Research Article

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Nonlinear fractional-order differential equations: New closed-form traveling-wave solutions

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Abstract: The fractional-order differential equations (FO-DEs) faithfully capture both physical and biological phenomena making them useful for describing nature. This work presents the stable and more effective closed-form traveling-wave solutions for the well-known nonlinear space–time fractional-order Burgers equation and Lonngren-wave equation with additional terms using the $\exp(-\Phi(\xi))$ expansion method. The main advantage of this method over other methods is that it provides more accuracy of the FO-DEs with less computational work. The fractional-order derivative operator is the Caputo sense. The transformation is used to reduce the space–time fractional differential equations (FDEs) into a standard ordinary differential equation. By putting the suggested strategy into practice, the new closed-form traveling-wave solutions for various values of parameters were obtained. The generated 3D graphical soliton wave solutions demonstrate the superiority and simplicity of the suggested method for the nonlinear space–time FDEs.

Keywords: Caputo fractional derivative, fractional-order Burger's equation, fractional-order Lonngren-wave equation, $\exp(-\Phi(\xi))$ method

1 Introduction

Numerous real-world applications of the constant-order fractional differential equations may be found in a variety of scientific and engineering disciplines, including structural mechanics, fluid dynamics, chemical kinematics, signal processing, and many more [1–4]. Numerous scholars have studied closed-form solutions for constant-order nonlinear fractional differential equations, including Ali *et al.* [5], who looked into the solution for the fractional-order (2 + 1)-dimensional breaking soliton problem. They successfully achieved the perfect solution in all conceivable forms. Another study [6] discovered the soliton solution for the variable-order fractional differential equation. Using the straightforward Hirota approach and the bilinear Banklund transformation, Wang [7] took the high-dimensional equation and discovered the soliton solution including kink and periodic soliton that were found by long wave-limited solutions, and the obtained data proved the solution. Kumar [8] considered the one, two, and three solitons for the modified KdV equation by applying Hirota's bilinear technique. They presented a graphic representation of the multi-soliton solution and its interactions. The fractional-order Schrödinger equation was covered by Das and Ray [9] using the extended projective Ricatti equation approach. The fractional derivative, which is in a conformable sense, was produced from the governing equation to construct several varieties of soliton, including kink soliton, single soliton, and dark soliton-type traveling-wave solutions. The Hirota bilinear technique [10], the sub-equation method [11], the exp-function method [12], the G'/G-expansion method [13], the extended rational sin/cos method [14], the $(G'/G,1/G)$-expansion method and the F-expansion method [15], and the new auxiliary method [16] are only a few examples of the numerous ways that have been used in the literature. Additionally, the traveling-wave solutions are a subclass of exact solutions that are crucial in determining the analytical solutions of NFDEs. These solutions can be transmitted with specific shapes and speeds using solitary waves, and they eventually reach zero at specific locations. Additionally, the closed-form solution can be observed in a variety of natural physical phenomena, including plasma, optical fiber, physics (solid-state and condensed matter), fluid dynamics, *etc.* Most importantly, finding traveling-wave solutions to the NFDEs is crucial to a deeper physical understanding of these equations.

In this study, two types of fractional differential equations (FDEs) are discussed: the fractional-order Burger's

