Research Article

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Analyzing fuzzy fractional Degasperis–Procesi and Camassa–Holm equations with the Atangana–Baleanu operator

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Abstract: This article presents a new approach for solving the fuzzy fractional Degasperis–Procesi (FFDP) and Camassa–Holm equations using the iterative transform method (ITM). The fractional Degasperis–Procesi (DP) and Camassa–Holm equations are extended from the classical DP and Camassa–Holm equations by incorporating fuzzy sets and fractional derivatives. The ITM is a powerful technique widely used for solving nonlinear differential equations. This approach transforms the fuzzy fractional differential equations into a series of ordinary differential equations, which are then solved iteratively using a recursive algorithm. Numerical simulations demonstrate the proposed approach's accuracy and effectiveness. The results show that the ITM provides an efficient and accurate method for solving the FFDP and Camassa–Holm equations. The proposed method can be extended to solve other fuzzy fractional differential equations.

Keywords: fractional fuzzy Degasperis–Procesi and Camassa–Holm equations, iterative transform method, Atangana–Baleanu operator

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1 Introduction

Fractional fuzzy differential equations (FFDEs) are important differential equations widely studied in recent years. They are used in many scientific and engineering applications, such as modeling of viscoelastic materials, signal processing, and control systems. FFDEs combine fuzzy sets and fractional calculus, providing a powerful tool for modeling systems with uncertain or imprecise information. The solutions of FFDEs are fuzzy functions that describe the behavior of the system under study. Several numerical methods have been proposed for solving FFDEs, including the fuzzy differential transform method, the fuzzy Laplace transform method, and the fuzzy adomian decomposition method [1,2]. The study of FFDEs is still an active area of research, and it is expected to lead to new insights into the modeling and control of complex systems [3–7].

The fuzzy fractional Degasperis–Procesi (FFDP) equations have recently emerged as a powerful tool for modeling complex systems in various fields, including fluid dynamics, quantum mechanics, and signal processing. The FFDP equations are a generalization of the well-known Degasperis–Procesi (DP) equations, which are nonlinear dispersive wave equations with cubic nonlinearity. The FFDP equations incorporate the concept of fractional calculus, which is a generalization of the traditional calculus to non-integer orders. Including the fuzzy logic concept in FFDP equations makes them even more versatile in modeling complex systems with uncertainties. Researchers have significantly contributed to the theoretical and numerical analysis of FFDP equations and their applications in various fields. The FFDP equations have proven to be a valuable tool for understanding the dynamics of complex systems with uncertainty and nonlinearities, and their potential for further exploration is significant [8–12]. This collection of recent publications spans a diverse array of topics within the realm of applied mathematics and physics. The first work by Kai *et al.* delves into the realm of nonlinear fourth-order time-fractional partial differential equations (PDEs), exploring exact solutions and dynamic properties [13].

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Meanwhile, Gao *et al.* focussed on anisotropic medium sensing controlled by bound states in the continuum, particularly in polarization-independent meta surfaces [14]. Another contribution by Yang and Kai explored dynamical properties, modulation instability analysis, and chaotic behaviors in nonlinear coupled Schrodinger equations within fiber Bragg gratings [15]. Zhou *et al.* presented an Iterative Threshold Algorithm of Log-Sum Regularization for Sparse Problems in the context of video technology [16]. Finally, Li *et al.* introduced a new improved fractional Tikhonov regularization method for moving force identification within the field of structures. These works collectively showcase the diverse applications and advancements in the intersection of mathematics and physics [17].

The fuzzy fractional Camassa–Holm equation is a variant of the well-known Camassa–Holm equation, which has been extensively studied in mathematical physics. This equation incorporates a fuzzy set-theoretic approach to the fractional derivative operator, making it particularly useful in modeling systems with uncertain or imprecise data. The fuzzy fractional Camassa–Holm equation has been shown to have several interesting properties, including the existence of multiple solutions and the potential for the formation of singularities. This equation has also been used in several real-world applications, such as modeling water waves and predicting the behavior of financial markets. Several studies have investigated the properties and behavior of the fuzzy fractional Camassa–Holm equation, including numerical simulations and analytical approximations. Overall, this equation represents a promising tool for modeling complex systems with imprecise or uncertain data and is likely to be the subject of ongoing research in the years to come [18,19].

The iterative transform method (ITM) is a recently developed method used in numerical analysis for solving PDEs. It combines the traditional ITM, which uses a nonlinear operator to transform the PDEs into a series of algebraic equations that can be solved iteratively. The new iterative transform method has been applied to various PDEs, including nonlinear reaction-diffusion equations, fractional differential equations, and wave equations. The method is effective, accurate, and computationally efficient in solving these types of PDEs [20,21].

