Research Article

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On the existence and numerical simulation of Cholera epidemic model

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Abstract: A model describing the transmission dynamics of cholera is considered in this article. The concerned model is investigated under the Caputo-Fabrizio fractal fractional derivative. The objective of this article is to study theoretical and numerical results for the model under our consideration. Classical fixed point approach is used to obtain sufficient conditions for the existence of solution to the proposed model. Adam's Bashforth numerical method is utilized for the numerical interpretation of the suggested model. The considered technique is a powerful mathematical tool, that provides a numerical solution for the concerned problem. To discuss the transmission dynamics of the considered model, several graphical presentations are given.

Keywords: transmission dynamics, CFFFD, fixed point approach, Adam's Bashforth numerical method

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1 Introduction

Fractional calculus (FC) is a mathematical discipline that investigates integrals and derivatives with orders that are not whole numbers. Leibnitz and L-Hospital have already wondered what would be a derivative of noninteger order [1]. Although the concept of FC was first developed in pure mathematics and is now considered to be a component of applied mathematics as well. The concept of FC has been used in various real-world problem investigations. FC has attracted attention in the fields of science and engineering, such as signal processing. Researchers have used tools of FC to model nonlinear systems in signal processing [2]. Many scholars have analyzed FC in control theory to design controllers for complex systems that cannot be modeled using classical techniques [3,4]. Awadalla and Yameni [5] have discussed FC in the field of physics to model the behavior of materials and systems that exhibit anomalous diffusion, such as porous media and biological tissues. Kumar *et al.* [6] have studied FC for designing of complex systems such as memristive and memcapacitive and nonlinear systems such as robotics and power systems. Some mathematicians have studied the behavior of stock prices, interest rates, the dynamics of market fluctuations and risk management using FC concepts [7]. Most researchers have investigated fractional derivatives to extract features and information from image contents such as edges, corners, and textures [8]. In the field of biology, scholars have examined FC to model the behavior of biological systems such as the cardiovascular system and neural networks. Researchers have presented a more comprehensive overview of the dynamics of these systems and help in the development of treatments and therapies [9–11]. Many researchers have applied tools of FC in chemistry to model the diffusion of molecules and chemical reactions in porous materials. In addition, they presented a more detailed description of the transport and reaction phenomena, which help in the design of catalysts and porous materials for chemical applications [9].

Khan *et al.* [13] introduced a ground-breaking concept regarding differential and integral operators, referred to as fractal fractional differential and integral operators. The aforesaid operators contain the traditional fractional order operators as special cases. Hence, the mentioned operators characterized by two parameters: first, the fractional order denoted as δ, followed by the fractal dimension denoted as κ. The motivation behind these novel operators lies in their capacity to address nonlocal phenomena in natural systems that also exhibit fractal behavior, as evidenced in previous works [14–16]. Numerous authors have studied these operators and used them in various fields. Qureshi and Atangana [17] have used fractal-fractional derivatives (FFDs) to model and analytically analyze the fluctuations in diarrheal transmission that occurred in Ghana during the period from 2008 to 2018. Likewise, Srivastava and Saad [18] have conducted a similar study. They used FFD to give the mathematical form to Ebola virus disease. In addition, the mentioned operators have potential applications in all other fields. Overall, the concept of fractal fractional provides a powerful tool for understanding and modeling complex systems that exhibit fractal patterns or properties. One of the most essential aspects of describing nonlinear physical phenomena is finding exact solutions for fractal fractional differential equations. The theory of derivatives and integrals of fractal fractional order can be used to successfully solve various physical phenomena [12].

In recent times, numerous researchers have directed their focus toward utilizing FFD for modeling real-world phenomena. The concept of FFD has significant applications in modeling and studying the dynamics of mentioned phenomena. Among the issues that have been recently investigated, cholera stands out prominently. Cholera, an infection causing gastroenteritis is acquired when an individual ingests an infectious dose or inoculum of the pathogenic *Vibrio cholera*. The transmission of cholera occurs through two main routes. Both the primary and secondary source in which people ingest contaminated food or water containing pathogenic virion that has come from an infected person, commonly known as person-to-person contact–involve people consuming the pathogen through contaminated seafood and water [19].

Formulations of real-world process in terms of mathematical models play a significant role. With the help of models, we can understand and predict the transmission of infectious diseases [20,21]. Cholera is also one of the major diseases due to which thousands of people lose their lives worldwide each year. Researchers have investigated the said disease *via* mathematical models involving classical differential equations extensively. Also, some researchers have used FC for investigations of various infectious diseases models [22,23]. The use of mathematical models has significantly contributed to our understanding of the dynamics of cholera epidemics and the effectiveness of control measures. Hailemariam Hntsa and Nerea Kahsay [25] have modeled the cholera transmission dynamic *via* mathematical formulation as follows:

$$\begin{cases} \frac{dS}{dt} = bN + (1-p)\zeta R - \left[(1-p)\varrho\frac{B}{B+K} + p\alpha + \rho\right]S, \\ \frac{dI}{dt} = (1-p)\varrho\frac{BS}{B+K} - (\gamma + d + \rho)I, \\ \frac{dR}{dt} = \gamma I - (\zeta + \rho)R, \\ \frac{dU}{dt} = p\alpha S + p\zeta R - \rho U, \\ \frac{dB}{dt} = \phi I - \phi\theta I - b(\rho_B - b_B)B, \end{cases} \tag{1}$$

where $\rho = \rho_i + \rho_d N$ and $S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, R(0) = R_0 \geq 0, U(0) = U_0 \geq 0$, and $B(0) = B_0 \geq 0$. At a given time t, the population denoted by $N(t)$ is categorized based on their infection status: $S(t)$ represents susceptible individuals, $I(t)$ denotes infected individuals, $R(t)$ signifies recovered individuals, and $U(t)$ refers to prevented individuals. In addition, $B(t)$ quantifies the concentration of *Vibrio cholera* in the aquatic environment at time t. In addition, the nomenclatures of the model is provided in Table 1.