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equation (FO-BE), which is used to model physical issues like acoustic wave propagation in gas tubes, water waves, and liquid with bubbles [17], and the fractional-order Lonngren-wave equation (FO-LWE), which describes the electric signals in telegraph lines based on the tunnel diode [18]. Numerous scholars have solved these kinds of fractional-order differential equations (FO-DEs) using various analytical techniques, such as the variational iterative method, as explained by Inc. [19]. To more effectively illustrate the accuracy, they presented graphical images and numerical results. Using the G'/G-expansion method, Bekir and Güner [20] were able to solve the FO-BE and FO population model. They demonstrate that the proposed method is more precise and efficient by using the fractional derivative, which is in the Riemann-Liouville sense. Similarly, Bulut *et al.* [21] modified the trial equation method used to solve the FO-BE. The space and time FO-BE analytical solution based on the variational iteration method with various initial conditions was also studied by Saad and Al-Sharif [22]. The outcome revealed greater agreement with the precise solutions. The aforementioned equation was presented by Esen and Tasbozan [23] using the cubic b-spline finite element method. By comparing the numerical results to the body of literature, they were validated. Esen *et al.* [24] examined the homotopy analysis method's approximate analytical solution for the FO-BE in another review. By adjusting the auxiliary parameter, they changed the convergence region. For the solution of FO-LWE, Wang [25] proposed a new mathematical technique called the fractional Yang wave transformed approach. The traveling-wave solution is represented in the 3D figures, and the fractional derivative is in the local fractional derivative form. Iqbal *et al.* [26] examined the Lonngren equation using the two variables (G'/G,$1/G$) expansion approach and observed that the solutions were in terms of trigonometric, hyperbolic, and rational functions. Graphic representations of the bell-shaped, singular soliton, singular periodic, and anti-bell-shaped are provided. Ali *et al.* [27] considered the space–time variable-order model and solved it by the analytical Khater method, which is a new concept in this field of research. The fractional-order operators are in the Caputo sense. They constructed the new exact soliton solution for the proposed model and demonstrated the efficiency of such types of variable-order models. Hussain *et al.* [28] worked on the Schamel KdV equation to discuss the effect of electron trapping in ion-acoustic waves. The obtained results included bell-shaped, double-periodic, shock-waves, and solitary waves, which are very effective in mathematical physics. Rehman *et al.* [29] considered the nonlinear fractional-order dispersive equation, which is used to describe wave propagation in an elastic, inhomogeneous Murnaghan's rod. They generated various solutions such as solitary, soliton as well and periodic wave solutions. Another survey [30] discussed two different fractional-order models and found their solution by the new auxiliary equation method. They investigated the physical meaning of the solution by the graphical representation. The two space–time fractional-order models are solved by the extended Kurdryashov method [31]. The beta-derivative operator is used for the fractional-order term, and the obtained solutions yield a variety of typical soliton shapes. The three-dimensional, two-dimensional, and contour graphs are plotted to confirm the capability and effectiveness of the method. Further similar comprehensive studies related to fractional-order models, their solutions by various techniques, and its physical investigation can be found in previous studies [32–39].

In this study, the proposed $\exp(-\Phi(\xi))$ approach is used to build the closed-form traveling-wave solutions for the two nonlinear FO-DEs named FO-BE and FO-LWE. The derived new closed-form traveling solutions are in the form of bell-shaped, solitary, and periodic solitons that are graphically represented in three dimensions. The fractional-order derivative is in the Caputo sense. The proposed approach and FO-DEs were found to be effective and valid by the newly created closed-form solutions. To the best of the author's knowledge, no research that used the closed-form $\exp(-\Phi(\xi))$ approach to solve the FO-BE and FO-LWE equations exists in the literature.

The remainder of this article is divided into the following sections: Section 2 explains the fundamental Caputo fractional derivative formula; Section 3 discusses the $\exp(-\Phi(\xi))$ method; Section 4 describes how the proposed method was applied to the FO-BE and FO-LWE; Section 4 includes results and discussion; and Section 5 contains the conclusion.

2 Caputo fractional derivative

The fractional derivative and the characteristics of values of algebraic types are covered in this section. A function with Caputo fractional derivative $F(x,y,\ldots,t)$ of order $0<\beta\le 1$ is defined as follows [40]:

$$
{}_0^cD_t^{\beta}F(x,y,\ldots,t) = \begin{cases} \dfrac{1}{\Gamma(1+\beta)}\displaystyle\int_0^t \dfrac{F'(x,y,..,t)}{\Gamma(t-\xi)^{\beta}}\mathrm{d}\xi, & 0<\beta<1, \\ F'(x,y,\ldots,t), & \beta=1. \end{cases} \tag{1}
$$

The property for algebraic type function is as follows:

$$ {}_0^cD_t^\beta t^\alpha = \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta+\alpha)} t^{\alpha-\beta},\ 0<\beta<1. \tag{2}$$

3 Methodology of exp(−Φ(ξ)) method

Consider the following nonlinear FO-DE of the order α:

$$ G(u, {}_0^cD_t^\alpha u, u, u, u, \ldots) = 0, \tag{3}$$

where G is a polynomial in u, and the fractional order is denoted by α, which is in the Caputo sense. Additionally, the fractional transformation is utilized as follows:

$$ u(x, y, \ldots, t) = U(\xi),\quad \xi = \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{y^\alpha}{\Gamma(1+\alpha)} + \ldots - \frac{ct^\alpha}{\Gamma(1+\alpha)}. \tag{4}$$

