Research Article

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Application of conserved quantities using the formal Lagrangian of a nonlinear integro partial differential equation through optimal system of one-dimensional subalgebras in physics and engineering

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Abstract: This research article analytically investigates a soliton equation of high dimensions, particularly with applications, and precisely in the fields of physical sciences and engineering. The soliton equation of high dimensions, particularly with applications, and precisely in the fields of physical sciences along with engineering, is examined with a view to securing various pertinent results of interest. For the first time, the conserved currents of an integrodifferential equation (especially those of higher dimensions) are calculated using a detailed optimal system of one-dimensional subalgebras. Infinitesimal generators of diverse structures ascribed to Lie point symmetries of the understudy model are first calculated *via* Lie group analysis technique. Additionally, we construct various commutations along Lie-adjoint representation tables connected to the nine-dimensional Lie algebra achieved. Further to that, detailed and comprehensive computation of the optimal system of one-dimensional subalgebras linked to the algebra is also unveiled for the under-investigated model. This, in consequence, engenders the calculation of abundant conserved currents for the soliton equation through Ibragimov's conserved vector theorem by utilizing its formal Lagrangian. Later, the applications of our results are highlighted.

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1 Introduction

Nonlinear equations with dispersive property picture a class of mathematical equations refereed usually to as partial differential equations (PDEs) [1–22]. These PDEs are key in delineating a number of physical models inclusive of waves residing in a shallow water channel, confinement of Bose–Einstein condensate, light propagation in an optical waveguide, and so forth.

Furthermore, it has been observed that these equations simply put are non-solvable explicitly, yet mathematical techniques with analysis, together with computational capabilities, are divulging the means of engendering these models as prognostic tools with a high level of efficiency. This provides a rich phenomenon, for instance, balancing nonlinearity alongside dispersion produces coherent structures like vortices, and solitons – metastable states that are long-lived, found out to be waves localized and propagates with little or no disfigurement. The applicability of such structures can be seen in optical communication, within which solitons are engaged in conveying information. Additionally, these structures also have physical interesting features as a result of their particle-like actions.

In the recent times, investigations have largely been turned to nonlinear partial differential equations (NLNPDEs) as well as exact travelling wave results associated with these NLNPDEs. As a result, diverse complex physical happenstances are depicted *via* these NLNPDEs. A few of these NLNPDEs including the Boussinesq–Burgers-type system recounting shallow water waves and also emerging near ocean beaches and lakes were given attention in this article [1]. Moreover, Adeyemo *et al.* [2] examined another generalized NLNPDEs called advection–diffusion equation with power law nonlinearity in fluid mechanics. This generalized equation characterized buoyancy-propelled plume movement embedded in a medium that is bent on nature. Additionally, the vector bright solitons, alongside their various interaction attributes related to the coupled Fokas–Lenells system [3], was studied in the given reference. The femtosecond optical pulses embedded in a double-refractive optical fiber, modeled into an NLNPDEs, were further investigated. Recently, Adeyemo *et al.* [4] examined a (3+1)-D nonlinear generalized type of potential Yu–Toda–Sasa–Fukuyama model existent in Physics alongside Engineering. Besides, Du *et al.* [5] investigated the modified as well as generalized Zakharov–Kuznetsov model, delineating the ion-acoustic meandering solitary waves resident in a magneto-plasma and possessive of electron–positron–ion observable in the autochthonous universe. This model was utilized in representing dust-magneto-acoustic, and ion-acoustic, together with dust-ion-acoustic waves in the laboratory dusty plasmas. Further to that, a generalized structure of the Korteweg–de Vries–Zakharov–Kuznetsov model was investigated by Khalique and Adeyemo [6]. The dilution of warm isentropic fluid alongside cold static framework species together with hot isothermal, applicable in fluid dynamics was recounted *via* the use of the model. The list continues unending, see more in previous studies [4,7–15].

Sophus Lie (1842–1899) with his quintessential work on Lie Algebras [17–20] which is essentially a unified approach for the treatment of a wide class of differential equations (DEs). With the inspiration of Galois theory, Sophus Lie, a Norwegian mathematician, established symmetry methods and demonstrated that many of the known ad hoc methods of integration of DEs could be obtained in a systematic manner. The approach has evolved into a helpful tool for solving DEs, classifying them, and preserving the solution set.

Furthermore, it has been observed that conservation laws are established and entrenched natural laws that have been studied by many researchers in various scientific fields. Conservation laws that are commonly used in this context include conservation of linear momentum in an isolated system, conservation of electric charge, conservation of energy, conservation of mechanical energy in the absence of dissipative forces, and many others. Conservation laws are deliberated to be basic laws of nature, with extensive application in physics and numerous other fields. Some of the important criteria of conservation laws are as follows [21]:

a) the stability analysis and the global behavior of solutions.
b) the development of numerical methods and provide an essential starting point for finding potential variables and nonlocally related systems.
c) the investigation of integrability and linearization mappings.