The authors have discussed the global and local stability analysis and also discussed boundedness and approximate solutions for various compartments using traditional derivative. Since the aforesaid model has not yet investigated *via* fractal FC to understand the complex geometry of the mentioned dynamical system. Since fractional differential operators are categorized in subbranches of local and nonlocal kernels. Those operators that involve power law kernels are called singular kernels. Moreover, those operators involve exponential and Mittag-Leffler kernels are called nonsingular kernels. Both kinds of operators have their own merits and demerits. Here, we remark that in general fractional differential operators are nonlocal because they involve integrals. On the other hand, time fractional derivatives have memory effects because these operators include information about the function at prior times. The mentioned operators take into account history and nonlocal dispersed effects, which are necessary for a more precise and accurate description of and understanding of the behavior of complex dynamical systems. A proper geometrical interpretation still do not exist for the aforesaid operators. Therefore, various definitions have been defined by researchers in which we cannot differentiate which one be the most notable and best. Due to this fact, researchers are continuously investigating the area for selecting the most better (we refer to the study by Rosales *et al.* [26]). In this regards, keeping in mind the

Table 1: Parameters for a system of (1)

Parameters	Parameters definition
b	A consistent rate of new individuals joining
ρ_i	Mortality rate of an individual unaffected by population density
ρ_d	Density-dependent death rate of an individual
d	Mortality rate of an individual resulting from a disease
ϱ	Individuals ingestion rates of *Vibrio cholera* from polluted water
γ	Ratio of infected class to the recovered class
α	Rate at which susceptible individuals transition from the susceptible class to the protected class as a result of applying a preventive method
ζ	The speed at which individuals who have recovered from an illness gradually lose their immunity
ϕ	The rate of individual infected from *Vibrio cholera* pathogens
K	The level of *Vibrio cholera* concentration in food and water
p	The proportion of hygienic compliance, ingestion of cholera bacterium
θ	The proportion of compliance with sanitation measures within the infected group
b_B	Rate at which new instances of *Vibrio cholera* are generated
ρ_B	Rate at which instances of *Vibrio cholera* are eliminated

importance of fractals fractional concept, researchers have used the said area to investigate epidemiological problems for more sophisticated analysis. To the best of our information, model (1) was studied under the classical derivatives for global and local stability analysis. But to understand the complex geometry of the adynamic of the aforesaid model, still the problem has not been investigated by using the concept of fractals FC. Therefore, keeping in mind the importance of non singular nonlocal FFD with exponential kernel, we investigate model (1) as follows:

$$\begin{cases} {}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}S(t) = \left\{bN + (1-p)\zeta R - \left[(1-p)\varrho\dfrac{B}{B+K} + p\alpha + \rho\right]S\right\}, \\ {}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}I(t) = \left\{(1-p)\varrho\dfrac{BS}{B+K} - (\gamma + d + \rho)I\right\}, \\ {}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}R(t) = \{\gamma I - (\zeta + \rho)R\}, \\ {}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}U(t) = \{p\alpha S + p\zeta R - \rho U\}, \\ {}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}B(t) = \{(1-\theta)\phi I + b(b_B - \rho_B)B\}, \end{cases} \tag{2}$$

where ${}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}$ stands for Caputo-Fabrizio fractal fractional derivative (CFFFD), δ is fractional order, κ is the fractal dimension, and $0 < \delta \le 1$, $0 < \kappa \le 1$. In addition, if we put $\delta = \kappa = 1$ in model (2), we obtain the traditional model (1). Hence, the model studied in (1) is a special case of our proposed model (2). We establish the existence theory and numerical results for the aforementioned model using some fixed point theorems [31]. In addition, in case of numerical analysis, we apply the method used already in the study by Khan and Atangana [32] for other problems. Several graphical presentations and CPU time for various fractals fractional orders are tabulated. In addition, this is remarkable that using exponential kernel instead of power law kernel makes the process easy for the theoretical analysis and numerical calculations in investigation of many practical applications.

Our article is organized as follows: Introduction is given in Section 1. In Section 2, we give some basic results. The existence theory is given in Section 3. The numerical scheme is developed in Section 4. The numerical simulations are given in Section 5. Section 6 is devoted to conclusion.

2 Background results

Here, we recollect some definitions and theorems that we use in our analysis in this article. If $0 \le t \le T < \infty$, and $\mathcal{Z} = C[0, T] = C(\mathscr{I})$ be the Banach space with norm $||G||_\infty = \max_{t\in\mathscr{I}}|G(t)|$.

Definition 2.1. [31] Suppose that Υ be continuous and differentiable both fractionally and in fractal sense on $(0, b)$, then we define CFFFD as follows:

$${}^{\mathrm{CFFFD}}{}_0\mathbf{D}_t^{\delta,\kappa}(\Upsilon(t)) = \frac{M(\delta)}{1-\delta}\frac{\mathrm{d}}{\mathrm{d}t^\kappa}\int_0^t \Upsilon(\eta)\exp\left(\frac{-\delta(t-\eta)}{1-\delta}\right)\mathrm{d}\eta,$$

where $0 < \delta, \kappa \le 1$, and $M(0) = M(1) = 1$.

Definition 2.2. [31] If Υ is continuous function on $(0, b)$, then fractals fractional integral (FFI) is given by

$$ {}^{\mathrm{FFI}}_{0}\mathbf{I}_t^{\delta,\kappa}(\Upsilon(t)) = \frac{\delta\kappa}{M(\delta)}\int_0^t \eta^{\delta-1}\Upsilon(\eta)\mathrm{d}\eta + \frac{\kappa(1-\delta)t^{\kappa-1}\Upsilon(t)}{M(\delta)}. $$

Lemma 2.2.1. [32] *If* Υ *be continuous and differentiable both fractionally and in fractal sense on* $(0, b)$, *and* $g \in L[0, b]$, *such that* g *vanishes when* $t \to 0$, *then the solution of*

$$ {}^{\mathrm{CFFFD}}_{0}\mathbf{D}_t^{\delta,\kappa}(\Upsilon(t)) = g(t), \quad \text{with } \Upsilon(0) = \Upsilon_0 $$

is given by

$$ \Upsilon(t) = \Upsilon_0 + \frac{\delta\kappa}{M(\delta)}\int_0^t \eta^{\delta-1}g(\eta)\mathrm{d}\eta + \frac{\kappa(1-\delta)t^{\kappa-1}g(t)}{M(\delta)}. $$

Theorem 2.3. [32] *If* $\mathbb{Q}_1$ *and* $\mathbb{Q}_2$ *be two operators such that the first one is contraction and the second one is completely continuous over a closed bounded subset* $\mathbf{B}$ *of a Banach space* $\mathcal{Z}$, *then the operator equation* $\mathbb{Q}G + \mathbb{Q}_2G = G$ *has at least one solution.*

