Research Article

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Analysis of nonlinear fractional-order Fisher equation using two reliable techniques

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Abstract: In this article, the solution to the time-fractional Fisher equation is determined using two well-known analytical techniques. The suggested approaches are the new iterative method and the optimal auxiliary function method, with the fractional derivative handled in the Caputo sense. The obtained results demonstrate that the suggested approaches are efficient and simple to use for solving fractional-order differential equations. The approximate and exact solutions of the partial fractional differential equations for integer order were compared. Additionally, the fractional-order and integer-order results are contrasted using simple tables. It has been confirmed that the solution produced using the provided methods converges to the exact solution at the appropriate rate. The primary advantage of the suggested method is the small number of computations needed. Moreover, it may be used to address fractional-order physical problems in a number of fields.

Keywords: OAFM, NIM, fractional-order Fisher equation, Caputo operator

1 Introduction

The use of mathematical modelling as part of a well-rounded mathematics education is essential. Through the process of mathematical modelling, an issue that exists in the real world is abstracted and represented in a mathematical format. Iyanda *et al.* [1] presented and applied the exponential matrix algorithm, differential transformation algorithm, and Runge–Kutta (RK5) to simulate the temperature distribution in five heating tanks in series for the successive preheating of multicomponent oil solutions. Shahzad *et al.* investigated the Darcy–Forchheimer effects in a micropolar nanofluid flow containing gyrotactic microorganisms between two coaxial, parallel, and radially stretching discs in the presence of gyrotactic motile microorganisms with convective thermal boundary conditions [2]. The intriguing nonlinear phenomenon known as chaos has grown considerably during the past 30 years. In a variety of fields, including biomedical design, secure communication, information encryption, and stream components, it is helpful or shows amazing potential [3]. Basit *et al.* [4] analysed the Darcy–Forchheimer flow of a hybrid nanofluid within two parallel discs. They combined gold (Au), silver (Ag), copper (Cu), and iron oxide (Fe_3O_4) nanoparticles with the base fluid, blood. In this way of looking at mathematics, mathematical models are seen to play a significant role in all subfields of the subject, whether they be algebra, arithmetic, geometry, or calculus. Everything in our universe goes through some kind of transformation or change throughout the course of time and space. Calculus is the primary tool for determining the significance of all of these changes, which all have profound repercussions for the physical world. The idea of a derivative is used to find the rate of change, and it is not wrong to say that almost every physical change in nature can be described by an equation involving derivatives. Differential equations can be used to describe all of these kinds of changes in a mathematical way. Differential equations can be either ordinary or partial. Most classical or fractional-order ordinary differential equations arise from modelling basic physical processes, while partial differential equations (PDEs) originate from modelling more intricate physical phenomena. Fractional calculus is a distinct sort of calculus from classical calculus. Fractional calculus helps explain difficult concepts such as memory and transmission. This discipline has drawn a large number of scholars since it is worldwide and has several applications in many fields of

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science, including physics, signal processing, modelling, control theory, economics, and chemistry [5–8]. Due to the fast-growing and extensive applications of fractional partial differential equations (FPDEs) in numerous scientific and technical domains, including medicine, chemistry, biology, electrical engineering, and viscoelasticity, much attention has been paid to the topic in recent decades. For more details on these and other applications, we refer to previous studies [9–12]. From this point of view, FPDE approximations and analytical solutions are essential components that must be included in order to correctly describe the dynamics of fundamental physical processes. In light of the reasoning presented earlier, mathematicians have developed and used a wide range of approximation and analytical techniques in order to solve a number of important mathematical models that are relevant to problems that occur in the real world. Mathematicians keep working in this field even though it is hard to find analytical or even close solutions to some nonlinear FPDEs and systems of FPDEs [13–17]. Different approaches to solving fractional-order PDEs have been utilized in the literature, including the residual power series method [18], the optimal homotopy analysis method [19], the Elzaki transformation [20], the finite difference method [21], Galerkin finite element methods [22], and Grünwald–Letnikov approximation [23]. The optimal auxiliary function method (OAFM) [24,25], which does not require any small or big parameter assumptions and does not require any discretization, and the new iterative method (NIM) [26,27] are used to solve nonlinear fractional-order Fisher equations.