Reduce Eq. (3) into the nonlinear ODE as follows:

$$ G(U, cU', U'', cU'', U''\ldots) = 0. \tag{5}$$

When l is constant, let the answer to the aforementioned equation have the following form:

$$ U(\xi) = a_0 + a_1\exp(-\Phi(\xi)) + \ldots + a_N\exp(-N\Phi(\xi)), \tag{6}$$

where N can be calculated by balancing the highest-order linear term with the highest-order nonlinear term and $\Phi(\xi)$ is a function that satisfies the first-order equation as:

$$ \Phi'(\xi) = \exp(-\Phi(\xi)) + A\exp(\Phi(\xi)) + B. \tag{7}$$

Many cases can be taken to obtain the solution, as follows:

Case 1. If $B^2-4A>0$ and $A\neq 0$, then

$$ \Phi_1(\xi) = \ln\left(\frac{-\sqrt{B^2-4A}\tanh\left(\frac{\sqrt{B^2-4A}}{2}(\xi+C)-B\right)}{2A}\right). $$

Case 2. If $B^2-4A>0$, $A=0$, then

$$ \Phi_2(\xi) = -\ln\left(\frac{B}{\cosh(B(\xi+C)) + \sinh(B\cosh(B(\xi+C))) - 1}\right). $$

Case 3. If $B^2-4A<0$, then

$$ \Phi_3(\xi) = \ln\left(\frac{\sqrt{4A-B^2}\tanh\left(\frac{\sqrt{B^2-4A}}{2}(\xi+C)-B\right)}{2A}\right). $$

Case 4. If $B^2-4A=0$, $A\neq 0$, and $B\neq 0$, then

$$ \Phi_4(\xi) = \ln\left(-\frac{2B(\xi+C)+4}{B^2(\xi+C)}\right). $$

Case 5. If $B^2-4A=0$, $A=0$, and $B=0$, then

$$ \Phi_5(\xi) = \ln(\xi+C). $$

Here, the polynomial expressed in terms of $\exp(-\Phi(\xi))$ was produced by substituting Eq. (5) for Eq. (4). Find the system of equations that simultaneously solves by equating the $\exp(-\Phi(\xi))$-based equation. The set of equations that follow exactly solve Eq. (3).

4 Applications

In this section, the $\exp(-\Phi(\xi))$ method is used to solve the fractional Burger's equations and the Lonngren-wave equation, which are nonlinear space–time FO-DEs. As following

4.1 The FO-BE

Consider the nonlinear space–time FO-BE, as follows:

$$ {}_0^cD_t^\gamma u + au\ {}_0^cD_t^\gamma u - v\ {}_0^cD_t^{2\gamma} u = 0, \tag{8}$$

where $0<\gamma\le 1$, and using the fractional transformation $\xi = \frac{x^\gamma}{\Gamma(1+\gamma)} + \frac{y^\gamma}{\Gamma(1+\gamma)} - \frac{ct^\gamma}{\Gamma(1+\gamma)}$ to reduce Eq. (3) into the ODE, as follows:

$$ -cU + \frac{a}{2}U^2 - v\frac{\mathrm{d}}{\mathrm{d}\xi}U = 0. \tag{9}$$

To obtain the value of $M=1$, balance the high-order linear term with the nonlinear term:

$$ U = a_0 + a_1\exp^{(-\phi(\xi))}. \tag{10}$$

Substituting Eq. (10) into (9), obtaining the algebraic equation in terms of $\exp(-\phi(\xi))$, and separating the same power of $(\exp(-\phi(\xi)))^n$, we obtain

$$ (\exp(-\phi(\xi)))^0 : aa_1^2 + 2va_1 = 0, $$

$$ (\exp(-\phi(\xi)))^1 : 2Bva_1 + 2aa_0a_1 - 2ca_1 = 0, $$

$$ (\exp(-\phi(\xi)))^2 : 2Ava_1 + aa_0^2 - 2ca_0 = 0. $$

Simultaneous solution of the aforementioned system of equations yielded the following results:

$$ a_0 = \frac{(-B\pm\sqrt{B^2-4A})v}{a},\ a_1 = -\frac{2v}{a},\ c = Bv + (-B\pm\sqrt{B^2-4A})v. $$