2 Basic definitions

2.1 Definition

If $\wp : \mathbb{R} \mapsto [0,1]$ represents a fuzzy set, it is considered to be a fuzzy set if the essential specifications hold [22–25]:

(1) $\wp$ is normal (for some $\eta_0 \in \mathbb{R}; \wp(\vartheta_0) = 1$);
(2) $\wp$ is upper semi-continuous;
(3)
$$\wp(\vartheta_1\omega + (1-\omega)\vartheta_2) \geq (\wp(\vartheta_1) \wedge \wp(\vartheta_2)) \forall \omega \in [0,1], \vartheta_1, \vartheta_2, \in \mathbb{R}$$
i.e., $\wp$ is convex;
(4) $cl\{\vartheta \in \mathbb{R}, \wp(\vartheta) > 0\}$ is compact.

2.2 Definition

The fuzzy number $\wp$, there is a κ-level set written as [22–25]

$$[\wp]^{\kappa} = \{\nu \in \mathbb{R} : \wp(\nu) \geq 1\},$$

where $\kappa \in [0,1]$ and $\nu \in \mathbb{R}$.

2.3 Definition

The generalized form of a fuzzy number is shown as $[\underline{\wp}(\kappa), \overline{\wp}(\kappa)]$ as long as $\kappa \in [0,1]$ meets the following condition [22–25]:

(1) $\wp(\kappa)$ is left continuous, left continuous at zero, nondecreasing and over bounded $(0,1]$;
(2) $\wp(\kappa)$ is right continuous, right continuous at zero, nonincreasing, and over bounded $(0,1]$;
(3) $\underline{\wp}(\kappa) \leq \overline{\wp}(\kappa)$.

2.4 Definition

Suppose that there are fuzzy set numbers $\kappa \in [0,1]$ and Y [22–25], $\tilde{\rho}_1 = (\underline{\rho_1}, \overline{\rho_1})$, $\tilde{\rho}_2 = (\underline{\rho_2}, \overline{\rho_2})$, then the additions, subtractions, and multiplications, respectively, are defined as:

(1) $\tilde{\rho}_1 \oplus \tilde{\rho}_2 = (\underline{\rho_1}(\kappa) + \underline{\rho_2}(\kappa), \overline{\rho_1}(\kappa) + \overline{\rho_2}(\kappa))$;
(2) $\tilde{\rho}_1 \ominus \tilde{\rho}_2 = (\overline{\rho_1}(\kappa) - \overline{\rho_2}(\kappa), \overline{\rho_1}(\kappa) - \overline{\rho_2}(\kappa))$;
(3) $\mathrm{Y} \odot \tilde{\rho}_1 = \{(\mathrm{Y}\underline{\rho_1}, \mathrm{Y}\overline{\rho_1})\mathrm{Y} \geq 0, (\mathrm{Y}\overline{\rho_1}, \mathrm{Y}\underline{\rho_1})\mathrm{Y} < 0$.

2.5 Definition

The fuzzy two sets of fuzzy mappings $\Theta : \tilde{E} \times \tilde{E} \mapsto \mathbb{R}$ [22–25], $\tilde{\rho}_1 = (\underline{\rho_1}, \overline{\rho_1})$, $\tilde{\rho}_2 = (\underline{\rho_2}, \overline{\rho_2})$, then Θ-distance among $\tilde{\rho}_1$ and $\tilde{\rho}_2$ is given as:

$$\Theta(\tilde{\rho}_1, \tilde{\rho}_2) = \sup_{\kappa \in [0,1]} [\max\{|\underline{\rho_1}(\kappa) - \underline{\rho_2}(\kappa)|, |\overline{\rho_1}(\kappa) - \overline{\rho_2}(\kappa)|\}].$$

2.6 Theorem

Suppose a fuzzy value function $\mathbf{E} : \mathbb{R} \mapsto \tilde{E}$ *such that* $\mathbf{E}(\gamma_0;\kappa) = [\underline{\mathbf{E}}(\gamma_0;\kappa), \bar{\mathbf{E}}(\gamma_0;\kappa)]$, *and* $\kappa \in [0,1]$. *Then* [22–25]:

(1) $(\gamma_0;\ \kappa)$ *and* $\mathbf{E}(\gamma_0;\kappa)$ *are differentiable, if* $\mathbf{E}$ *is a (1)-differentiate, and*:

$$[\mathbf{E}'(\gamma_0)]^{\kappa} = [\underline{\mathbf{E}}'(\gamma_0;\ \kappa), \bar{\mathbf{E}}'(\gamma_0;\ \kappa)].$$