The main motivation of this study is to explore innovative mathematical techniques for solving complex nonlinear fractional differential equations, particularly the Fisher equation. This equation has a wide range of applications in biology, ecology, and population dynamics, making it an important subject of study. Fisher's equation, sometimes known as the reaction–diffusion equation, has a simple and classic example provided by

$$D_{\mu}^{\sigma}\Theta(\psi,\mu) = \lambda\Theta_{\zeta\zeta}(\zeta,\nu) + \mu\Theta(\zeta,\nu)(1-\Theta^{2}(\zeta,\nu)), \quad (1)$$

which essentially combines the diffusion equation with the diffusion factor λ, the logistic equation, and the birth rate μ.

The objective of this study is to solve fractional-order nonlinear Fisher equation using two analytical techniques, *i.e.* OAFM and NIM, in conjunction with a fractional operator of Caputo type. In order to facilitate a comparison of the outcomes produced by these two approaches, graphs and tables will be constructed for each problem.

The rest of this article is arranged as follows: Section 2 goes over the OAFM and NIM processes. Section 3 discusses OAFM and NIM implementation with examples. The final portion is dedicated to the conclusion.

2 Preliminaries

In this section, we will discuss several basic definitions and conclusions relating to the Caputo fractional derivative.

Definition 1. The formula for the Riemann fractional integral is as follows [28]:

$$J_t^{\sigma}\,\omega(x,t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-r)^{\sigma-1}\omega(x,r)\mathrm{d}r. \quad (2)$$

Definition 2. The fractional derivative of f according to the Caputo formula is defined as [29]

$$^{C}D_t^{\sigma}\omega(x,t) = \frac{1}{\Gamma(m-\sigma)}\int_0^t (t-r)^{m-\sigma-1}\omega(x,r)\mathrm{d}r, \quad m-1 < \sigma \le m,\ t>0. \quad (3)$$

Lemma 1. *For* $n-1<\sigma\le n,\ p>-1,\ t\ge 0,$ *and* $\lambda\in R$, *we have*

1) $D_t^{\sigma}t^{p} = \frac{\Gamma(\sigma+1)}{\Gamma(p-\sigma+1)}t^{p-\sigma}$,
2) $D_t^{\sigma}\lambda = 0$,
3) $D_t^{\sigma}I_t^{\sigma}\omega(x,t) = \omega(x,t)$,
4) $I_t^{\sigma}D_t^{\sigma}\omega(x,t) = \omega(x,t) - \sum_{i=0}^{n-1}\partial^{i}\omega(x,0)\frac{t^{i}}{i!}$,

3 General procedure for the proposed methods

3.1 General procedure of OFAM

This section will cover how to solve general fractional-order PDEs using OAFM.

$$\frac{\partial^{\sigma}\Theta(\zeta,\nu)}{\partial\nu^{\sigma}} = g(\zeta,\nu) + N(\Theta(\zeta,\nu)). \quad (4)$$

Subject to the initial conditions:

$$\begin{aligned} &D_t^{\sigma-r}(\zeta,0) = \phi_r(\zeta), && r = 0,1,\ldots,s-1,\\ &D_t^{\sigma-s}(\zeta,0) = 0, && s=[\sigma],\\ &D_t^{r}(\zeta,0) = \psi_r(\zeta), && r=0,1,\ldots,s-1,\\ &D_t^{s}(\zeta,0) = 0, && s=[\sigma], \end{aligned} \quad (5)$$

$\frac{\partial^\sigma}{\partial v^\sigma}$ represents the Caputo operater $\Theta(\zeta, v)$ is an unknown function, and $g(\zeta, v)$ is a known analytic function

Step 1: To solve Eq. (4), we will use an approximate solution that has two components, such as:

$$\tilde{\Theta}(\zeta, v) = \Theta_0(\zeta, v) + \Theta_1(\zeta, v, C_i), \quad i = 1, 2, 3, \ldots, p. \tag{6}$$

Step 2: To determine the zero- and first-order solutions, we substitute Eq. (6) into Eq. (4), which results in

$$\begin{aligned}&\frac{\partial^\sigma\Theta_0(\zeta, v)}{\partial v^\sigma} + \frac{\partial^\sigma\Theta_1(\zeta, v)}{\partial v^\sigma} + g(\zeta, v)\\&+ N\left[\left(\frac{\partial^\sigma\Theta_0(\zeta, v)}{\partial v^\sigma}\right) + \left(\frac{\partial^\sigma\Theta_1(\zeta, v, C_i)}{\partial v^\sigma}\right)\right] = 0.\end{aligned} \tag{7}$$

Step 3: For the purpose of determining the first approximation $\Theta_0(\zeta, v)$ based on the linear equation,

$$\frac{\partial^\sigma\Theta_0(\zeta, v)}{\partial v^\sigma} + g(\zeta, v) = 0. \tag{8}$$