3 Existence and uniqueness of solution

This section is devoted to the main results of the article. We use Banach and Krasnoselskii's fixed point theorem [31] to elaborate the required theory of existence of solution. To proceed further, we can write the proposed model (2) in the sense of Caputo–Fabrizio fractional (CF) differential equations form as follows:

$$ \begin{cases} {}^{CF}_{0}\mathbf{D}_t^{\delta}S(t) = \kappa t^{\kappa-1}\Phi_1(S, I, R, U, B, t), \\ {}^{CF}_{0}\mathbf{D}_t^{\delta}I(t) = \kappa t^{\kappa-1}\Phi_2(S, I, R, U, B, t), \\ {}^{CF}_{0}\mathbf{D}_t^{\delta}R(t) = \kappa t^{\kappa-1}\Phi_3(S, I, R, U, B, t), \\ {}^{CF}_{0}\mathbf{D}_t^{\delta}U(t) = \kappa t^{\kappa-1}\Phi_4(S, I, R, U, B, t), \\ {}^{CF}_{0}\mathbf{D}_t^{\delta}B(t) = \kappa t^{\kappa-1}\Phi_5(S, I, R, U, B, t), \\ S(0) = S_0, I(0) = I_0, R(0) = R_0, U(0) = U_0, B(0) = B_0, \end{cases} \tag{3} $$

where the right-hand sides of proposed model (2) can be written as follows:

$$ \begin{cases} \Phi_1(S, I, R, U, B, t) = bN + (1-p)\zeta R - \left((1-p)\varrho\frac{B}{B+K} + p\alpha \right.\\ \qquad\qquad \left. + \rho\right)S, \\ \Phi_2(S, I, R, U, B, t) = (1-p)\varrho\frac{BS}{B+K} - (\gamma + d + \rho)I, \\ \Phi_3(S, I, R, U, B, t) = \gamma I - (\zeta + \rho)R, \\ \Phi_4(S, I, R, U, B, t) = p\alpha S + p\zeta R - \rho U, \\ \Phi_5(S, I, R, U, B, t) = (1-\theta)\phi I + b(b_B - \rho_B)B. \end{cases} $$

One of the alternative forms of (3) by using $G = (S, I, R, U, B)$ and $G_0 = (S_0, I_0, R_0, U_0, B_0)$ to develop the existence theory can be obtained by considering the following

$$ {}^{CF}_{0}\mathbf{D}_t^{\delta}G(t) = \kappa t^{\kappa-1}F(t, G(t)), \quad G(0) = G_0. \tag{4} $$

Equivalently, we can write the integral form of (4) as follows:

$$ \begin{aligned} G(t) = G_0 &+ \frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}F(t, G(t)) \\ &+ \frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}F(\xi, G(\xi))\mathrm{d}\xi, \end{aligned} \tag{5} $$

where $G(t)$ is given as follows:

$$ G(t) = \begin{bmatrix} S(t) \\ I(t) \\ R(t) \\ U(t) \\ B(t) \end{bmatrix}, \quad F(t, G(t)) = \begin{bmatrix} \Phi_1(S, I, R, U, B, t) \\ \Phi_2(S, I, R, U, B, t) \\ \Phi_3(S, I, R, U, B, t) \\ \Phi_4(S, I, R, U, B, t) \\ \Phi_5(S, I, R, U, B, t) \end{bmatrix}. $$

The following hypothesis hold.

(D1) For $G, \bar{G} \in \mathcal{Z}$, one has $\mathscr{K}_F > 0$, such that

$$ |F(t, G(t)) - F(t, \bar{G}(t))| \le \mathscr{K}_F|G(t) - \bar{G}(t)|. $$

(D2) For real values $\mathscr{M}_0, \mathscr{M}_1 > 0$, one has

$$ |F(t, G(t))| \le \mathscr{M}_0 + \mathscr{M}_1|G(t)|. $$

Let define the operator by

$$ \begin{aligned} \mathbb{P}G(t) = G_0 &+ \frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}F(t, G(t)) \\ &+ \frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}F(\xi, G(\xi))\mathrm{d}\xi. \end{aligned} \tag{6} $$

Theorem 3.1. *Under the assumption* (D_1) *and if* $\frac{T^{\kappa-1}(\kappa+\delta T)\mathscr{K}_F}{M(\delta)} < 1$ *holds, then problem* (4) *has a unique solution.*

Proof. Taking $G, \bar{G} \in \mathcal{Z}$, we have from (6)

$$\begin{aligned}\|\mathbb{P}G(t) - \bar{G}(t)\|_\infty &= \max_{t\in\mathscr{I}}\left|\frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}[F(t,G(t)) - F(t,\bar{G}(t))] + \frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}[F(\xi,G(\xi)) - F(\xi,\bar{G}(\xi))]\mathrm{d}\xi\right| \\ &\le \frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}\mathscr{K}_F\|G-\bar{G}\|_\infty + \frac{\delta\kappa}{M(\delta)}\int_0^T \xi^{\kappa-1}\mathscr{K}_F\|G-\bar{G}\|_\infty\mathrm{d}\xi \\ &\le \left[\frac{\kappa T^{\kappa-1}(1-\delta)}{M(\delta)}\mathscr{K}_F + \frac{\delta T^{\kappa}}{M(\delta)}\mathscr{K}_F\right]\|G-\bar{G}\|_\infty \\ &\le \frac{T^{\kappa-1}(\kappa+\delta T)\mathscr{K}_F}{M(\delta)}\|G-\bar{G}\|_\infty.\end{aligned} \tag{7}$$

Therefore, one concludes that $\mathbb{P}$ fulfills the criteria of Banach contraction. Hence, $\mathbb{P}$ has a unique fixed point. Consequently, we can claim that the considered model (2) has a unique solution. □

Theorem 3.2. *If assumptions* (D_1, D_2)*, and the condition* $\frac{\kappa T^{\kappa-1}}{M(\delta)}\mathscr{K}_F < 1$ *hold, then the problem* (4) *has at least one fixed point. From which we conclude that the proposed model* (2) *has at least one solution.*

Proof. From (5), we define two operators $\mathbb{Q}_1$ and $\mathbb{Q}_2$ as follows:

$$\mathbb{Q}_1[G(t)] = G_0 + \frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}F(t,G(t)) \tag{8}$$

and

$$\mathbb{Q}_2[G(t)] = \frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}F(\xi,G(\xi))\mathrm{d}\xi. \tag{9}$$

Considering $G, \bar{G} \in \mathcal{Z}$, and from (10), one has

$$\begin{aligned}\|\mathbb{Q}_1(G) - \mathbb{Q}_1(\bar{G})\|_\infty &= \max_{t\in\mathscr{I}}\left|\frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}[F(t,G(t)) - F(t,\bar{G}(t))]\right| \\ &\le \frac{\kappa T^{\kappa-1}}{M(\delta)}\mathscr{K}_F\|G-\bar{G}\|_\infty.\end{aligned} \tag{10}$$

Thus, $\mathbb{Q}_1$ is contraction. Also, if $\mathbf{B} = \{G \in \mathcal{Z} : \|G\|_\infty \le \mathcal{R}\}$, be closed bounded subset of $\mathcal{Z}$, where $\mathcal{R} \ge \frac{\delta.\mathscr{M}_0 T^\kappa}{M(\delta) - \delta.\mathscr{M}_1 T^\kappa}$, then $\mathbb{Q}_2$ is completely continuous operator over $\mathbf{B}$. Obviously $\mathbb{Q}_2$ is continuous as F is continuous. Also

$$\begin{aligned}\|\mathbb{Q}_2(G)\| &= \max_{t\in\mathscr{I}}\left|\frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}F(\xi,G(\xi))\mathrm{d}\xi\right| \\ &\le \frac{\delta\kappa}{M(\delta)}\max_{t\in\mathscr{I}}\int_0^T |T^{\kappa-1}F(\xi,G(\xi))|\mathrm{d}\xi \\ &\le \frac{\delta\kappa}{M(\delta)}\left[\frac{(\mathscr{M}_0+\mathscr{M}_1\mathcal{R})T^\kappa}{\kappa}\right] \\ &\le \mathcal{R}.\end{aligned}$$