Securing soliton solutions to NLNPDEs, as a result of its pertinence, is thus becoming a crucial point of interest and active space of investigation to scientists. Consequently, in a bid to gain the soliton solutions, travelling wave solutions, and other interesting exact solutions to NLNPDEs, sturdy approaches have been developed in the literature by scientists. We have some of them as power series solution method [22], simplest equation method [23], Darboux transformation [24], multiple exponential function method [25], just to mention a few. Others include bifurcation technique [26], Painlevé expansion [27], homotopy perturbation technique [28], tanh–coth approach [29], extended homoclinic test approach [30], Cole–Hopf transformation technique [31], Adomian decomposition approach [32], Bäcklund transformation [33], F-expansion technique [34], rational expansion technique [35], extended simplest equation approach [36], Kudryashov's technique [37], Hirota technique [38], Darboux transformation [39], tanh-function technique [40], $\left(\frac{G'}{G}\right)$-expansion technique [41], sine-Gordon equation expansion technique [42], generalized unified technique [43], exponential function technique [44], and so on. Since the inception of Kadomtsev and Petviashvili's hierarchy of equations a little more than half a century ago, dozens of research papers have emerged, each exploring an aspect of this rich domain of equations, see for example previous studies [45–51].

There have emerged NLNPDEs that have been solved by using already arisen mathematical techniques. Nevertheless, not all the emergent modeled equations are solvable. Examination of analytic explicit outcomes to soliton equations is already in the limelight and as such highly significant with influence in physics of mathematics among others [9–11]. Such soliton model includes the 3D soliton-modeled Jimbo–Miwa-type [52] given as

$$2q_{yt} + 3q_{xx}v_y + 3q_xq_{yx} - 3q_{zz} + q_{xxxy} = 0, \tag{1.1}$$

investigated under the variable-dependent-Cole–Hopf transformation that reads

$$u = 2(\log_e\tau)_x,\, q = 2(\log_e\omega)_x.$$

Additionally, mapping (1.1) and Hirota bilinear relation to each other results in

$$(2D_tD_y + D_x^3D_y - 3D_z^2)\omega\cdot\omega = 0. \tag{1.2}$$

Thus, bilinear differential operators unveiled in (1.2), that is, $D_x, D_y : \Omega\times\Omega\to\Omega$ are computed as

$$D_y^n D_x^m[h(y,x)\cdot s(y,x)] = \left(\frac{\partial}{\partial y}-\frac{\partial}{\partial y'}\right)^n\left(\frac{\partial}{\partial x}-\frac{\partial}{\partial x'}\right)^m \times [h(y,x)s(y',x')]|_{y=y',x=x'}, \tag{1.3}$$

where $m, n \geq 0$, $h, s \in \Omega : C^\infty \to C$ as well as C^∞ standing for differential functions space. Thus, we give Eq. (1.2) regarding ω as

$$\omega\omega_{xxxy} + 2\omega\omega_{ty} - 3\omega\omega_{zz} - 3\omega_{xxy}\omega_x + 3\omega_{xx}\omega_{xy} - 2\omega_t\omega_y - \omega_{xxx}\omega_y + 3\omega_z^2 = 0. \tag{1.4}$$

Furthermore, Asaad [52] engaged the Pfaffian technique to achieve closed-form results to the soliton equation (1.4).

Next, the high dimensions soliton model divulged as

$$3\Psi_{xz} - (2\Psi_t + \Psi_{xxx} - 2\Psi\Psi_x)_y + 2(\Psi_x\partial_x^{-1}\Psi_y)_x = 0, \tag{1.5}$$

where $\Psi = \Psi(t, x, y, z)$, was at first gleaned by Geng [53]. The author went ahead to decompose (1.5) *via* the infuse of bi-dimensional Ablowitz–Kaup–Newell–Segur equations into solvable ordinary differential equations system. Liu and Zhang [54] examined (1.5) and thus gained mixed lump strip results together with solitonic-lump outcomes satisfying the equation through popular Hirota bilinear approach. Besides, investigations on them reveal that solitonic lump outcomes localized rationally in every observed direction within the space. Not only that diverse copious solutions that are periodic to the model (1.5) were secured under three-wave solution approach alongside the Hirota bilinear technique, see also the study of Liu *et al.* [55]. Geng and Ma [56] engaged a non-linearized-Lax-pair-approach in fetching geometric algebraic results of (1.5) explicitly and in Riemann theta function structure. Moreover, the Wronskian approach alongside Hirota technique was utilized to gain N-solitonic-solutions of the equation by the authors. Based on Pfaffian derivative relation, Jian-Ping and Xian-Guo [57] found Grampian determinant solutions to (1.5). Meanwhile, on invoking the bilinear Bäcklund transformation, more viable results of the equation have been explicitly gained by Jian-Ping [58]. Additionally, Wang *et al.* [59] broke down (1.5) into three integrable bi-dimensional models. They achieved this on the basis of a quartet Lax pair condition. These three models are as follows: nonlinear Schrödinger model, Lakshmanan–Porsezian–Daniel model, and complex modified Korteweg–de Vries equation in dimensions that are dissimilar. General rational Nth-order outcome given in a compact structure way was achieved for (1.5) using the Darboux transformation technique together with the limit approach. Not only that, in the study of Wang and Wei [60], the decomposition technique was invoked to secure solution regarding N-anti-dark soliton of the equation under contemplation on a finite background *via* the application of the limit technique and the Darboux transformation. Meanwhile, for the N-anti-dark soliton solution, the asymptotic analysis was furnished. Additionally, the authors proved that elastic-nature collision existed between multiple antidark solitonic outcome.