Case 1. If $B^2 - 4A > 0$ and $A \neq 0$,

then

$$u_1(x,t) = -\frac{\left(k^2 v\left(\begin{array}{l}\sqrt{B^2-4A}\tanh\left(\frac{1}{2}\sqrt{B^2-4A}\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + \frac{1}{2}\sqrt{B^2-4A}\,C - B\right)B \\ -\tanh\left(\frac{1}{2}\sqrt{B^2-4A}\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + \frac{1}{2}\sqrt{B^2-4A}\,C - B\right)B^2 \\ +4\tanh\left(\frac{1}{2}\sqrt{B^2-4A}\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + \frac{1}{2}\sqrt{B^2-4A}\,C - B\right)A - 4A\end{array}\right)\right)}{\left(\sqrt{B^2-4A}\,a\tanh\left(\frac{1}{2}\sqrt{B^2-4A}\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + \frac{1}{2}\sqrt{B^2-4A}\,C - B\right)\right)}.$$

Case 2. If $B^2 - 4A > 0, A = 0$, then

$$u_2(x,t) = \frac{\begin{array}{l}v\left(\cosh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) + \sinh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) - 1\right)\sqrt{B^2-4A} \\ - B\left(\cosh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) + \sinh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) + 1\right)\end{array}}{a\left(\cosh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) + \sinh\left(B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right)\right) - 1\right)}.$$

Case 3. If $B^2 - 4A < 0$, then

$$u_3(x,t) = \frac{\left(\left(\sqrt{-B^2+4A}\,(-B+\sqrt{-B^2+4A})\tanh\left(\frac{1}{2}\sqrt{-B^2+4A}\left(\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + C\right) - B\right) - 4A\right)v\right)}{\left(\sqrt{-B^2+4A}\,a\tanh\left(\frac{1}{2}\sqrt{-B^2+4A}\left(\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + C\right) - B\right)\right)}.$$

Case 4. If $B^2 - 4A > 0, A \neq 0$, then

$$u_4(x,t) = \frac{\left(\left(2 + B\left(\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + C\right)\right)\sqrt{B^2-4A} - 2B\right)v}{a\left(2 + B\left(\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + C\right)\right)}.$$

Case 5. If $B^2 - 4A = 0$, $A = 0$, and $B \neq 0$, then

$$u_5(x,t) = -\frac{v\left(-C - \left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right)\right)\sqrt{B^2-4A} + BC + B\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)} + C\right) + 2}{a\left(\left(\frac{x^\gamma + y^\gamma - ct^\gamma}{\Gamma(1+\gamma)}\right) + C\right)}.$$

4.2 The FO-LWE

The traveling-wave solution based on the nonlinear FO-LWE is being studied in this case:

$$ {}_0^cD_t^{2\rho}({}_0^cD_x^{2\rho}u - \alpha u + \beta) + {}_0^cD_x^{2\rho}u = 0, 0 < \rho \le 1, \quad (11)$$

where α and β are the constants, and using the fractional transformation $\xi = \frac{x^\rho}{\Gamma(1+\rho)} - \frac{ct^\rho}{\Gamma(1+\rho)}$ and integrating two times to reduce Eq. (11) into the ODE, as follows:

$$c^2U'' + (1 - \alpha c^2)U + \beta c^2U^2 = 0. \quad (12)$$

To obtain the value of $M = 2$, balance the high-order linear term with the nonlinear term:

$$U = a_0 + a_1 \exp^{(-\phi(\xi))} + a_2 \exp^{(-2\phi(\xi))}. \quad (13)$$

Substituting Eq. (13) into (12), obtaining algebraic equation in terms of $\exp(-\phi(\xi))$, and separating the same power of $(\exp(-\phi(\xi)))$, we obtain

$$(\exp(-\phi(\xi)))^0 = 6c^2a_2 + c^2\beta a_2^2 = 0,$$

$$(\exp(-\phi(\xi)))^1 = 2c^2a_1 + 10c^2a_2B + 2c^2\beta a_1a_2 = 0,$$

$$(\exp(-\phi(\xi)))^2 = 3c^2a_1B + 8c^2a_2\mu + 4c^2a_2B^2 + a_2 - c^2\alpha a_2 + 2c^2\beta a_1a_0 + c^2\beta a_1^2 = 0,$$

$$(\exp(-\phi(\xi)))^3 = 2c^2a_1A + c^2a_1B^2 + 6c^2a_2AB + a_1 - c^2\alpha a_1 + 2c^2\beta a_0a_1 = 0,$$

$$(\exp(-\phi(\xi)))^4 = c^2a_1BA + 2c^2a_2A^2 + a_0 - c^2\alpha a_0 + c^2\beta a_0^2 = 0.$$