Hence, $\mathbb{Q}_2$ is bounded. Let $t_1 < t_2 \in \mathscr{I}$, then

$$\begin{aligned}&|\mathbb{Q}_2(G(t_2)) - \mathbb{Q}_2(G(t_1))| \\ &= \left|\frac{\delta\kappa}{M(\delta)}\int_0^{t_2}\xi^{\kappa-1}F(\xi,G(\xi))\mathrm{d}\xi - \frac{\delta\kappa}{M(\delta)}\int_0^{t_1}\xi^{\kappa-1}F(\xi,G(\xi))\mathrm{d}\xi\right| + \left|\frac{\delta\kappa}{M(\delta)}\int_{t_1}^{t_2}\xi^{\kappa-1}F(\xi,G(\xi))\mathrm{d}\xi\right| \\ &\le \frac{\delta\kappa}{M(\delta)}[\mathscr{M}_0+\mathscr{M}_1\mathcal{R}]\int_{t_1}^{t_2}\xi^{\kappa-1}\mathrm{d}\xi \\ &\le \frac{\delta\kappa}{M(\delta)}[\mathscr{M}_0+\mathscr{M}_1\mathcal{R}]\frac{t_2^\kappa - t_1^\kappa}{\kappa} \\ &= \frac{\delta[\mathscr{M}_0+\mathscr{M}_1\mathcal{R}]}{M(\delta)}[t_2^\kappa - t_1^\kappa].\end{aligned} \tag{11}$$

With the use of $t_2 \to t_1$ in right-hand side of (11) implies that $|\mathbb{Q}_2(G(t_2)) - \mathbb{Q}_2(G(t_1))| \to 0$. Also boundedness of $\mathbb{Q}_2$ yields that

$$\|\mathbb{Q}_2(G(t_2)) - \mathbb{Q}_2(G(t_1))\|_\infty \to 0, \quad \text{if } t_2 \to t_1.$$

Thus, all conditions of Arzelá–Ascoli theorem hold. Therefore, by using Krasnoselskii's fixed point theorem, problem (4) has at least one fixed point. Consequently, we can claim that the proposed model (4) has at least one solution. □

4 Numerical method

Following the numerical scheme constructed for general system in the study by Khan and Atangana [32], the solution of (4) can be expressed as follows:

$$G(t) = G_0 + \frac{\kappa t^{\kappa-1}(1-\delta)}{M(\delta)}F(t, G(t)) + \frac{\delta\kappa}{M(\delta)}\int_0^t \xi^{\kappa-1}F(\xi, G(\xi))\mathrm{d}\xi,$$

which on using $t = t_{i+1}$ can be expressed as follows:

$$G(t_{i+1}) = G(t_i) + \frac{\kappa t_i^{\kappa-1}(1-\delta)}{M(\delta)}F(t_i, G(t_i)) + \frac{\delta\kappa}{M(\delta)}\int_0^{t_{i+1}} \xi^{\kappa-1}F(\xi, G(\xi))\mathrm{d}\xi. \quad (12)$$

From (4) implies that

$$G(t_{i+1}) = G(t_i) + \frac{\kappa t_i^{\kappa-1}(1-\delta)}{M(\delta)}F(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}(1-\delta)}{M(\delta)}F(t_{i-1}, G(t_{i-1})) + \frac{\delta\kappa}{M(\delta)}\int_0^{t_{i+1}} \xi^{\kappa-1}F(\xi, G(\xi))\mathrm{d}\xi.$$

On simplification of the integral, we have

$$G(t_{i+1}) = G(t_i) + \frac{\kappa t_i^{\kappa-1}(1-\delta)}{M(\delta)}F(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}(1-\delta)}{M(\delta)}F(t_{i-1}, G(t_{i-1})) + \frac{h}{2}\frac{\delta\kappa}{M(\delta)}[3t_i^{\kappa-1}F(t_i, G(t_i)) - t_{i-1}^{\kappa-1}F(t_{i-1}, G(t_{i-1}))]. \quad (13)$$

$$Q_1(\xi) = \frac{\xi - t_{i-1}}{t_i - t_{i-1}}t_n^{\kappa-1}F(t_i, G(t_i)) - \frac{\xi - t_i}{t_i - t_{i-1}}t_{n-1}^{\kappa-1}F(t_{i-1}, G(t_{i-1})). \quad (14)$$

Finally, we obtain the formula for numerical simulation on further simplification by using the interpolation formula (14) and evaluating the integral of (13):

$$G(t_{i+1}) = G(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]F(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]F(t_{i-1}, G(t_{i-1})). \quad (15)$$

Now, in view of formula (15), we deduce the numerical scheme for our proposed model as follows:

$$\begin{aligned}
S(t_{i+1}) &= S(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]\Phi_1(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]\Phi_1(t_{i-1}, G(t_{i-1})),\\
I(t_{i+1}) &= I(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]\Phi_2(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]\Phi_2(t_{i-1}, G(t_{i-1})),\\
R(t_{i+1}) &= R(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]\Phi_3(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]\Phi_3(t_{i-1}, G(t_{i-1})),\\
U(t_{i+1}) &= U(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]\Phi_4(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]\Phi_4(t_{i-1}, G(t_{i-1})),\\
B(t_{i+1}) &= B(t_i) + \frac{\kappa t_i^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{3\delta\Delta t}{2}\right]\Phi_5(t_i, G(t_i)) - \frac{\kappa t_{i-1}^{\kappa-1}}{M(\delta)}\left[1 - \delta + \frac{\delta\Delta t}{2}\right]\Phi_5(t_{i-1}, G(t_{i-1})).
\end{aligned} \quad (16)$$

The proposed numerical method has some advantages, and for instance, it evaluates one extra function per step and produces high-order accuracy. The aforesaid numerical method also called the explicit type numerical scheme. In addition, the Adams–Bashforth method demonstrates excellent computational efficiency in low-dimensional systems simulation. Recently, some researchers have confirmed experimentally

Table 2: Parameters values taken from [25]

Parameters	Values	Parameters	Values
b	0.00082	ρ_i	4.21×10^{-5}
ρ_d	3.245×10^{-8}	d	0.01
ϱ	1	γ	0.2
α	0.1	ζ	0.01
ϕ	10	K	10^6
p	0.7	θ	0.8
b_B–ρ_B	−0.33		