Hence, in our study, we seek to explicitly examine the optimal solutions assented to the three-dimensional soliton equation (1.5) *via* robust Lie group theoretic technique. Consequently, the integral function emergent in (1.5) is first eliminated *via* the representation $v = \int u_y \mathrm{d}x$. Thus, (1.5) alters the equation system that reads

$$3u_{xz} - 2u_{ty} + 2u_xu_y + 2uu_{xy} + 2vu_{xx} + 2v_xu_x - u_{xxxy} = 0, \tag{1.6a}$$

$$u_y - v_x = 0, \tag{1.6b}$$

denoted as HD-SOLeqn for short. We observed that recently Khalique and Adeyemo [61,62] bought into play Lie-symmetric approach to gain various abundant invariant solutions of system (1.6). In addition, copious solutions in terms of closed-formed travelling waves of the under-study model *via* the systematic polynomial complete discriminant alongside elementary integral approaches. Meanwhile, homotopy formula was employed in computing some conserved quantities of (1.6).

Nonetheless, this study invokes the optimal Lie algebraic systematic approach to compute various vectors *via* a nine-dimensional Lie subalgebras ascribed to (1.6) to gain more extensive conserved quantities of the system with various applications in sciences and engineering. We state for the purpose of emphasis that this research engage a detailed computation of one-dimensional optimal system of Lie subalgebras, obtained from a nine-dimensional Lie algebra, to generate abundant conservation laws to (1.6). Moreover, for the first time, the significance of the associated conserved quantities are highlighted within the fields of physical sciences. All these attest to the fact that the work is novel and original.

Now, we catalog the rest of the research article as follows. Section 2 supplies well-thought-out steps adhered-to in calculating the Lie symmetries ascribed to the soliton equation. Moreover, one parametric transformation groups alongside a Lie-sub-algebraic optimal system is computed for the gained Lie algebra. In addition, Section 3 achieves conserved current calculations for optimal system of Lie subalgebras of solitonic system (1.6) in conjunction with the formal Lagrangian using Ibragimov's conserved quantities theorem after which conclusions are furnished.

2 Lie algebra and optimal system of the HD-SOLeqn (1.6)

Computations of Lie-point symmetries of HD-SOLeqn (1.6) are first done from where optimal systems of sub-algebras are contrived. In consequence, engagement of the gained symmetries is taken into consideration to attain diverse possible conserved quantities ascribed to system (1.6).

2.1 Conspectus of infinitesimal generators of (1.6)

We suppose first that Lie transformation group of infinitesimal generators be explicated in the following format:

$$\begin{aligned} \bar{t} &\approx t + \varepsilon\xi^1(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{z} &\approx z + \varepsilon\xi^4(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{x} &\approx x + \varepsilon\xi^2(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{u} &\approx u + \varepsilon\phi^1(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{y} &\approx y + \varepsilon\xi^3(t,x,y,z,u,v) + O(\varepsilon^2),\\ \bar{v} &\approx v + \varepsilon\phi^2(t,x,y,z,u,v) + O(\varepsilon^2). \end{aligned} \tag{2.1}$$

We now contemplate an infinitesimal dimensional Lie algebra covered by vector fields

$$\begin{aligned} w = \xi^1\frac{\partial}{\partial x} + \xi^2\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \xi^4\frac{\partial}{\partial t}\\ + \phi^1\frac{\partial}{\partial u} + \phi^2\frac{\partial}{\partial v}, \end{aligned} \tag{2.2}$$

where coefficient functions $\xi^1, \xi^2, \xi^3, \xi^4, \phi^1, \phi^2$, depending on t, x, y, z, u, and v. Hence, one examines an appurtenant theorem:

Theorem 2.1. *Suppose vector w is assumed to be the infinitesimal generators ascribed to classical point symmetric group of HD-SOLeqn* (1.6), *where ξ^i, $i = 1, 2, 3, 4$ alongside ϕ^1 and ϕ^2, regarded as smooth functions of variables t, x, y, z, u, and v. Thus, one engenders results which are formatted as*:

$$\begin{aligned} \xi^1 &= \mathbf{c}_1 + \frac{3}{2}(-\mathbf{c}_2 + \mathbf{c}_4)t,\\ \xi^2 &= \frac{1}{2}(-\mathbf{c}_2 + \mathbf{c}_4)x + H^1(z,t) + Q(t),\\ \xi^3 &= \mathbf{c}_2 y + H(z),\\ \xi^4 &= \mathbf{c}_3 + \mathbf{c}_4 z,\\ \phi^1 &= (\mathbf{c}_2 - \mathbf{c}_4)u + \frac{3}{2}H'(z) - Q'(t) - H^1_t(z,t),\\ \phi^2 &= -\frac{1}{2}v(\mathbf{c}_2 + \mathbf{c}_4) + \frac{3}{2}H^1_z(z,t), \end{aligned} \tag{2.3}$$

where constants $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4$ are arbitrary along with functions $H(z)$, $H^1(z,t)$, and $Q(t)$ being observed to be, respectively, depending on their respective arguments.

