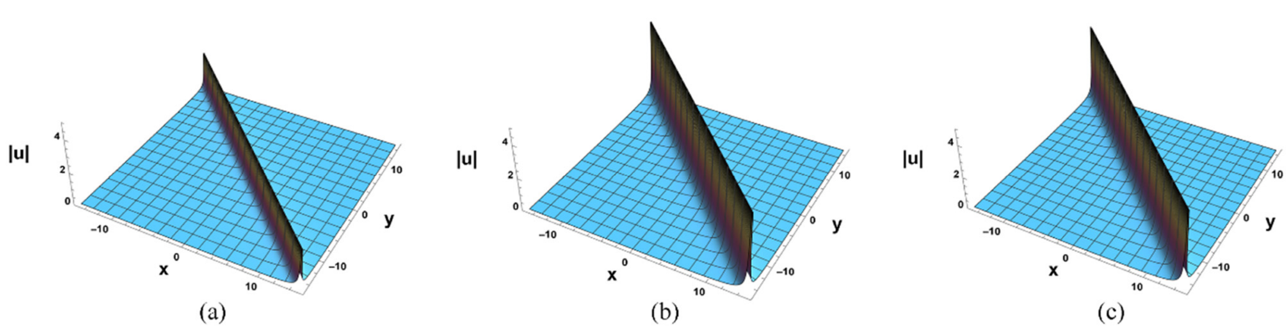

Figure 1: One-soliton solution with $\alpha = 1$, $k_1 = 0.1$, $\varepsilon = 1$, and $t = 0$. (a) $\gamma = 0.3$, $\beta_2 = -12$; (b) $\gamma = 0.3$, $\beta_2 = -3$; (c) $\gamma = 1.2$, $\beta_2 = -12$.

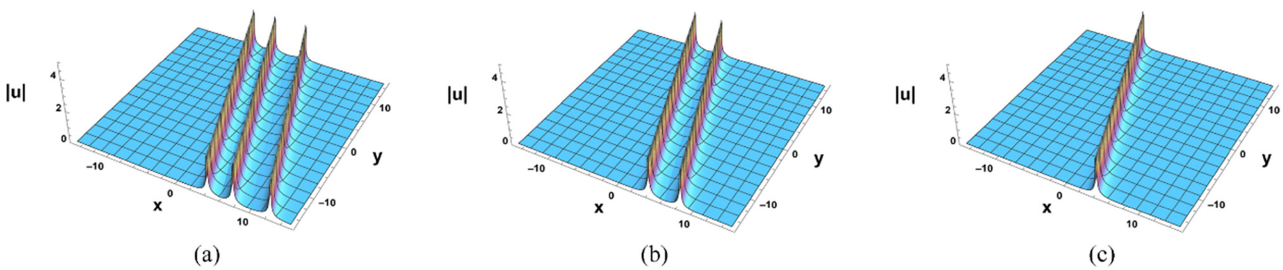

Figure 2: Three-soliton solution with $\alpha = 0.1$, $\gamma = 1.5$, $\beta_2 = -12$, $\varepsilon = 1$, and $t = 0$. (a) $k_1 = 1.5,\ k_2 = 1.7,\ k_3 = 1.9$; (b) $k_1 = 1.5, k_2 = k_3 = 1.7$; (c) $k_1 = k_2 = k_3 = 1.7$.

amplitudes of two-soliton increase, but the shapes and widths of two-soliton remain unchanged.

Figure 2(a) presents the figure of the three-soliton. It can be found that the three-soliton can become two-soliton when any two of k_1, k_2, and k_3 are equal, and three-soliton can become one-soliton when $k_1 = k_2 = k_3$. Figure 2(b) presents the two-soliton result of $k_2 = k_3$, and Figure 2(c) presents the one-soliton results of $k_1 = k_2 = k_3$.