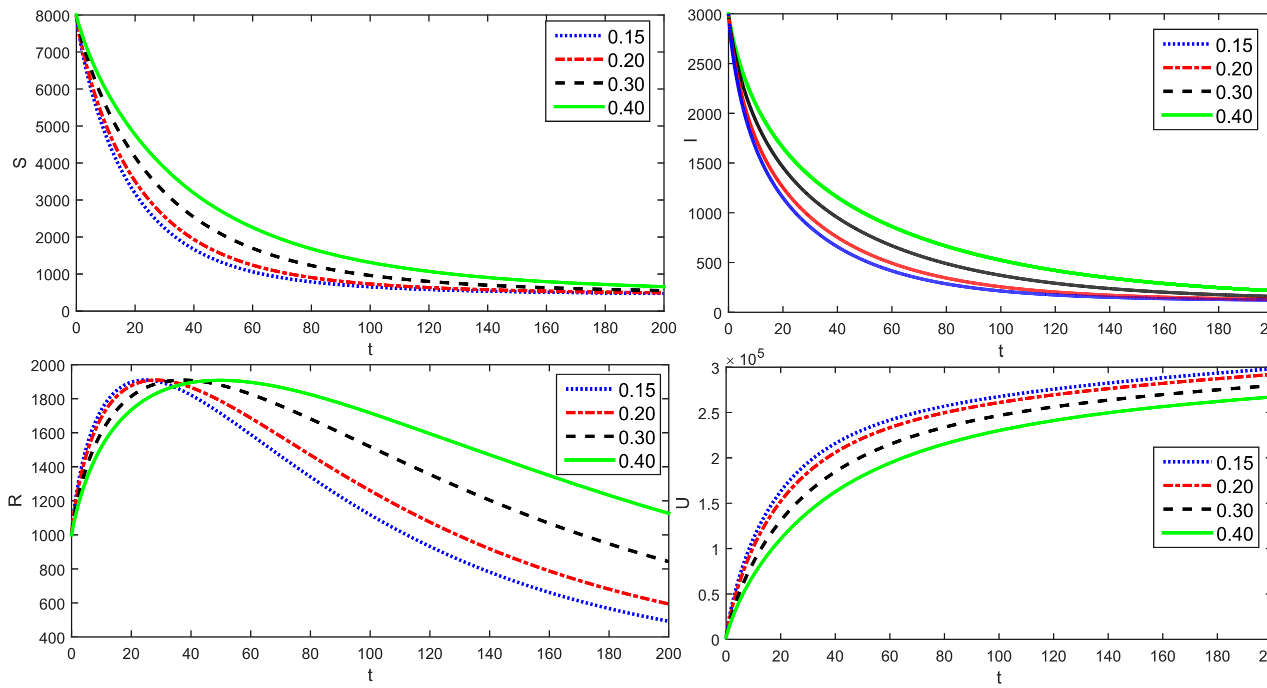

Figure 1: Presentation of numerical solution of S, I, R, U for different values of δ and $\kappa = 0.82$.

and theoretically that the aforesaid numerical method poses better numerical stability as compared to original predictor–corrector numerical method [33].

5 Numerical simulations

In the preceding section, we apply the previous section numerical scheme and use the parameters values given in Table 2. Moreover, taking $S(0) = 8{,}000$, $I(0) = 3{,}000$, $R(0) = 1{,}000$, $U(0) = 4{,}000$, and $B(0) = 25{,}000$ as initial data from [25].

The solution are presented graphically in Figures 1–6 using various values of fractals and fractional orders.

In Figures 1–6, we have presented the approximate solution for the proposed model using distinct values of fractals-fractional orders. The concerned dynamics have been demonstrated for very small as for large values

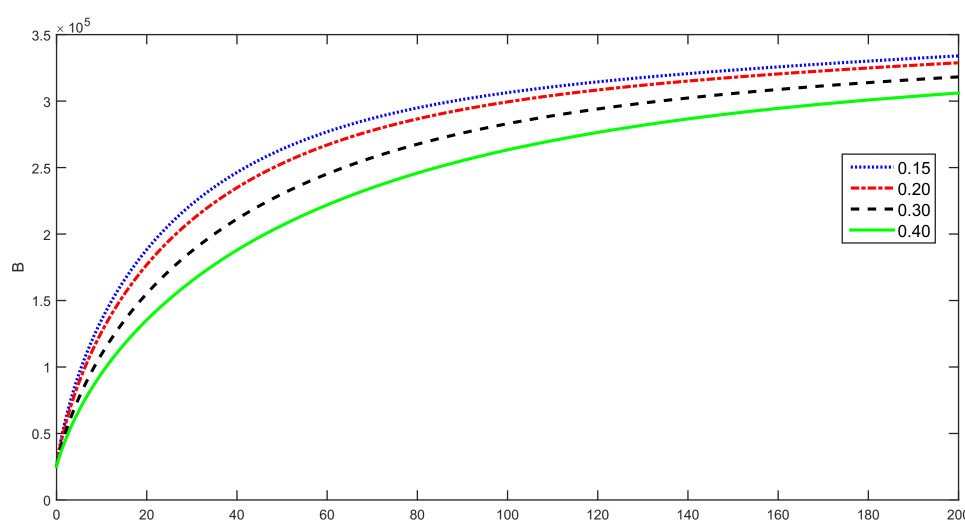

Figure 2: Presentation of numerical solution of B for different values of δ and $\kappa = 0.82$.