Using the inverse operator, we arrive to $\Theta_0(\zeta, v)$, which is expressed as follows:

$$\Theta_0(\zeta, v) = g(\zeta, v). \tag{9}$$

Step 4: The nonlinear term seen in expanding form (7) is

$$\begin{aligned}&N\left[\frac{\partial^\sigma\Theta_0(\zeta, v)}{\partial v^\sigma} + \frac{\partial\Theta_1^\sigma(\zeta, v, C_i)}{\partial v^\alpha}\right]\\&= N[\Theta_0(\zeta, v)] + \sum_{k=1}^{\infty}\frac{\Theta_1^k}{k!}N^{(k)}[\Theta_0(\zeta, v)].\end{aligned} \tag{10}$$

Step 5: In order to quickly solve Eq. (10) and speed up the convergence of the first-order approximation $\tilde{\Theta}(\zeta, v)$, we offer an alternative equation that may be written as follows:

$$\begin{aligned}\frac{\partial^\sigma\Theta_1(\zeta, v, C_i)}{\partial v^\sigma} &= -\Delta_1[\Theta_0(\zeta, v)]N[\Theta_0(\zeta, v)]\\&\quad - \Delta_2[\Theta_0(\zeta, v), C_j].\end{aligned} \tag{11}$$

Step 6: We obtain a first-order solution $\Theta_1(\zeta, v)$ using the inverse operator after substituting the auxiliary function into Eq. (11) with the auxiliary function.

Step 7: The numerical values of the convergence control parameters C_i are determined using several approaches, such as least-squares, Galerkin's, Ritz, and collocation. We use the least-squares technique to reduce errors

$$J(C_i, C_j) = \int_0^v\int_\Omega R^2(\zeta, v;\, C_i, C_j)\mathrm{d}\zeta\mathrm{d}v, \tag{12}$$

where R denotes the residual:

$$\begin{aligned}R(\zeta, v, C_i, C_j) = &\frac{\partial\tilde{\Theta}(\zeta, v, C_i, C_j)}{\partial v}\\&+ g(\zeta, v) + N[\tilde{\Theta}(\zeta, v, C_i, C_j)],\\&i = 1, 2, \ldots, s,\ j = S + 1, S + 2, \ldots, p.\end{aligned} \tag{13}$$

3.2 Analysis of NIM [30]

For the basic idea of NIM, we consider the general functional equation:

$$\Theta(\zeta) = f(\zeta) + \aleph(\Theta(\zeta)), \tag{14}$$

where $\aleph$ is the nonlinear operator from a Banach space B to B and f is the known function. We have been looking for a solution of (14) having the series form

$$\Theta(\zeta) = \sum_{i=0}^{\infty}\Theta_i(\zeta). \tag{15}$$

The nonlinear term can be decomposed as

$$\begin{aligned}\aleph\left(\sum_{i=0}^{\infty}\Theta_i(\zeta)\right) = \aleph(\Theta_0) + \sum_{i=0}^{\infty}\left[\aleph\left(\sum_{j=0}^{i}\Theta_j(\zeta)\right)\right.\\\left. - \aleph\left(\sum_{j=0}^{i-1}\Theta_j(\zeta)\right)\right].\end{aligned} \tag{16}$$

From (15) and (16), (14) is equivalent to

$$\begin{aligned}\sum_{i=0}^{\infty}\Theta_i(\zeta) = f + \aleph(\Theta_0) + \sum_{i=0}^{\infty}\\\times\left[\aleph\left(\sum_{j=0}^{i}\Theta_j(\zeta)\right) - \aleph\left(\sum_{j=0}^{i-1}\Theta_j(\zeta)\right)\right].\end{aligned} \tag{17}$$

We define the following recurrence relations:

$$\begin{aligned}&\Theta_0 = f,\\&\Theta_1 = \aleph(\Theta_0),\\&\Theta_2 = \aleph(\Theta_0 + \Theta_1) - \aleph(\Theta_0),\\&\Theta_{n+1} = \aleph(\Theta_0 + \Theta_1 + \cdots\Theta_n) - \aleph(\Theta_0 + \Theta_1 + \cdots\Theta_{n-1}),\\&\qquad n = 1, 2, 3, \ldots.\end{aligned} \tag{18}$$

Then,

$$\begin{aligned}&(\Theta_0 + \Theta_1 + \cdots\Theta_n) = \aleph(\Theta_0 + \Theta_1 + \cdots\Theta_n), n = 1, 2, 3\cdots,\\&\Theta = \sum_{i=0}^{\infty}\Theta_i(\zeta) = f + \aleph\left(\sum_{i=0}^{\infty}\Theta_i(\zeta)\right).\end{aligned} \tag{19}$$