Solving the above system of equations simultaneously and the obtained values are as follows:

Set 1:

$$a_0 = \frac{1}{24}\frac{36c^2\alpha - 36 - c^2\beta^2a_1^2}{c^2\beta},$$

$$a_2 = -\frac{6}{\beta}, B = -\frac{1}{6}\beta a_1, A = -\frac{1}{44}\frac{36c^2\alpha - 36 - c^2\beta^2a_1^2}{c^2},$$

Set 2:

$$a_0 = -\frac{1}{24}\frac{12c^2\alpha - 12 + \beta^2a_1^2}{c^2\beta},$$

$$a_2 = -\frac{6}{\beta}, B = -\frac{1}{6}\beta a_1, A = \frac{1}{44}\frac{36c^2\alpha - 36 + c^2\beta^2a_1^2}{c^2}.$$

Case 1. If $B^2 - 4A > 0$, and $A \neq 0$, then

$$u_1(x,t) = -\frac{1}{24}\frac{12c^2\alpha - 12 + \beta^2a_1^2}{c^2\beta} - \frac{2a_1A}{\sqrt{B^2 - 4A}\tanh\left(\frac{1}{2}\sqrt{B^2 - 4A}\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) - \lambda\right)} - \frac{24A^2}{\beta(B^2 - 4A)\tanh\left(\frac{1}{2}\sqrt{B^2 - 4A}\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) - \lambda\right)^2}.$$

Case 2. If $B^2 - 4A > 0, A = 0$, then

$$u_2(x,t) = -\frac{1}{24}\frac{12c^2\alpha - 12 + c^2\beta^2a_1^2}{c^2\beta} + \frac{a_1A}{\cosh\left(B\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)\right) + \sinh\left(B\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)\right) - 1} - \frac{6B^2}{\beta\left(\cosh\left(B\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)\right) + \sinh\left(B\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)\right) - 1\right)^2}.$$

Case 3. If $B^2 - 4A < 0$, then

$$u_3 = -\frac{1}{24}\frac{12c^2\alpha - 12 + c^2\beta^2a_1^2}{c^2\beta} + \frac{2a_1A}{\sqrt{4A - B^2}\tanh\left(\frac{1}{2}\sqrt{4A - B^2}\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) - \lambda\right)} - \frac{24A^2}{\beta(4A - B^2)\tanh\left(\frac{1}{2}\sqrt{4A - B^2}\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) - \lambda\right)^2}.$$

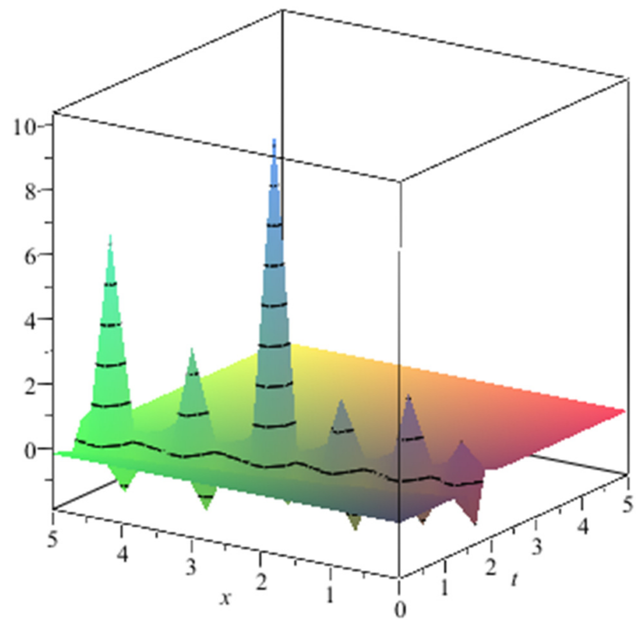

Figure 1: Dynamical shape of the u_1 shows the singular soliton solution at the values: at $c = -10$, $B = 5$, $A = 6$, $\alpha = 1$, $v = 1$, $C = 1$, $a_0 = 2$, $a_1 = 4$, $\gamma = 0.25$, and $y = 2$.