(11) $\underline{\mathbf{E}}(\gamma_0;\ \kappa)$ *and* $\bar{\mathbf{E}}(\gamma_0;\ \kappa)$ *are differentiates, if* $\mathbf{E}$ *is a (2)-differentiate, and*:

$$[\mathbf{E}'(\gamma_0)]^{\kappa} = [\bar{\mathbf{E}}'(\gamma_0;\ \kappa), \underline{\mathbf{E}}'(\gamma_0;\ \kappa)].$$

2.7 Definition

Consider that a fuzzy mapping $v^{(\gamma)}_{g\mathcal{H}} = v^{(\gamma)} \in \mathbb{C}^F[0,s] \cap [0,s]\mathbb{L}^F$, the function $v : F \to \mathbf{E}, F \in \mathbb{R}$ is called a fuzzy function. The fuzzy $g\mathcal{H}$-fractional Caputo differentiability of the fuzzy value maps nu is represented as

$$(g\mathcal{H}\mathcal{D}^{ß}v)(\boldsymbol{\varepsilon}) = \mathcal{J}^{\gamma-ß}_{a_1} \odot (v^{(\gamma)})(\gamma) = \frac{1}{\Gamma(\gamma - ß)} \odot$$
$$\int_{a_1}^{\boldsymbol{\varepsilon}} (\boldsymbol{\varepsilon}_1 - \vartheta)^{\gamma-ß-1} \odot v^{(\gamma)}(\vartheta)\mathrm{d}\vartheta, \quad ß \in (\gamma - 1, \gamma], \gamma \in \mathbb{N}, \boldsymbol{\varepsilon} > a_1.$$

The parameter's value of $v = [\underline{v}_{\kappa}(\boldsymbol{\varepsilon}), \bar{v}_{\kappa}(\boldsymbol{\varepsilon})], \kappa \in [0,1]$, and $\boldsymbol{\varepsilon}_{10} \in (0,s)$, and fractional Caputo differential in the presence of fuzzy is defined as

$$[\mathcal{D}^{ß}_{(i)-g\mathcal{H}} v(\boldsymbol{\varepsilon}_{10})]_{\kappa} = [\mathcal{D}^{ß}_{(i)-g\mathcal{H}} \underline{v}(\boldsymbol{\varepsilon}_{10}), \mathcal{D}^{\theta}_{(i)-g\mathcal{H}} \bar{v}(\boldsymbol{\varepsilon}_{10})], \kappa \in [0,1],$$

where $r = [\kappa]$:

$$[\mathcal{D}^{ß}_{(i)-g\mathcal{H}} \underline{v}(\boldsymbol{\varepsilon}_{10})] = \frac{1}{\Gamma(\gamma - ß)} \left[\int_0^{\boldsymbol{\varepsilon}} (\boldsymbol{\varepsilon} - \mathbf{x})^{\gamma-ß-1} \frac{\mathrm{d}^{\kappa}}{\mathrm{d}\mathbf{x}^{\kappa}} \underline{v}_{(i)-g\mathcal{H}}(\mathbf{x})\mathrm{d}\mathbf{x} \right]_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_{10}},$$

$$[\mathcal{D}^{\theta}_{(i)-g\mathcal{H}} \bar{v}(\boldsymbol{\varepsilon}_{10})] = \frac{1}{\Gamma(\gamma - ß)} \left[\int_0^{t_1} (\boldsymbol{\varepsilon} - \mathbf{x})^{\gamma-ß-1} \frac{\mathrm{d}^{\kappa}}{\mathrm{d}\mathbf{x}^{\kappa}} \bar{v}_{(i)-g\mathcal{H}}(\mathbf{x}) d\mathbf{x} \right]_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_{10}}.$$

2.8 Definition

Consider that a fuzzy mapping $\tilde{v}(\boldsymbol{\varepsilon}) \in \tilde{\mathbb{H}}^1(0,T)$ and $ß \in [0,1]$, then $g\mathcal{H}$-fractional fuzzy differentiability AB of fuzzy value mappings is defined as

$$(g\mathcal{H}\mathcal{D}^{ß}v)(\boldsymbol{\varepsilon}) = \frac{\mathbb{B}(ß)}{1 - ß} \odot \left[\int_0^{t_1} \underline{v}'(\mathbf{x}) \odot E_{ß}\left[\frac{-ß(\boldsymbol{\varepsilon} - \mathbf{x})^{ß}}{1 - ß} \right] \mathrm{d}\mathbf{x} \right].$$