See the comprehensive proof of Theorem 2.1 in the study of Khalique and Adeyemo [61].

Suppose one assumes in gained solution (2.3) that arbitrary $H^1(z,t) = \mathbf{c}_5 z + \mathbf{c}_6 t + \mathbf{c}_7$, $H(z) = \mathbf{c}_8 z + \mathbf{c}_9$, as well as $Q(t) = 0$, we institute more symmetries of (1.6). Hence, the arrival at the subsequent corollary:

Corollary 2.1. *Lie algebra represented by $\mathfrak{g}$ for infinitesimal symmetries ascribed to soliton system* (1.6) *given in three dimensions consists of vector fields alongside their respective names, viz,*

$$\begin{aligned} w_1 &= \frac{\partial}{\partial t}, \text{ time translation},\\ w_2 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}, \text{ Scaling},\\ w_3 &= \frac{\partial}{\partial z}, \text{ z space translation},\\ w_4 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2z\frac{\partial}{\partial z} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \text{ Scaling},\\ w_5 &= 2z\frac{\partial}{\partial x} + 3\frac{\partial}{\partial v}, \text{ Galilean boost},\\ w_6 &= t\frac{\partial}{\partial x} - \frac{\partial}{\partial u}, \text{ Galilean boost},\\ w_7 &= \frac{\partial}{\partial x}, \text{ x space translation},\\ w_8 &= 2z\frac{\partial}{\partial y} + 3\frac{\partial}{\partial u}, \text{ Galilean boost},\\ w_9 &= \frac{\partial}{\partial y}. \text{ y space translation}. \end{aligned} \tag{2.4}$$

Thus, HD-SOLeqn system (1.6) *admits Lie algebra of nine dimensions whose basis are formatted as* $\{w_1, w_2, w_3, \ldots, w_9\}$.

One observes quickly here that the instituted infinitesimal generators can be explicated as a linearly combined vectors formatted as

$$\begin{aligned} w = a_1w_1 + a_2w_2 + a_3w_3 + a_4w_4 + a_5w_5\\ + a_6w_6 + a_7w_7 + a_8w_8 + a_9w_9. \end{aligned} \tag{2.5}$$

The next phase of the research engenders the Lie transformation-groups connected to Lie-algebra $\mathfrak{g}$.

2.2 One-parameter Lie group transformation of (2.4)

On involving the Lie equations in previous studies [17,63] alongside the related initial conditions in the calculation of one parameteric transformation group ascribed to the gained generators (2.4). Thus, we institute a theorem given shortly:

Theorem 2.2. *Suppose that transformation group* $G^i_\varepsilon(t, x, y, z, u, v),\ i = 1, 2, 3, \dots, 9$ *of one parameter is engendered by infinitesimal generators* $w_1, w_2, w_3 \dots, w_9$ *in* (2.4), *thus, for each generator, we secure accordingly*

$$\begin{aligned}
G^1_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t+\varepsilon, x, y, z, u, v),\\
G^2_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (te^{3\varepsilon}, xe^{\varepsilon}, ye^{-2\varepsilon}, z, ue^{-2\varepsilon}, ve^{\varepsilon}),\\
G^3_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x, y, z+\varepsilon, u, v),\\
G^4_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (te^{3\varepsilon}, xe^{\varepsilon}, y, ze^{2\varepsilon}, ue^{-2\varepsilon}, ve^{-\varepsilon}),\\
G^5_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x+2\varepsilon z, y, z, u, v+3\varepsilon),\\
G^6_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x+\varepsilon t, y, z, u-\varepsilon, v),\\
G^7_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x+\varepsilon, y, z, u, v),\\
G^8_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x, y+2\varepsilon z, z, u+3\varepsilon, v),\\
G^9_\varepsilon &: (\bar t, \bar x, \bar y, \bar z, \bar u, \bar v) \longrightarrow (t, x, y+\varepsilon, z, u, v),
\end{aligned}$$

with $\varepsilon \in \mathbb{R}$ *admired as the parametric group.*

We direct the reader to the designated references with a view to gaining a much better under-standing of the ascribed proof of Theorem 2.2. Now, by extension, the subsequent theorem suffices, that is:

Theorem 2.3. *If* $u = h(t, x, y, z)$ *alongside* $v = g(t, x, y, z)$ *fulfills HD-SOLeqn* (1.6), *so are the functions explicated as*

$$\begin{aligned}
&u_1 = h(t-\varepsilon, x, y, z), v_1 = g(t-\varepsilon, x, y, z),\\
&u_2 = ue^{2\varepsilon}h(te^{-3\varepsilon}, xe^{-\varepsilon}, ye^{2\varepsilon}, z),\\
&v_2 = ve^{-\varepsilon}g(te^{-3\varepsilon}, xe^{-\varepsilon}, ye^{2\varepsilon}, z),\\
&u_3 = h(t, x, y, z-\varepsilon), v_3 = g(t, x, y, z-\varepsilon),\\
&u_4 = ue^{2\varepsilon}h(te^{-3\varepsilon}, xe^{-\varepsilon}, y, ze^{-2\varepsilon}),\\
&v_4 = ve^{\varepsilon}g(te^{-3\varepsilon}, xe^{-\varepsilon}, y, ze^{-2\varepsilon}),\\
&u_5 = h(t, x-2\varepsilon z, y, z),\\
&v_5 = g(t, x-2\varepsilon z, y, z) - 3\varepsilon,\\
&u_6 = h(t, x-\varepsilon t, y, z) + \varepsilon,\\
&v_6 = g(t, x-\varepsilon t, y, z),\\
&u_7 = h(t, x-\varepsilon, y, z),\\
&v_7 = g(t, x-\varepsilon, y, z),\\
&u_8 = h(t, x, y-2\varepsilon z, z) - 3\varepsilon,\\
&v_8 = g(t, x, y-2\varepsilon z, z),\\
&u_9 = h(t, x, y-\varepsilon, z) + \varepsilon,\\
&v_9 = g(t, x, y-\varepsilon, z),
\end{aligned}$$

with $u^i(t, x, y, z) = G^i_\varepsilon \cdot h(t, x, y, z),\ \forall i = 1, 2, \dots, 9,$ *where* $\varepsilon \gg 1$ *is conjectured as any real-positive number.*

2.3 Optimal system of Lie subalgebras of (1.6)

This section tends to utilize the accessible chance of a symmetry group in the computation of one-dimensional Lie optimal system ascribed to Lie subalgebra [17,64] concerning (1.6). Therefore, deciding on subgroups of a symmetry group allows one to gain various kinds of solutions with a well-standardized approach which is stressed. The piece of work involved in sub-algebraic classification that is single-dimensional is a connective factor compared to the signalized one with orbit codification of the Lie adjoint-representation [17]. Now, an subalgebraic optimal set is achieved through the choice of single representative of any given equivalent subalgebraic class. Additionally, it is possible to decipher the involved issue in orbit classification *via* the engagement of a Lie algebra general member. Thereafter, simplification is done *via* diverse transformations of adjoint. So, grounded on a well-known algorithm outlined by Hu *et al.* [64], we institute an optimal system of one dimension for HD-SOLeqn (1.6). Now, we seek first the principal invariant function associated with (1.6), next, we calculate the transformation matrix and on the final analysis, classification of the Lie algebraic finite dimension [17,64] is computed for (1.6) under consideration.

2.3.1 Principal invariants of System (1.6)

In achieving the one dimensional sub-algebra optimal system of Lie algebra $\mathfrak{g}$ associated with $\mathbb{R}^9$, it is obligatory to construct the basic invariant in order to aid the selection of representative elements. The table of commutation relations, explicated in Table 1, unveils various combinations of relations ascribed to Lie brackets concerning $\mathfrak{g}$. The (*i;j*)th entry of Table 1 is then computed *via* Lie bracket occasioned by $[w_i, w_j] = w_iw_j - w_jw_i$.

We observe that Table 1 is skew symmetric having got zero diagonal elements. The real function Φ is taken as the invariant and it satisfies the relation $\Phi(w) = \Phi(Ad_g(w))$ for all $w \in \mathfrak{g}$ alongside any subgroup g. Thus, we let $w = \sum_{i=1}^{9}$ as well as $g = e^v(v = \sum_{j=1}^{9} b_jw_j)$, where for g, we have

$$\begin{aligned}
Ad_{\exp(\varepsilon v)}(w) &= e^{-\varepsilon v}we^{\varepsilon v}\\
&= w - \varepsilon[v, w] + \frac{1}{2!}\varepsilon^2[v, [v, w]] - \dots\\
&= (a_1w_1 + \dots + a_9w_9) - \varepsilon[b_1w_1 \\
&\quad + \dots + b_9w_9, a_1w_1 + \dots + a_9w_9] + O(\varepsilon^2) \\
&= (a_1w_1 + \dots + a_9w_9) - \varepsilon(\Sigma_1w_1 + \dots + \Sigma_9w_9)\\
&\quad + O(\varepsilon^2),
\end{aligned} \tag{2.6}$$

Table 1: Lie algebra commutator table for HD-SOLeqn (1.6)