4 Quasi-soliton solutions

4.1 One-quasi-soliton solutions

It is known from the study of Zhou *et al.* [47] that the one-quasi-soliton solution can be deduced from the two-soliton solution (18), in which we assume that

$$k_1 = s + ci,\ k_2 = k_1^* = s - ci, \tag{22}$$

where s and c are constants, and the symbol "*" indicates complex conjugate. By substituting Eq. (22) into Eq. (18), we obtain

$$u_1 = \begin{cases} \pm\sqrt{2}\sqrt{\dfrac{-\alpha\gamma}{\beta_2}}\dfrac{2[(1+Q_2)Q_{1,x} - Q_{2,x}Q_1]}{(1+Q_2)^2 - Q_1^2}, & \varepsilon = \pm 1, \\ \alpha\gamma\beta_2 < 0 & \\ \mp\sqrt{2}\sqrt{\dfrac{\alpha\gamma}{\beta_2}}\dfrac{2[(1-Q_2)Q_{1,x} + Q_{2,x}Q_1]}{(1-Q_2)^2 + Q_1^2}, & \varepsilon = \pm i, \\ \alpha\gamma\beta_2 > 0 & \end{cases}, \tag{23}$$

where

$$Q_1 = 2e^{sx+\sqrt{\alpha}sy+(-\alpha)st}\cos[cx + \sqrt{\alpha}cy + (-\alpha)ct],$$
$$Q_2 = -\frac{c^2}{s^2}e^{2sx+2\sqrt{\alpha}sy+(-2\alpha)st}.$$

Figure 3 shows one-quasi-soliton wave and exhibits the quasi-soliton waves affected by β_2 and γ. Compared with Figure 3(a), when the β_2 and γ change, the amplitudes of the one-quasi-solitons are varied in Figure 3(b) and (c), but the periods and velocities of the one-quasi-solitons remain unchanged. Figure 4 shows one-quasi-soliton with different values of the scaled time t. It can be found that the positions of the one-quasi-soliton are varied, but the amplitudes, periods, and velocities of one-quasi-solitons remain unchanged.

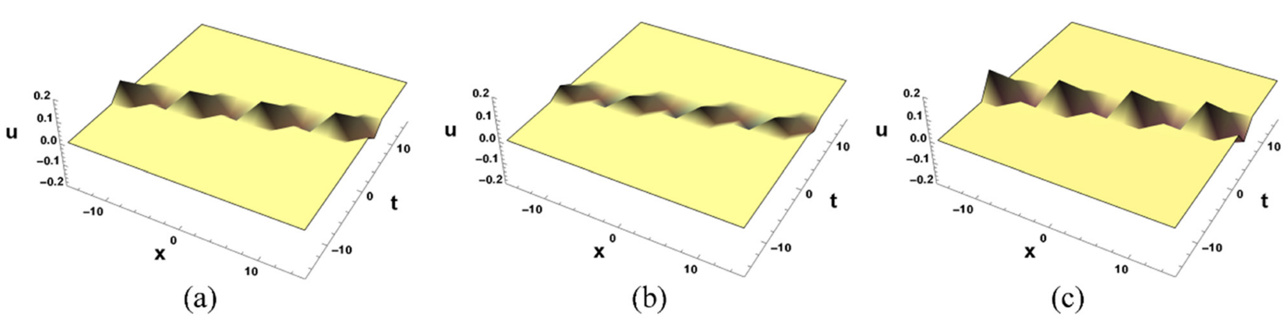

Figure 3: One-quasi-soliton solution with $\alpha = 4$, $\varepsilon = i$, $s = 1.2$, and $c = 1$, except that (a) $\gamma = 0.4$, $\beta_2 = 5$; (b) $\gamma = 0.4$, $\beta_2 = 20$; (c) $\gamma = 1$, $\beta_2 = 5$.

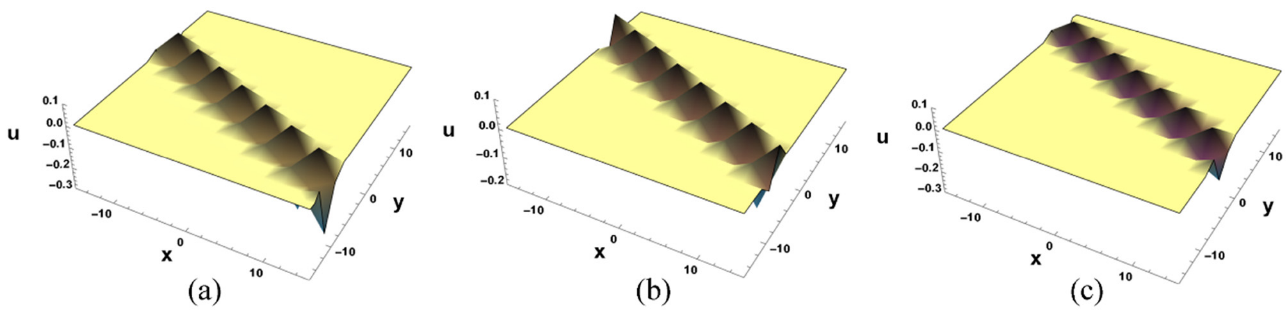

Figure 4: One-quasi-soliton solution with $\alpha = 4$, $\gamma = 0.4$, $\beta_2 = 5$, $\varepsilon = i$, $s = 1.2$, and $c = 1$, except that (a) $t = -1.6$; (b) $t = 0$; (c) $t = 1.6$.