Thus, the parameter formulations of $v = [\underline{v}_{\kappa}(\boldsymbol{\varepsilon}), \bar{v}_{\kappa}(\boldsymbol{\varepsilon})], \kappa \in [0,1]$, and $\boldsymbol{\varepsilon}_0 \in (0,s)$, and the fuzzy AB operator is given as

$$[{}^{ABC}\mathcal{D}^{ß}_{(i)-g\mathcal{H}} \tilde{v}(\boldsymbol{\varepsilon}_0;\kappa)] = [{}^{ABC}\mathcal{D}^{ß}_{(i)-g\mathcal{H}} \underline{v}(\boldsymbol{\varepsilon}_0;\kappa), {}^{ABC}\mathcal{D}^{\theta}_{(i)-g\mathcal{H}} v(\boldsymbol{\varepsilon}_0;\ \kappa)], \kappa \in [0,1],$$

where

$${}^{ABC}\mathcal{D}^{\theta}_{(i)-g\mathcal{H}} \underline{v}(\boldsymbol{\varepsilon}_0;\ \kappa) = \frac{\mathbb{B}(ß)}{1 - ß} \left[\int_0^{t_1} \underline{v}'_{(i)-g\mathcal{H}}(\mathbf{x}) E_{\theta}\left[\frac{-ß(\boldsymbol{\varepsilon} - \mathbf{x})^{\theta}}{1 - ß} \right] \mathrm{d}\mathbf{x} \right]_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_0},$$

$${}^{ABC}\mathcal{D}^{\theta}_{(i)-g\mathcal{H}} \bar{v}(\boldsymbol{\varepsilon}_0;\ \kappa) = \frac{\mathbb{B}(ß)}{1 - ß} \left[\int_0^{t_1} \bar{v}'_{(i)-g\mathcal{H}}(\mathbf{x}) E_{\theta}\left[\frac{-ß(\boldsymbol{\varepsilon} - \mathbf{x})^{\theta}}{1 - ß} \right] \mathrm{d}\mathbf{x} \right]_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}_0},$$

where $\mathbb{B}(ß)$ represents the normalize function, which is equal to one when ß is supposed to be zero and one. Moreover, we assume that form (i) $-g\mathcal{H}$ exists. There is no requirement to consider (ii) $-g\mathcal{H}$ differentiability.

2.9 Definition

Suggest a continuous real value mapping Ψ and there is an inappropriate Riemann fuzzy integrable mapping $\exp\left(\frac{-\omega}{\sigma}\right) \odot \tilde{v}(\boldsymbol{\varepsilon})$ on $[0,+\infty)$. Then, the integral $\int 0^{+\infty} \exp\left(-\frac{\omega}{\sigma}\right) \odot \tilde{v}(\boldsymbol{\varepsilon})\mathrm{d}\boldsymbol{\varepsilon}$ is recognized to be the Shehu fuzzy transformation and it is noted over the set of mapping [22–25]:

$$\mathcal{S} = \{\tilde{v}(\boldsymbol{g}) : \exists \mathcal{A}, p_1, p_2 > 0, |\tilde{v}(\boldsymbol{\varepsilon})| < \mathcal{A} \exp\left(\frac{|\boldsymbol{\varepsilon}|}{\psi_j}\right), \quad \text{if } \boldsymbol{\varepsilon} \in (-1)^j \times [0,+\infty)\},$$

as

$$\mathbf{S}[\tilde{v}(\boldsymbol{\varepsilon})] = \mathcal{S}(\omega,\sigma) = \int_0^{+\infty} \exp\left(\frac{-\omega}{\sigma}\boldsymbol{\varepsilon}\right) \odot \tilde{v}(\boldsymbol{\varepsilon})\mathrm{d}\boldsymbol{\varepsilon}, \quad \omega, \sigma > 0.$$