Case 4. If $B^2 - 4A = 0, B \neq 0$, and $A \neq 0$, Then

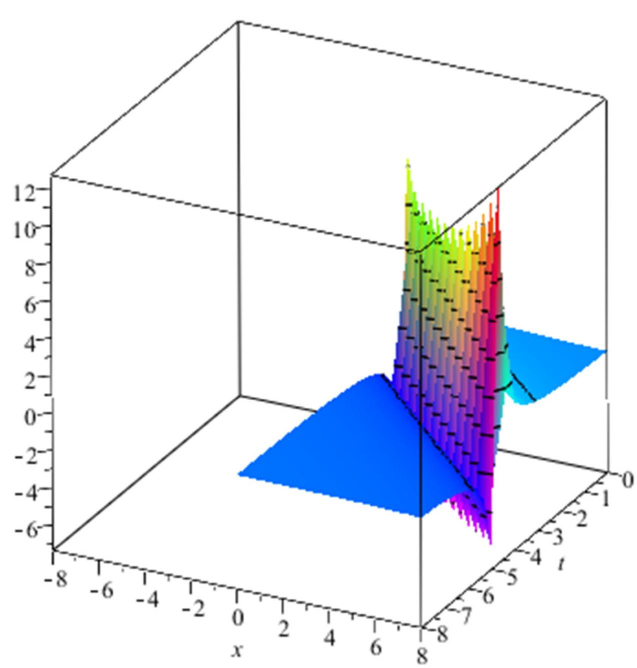

Figure 2: Dynamical shape of the u_4 shows the periodic kink waves at he values: $c = 2$, $B = -5$, $A = 6$, $\alpha = 10$, $v = -11$, $C = 1$, $a_0 = 2$, $a_1 = 4$, $\gamma = 0.95$, and $y = 1$.

$$u_4 = -\frac{1}{24}\frac{12c^2\alpha - 12 + c^2\beta^2 a_1^2}{c^2\beta} - \frac{a_1\lambda^2\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)}{2\lambda\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) + 4} - \frac{a_1\lambda^4\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)^2}{\beta\left(2\lambda\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right) + 4\right)^2}.$$