$$E_{11} = \frac{2(-2s_1c_2 + 2s_2c_1 + s_1^2 - s_2^2 + c_1^2 - c_2^2)(2s_1c_2 - 2s_2c_1 + s_1^2 - s_2^2 + c_1^2 - c_2^2)}{[(s_1 + s_2)^2 + (c_1 + c_2)^2]^2},$$

$$E_{12} = \frac{2(-2s_1c_2 - 2s_2c_1 + s_1^2 - s_2^2 + c_1^2 - c_2^2)(2s_1c_2 + 2s_2c_1 + s_1^2 - s_2^2 + c_1^2 - c_2^2)}{[(s_1 + s_2)^2 + (c_1 - c_2)^2]^2},$$

4.2 Two-quasi-soliton solutions

For the four-soliton solution, we assume that

$$\begin{aligned} k_1 &= s_1 + c_1 i,\ k_2 = s_1 - c_1 i,\ k_3 = s_2 + c_2 i, \\ k_4 &= s_2 - c_2 i, \end{aligned} \tag{24}$$

where s_1, s_2, c_1, and c_2 are the real constants. By using Eq. (24) and four-soliton solutions, we obtain two-quasi-solitons as follows:

$$E_{21} = \frac{8(s_2c_1 - s_1c_2)(-s_1^2 + s_2^2 - c_1^2 + c_2^2)}{[(s_1 + s_2)^2 + (c_1 + c_2)^2]^2},$$

$$E_{22} = \frac{8(s_2c_1 + s_1c_2)(-s_1^2 + s_2^2 - c_1^2 + c_2^2)}{[(s_1 + s_2)^2 + (c_1 - c_2)^2]^2},$$

$$Y_{11} = \frac{-c_1^2}{2s_1^2}(E_{11}E_{12} + E_{21}E_{22}),\ Y_{12} = \frac{-c_1^2}{2s_1^2}(E_{21}E_{12} - E_{11}E_{22}),$$

$$Y_{21} = \frac{-c_2^2}{2s_2^2}(E_{11}E_{12} - E_{21}E_{22}),\ Y_{22} = \frac{-c_2^2}{2s_2^2}(E_{21}E_{12} + E_{11}E_{22}),$$

$$u_2 = \begin{cases} \pm\sqrt{2}\sqrt{\dfrac{-\alpha\gamma}{\beta_2}}\,\dfrac{2[(1 + \Psi_2 + \Psi_4)(\Psi_{1,x} + \Psi_{3,x}) - (\Psi_{2,x} + \Psi_{4,x})(\Psi_1 + \Psi_3)]}{(1 + R_2 + R_4)^2 - (R_1 + R_3)^2}, & \varepsilon = \pm 1,\ \alpha\gamma\beta_2 < 0 \\ \mp\sqrt{2}\sqrt{\dfrac{\alpha\gamma}{\beta_2}}\,\dfrac{2[(1 - \Psi_2 + \Psi_4)(\Psi_{1,x} - \Psi_{3,x}) - (-\Psi_{2,x} + \Psi_{4,x})(\Psi_1 - \Psi_3)]}{(1 - \Psi_2 + \Psi_4)^2 + (\Psi_1 - \Psi_3)^2}, & \varepsilon = \pm i,\ \alpha\gamma\beta_2 > 0, \end{cases} \tag{25}$$

where

$$\Psi_1 = 2[e^{\varphi_1}\cos(T_1) + e^{\varphi_2}\cos(T_2)],$$

$$\begin{aligned} \Psi_2 = e^{\varphi_1+\varphi_2}[&E_{11}\cos(T_1 + T_2) + E_{12}\cos(T_1 - T_2) + E_{21}\sin(T_1 \\ &+ T_2) + E_{22}\sin(T_1 - T_2)] - \left(\frac{c_1^2}{s_1^2}e^{2\varphi_1} + \frac{c_2^2}{s_2^2}e^{2\varphi_2}\right), \end{aligned}$$