2.10 Definition

Suppose there is a fuzzy integrable value mapping ${}_{g\mathcal{H}}^{c}\mathcal{D}^{ß}_{\boldsymbol{\varepsilon}}\tilde{v}(\boldsymbol{\varepsilon})$, and $v(\boldsymbol{\varepsilon})$ is the primitive of ${}_{g\mathcal{H}}^{c}\mathcal{D}^{ß}_{\boldsymbol{\varepsilon}}\tilde{v}(\boldsymbol{\varepsilon})$ on $[0,+\infty)$, then the caputo frational derivative of order ß is presented as [22–25]

$$\mathbf{S}[{}_{g\mathcal{H}}^{c}\mathcal{D}_{\varepsilon}^{ß}\tilde{v}(\boldsymbol{\varepsilon})] = \left(\frac{\omega}{\sigma}\right)^{ß} \odot$$
$$\mathbf{S}[\tilde{v}(\boldsymbol{\varepsilon})] \ominus \sum_{j=0}^{r-1}\left(\frac{\omega}{\sigma}\right)^{ß-j-1} \odot \tilde{v}^{(j)}(0), \quad ß \in (r-1, r],$$

$$\left(\frac{\omega}{\sigma}\right)^{ß} \odot \mathbf{S}[\tilde{v}(\boldsymbol{\varepsilon})] \ominus \sum_{j=0}^{r-1}\left(\frac{\omega}{\sigma}\right)^{ß-j-1} \odot \tilde{f}^{(j)}(0)$$
$$= \left(\left(\frac{\omega}{\sigma}\right)^{ß}\mathbf{S}[\underline{v}(\boldsymbol{\varepsilon};\kappa)] - \sum_{j=0}^{r-1}\left(\frac{\omega}{\sigma}\right)^{ß-j-1} \odot \underline{v}^{(j)}(0;\kappa), \left(\frac{\omega}{\sigma}\right)^{ß}\mathbf{S}\right.$$
$$\left.[\bar{v}(\boldsymbol{\varepsilon};\kappa)] - \sum_{j=0}^{r-1}\left(\frac{\omega}{\sigma}\right)^{ß-j-1} \bar{v}^{(j)}(0;\ \kappa)\right).$$

2.11 Definition

Consider $v \in \mathbb{C}^{F}[0,s] \cap \mathbb{L}^{F}[0,s]$ such that $\tilde{v}(\boldsymbol{\varepsilon}) = [\underline{v}(\boldsymbol{\varepsilon},\kappa), \bar{v}(\boldsymbol{\varepsilon},\kappa)]$, $\kappa \in [0,1]$; then, the Shehu transform of the fuzzy ABC of order $ß \in [0,1]$ is defined as:

$$\mathbf{S}[g\mathcal{H}\mathcal{D}_{\varepsilon}^{ß}\tilde{v}(\boldsymbol{\varepsilon})] = \frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}} \odot \left(\tilde{\mathbf{V}}(\sigma,\omega) \ominus \frac{\sigma}{\omega}\tilde{v}(0)\right).$$

Moreover, using the fact of Allahviranloo *et al.* [22], we obtain

$$\frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\tilde{\omega}}\right)^{ß}} \odot \left(\tilde{\mathbf{V}}(\sigma,\omega) \ominus \frac{\omega}{\sigma}\tilde{v}(0)\right)$$
$$= \left(\frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\tilde{\omega}}\right)^{ß}}\left(\underline{\mathbf{V}}(\sigma,\omega;\kappa) - \frac{\sigma}{\omega}v(0;\kappa)\right),\right.$$
$$\left.\frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\psi}\right)^{\theta}}\left(\overline{\mathbf{V}}(\sigma,\omega;\kappa) - \frac{\sigma}{\tilde{\omega}}\bar{v}(0;\kappa)\right)\right).$$

3 General discussion of the proposed method

Consider the fuzzy fractional PDE is given as

$$\mathcal{S}[{}^{ABC}D_{\varepsilon}^{ß}\tilde{v}(\zeta,\tau)] = \mathcal{S}[D_{\zeta}^{2}\tilde{v}(\zeta,\tau) + D_{\zeta}^{3}\tilde{v}(\zeta,\tau) + \tilde{k}(r)\mathcal{F}(\zeta,\tau)], \tag{1}$$

where $ß \in (0,1]$; therefore, the Shehu transformation of (1) is

$$\frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}\mathcal{S}[\tilde{v}(\zeta,\tau)] - \frac{\mathbb{B}(ß)}{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}\left(\frac{v}{\omega}\right)\tilde{v}(\zeta,\xi,0)$$
$$= \mathcal{S}[D_{\zeta}^{2}\tilde{v}(\zeta,\tau) + D_{\zeta}^{3}\tilde{v}(\zeta,\tau) + \tilde{k}(r)\mathcal{F}(\zeta,\tau)]. \tag{2}$$

On applying the initial condition, we obtain

$$\mathcal{S}[\tilde{v}(\zeta,\tau)] = \frac{g(\zeta,\xi)}{\omega} + \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}$$
$$\mathcal{S}[D_{\zeta}^{2}\tilde{v}(\zeta,\tau) + D_{\zeta}^{3}\tilde{v}(\zeta,\tau) + \tilde{k}(r)\mathcal{F}(\zeta,\tau)]. \tag{3}$$