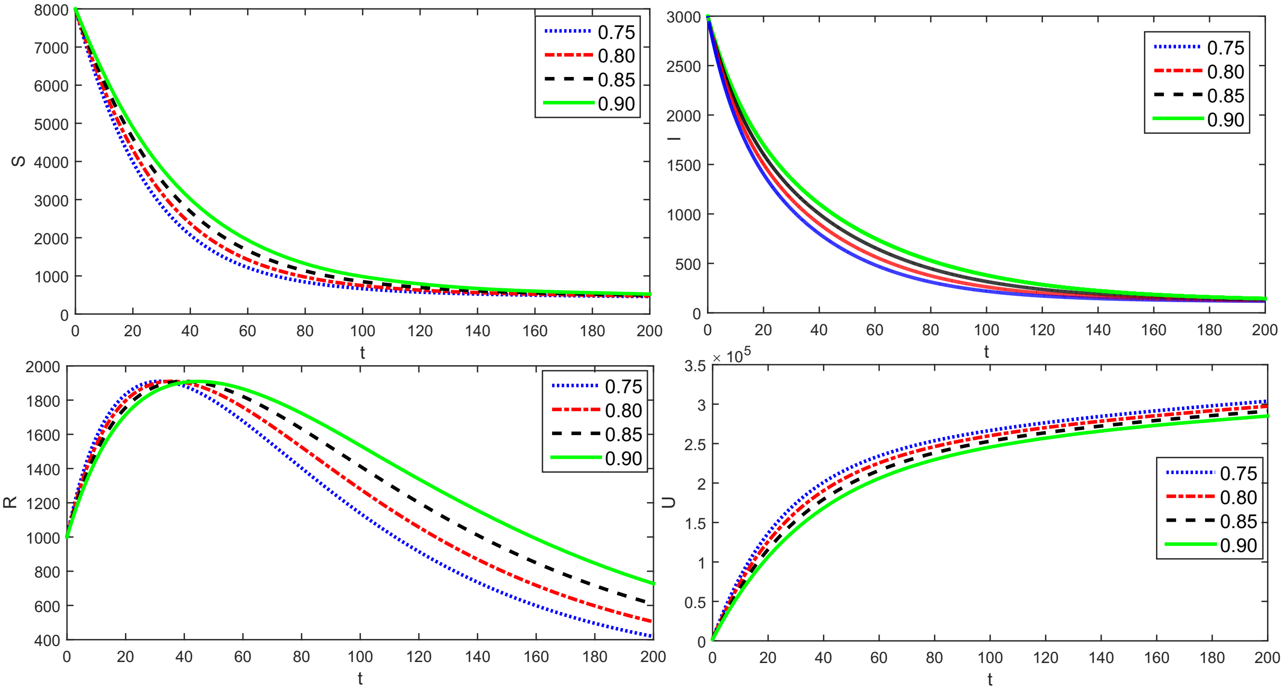

Figure 3: Presentation of numerical solution of S, I, R, U for different values of δ and $\kappa = 0.99$.

of fractals order different fractional orders. The fractals orders have a significant impact on the dynamics of different classes. In the same way, smaller fractional order derivatives play significant roles in the decay process as with the mentioned the process become faster than greater orders. Moreover, here in Figures 7–10, we simulate the results for the proposed model using various fractals fractional orders. Here, one thing we can see that when $\kappa \to 1$ and $\delta \to 1$, the convergence in curves of solution is obtained.

Here, in Table 3, we compute the CPU time for different fractals fractional orders of various compartments. The time computational cost is much more small although the system is nonlinear. This indicates the numerical efficiency of the proposed method.

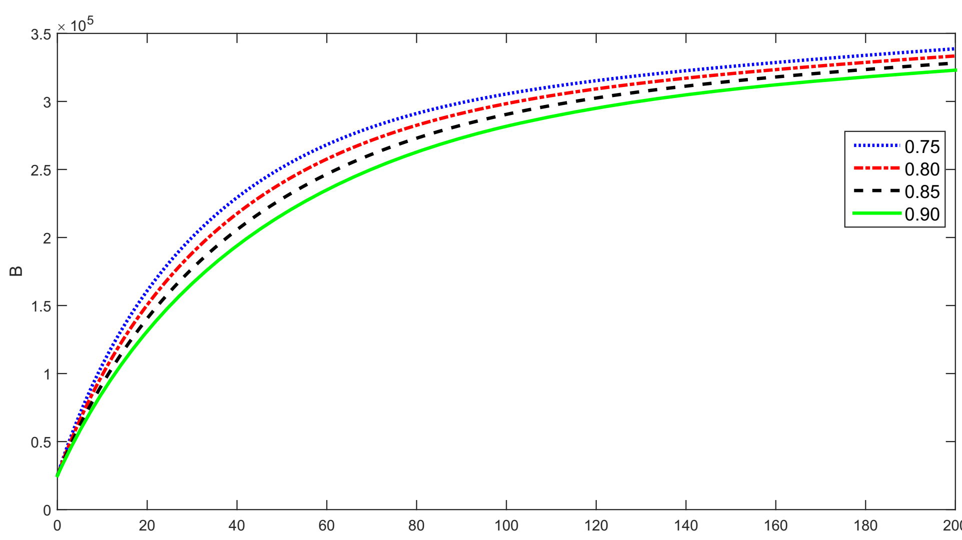

Figure 4: Presentation of numerical solution of B for different values of δ and $\kappa = 0.99$.

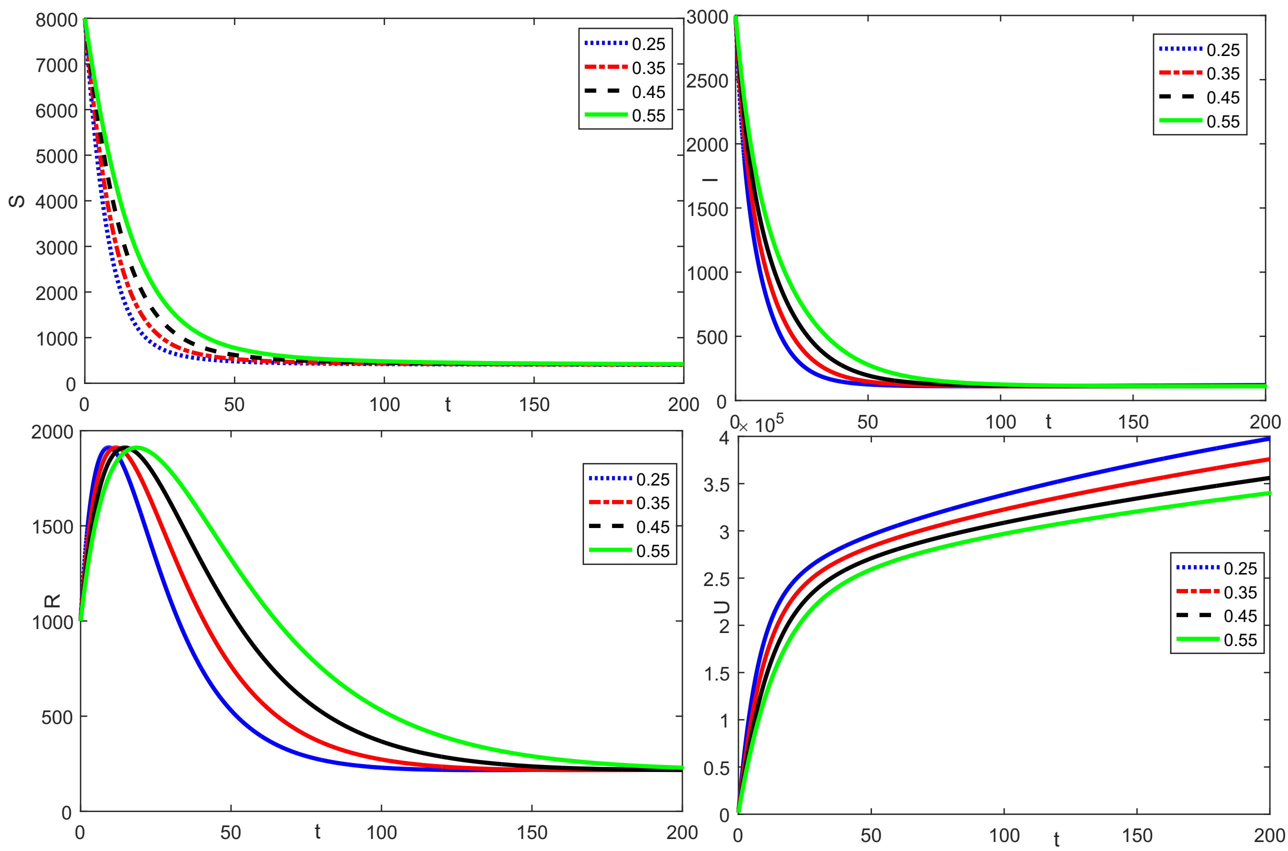

Figure 5: Presentation of numerical solutions of S, I, R, U for different values of δ and $\kappa = 1.0$.