3.2.1 Basic road map of NIM

In this section, we discuss basic idea for solving fractional-order nonlinear PDE using the NIM. Consider the following fractional-order PDE:

$$D_t^{\alpha}\Theta(\zeta,t) = A(\Theta,\partial\Theta) + B(\zeta,t),\ \ m-1<\alpha\le m, m\in N, \tag{20}$$

$$\frac{\partial^k}{\partial t^k}\Theta(\zeta,0) = h_k(\zeta),\ \ k = 0,1,2,3\cdots m-1, \tag{21}$$

where A is nonlinear function of Θ and $\partial\Theta$ and B is the source function. In view of the NIM, the initial value problem (20), (21) is equivalent to the integral equation

$$\Theta(\zeta,t) = \sum_{k=0}^{m-1} h_k(\zeta)\frac{t^k}{k!} + I_t^{\sigma}(A) + I_t^{\sigma}(B) = f + N(\zeta), \tag{22}$$

where

$$f = \sum_{k=0}^{m-1} h_k(\zeta)\frac{t^k}{k!} + I_t^{\sigma}(B), \tag{23}$$

$$N(\Theta) = I_t^{\sigma}(A). \tag{24}$$

4 Numerical problem

4.1 Problem 1

Consider the fractional-order Fisher equation, which is represented by

$$D_v^{\sigma}\Theta(\zeta,v) - \Theta_{\zeta\zeta}(\zeta,v) - \Theta(\zeta,v)(1-\Theta^2(\zeta,v)) = 0, \tag{25}$$

subject to the initial condition

$$\Theta(\zeta,0) = \delta. \tag{26}$$

4.1.1 Implementation of OAFM

From Eq. (25), linear and nonlinear terms

$$\begin{aligned} Ł(\Theta) &= \frac{\partial^{\sigma}\Theta(\zeta,v)}{\partial v^{\sigma}},\\ N(\Theta) &= -\Theta(\zeta,v)(1-\Theta(\zeta,v)),\\ g(\zeta,v) &= 0. \end{aligned} \tag{27}$$

The initial approximate $\Theta_0(\zeta,v)$ is obtained from Eq. (9)

$$\frac{\partial^{\sigma}\Theta_0(\zeta,v)}{\partial v^{\sigma}} = 0. \tag{28}$$

By making use of the inverse operator, we obtain the following solution:

$$\Theta_0(\zeta,v) = \delta. \tag{29}$$

Substituting Eq. (29) into Eq. (27), the nonlinear term becomes

$$N[\Theta_0(\zeta,v)] = -\delta(1-\delta). \tag{30}$$

The first approximation $\Theta_1(\zeta,v)$ is given by Eq. (11):

$$\begin{aligned} \frac{\partial^{\sigma}\Theta_1(\zeta,v)}{\partial v^{\sigma}} &= \Delta_1[\Theta_0(\zeta,v)]N[\Theta_0(\zeta,v)]\\ &\quad + \Delta_2[\Theta_0(\zeta,v),C_j]. \end{aligned} \tag{31}$$

We choose the auxiliary functions Δ_1 and Δ_2 as

$$\begin{aligned} \Delta_1 &= c1(\delta)^2 + c2(\delta)^4,\\ \Delta_2 &= c3(\delta)^4 + c4(\delta)^6. \end{aligned} \tag{32}$$

$$\Theta_1(\zeta,v) = \frac{(c1\delta^4 - c1\delta^3 + \delta^5(c3 + c2(-1+\delta) + c4\delta))}{\Gamma[1+\sigma]}v^{\sigma}. \tag{33}$$

Adding Eq. (33) and (30), we obtain the OAFM solution:

$$\begin{aligned} &\Theta(\zeta,v)\\ &= \delta + \frac{(c1\delta^4 - c1\delta^3 + \delta^5(c3 + c2(-1+\delta) + c4\delta))}{\Gamma[1+\sigma]}v^{\sigma}. \end{aligned} \tag{34}$$

4.1.2 Implementation of NIM

Applying Riemman–Levelli integral to Eq. (25) and making use of Eq. (26), we obtain

$$\begin{aligned} \Theta(\zeta,v) &= \frac{1}{(1+e^{\zeta})^2} + J_v^{\sigma}[\Theta_{\zeta\zeta}(\zeta,\gamma) + \Theta(\zeta,\gamma)\\ &\quad - \Theta(\zeta,\gamma)^2]. \end{aligned} \tag{35}$$