$[w_i, w_j]$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
w_1	0	$3w_1$	0	$3w_1$	0	w_7	0	0	0
w_2	$-3w_1$	0	0	0	$-w_5$	$2w_6$	$-w_7$	$2w_8$	$2w_9$
w_3	0	0	0	$2w_3$	$2w_7$	0	0	$2w_9$	0
w_4	$-3w_1$	0	$-2w_3$	0	w_5	$2w_6$	$-w_7$	$2w_8$	0
w_5	0	w_5	$-2w_7$	$-w_5$	0	0	0	0	0
w_6	$-w_7$	$-2w_6$	0	$-2w_6$	0	0	0	0	0
w_7	0	w_7	0	w_7	0	0	0	0	0
w_8	0	$-2w_8$	$-2w_9$	$-2w_8$	0	0	0	0	0
w_9	0	$-2w_9$	0	0	0	0	0	0	0

with function $\Sigma_i \equiv \Sigma_i(a_1, \ldots, a_9, b_1, \ldots, b_9)$ secured *via* basic computations with the aid of Table 1. In consequence, we have the values of Σ_i as

$$
\begin{aligned}
&\Sigma_1 = -3a_1b_2 - 3a_1b_4 + 3a_2b_1 + 3a_4b_1,\\
&\Sigma_2 = 0, \ \Sigma_3 = -2a_3b_4 + 2a_4b_3,\\
&\Sigma_4 = 0, \Sigma_5 = a_2b_5 - a_4b_5 - a_5b_2 + a_5b_4,\\
&\Sigma_6 = -2a_2b_6 - 2a_4b_6 + 2a_6b_2 + 2a_6b_4,\\
&\Sigma_7 = -a_1b_6 + a_2b_7 - 2a_3b_5 + a_4b_7 + 2a_5b_3 + a_6b_1\\
&\qquad - a_7b_2 - a_7b_4,\\
&\Sigma_8 = -2a_2b_8 - 2a_4b_8 + 2a_8b_2 + 2a_8b_4,\\
&\Sigma_9 = -2a_2b_9 - 2a_3b_8 + 2a_8b_3 + 2a_9b_2.
\end{aligned} \tag{2.7}
$$

Now, for any b_j, where $1 \leq j \leq 9$, it is required that we have

$$
\begin{aligned}
&\Sigma_1\frac{\partial\Phi}{\partial a_1} + \Sigma_2\frac{\partial\Phi}{\partial a_2} + \Sigma_3\frac{\partial\Phi}{\partial a_3} + \Sigma_4\frac{\partial\Phi}{\partial a_4} + \Sigma_5\frac{\partial\Phi}{\partial a_5}\\
&\quad + \Sigma_6\frac{\partial\Phi}{\partial a_6} + \Sigma_7\frac{\partial\Phi}{\partial a_7} + \Sigma_8\frac{\partial\Phi}{\partial a_8} + \Sigma_9\frac{\partial\Phi}{\partial a_9} = 0.
\end{aligned} \tag{2.8}
$$

Thus, by equating the coefficients of all same powers of β_j in (2.8), one achieves the needed nine DEs with regards to real-valued function invariant $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$ as

$$
\begin{aligned}
&-2a_2\frac{\partial\Phi}{\partial a_9} = 0, \ (a_2 + a_4)\frac{\partial\Phi}{\partial a_7} = 0,\\
&(a_2 - a_4)\frac{\partial\Phi}{\partial a_5} - 2a_3\frac{\partial\Phi}{\partial a_7} = 0\\
&(3a_2 + 3a_4)\frac{\partial\Phi}{\partial a_1} + a_6\frac{\partial\Phi}{\partial a_7} = 0,\\
&(-2a_2 - 2a_4)\frac{\partial\Phi}{\partial a_8} - 2a_3\frac{\partial\Phi}{\partial a_9} = 0,\\
&(-2a_2 - 2a_4)\frac{\partial\Phi}{\partial a_6} - a_1\frac{\partial\Phi}{\partial a_7} = 0,\\
&2a_4\frac{\partial\Phi}{\partial a_8} + 2a_5\frac{\partial\Phi}{\partial a_7} + 2a_8\frac{\partial\Phi}{\partial a_9} = 0,\\
&-3a_1\frac{\partial\Phi}{\partial a_1} - a_5\frac{\partial\Phi}{\partial a_5} + 2a_6\frac{\partial\Phi}{\partial a_6}\\
&\quad - a_7\frac{\partial\Phi}{\partial a_7} + 2a_8\frac{\partial\Phi}{\partial a_8} + 2a_9\frac{\partial\Phi}{\partial a_9} = 0,\\
&-3a_1\frac{\partial\Phi}{\partial a_1} - 2a_3\frac{\partial\Phi}{\partial a_3} + a_5\frac{\partial\Phi}{\partial a_5}\\
&\quad + 2a_6\frac{\partial\Phi}{\partial a_6} - a_7\frac{\partial\Phi}{\partial a_7} + 2a_8\frac{\partial\Phi}{\partial a_8} = 0.
\end{aligned} \tag{2.9}
$$

Table 2: Adjoint representation table of Lie algebra for HD-SOLeqn (1.6)