Case 5. If $B^2 - 4A = 0, B = 0$, and $A = 0$, then

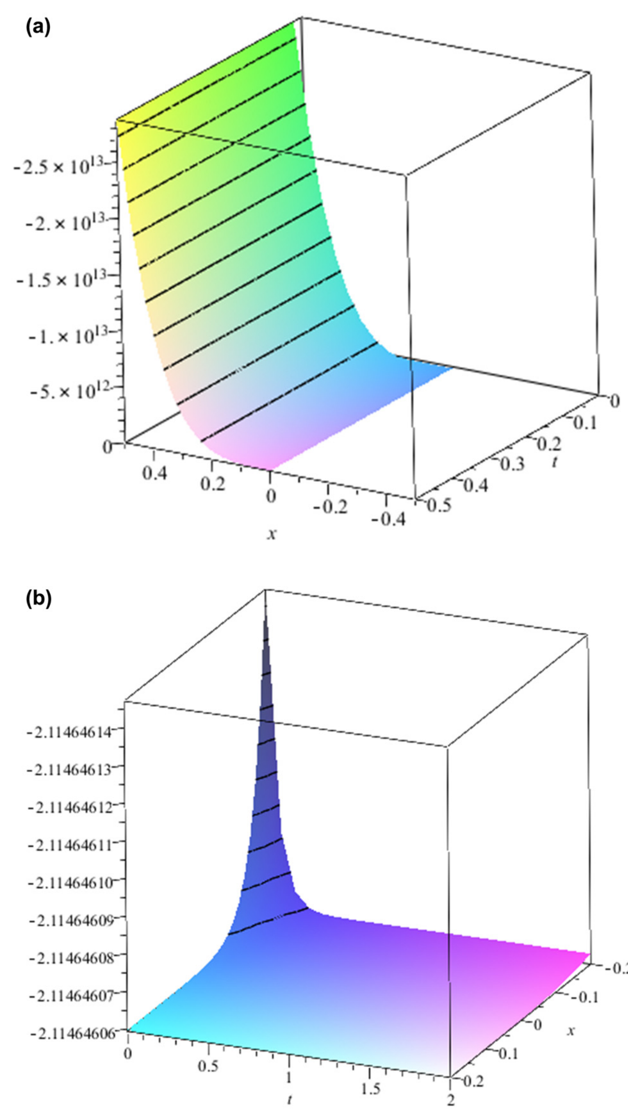

Figure 3: The dynamical shape of the u_1 show the soliton waves solution at the mentioned values of pararmeters. (a) at $c = 4$, $a_0 = 2$, $a_1 = 4$, $a_2 = 2$, $\rho = 0.25$, $\alpha = 2$, $\beta = 2$, $C = 2$, $B = 10$, $A = 2$. at $c = -1$, $a_0 = 20$, $a_1 = 4$, $a_2 = 10$, $\rho = 0.95$, $\alpha = 3$, $\beta = 2$, $C = \frac{1}{20}$, $B = -15$, and $A = \frac{1}{2}$.

$$u_5 = -\frac{1}{24}\frac{12c^2\alpha - \ 12 + c^2\beta^2 a_1^2}{c^2\beta} + \frac{a_1}{\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C} - \frac{6}{\beta\left(\left(\frac{x^\rho - ct^\rho}{\Gamma(1+\rho)}\right) + C\right)^2}.$$

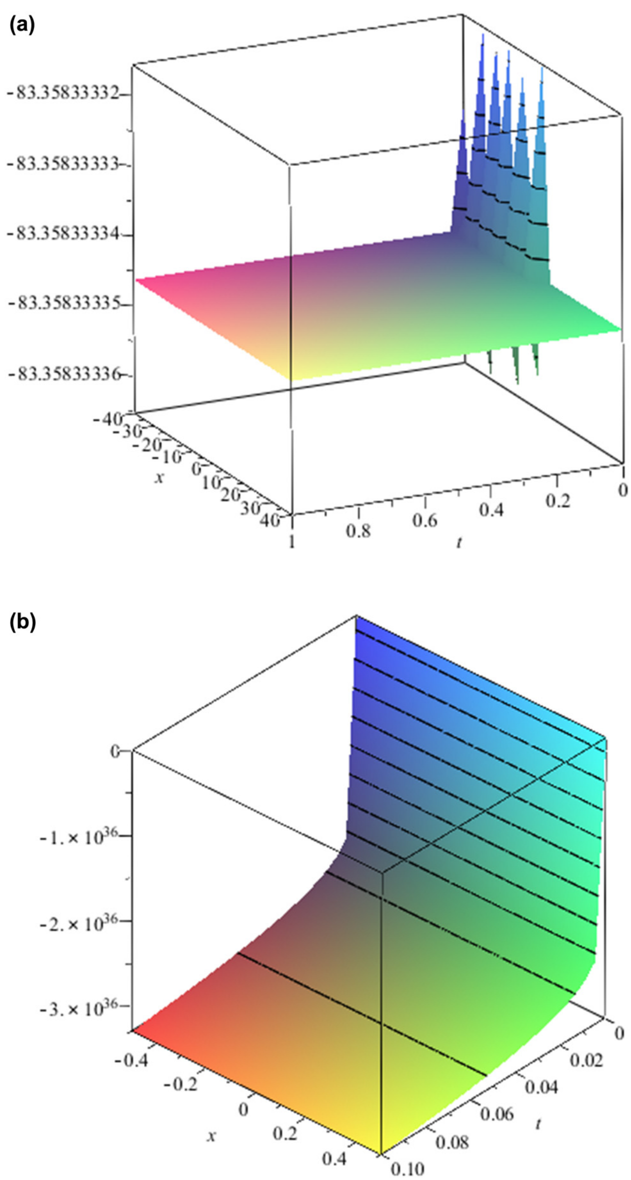

Figure 4: The dynamical shape of the u_2 show the singular soliton and the soliton waves soliton solution at the mentioned parameters values. (a) at $c = -1$, $a_0 = 13$, $a_1 = 10$, $a_2 = 10$, $\rho = 0.5$, $\alpha = 2$, $\beta = 20$, $C = 2$, $B = 30$, $A = -1.339$. (b) at $c = -4$, $a_0 = 2$, $a_1 = 40$, $a_2 = 20$, $\rho = 0.001$, $\alpha = 12$, $\beta = 2$, $C = 2$, $B = -10$, and $A = 0$.