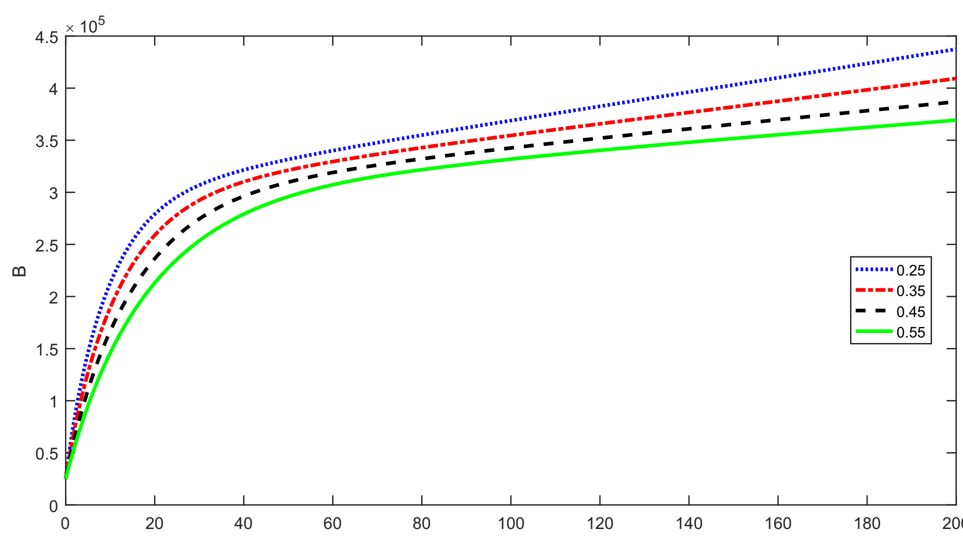

Figure 6: Presentation of numerical solution of B for different values of δ and $\kappa = 1.0$.

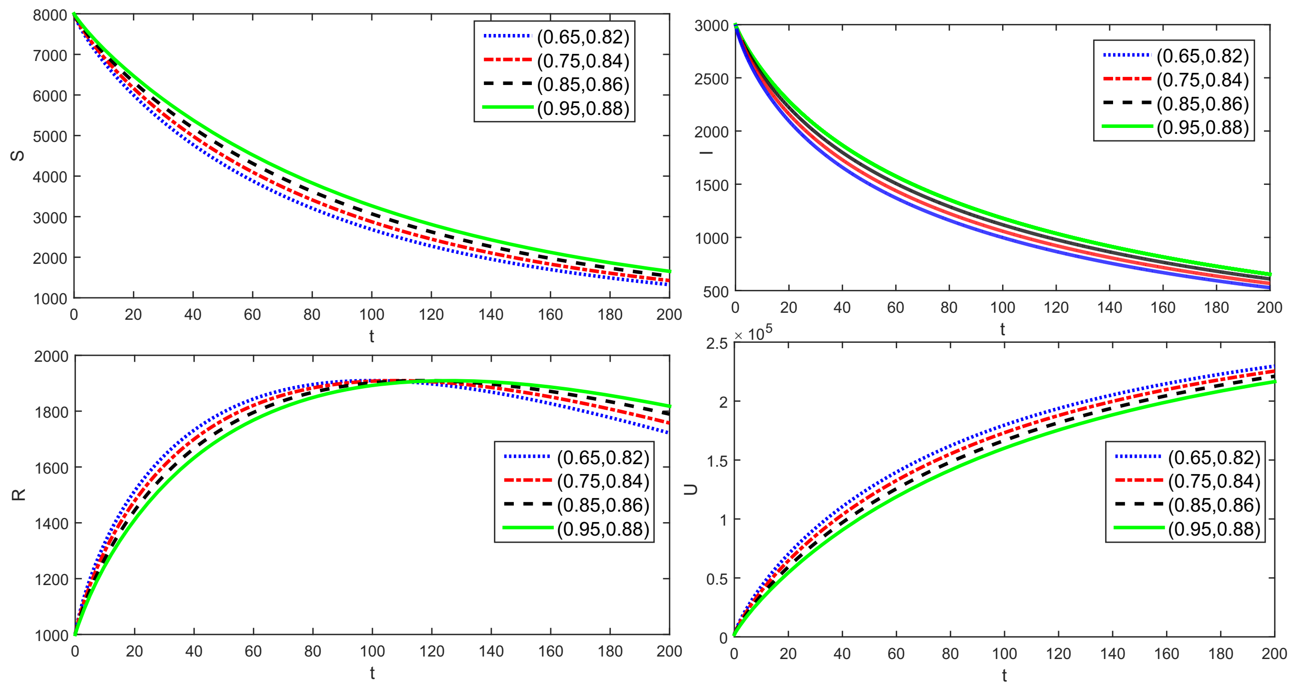

Figure 7: Presentation of numerical solutions of S, I, R, U for different values of δ and κ.

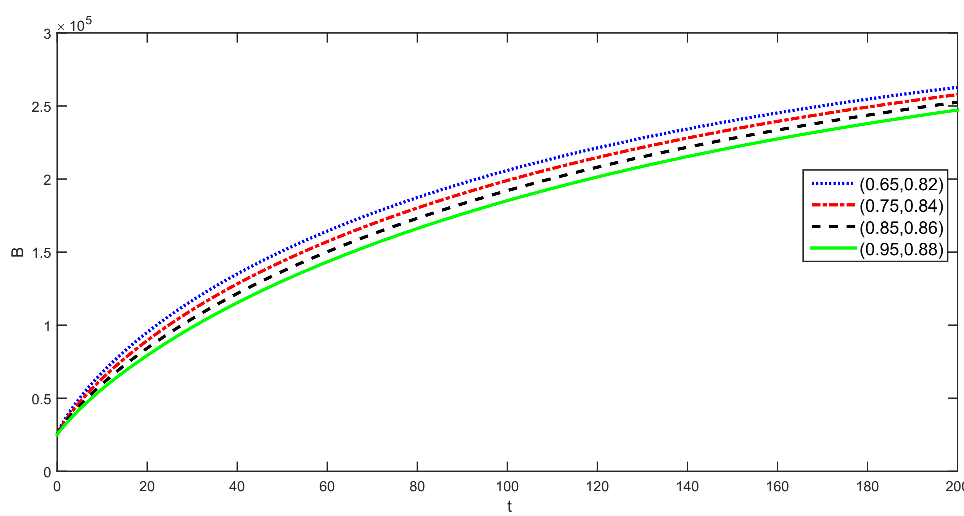

Figure 8: Presentation of numerical solution of B for different values of δ and κ.