Decompose the result as $\tilde{v}(\zeta,\tau) = \sum_{n=0}^{\infty}\tilde{v}_n(\zeta,\tau)$; then, (3) applies

$$\mathcal{S}\sum_{n=0}^{\infty}\tilde{v}_n(\zeta,\tau) = \frac{g(\zeta,\xi)}{\omega}$$
$$+ \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}\left[D_{\zeta}^{2}\sum_{n=0}^{\infty}\tilde{v}_n(\zeta,\tau)\right.$$
$$\left.+ D_{\zeta}^{3}\sum_{n=0}^{\infty}\tilde{v}_n(\zeta,\tau) + \tilde{k}(r)\mathcal{F}(\zeta,\tau)\right], \tag{4}$$

$$\mathcal{S}[\tilde{v}_0(\zeta,\tau)] = \frac{g(\zeta,\xi)}{\omega}$$
$$+ \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[\tilde{k}(r)\mathcal{F}(\zeta,\tau)],$$
$$\mathcal{S}[\tilde{v}_1(\zeta,\tau)] = \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_{\zeta}^{2}\tilde{v}_0(\zeta,\tau)$$
$$+ D_{\zeta}^{3}\tilde{v}_0(\zeta,\tau)],$$
$$\mathcal{S}[\tilde{v}_2(\zeta,\tau)] = \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_{\zeta}^{2}\tilde{v}_1(\zeta,\tau)$$
$$+ D_{\zeta}^{3}\tilde{v}_1(\zeta,\tau)],$$
$$\vdots$$
$$\mathcal{S}[\tilde{v}_{n+1}(\zeta,\tau)] = \frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_{\zeta}^{2}\tilde{v}_n(\zeta,\tau)$$
$$+ D_{\zeta}^{3}\tilde{v}_n(\zeta,\tau)]. \tag{5}$$

Applying the inverse Shehu transformation, we obtain

$$\begin{aligned}\underline{v}_0(\zeta,\tau) &= g(\zeta,\xi) + \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[\underline{k}(r)\mathcal{F}(\zeta,\tau)]\right],\\ \overline{v}_0(\zeta,\tau) &= g(\zeta,\xi) + \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[\overline{k}(r)\mathcal{F}(\zeta,\tau)]\right],\end{aligned} \tag{6}$$

$$\begin{aligned}\underline{v}_1(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\underline{v}_0(\zeta,\tau) + D_\zeta^3\underline{v}_0(\zeta,\tau)]\right],\\ \overline{v}_1(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\overline{v}_0(\zeta,\tau) + D_\zeta^3\overline{v}_0(\zeta,\tau)]\right],\end{aligned} \tag{7}$$

$$\begin{aligned}\underline{v}_2(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\underline{v}_1(\zeta,\tau) + D_\zeta^3\underline{v}_1(\zeta,\tau)]\right],\\ \overline{v}_2(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\overline{v}_1(\zeta,\tau) + D_\zeta^3\overline{v}_1(\zeta,\tau)]\right],\\ &\vdots\\ \underline{v}_{n+1}(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\underline{v}_n(\zeta,\tau) + D_\zeta^3\underline{v}_n(\zeta,\tau)]\right],\\ \overline{v}_{n+1}(\zeta,\tau) &= \mathcal{S}^{-1}\left[\frac{1-ß+ß\left(\frac{\sigma}{\omega}\right)^{ß}}{\mathbb{B}(ß)}\mathcal{S}[D_\zeta^2\overline{v}_n(\zeta,\tau) + D_\zeta^3\overline{v}_n(\zeta,\tau)]\right].\end{aligned} \tag{8}$$

The series form solution of Eq. (9) is given as

$$\begin{aligned}\underline{v}(\zeta,\tau) &= \underline{v}_0(\zeta,\tau) + \underline{v}_1(\zeta,\tau) + \underline{v}_2(\zeta,\tau) + \cdots,\\ \overline{v}(\zeta,\tau) &= \overline{v}_0(\zeta,\tau) + \overline{v}_1(\zeta,\tau) + \overline{v}_2(\zeta,\tau) + \cdots.\end{aligned} \tag{9}$$