By the NIM algorithm, the *zero*th-order problem of $\Theta(\zeta,v)$

$$\Theta_0(\zeta,v) = \delta. \tag{36}$$

The first-order component of solution is as follows:

$$\Theta_1(\zeta,v) = \frac{\delta(-1+\delta)}{\Gamma[1+\sigma]}v^{\sigma}. \tag{37}$$

The second-order component of solution is as follows:

$$\Theta_2(\zeta,v) = \frac{\delta^2(v^{\sigma} - \delta v^{\sigma} + \Gamma[1+\sigma])^2}{\Gamma^3[1+\sigma]}v^{\sigma}. \tag{38}$$

Therefore, three-term approximate solution of $\Theta(\zeta,v)$

$$\begin{aligned} \Theta(\zeta,v) &= \delta + \frac{\delta(-1+\delta)}{\Gamma[1+\sigma]}v^{\sigma}\\ &\quad + \frac{\delta^2(v^{\sigma} - \delta v^{\sigma} + \Gamma[1+\sigma])^2}{\Gamma^3[1+\sigma]}v^{\sigma}. \end{aligned} \tag{39}$$

Table 1: Convergence-control parameter values obtained by the least-squares method for ci

c1	c2	c3	c4
0.7904609992608322	−0.651906484683587	1.0364255516293996	−0.3044300441851726

4.2 Problem 2

Consider the fractional-order Fisher equation, which is represented by

$$D_{\nu}^{\sigma}\Theta(\zeta,\nu) = \Theta_{\zeta\zeta}(\zeta,\nu) + 6\Theta(\zeta,\nu) - 6(\Theta(\zeta,\nu))^2, \quad (40)$$

subject to the initial condition

$$\Theta(\zeta,0) = \frac{1}{(1+e^{\zeta})^2}. \quad (41)$$

4.2.1 Implementation of OAFM

From Eq. (40), we assume linear and nonlinear terms

$$\begin{aligned} Ł(\Theta) &= \frac{\partial^{\sigma}\Theta(\zeta,\nu)}{\partial\nu^{\sigma}}, \\ N(\Theta) &= 6\Theta(\zeta,\nu)(1-\Theta(\zeta,\nu)). \\ g(\zeta,\nu) &= 0. \end{aligned} \quad (42)$$

The initial approximate $\Theta_0(\zeta,\nu)$ is obtained from Eq. (9)

$$\frac{\partial^{\sigma}\Theta_0(\zeta,\nu)}{\partial\nu^{\sigma}} = 0. \quad (43)$$

By making use of the inverse operator, we obtain the following solution:

$$\Theta_0(\zeta,\nu) = \frac{1}{(1+e^{\zeta})^2}. \quad (44)$$

Substituting Eq. (44) into Eq. (42), the nonlinear term becomes

$$N[\Theta_0(\zeta,\nu)] = \frac{6}{(1+e^{\zeta})^4}. \quad (45)$$

The first approximation $\Theta_1(\zeta,\nu)$ is given by Eq. (11):

$$\begin{aligned} \frac{\partial^{\sigma}\Theta_1(\zeta,\nu)}{\partial\nu^{\sigma}} &= \Delta_1[\Theta_0(\zeta,\nu)]N[\Theta_0(\zeta,\nu)] \\ &+ \Delta_2[\Theta_0(\zeta,\nu), C_j]. \end{aligned} \quad (46)$$

We choose the auxiliary functions Δ_1 and Δ_2 as

$$\begin{aligned} \Delta_1 &= c1\left(\frac{1}{(1+e^{\zeta})^3}\right) + c2\left(\frac{1}{(1+e^{\zeta})^4}\right), \\ \Delta_2 &= c3\left(\frac{10e^{\zeta}}{(1+e^{\zeta})^5}\right) + c4\left(\frac{50e^{\zeta}}{(1+e^{\zeta})^7}\right), \end{aligned} \quad (47)$$

$$\begin{aligned} &\Theta_1(\zeta,\nu) \\ &= \frac{2(3c2 + (1+e^{\zeta})(3c1 + 5e^{\zeta}(5c4 + c3(1+e^{\zeta})^2)))}{(1+e^{\zeta})^8\Gamma[1+\sigma]}\nu^{\sigma}. \end{aligned} \quad (48)$$

By adding Eq. (48) and (44), we obtain the OAFM solution:

Table 2: Convergence-control parameter values obtained by the least-squares method for ci

c1	c2	c3	c4
800.541870620842	−926.8892128741605	19.985728918740982	−56.30059353901681

Table 3: Numerical compression of the OAFM and NIM solutions with exact solution as well as absolute error (AE) at $\delta = 0.005$

ζ	OAFM solution	NIM solution	Exact solution	AE (OAFM)	AE (NIM)
0.1	0.00499999	0.00509846	0.00052296	5.2296×10^{-4}	4.24495×10^{-4}
0.2	0.00499998	0.00539362	0.00110028	1.10028×10^{-3}	7.06637×10^{-4}
0.3	0.00499997	0.00588521	0.00173754	1.73754×10^{-3}	8.523×10^{-4}
0.4	0.00499996	0.00657291	0.00244086	2.44086×10^{-3}	8.67914×10^{-4}
0.5	0.00499995	0.00745644	0.003217	3.217×10^{-3}	7.60516×10^{-4}
0.6	0.00499994	0.00853549	0.00407336	4.07336×10^{-3}	5.3781×10^{-4}

Table 4: Numerical compression of the OAFM and NIM solutions with exact solution as well as absolute error (AE) at $\nu = 0.001$

ζ	OAFM solution	NIM solution	Exact solution	AE (OAFM)	AE (NIM)
0.5	0.142946	0.141959	0.143426	4.80×10^{-4}	1.4675×10^{-3}
0.6	0.125983	0.125053	0.126372	3.89×10^{-4}	1.3185×10^{-3}
0.7	0.110518	0.10966	0.110837	3.19×10^{-4}	1.1768×10^{-3}
0.8	0.096513	0.095737	0.096781	2.68×10^{-4}	1.0437×10^{-3}
0.9	0.083914	0.083226	0.084146	2.32×10^{-4}	9.2001×10^{-4}
1	0.072652	0.072054	0.07286	2.08×10^{-4}	8.0634×10^{-4}

Table 5: Comparison of NIM and OAFM solutions at different values of fractional-order

ζ	NIM at $\sigma = 0.65$	OAFM at $\sigma = 0.65$	NIM at $\sigma = 0.85$	OAFM at $\sigma = 0.85$
0	0.304325	0.250746	0.267645	0.250746
0.1	0.277865	0.227829	0.242451	0.227829
0.2	0.252461	0.20603	0.218536	0.20603
0.3	0.228248	0.185465	0.196005	0.185465
0.4	0.205339	0.166165	0.174933	0.166165
0.5	0.18382	0.148128	0.155371	0.148128
0.6	0.163755	0.131347	0.13734	0.131347

$$\Theta(\zeta, \nu) = \frac{1}{(1 + e^{\zeta})^2} + \frac{2(3c2 + (1 + e^{\zeta})(3c1 + 5e^{\zeta}(5c4 + c3(1 + e^{\zeta})^2)))}{(1 + e^{\zeta})^8\Gamma[1 + \sigma]}\nu^{\sigma}. \tag{49}$$

4.2.2 Implementation of NIM

Applying Riemman–Levelli integral to Eq. (40) and making use of Eq. (41), we obtain

$$\Theta(\zeta, \nu) = \frac{1}{(1 + e^{\zeta})^2} + J_{\nu}^{\sigma}[\Theta_{\zeta\zeta}(\zeta, \nu) + 6\Theta(\zeta, \nu) - 6\Theta(\zeta, \nu)^2]. \tag{50}$$

By the NIM algorithm, the *zero*th-order problem of $\Theta(\zeta, \nu)$

$$\Theta_0(\zeta, \nu) = \frac{1}{(1 + e^{\zeta})^2}. \tag{51}$$

The first-order component of solution is as follows:

$$\Theta_1(\zeta, \nu) = \frac{10e^{\zeta}}{(1 + e^{\zeta})^3\Gamma(\sigma + 1)}. \tag{52}$$

The second-order component of solution is as follows:

$$\Theta_2(\zeta, \nu) = \frac{50e^{\zeta}\nu^{2\sigma}((1 + e^{\zeta})^2(-1 + 2e^{\zeta}))}{(1 + e^{\zeta})^6\Gamma(2\sigma + 1)} + \frac{50e^{\zeta}\nu^{2\sigma}(12(4)^{\sigma}e^{\zeta}\nu^{\sigma}\Gamma[\frac{1}{2} + \sigma])}{\sqrt{\pi}(1 + e^{\zeta})^6\Gamma(\sigma + 1)\Gamma(3\sigma + 1)}. \tag{53}$$