Ad_g	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9
w_1	w_1	Q_0	w_3	$w_4 - 3\varepsilon_1 w_1$	w_5	Q_{12}	w_7	w_8	w_9
w_2	$e^{3\varepsilon_2}w_1$	w_2	w_3	w_4	$e^{\varepsilon_2}w_5$	$e^{-2\varepsilon_2}w_6$	$e^{\varepsilon_2}w_7$	$e^{-2\varepsilon_2}w_8$	$e^{-2\varepsilon_2}w_9$
w_3	w_1	w_2	w_3	$w_4 - 2\varepsilon_3 w_3$	Q_{10}	w_6	w_7	Q_{11}	w_9
w_4	$e^{3\varepsilon_4}w_1$	w_2	$e^{2\varepsilon_4}w_3$	w_4	$e^{-\varepsilon_4}w_5$	$e^{-2\varepsilon_4}w_6$	$e^{\varepsilon_4}w_7$	$e^{-2\varepsilon_4}w_8$	w_9
w_5	w_1	Q_1	Q_8	$w_4 + \varepsilon_5 w_5$	w_5	w_6	w_7	w_8	w_9
w_6	Q_3	Q_4	w_3	$w_4 + 2\varepsilon_6 w_6$	w_5	w_6	w_7	w_8	w_9
w_7	w_1	Q_5	w_3	$w_4 - \varepsilon_7 w_7$	w_5	w_6	w_7	w_8	w_9
w_8	w_1	Q_6	Q_9	$w_4 + 2\varepsilon_8 w_8$	w_5	w_6	w_7	w_8	w_9
w_9	w_1	Q_7	w_3	w_4	w_5	w_6	w_7	w_8	w_9

$Q_0 = w_2 - 3\varepsilon_1 w_1$, $Q_1 = w_2 - \varepsilon_5 w_5$, $Q_3 = w_1 + \varepsilon_6 w_7$, $Q_4 = w_2 + 2\varepsilon_6 w_6$, $Q_5 = w_2 - \varepsilon_7 w_7$, $Q_6 = w_2 + 2\varepsilon_8 w_8$, $Q_7 = w_2 + 2\varepsilon_9 w_9$, $Q_8 = w_3 + 2\varepsilon_5 w_7$, $Q_9 = w_3 + 2\varepsilon_8 w_9$, $Q_{10} = w_5 - 2\varepsilon_3 w_7$, $Q_{11} = w_8 - 2\varepsilon_3 w_9$, $Q_{12} = w_6 - \varepsilon_1 w_7$.

On solving the system of equations displayed in (2.9), one secures the value of invariant $\Phi(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) = G(a_2, a_4)$, which we call the principal invariant function of Lie algebra $\mathfrak{g}$ associated with $\mathbb{R}^9$. In this case, function G is an arbitrary function depending on a_2 and a_4. In consequence, HD-SOLeqn (1.6) has only two basic invariants representing the killing form as depicted by Olver [17].

2.3.2 Adjoint transformation matrix of (1.6)

Let $\mathfrak{g}$ be the symmetry Lie algebra having basis $\{w_1, w_2, w_3, \dots, w_9\}$ of Section 2.1 and also identifying with $\mathbb{R}^9$ as a vector space imploring the map $w_i \longrightarrow e_i$ with $\{e_1, e_2, e_3, \dots, e_9\}$ regarded as the standard basis of $\mathbb{R}^9$. Suppose $\mathfrak{g} = \exp(\varepsilon_i w_i)$ for real constants ε_i, $i = 1, 2, 3, \dots, 9$, by reconing Table 2 and calculating the exponential of matrices $\varepsilon_i Ad(w_i)$, we gain adjoint matrix representations of $Ad(w_i)$. For instance, from Table 2, whose $(i;j)$th entry is $Ad_{\exp \varepsilon w_i}(w_j)$, we use the relation

$$
\begin{aligned}
Ad_{\exp(\varepsilon_1 w_1)} w &= a_1 Ad_{\exp(\varepsilon_1 w_1)} w_1 + a_2 Ad_{\exp(\varepsilon_1 w_1)} w_2 \\
&\quad + \dots + a_9 Ad_{\exp(\varepsilon_1 w_1)} w_9 \\
&= (a_1 - 3\varepsilon_1 a_2 - 3\varepsilon_1 a_4) w_1 + a_2 w_2 + \dots + a_5 w_5 \\
&\quad - \varepsilon_1 a_6 w_6 + \dots + a_9 w_9, \\
&= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9) \cdot A_1^{\varepsilon} \\
&\quad \cdot (w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, w_9)^{\mathrm{T}},
\end{aligned}
$$

whereby we have A_1^{ε} to be

$$
A_1^{\varepsilon} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\varepsilon_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3\varepsilon_1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -\varepsilon_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

Therefore, by adopting the same process, one secures the other eight adjoint transformation matrices as

$$
A_2^{\varepsilon} = \begin{pmatrix}
e^{3\varepsilon_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{\varepsilon_2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{\varepsilon_2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2}
\end{pmatrix},
$$

$$
A_3^{\varepsilon} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2\varepsilon_3 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -2\varepsilon_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2\varepsilon_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
A_4^{\varepsilon} = \begin{pmatrix}
e^{3\varepsilon_4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & e^{2\varepsilon_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & e^{-\varepsilon_4} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^{\varepsilon_4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_4} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
A_5^{\varepsilon} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & -\varepsilon_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 2\varepsilon_5 & 0 & 0 \\
0 & 0 & 0 & 1 & \varepsilon_5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
A_6^{\varepsilon} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & \varepsilon_6 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 2\varepsilon_6 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 2\varepsilon_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$
A_7^{\varepsilon} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -\varepsilon_7 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -\varepsilon_7 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$$A_8^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2\varepsilon_8 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2\varepsilon_8 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2\varepsilon_8 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$A_9^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2\varepsilon_9 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In consequence, we secure the general adjoint transformation matrix as