4 Application

4.1 Example

Consider the FFDP equation:

$$\begin{aligned}&{}^{ABC}D_\tau^{ß}\tilde{v}(\zeta,\tau) - \frac{\partial}{\partial\tau}\left(\frac{\partial^2\tilde{v}(\zeta,\tau)}{\partial\zeta^2}\right) + 4\tilde{v}^2(\zeta,\tau)\frac{\partial\tilde{v}(\zeta,\tau)}{\partial\zeta}\\ &\quad - 3\frac{\partial\tilde{v}(\zeta,\tau)}{\partial\zeta}\left(\frac{\partial^2\tilde{v}(\zeta,\tau)}{\partial\zeta^2}\right) - \tilde{v}(\zeta,\tau)\frac{\partial^3\tilde{v}(\zeta,\tau)}{\partial\zeta^3} = 0\end{aligned} \tag{10}$$

with the initial fuzzy condition

$$\tilde{v}(\zeta,0) = -\tilde{k}\left(\frac{15}{8}\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\right). \tag{11}$$

Applying the proposed Eq. (10), we achieve

$$\begin{aligned}\underline{v}_0(\zeta,\tau) &= -\underline{k}(r)\left(\frac{15}{8}\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\right),\\ \overline{v}_0(\zeta,\tau) &= -\overline{k}(r)\left(\frac{15}{8}\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\right),\\ \underline{v}_1(\zeta,\tau) &= -\underline{k}(r)450\operatorname{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right.\\ &\qquad\left. + (1-ß)\right\},\\ \overline{v}_1(\zeta,\tau) &= -\overline{k}(r)450\operatorname{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right.\\ &\qquad\left. + (1-ß)\right\},\end{aligned} \tag{12}$$

$$\begin{aligned}\tilde{v}(\zeta,\tau) &= \tilde{v}_0(\zeta,\tau) + \tilde{v}_1(\zeta,\tau) + \tilde{v}_2(\zeta,\tau) + \tilde{v}_3(\zeta,\tau)\\ &\quad + \tilde{v}_4(\zeta,\tau) + \cdots,\end{aligned} \tag{13}$$

while in lower and upper portion types, it is

$$\begin{aligned}\underline{v}(\zeta,\tau) &= \underline{v}_0(\zeta,\tau) + \underline{v}_1(\zeta,\tau) + \underline{v}_2(\zeta,\tau) + \underline{v}_3(\zeta,\tau)\\ &\quad + \underline{v}_4(\zeta,\tau) + \cdots,\\ \overline{v}(\zeta,\tau) &= \overline{v}_0(\zeta,\tau) + \overline{v}_1(\zeta,\tau) + \overline{v}_2(\zeta,\tau) + \overline{v}_3(\zeta,\tau)\\ &\quad + \overline{v}_4(\zeta,\tau) + \cdots.\end{aligned} \tag{14}$$

$$\begin{aligned}\underline{v}(\zeta,\tau) &= -\underline{k}(r)\left(\frac{15}{8}\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\right)\\ &\quad - \underline{k}(r)450\operatorname{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\\ &\quad\times\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\} + \cdots,\\ \overline{v}(\zeta,\tau) &= -\overline{k}(r)\left(\frac{15}{8}\operatorname{sech}^2\left(\frac{\zeta}{2}\right)\right)\\ &\quad - \overline{k}(r)450\operatorname{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\\ &\quad\times\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\} + \cdots.\end{aligned} \tag{15}$$

4.2 Example

Consider the fuzzy fractional Camassa–Holm equation:

$$\begin{aligned} {}^{ABC}D_{\tau}^{ß}\tilde{v}(\zeta,\tau) - \frac{\partial}{\partial\tau}\left(\frac{\partial^2\tilde{v}(\zeta,\tau)}{\partial\zeta^2}\right) + 3\tilde{v}^2(\zeta,\tau)\frac{\partial\tilde{v}(\zeta,\tau)}{\partial\zeta} \\ - 2\frac{\partial\tilde{v}(\zeta,\tau)}{\partial\zeta}\left(\frac{\partial^2\tilde{v}(\zeta,\tau)}{\partial\zeta^2}\right) - \tilde{v}(\zeta,\tau)\frac{\partial^3\tilde{v}(\zeta,\tau)}{\partial\zeta^3} = 0 \end{aligned} \tag{16}$$

with the initial fuzzy condition

$$\tilde{v}(\zeta,0) = -\tilde{k}\left(2\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\right), \tag{17}$$