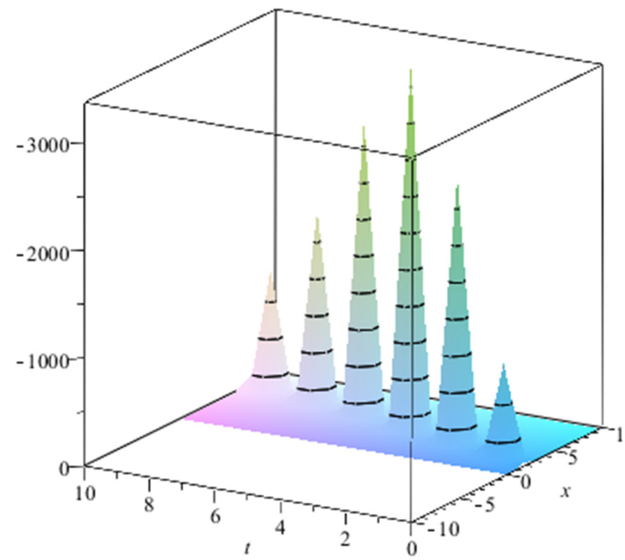

Figure 5: Dynamical shape of the u_4 show the single-wave soliton solution at $c = \frac{1}{2}$, $a_0 = 20$, $a_1 = 4$, $a_2 = 10$, $\rho = 0.95$, $\alpha = 3$, $\beta = 2$, $C = \frac{1}{20}$, $B = -1$, and$A = -10$.

5 Discussion

In this section, discuss about the graphical representation of the proposed FO-DEs solved through the $\exp(-\Phi(\xi))$ method. The graphical representation of FO-BE is presented in Figures 1 and 2 in the form of a 3D plot, and the FO-LWE is represented from Figures 3–5. In Figure 1, obtain the 3D plot in the form of singular shaped-soliton solution at $c = -10, B = 5, A = 6, \alpha = 1, \nu = 1, C = 1, a_0 = 2, a_1 = 4, \gamma = 0.25$, and $\gamma = 2$. Figure 2 confirms the periodic kink shape soliton solution in the form of the 3D plot at $c = 2, B = -5, A = 6, \alpha = 10, \nu = -11, C = 1, a_0 = 2, a_1 = 4, \gamma = 0.95$, and $\gamma = 1$. The dynamical shape in Figure 3 obtained the soliton solution for the parametric values at $c = 4, a_0 = 2, a_1 = 4, a_2 = 2, \rho = 0.25, \alpha = 2, \beta = 2, C = 2, B = 10, A = 2$, and $c = -1$, $a_0 = 20$, $a_1 = 4$, $a_2 = 10, \rho = 0.95, \alpha = 3, \beta = 2, C = \frac{1}{20}, B = -15, A = \frac{1}{2}$, respectively, in 3D plot. Figure 4(a) and (b) represents the singular soliton and soliton wave solution $c = -1$, $a_0 = 13$, $a_1 = 10$, $a_2 = 10$, $\rho = 0.5, \alpha = 2$, $\beta = 20$, $C = 2, B = 30$, $A = -1$, and the soliton solution waves solution at $c = -4$, $a_0 = 2$, $a_1 = 40$, $a_2 = 20, \rho = 0.001$, $\alpha = 12$, $\beta = 2$, $C = 2$, $B = -10$, $A = 0$. Figure 5 represents the singular soliton wave solution in 3D plot at $c = \frac{1}{2}$, $a_0 = 20$, $a_1 = 4$, $a_2 = 10$, $\rho = 0.95$, $\alpha = 3, \beta = 2, C = \frac{1}{20}, B = -1, A = -10$.

6 Conclusion

This article successfully implemented the proposed approach to the nonlinear space–time FO-BE and FO-LWE. The approach delivered a variety of new exact solutions of different physical structures for different values of some free parameters and of fractional order. We get various types of closed-form solutions in the form of singular soliton, periodic kink soliton, and other new types of solitons waves for various values of parameters, which are new and valid. Finally, the proposed approach provides a powerful mathematical tool to obtain more exact solutions to the FO-DEs that arise in mathematical physics.

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