6 Conclusion

Mathematical models have been considered powerful tools to investigate various natural and environmental phenomenons from different perspectives. Therefore, epidemiology has been very well considered under the mentioned tools for further explorations and investigations. The bacterial illness cholera is typically transmitted by tainted water that cause dehydration from infected human. If this is not properly treated, then cause death within few hours. By keeping in mind the importance of the aforesaid illness, we have considered a compartmental mathematical model for the aforesaid disease to investigate it from mathematical perspectives. We have used the concept of nonlocal fractals FC concept to elaborate some theoretical and numerical results. By considering the proposed model under the CFFFD, we have deduced necessary and sufficient conditions for the existence theory

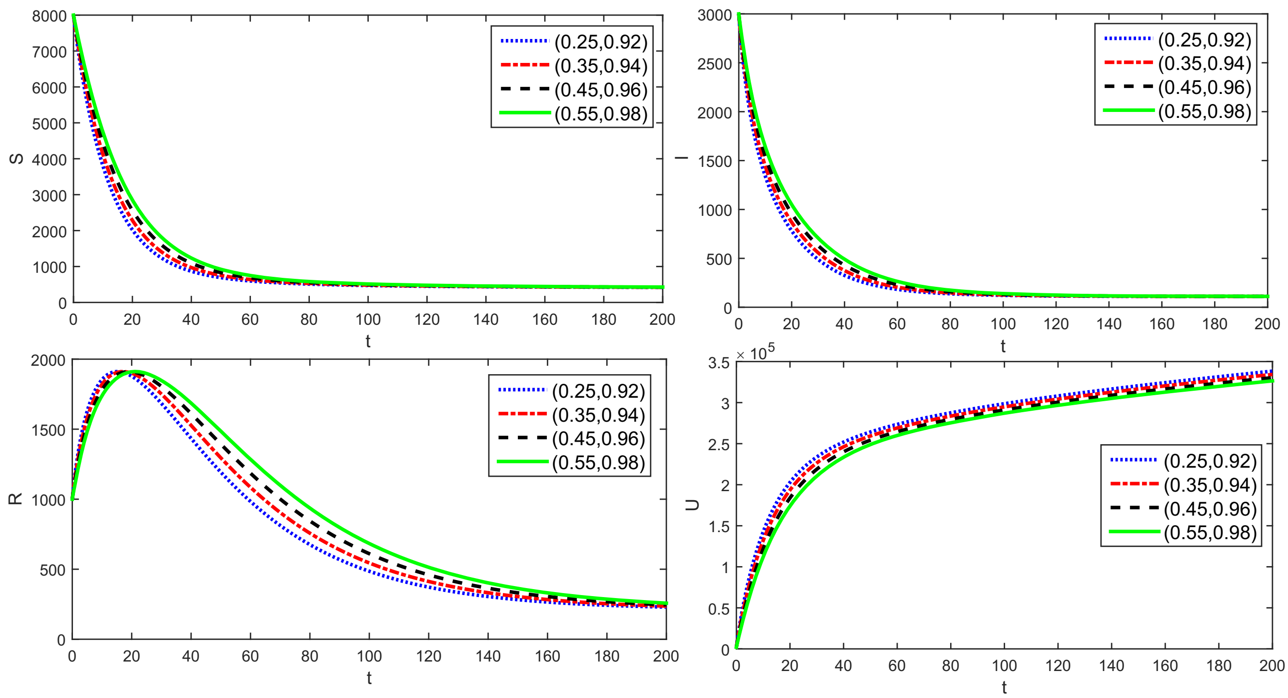

Figure 9: Presentation of numerical solutions of S, I, R, U for different values of δ and κ.

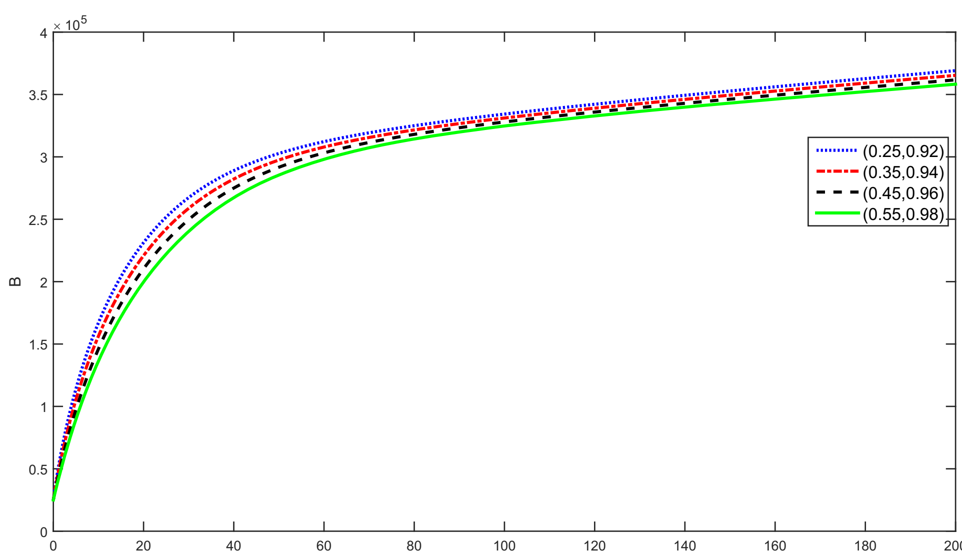

Figure 10: Presentation of numerical solution of B for different values of δ and κ.

of solution using the fixed point theory due to Krasnoselskii and Banach. In addition, for numerical simulation, we have extended the Adam–Bashforth method and constructed a numerical algorithm to present our results graphically. We have presented the numerical results graphically for various fractals and fractional orders. Moreover, the CPU time to record the efficiency of the method has also been computed and tabulated. We observed that the two-step Adams–Bashforth approach has the ability to produce best numerical results for fractals fractional problems. Moreover, the mentioned scheme is also better in cost computation compared to other such type numerical method. On the other hand, FFDs have significance applications in the description of real-world problems. In the future, the concept and methodology we have used can be

Table 3: CPU time time for different fractals fractional orders of various compartments taking $t = 100$

Class	$\delta = 0.25$, $\kappa = 0.05$	$\delta = 0.45$, $\kappa = 0.25$	$\delta = 0.65$, $\kappa = 0.75$	$\delta = 0.95$, $\kappa = 0.99$	$\delta = 1.0$, $\kappa = 1.0$
S	51 s	60 s	90 s	123 s	100 s
I	53 s	62 s	89 s	124 s	99 s
R	55 s	63 s	91 s	122 s	98 s
U	56 s	59 s	88 s	120 s	101 s
B	57 s	58 s	85 s	118 s	102 s

extended to more complex dynamical systems in physical as well as biological sciences.

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