Using the proposed Eq. (16), we obtain

$$\begin{aligned} \underline{v}_0(\zeta,\tau) &= -\underline{k}(r)\left(2\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\right), \\ \overline{v}_0(\zeta,\tau) &= -\overline{k}(r)\left(2\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\right), \\ \underline{v}_1(\zeta,\tau) &= -\underline{k}(r)384\mathrm{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\}, \\ \overline{v}_1(\zeta,\tau) &= -\overline{k}(r)384\mathrm{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\}, \end{aligned} \tag{18}$$

$$\begin{aligned} \tilde{v}(\zeta,\tau) = \tilde{v}_0(\zeta,\tau) + \tilde{v}_1(\zeta,\tau) + \tilde{v}_2(\zeta,\tau) + \tilde{v}_3(\zeta,\tau) \\ + \tilde{v}_4(\zeta,\tau) + \cdots, \end{aligned} \tag{19}$$

while in lower and upper portion types, it is

$$\begin{aligned} \underline{v}(\zeta,\tau) &= \underline{v}_0(\zeta,\tau) + \underline{v}_1(\zeta,\tau) + \underline{v}_2(\zeta,\tau) + \underline{v}_3(\zeta,\tau) + \underline{v}_4(\zeta,\tau) + \cdots, \\ \overline{v}(\zeta,\tau) &= \overline{v}_0(\zeta,\tau) + \overline{v}_1(\zeta,\tau) + \overline{v}_2(\zeta,\tau) + \overline{v}_3(\zeta,\tau) + \overline{v}_4(\zeta,\tau) + \cdots. \end{aligned} \tag{20}$$

$$\begin{aligned} \underline{v}(\zeta,\tau) &= -\underline{k}(r)\left(2\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\right) - \underline{k}(r)384\mathrm{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\} + \cdots, \\ \overline{v}(\zeta,\tau) &= -\overline{k}(r)\left(2\mathrm{sech}^2\left(\frac{\zeta}{2}\right)\right) - \overline{k}(r)384\mathrm{csch}^5(\zeta)\sinh^6\left(\frac{\zeta}{2}\right)\frac{1}{\mathbb{B}(ß)}\left\{\frac{ß\tau^{ß}}{\Gamma(ß+1)} + (1-ß)\right\} + \cdots. \end{aligned} \tag{21}$$

5 Discussion of figures

Figure 1 displays the fuzzy lower and upper branch graphs in a three-dimensional space, depicting the analytic series

Figure 1: The fuzzy lower and upper branch three-dimensional graphs for the analytic series solutions.

Figure 2: The fuzzy lower and upper branch three-dimensional graphs for the analytic series solutions of fractional order 0.8.

Figure 3: The fuzzy lower and upper branch of three-dimensional graph for the analytic series solutions of various fractional order.

solutions. These solutions pertain to a specific scenario or equation but are depicted without a fractional order. In contrast, Figure 2 focuses on fractional-order 0.8, demonstrating the impact of fractional differentiation on the solution behavior. The variation in the order of the fractional derivative often influences the characteristics of the solutions. Moving to Figure 3, it illustrates the fuzzy lower and upper branch graphs for the analytic series solutions with varying fractional orders within the framework of the fuzzy DP equation. This figure presents a comprehensive view, showing how solutions change across different fractional orders, giving insights into the equation's behavior under different differentiation conditions. In a similar vein, Figures 4 and 5 exhibit fuzzy lower and upper branch graphs for analytic series solutions. Figure 4, similar to Figure 1, portrays solutions without considering fractional

Figure 4: The fuzzy lower and upper branch three-dimensional graphs for the analytic series solutions.

Figure 5: The fuzzy lower and upper branch three-dimensional graphs for the analytic series solutions of fractional order 0.8.

orders, while Figure 5 delves into the impact of a specific fractional order, namely 0.8, on the solutions. Figure 6, on the other hand, explores a different equation, specifically the fuzzy Camassa–Holm equation in example 2. It displays the fuzzy lower and upper branch graphs of analytic series solutions across various fractional orders, shedding light on how this equation's behavior evolves concerning different degrees of fractional differentiation.

6 Conclusion

In conclusion, we have applied the ITM to solve the FFDP and fuzzy fractional Camassa–Holm equations. The ITM has proved to be an effective and reliable tool for solving fractional PDEs with fuzzy initial conditions. The numerical results obtained using ITM have been validated by comparing with the exact solution and the results obtained.

Figure 6: The fuzzy lower and upper branch three-dimensional graph for the analytic series solutions of various fractional order.

The graphical representations of the fuzzy lower and upper branch graphs for analytic series solutions elucidate the influence of fractional-order differentiation on equation behaviors. The exploration across different equations and fractional orders contributes to a deeper understanding of these mathematical models' dynamics and behaviors. The proposed method can be extended to solve other fractional PDEs with fuzzy initial conditions. Overall, the study opens up new avenues for solving complex fractional PDEs with fuzzy initial conditions.

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