$$\begin{aligned} \Psi_3 = e^{\varphi_1+2\varphi_2}[Y_{11}\cos(T_1) + Y_{12}\sin(T_1)] + e^{2\varphi_1+\varphi_2}[E_{21}\cos(T_2) \\ + E_{22}\sin(T_2)],\ \Psi_4 = Ae^{2\varphi_1+2\varphi_2}, \end{aligned}$$

$$\varphi_1 = s_1x + \sqrt{\alpha}s_1y - \alpha s_1t,\ \varphi_2 = s_2x + \sqrt{\alpha}s_2y - \alpha s_2t,$$

$$T_1 = c_1x + \sqrt{\alpha}c_1y - \alpha c_1t,\ T_2 = c_2x + \sqrt{\alpha}c_2y - \alpha c_2t,$$

$$A = \frac{n_1^2n_2^2[(s_1 - s_2)^2 + (c_1 - c_2)^2]^2[(s_1 - s_2)^2 + (c_1 + c_2)^2]^2}{m_1^2m_2^2[(s_1 + s_2)^2 + (c_1 - c_2)^2]^2[(s_1 + s_2)^2 + (c_1 + c_2)^2]^2}.$$

The effect of parameters β_2 and γ in the two-quasi-soliton solution is the same as in the one-quasi-solitons; when β_2 and γ change, the amplitudes of the two-quasi-solitons are varied, but the periods and velocities of the two-quasi-solitons remain unchanged.

Figure 5 shows the two-quasi-soliton solution with different values of t. Although the positions of the two-quasi-soliton varied, the amplitudes, periods, and velocities of the two-quasi-solitons remained unchanged.

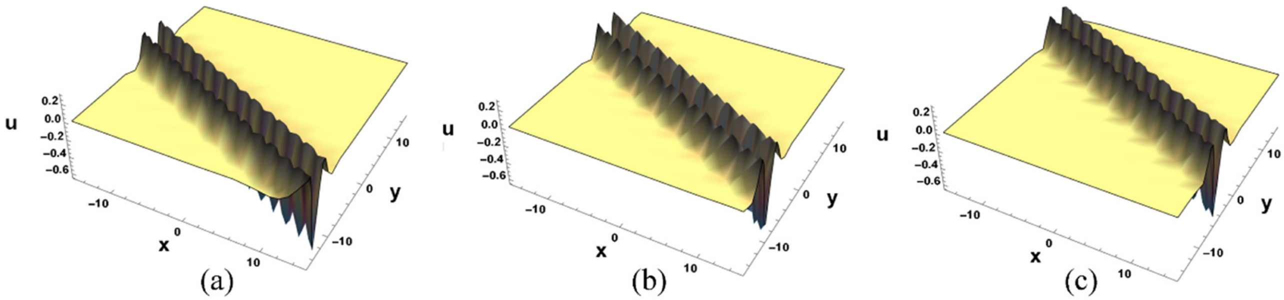

Figure 5: Two-quasi-soliton solution with $\alpha = 4$, $\gamma = 0.4$, $\beta_2 = 5$, $s_1 = 0.4$, $c_1 = 0.6$, $s_2 = 0.4$, $c_2 = -0.6$ and $\varepsilon = i$, except that (a) $t = -2$; (b) $t = 0$; (c) $t = 2$.

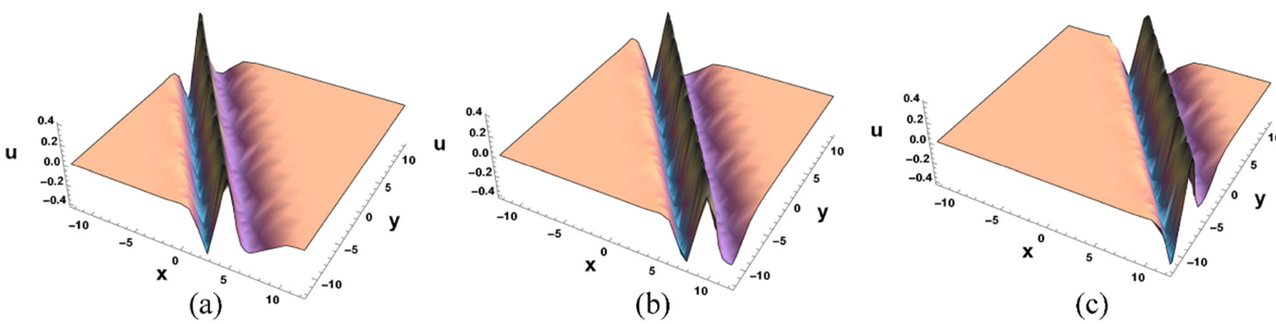

Figure 6: Mixed solution with $\alpha = 0.5$, $\gamma = 0.4$, $\beta_2 = 5$, $s_3 = 1.2$, $c_3 = 1$, $R_0 = 1$, and $\varepsilon = i$, except that (a) $t = -10$; (b) $t = 0$; (c) $t = 10$.