Therefore, three-term approximate solution of $\Theta(\zeta, \nu)$

$$\Theta(\zeta, \nu) = \frac{1}{(1 + e^{\zeta})^2} + \frac{10e^{\zeta}}{(1 + e^{\zeta})^3\Gamma(\sigma + 1)} + \frac{50e^{\zeta}\nu^{2\sigma}((1 + e^{\zeta})^2(-1 + 2e^{\zeta}))}{(1 + e^{\zeta})^6\Gamma(2\sigma + 1)} + \frac{50e^{\zeta}\nu^{2\sigma}12(4)^{\sigma}e^{\zeta}\nu^{\sigma}\Gamma\left[\frac{1}{2} + \sigma\right]}{\sqrt{\pi}(1 + e^{\zeta})^6\Gamma(\sigma + 1)\Gamma(3\sigma + 1)}. \tag{54}$$

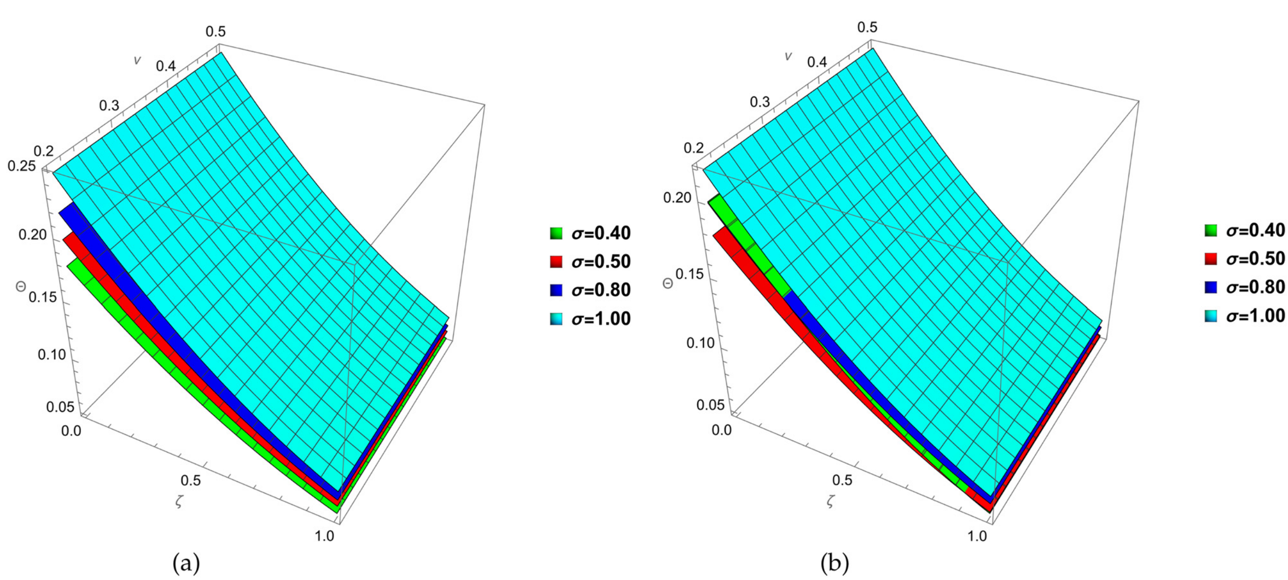

Figure 1: 3D plots of NIM and OAFM solution of $\Theta(\zeta, \nu)$ at various values of fractional order for Problem (1): (a) 3D plots of $\Theta(\zeta, \nu)$ using NIM and (b) 3D plots of $\Theta(\zeta, \nu)$ using OAFM.