4.3 Coaction of one-soliton and one-quasi-soliton wave

For three-soliton solutions (21), let

$$k_1 = s_3 + c_3 i,\ k_2 = k_1^* = s_3 - c_3 i,\ k_3 = R_0, \tag{26}$$

where s_3, c_3, and R_0 are real constants, we can obtain the following mixed solutions:

$$u_3 = \begin{cases} \pm\sqrt{\dfrac{-2\alpha\gamma}{\beta_2}} \dfrac{2[(1 + W_2)(W_{1,x} + W_{3,x}) - W_{2,x}(W_1 + W_3)]}{(1 + W_2)^2 - (W_1 + W_3)^2}, & \varepsilon = \pm 1,\ \alpha\gamma\beta_2 < 0 \\ \mp\sqrt{\dfrac{-2\alpha\gamma}{\beta_2}} \dfrac{2[(1 - W_2)(W_{1,x} - W_{3,x}) + W_{2,x}(W_1 - W_3)]}{(1 - W_2)^2 - (W_1 - W_3)^2}, & \varepsilon = \pm i,\ \alpha\gamma\beta_2 > 0 \end{cases}, \tag{27}$$

where

$$W_1 = 2e^{\Phi_1}\cos F + e^{\Phi_2},$$

$$W_2 = -\frac{c_3^2}{s_3^2}e^{2\Phi_1} + e^{\Phi_1+\Phi_2}[M_1\cos F + M_2\sin F],$$

$$W_3 = M_3 e^{2\Phi_1+\Phi_2},$$

$$\Phi_1 = s_3 x + \sqrt{\alpha}s_3 y - \alpha s_3 t,\ \Phi_2 = R_0 x + \sqrt{\alpha}R_0 y - \alpha R_0 t,$$

$$F = c_3 x + \sqrt{\alpha}c_3 y - \alpha c_3 t,$$

$$M_1 = \frac{2(s_3^2 - 2c_3R_0 + c_3^2 - R_0^2)(s_3^2 + 2c_3R_0 + c_3^2 - R_0^2)}{[(s_3 + R_0)^2 + c_3^2]^2},$$

$$M_2 = \frac{8c_3R_0(s_3^2 + c_3^2 - R_0^2)}{[(s_3 + R_0)^2 + c_3^2]^2},$$

$$M_3 = -\frac{c_3^2[(s_3 - R_0)^2 + c_3^2]^2}{s_3^2[(s_3 + R_0)^2 + c_3^2]^2}.$$

The effect of parameters β_2 and γ in the co-actions of the one-soliton and one-quasi-soliton waves is the same as in the one-quasi-solitons; when β_2 and γ changed, the amplitudes of the mixed solutions varied, the periods and velocities of the mixed solutions remained unchanged.

Figure 6 shows the mixed solution with different values of scaled time t. Although the positions of the mixed solutions were varied, the amplitudes and velocities of the mixed solutions remained unvaried.

5 Conclusion

In this study, soliton solutions, quasi-soliton solutions, and mixed solutions of the ZK–mZK–BBM equation were obtained using the Hirota method and complex conjugate transformations. One-soliton figures for solution (15), parallel two-soliton figures for solution (18), and parallel three-soliton figures for solution (21) are shown in Figures 1 and 2. Figure 2 also presents that three-soliton can become two-soliton when any two of k_1, k_2, and k_3 are equal, and three-soliton become one-soliton when $k_1 = k_2 = k_3$. Figures 3 and 4 show one-quasi-soliton figures of solution (23), while Figure 5 shows parallel two-quasi-soliton figures of solution (25), and Figure 6 shows the figures of mixed solutions (27).

We found that the amplitudes of solitons, quasi-solitons, and mixed solutions change when β_2 and γ change, but the shapes and widths remain unchanged. The positions of the quasi-soliton solutions and mixed solutions varied with different values of scaled time t, but the amplitudes, periods, and velocities of the mixed solutions remain unchanged.

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