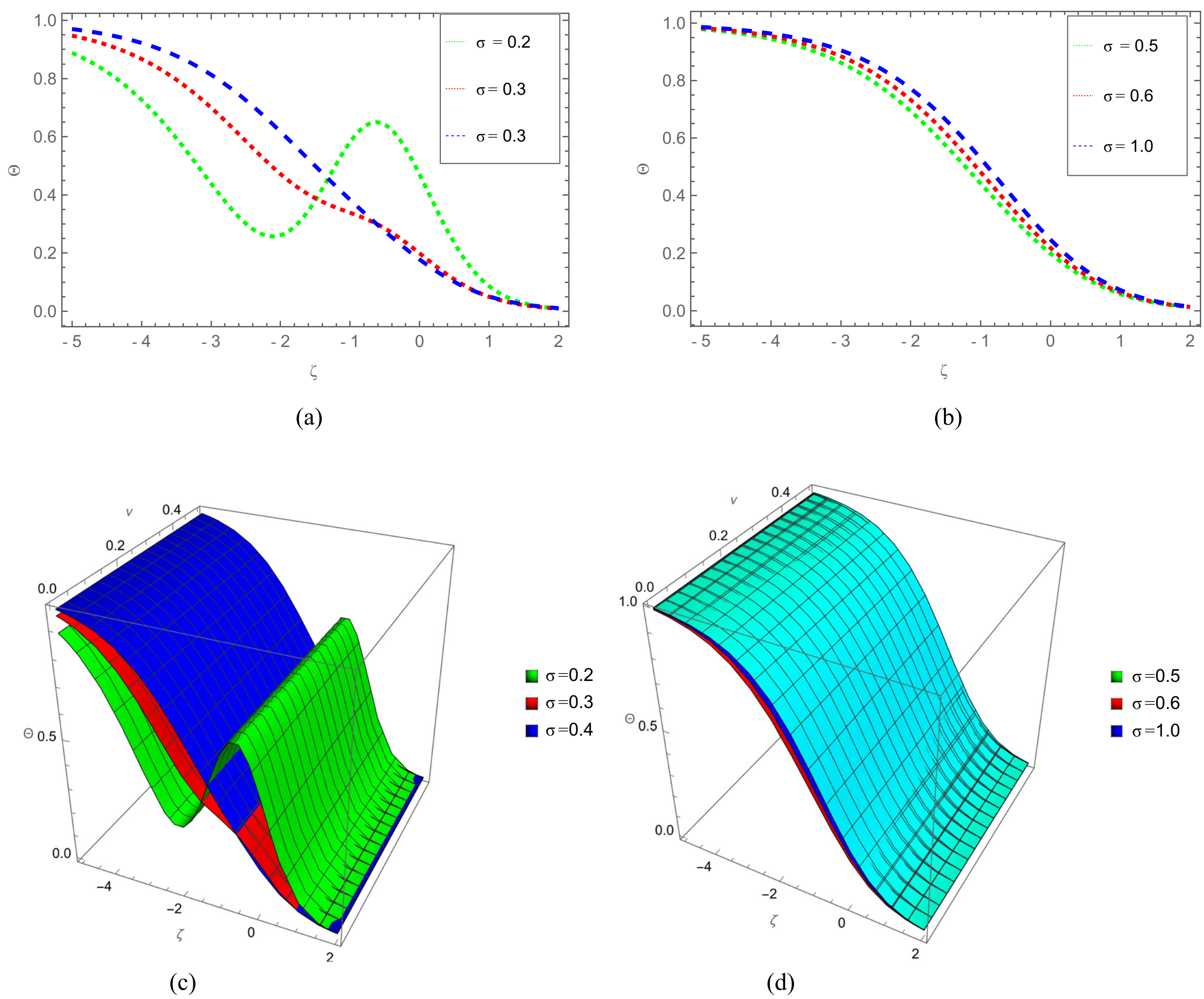

Figure 2: 2D-plots and 3D-plots of $\Theta(\zeta, \nu)$ at various values of fractional order for problem (2). (a) 2D-plots of $\Theta(\zeta, \nu)$ using NIM, (b) 2D-plots of $\Theta(\zeta, \nu)$ using NIM, (c) 3D-plots of $\Theta(\zeta, \nu)$ using NIM, and (d) 3D-plots of $\Theta(\zeta, \nu)$ using NIM.

5 Numerical results and discussion

Tables 1 and 2 represent the numerical values of auxiliary constant for Examples 1 and 2, respectively. Tables 3 and 4 provide the comparison of the OAFM and NIM solutions with the exact solution as well as absolute error at $\sigma = 14$ and $\nu = 0.001$ for Problems 1 and 2, respectively. Table 5 shows the comparison of NIM and OAFM solutions at fractional-order $\sigma = 0.65$ and $\sigma = 0.85$.

Figure 1 shows the closed contact of NIM and OAFM solutions for different values of fractional order. Figure 2 represents the 2D and 3D plots of $\Theta(\zeta, \nu)$ using NIM.

6 Conclusion

In this study, we provide OAFM and NIM for solving nonlinear fractional-order Fisher equation. The derivative is regarded in the sense of Caputo. The solution obtained using the given approaches reveals that our results agree closely with the exact solution. Finally, we can conclude that the provided approaches are sufficiently consistent and can be used to analyse a broad variety of fractional-order nonlinear mathematical models that help explain the behaviour of highly nonlinear, complicated phenomena in important scientific and engineering fields.

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