$$A^{\varepsilon} = \begin{pmatrix} e^{3\varepsilon_2+3\varepsilon_4} & 0 & 0 & 0 & 0 & 0 & e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_6 & 0 & 0 \\ A_{21}^{\varepsilon} & 1 & 0 & 0 & -\varepsilon_5 & 2\varepsilon_6 & A_{27}^{\varepsilon} & 2\varepsilon_8 & 2\varepsilon_9 \\ 0 & 0 & e^{2\varepsilon_4} & 0 & 0 & 0 & 2e^{2\varepsilon_4}\varepsilon_5 & 0 & 2e^{2\varepsilon_4}\varepsilon_8 \\ A_{41}^{\varepsilon} & 0 & -2e^{2\varepsilon_4}\varepsilon_3 & 1 & \varepsilon_5 & 2\varepsilon_6 & A_{47}^{\varepsilon} & 2\varepsilon_8 & -4e^{2\varepsilon_4}\varepsilon_3\varepsilon_8 \\ 0 & 0 & 0 & 0 & e^{\varepsilon_2-\varepsilon_4} & 0 & -2e^{\varepsilon_2+\varepsilon_4}\varepsilon_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2-2\varepsilon_4} & -e^{\varepsilon_2+\varepsilon_4}\varepsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{\varepsilon_2+\varepsilon_4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2-2\varepsilon_4} & -2e^{-2\varepsilon_2}\varepsilon_3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & e^{-2\varepsilon_2} \end{pmatrix},$$

where $A_{21}^{\varepsilon} = -3\varepsilon_1 e^{3\varepsilon_2+3\varepsilon_4}$, $A_{41}^{\varepsilon} = -3\varepsilon_1 e^{3\varepsilon_2+3\varepsilon_4}$, $A_{27}^{\varepsilon} = -3\varepsilon_1 \varepsilon_6 e^{3\varepsilon_2+3\varepsilon_4} - \varepsilon_7$, $A_{47}^{\varepsilon} = -4\varepsilon_3\varepsilon_5 e^{2\varepsilon_4} - 3\varepsilon_1\varepsilon_6 e^{3\varepsilon_2+3\varepsilon_4} - \varepsilon_7$.

2.3.3 Adjoint transformation equation and Lie subalgebras of (1.6)

Here, the adjoint transformation equation associated with parameters $a_1, a_2, a_3, \ldots, a_9$ is presented as

$$\begin{aligned} &(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3, \tilde{a}_4, \tilde{a}_5, \tilde{a}_6, \tilde{a}_7, \tilde{a}_8, \tilde{a}_9) \\ &\quad = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)A^{\varepsilon}, \end{aligned} \tag{2.10}$$

where A^{ε} is the global adjoint matrix. One can assert here, that $v = \sum_{i=1}^{9}\tilde{a}_i w_i$ as well as $w = \sum_{i=1}^{9} a_i w_i$ coincide under the adjoint action. So, the involved adjoint equations associated with (1.6) with regards to system (2.10) are computed as

$$\begin{aligned} \tilde{a}_1 &= a_1 e^{3\varepsilon_2+3\varepsilon_4} - 3a_2 e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_1 - 3a_4 e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_1, \\ \tilde{a}_2 &= a_2, \\ \tilde{a}_3 &= a_3 e^{2\varepsilon_4} - 2a_4 e^{2\varepsilon_4}\varepsilon_3, \\ \tilde{a}_4 &= a_4, \\ \tilde{a}_5 &= a_5 e^{\varepsilon_2-\varepsilon_4} - a_2\varepsilon_5 + a_4\varepsilon_5, \\ \tilde{a}_6 &= a_6 e^{-2\varepsilon_2-2\varepsilon_4} + 2a_2\varepsilon_6 + 2a_4\varepsilon_6, \\ \tilde{a}_7 &= a_7 e^{\varepsilon_2+\varepsilon_4} - a_6 e^{\varepsilon_2+\varepsilon_4}\varepsilon_1 - 2a_5 e^{\varepsilon_2+\varepsilon_4}\varepsilon_3 + 2a_3 e^{2\varepsilon_4}\varepsilon_5 \\ &\quad + a_1 e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_6 - a_2(3e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_1\varepsilon_6 + \varepsilon_7) \\ &\quad - a_4(4e^{2\varepsilon_4}\varepsilon_3\varepsilon_5 + 3e^{3\varepsilon_2+3\varepsilon_4}\varepsilon_1\varepsilon_6 + \varepsilon_7), \\ \tilde{a}_8 &= a_8 e^{-2\varepsilon_2-2\varepsilon_4} + 2a_2\varepsilon_8 + 2a_4\varepsilon_8, \\ \tilde{a}_9 &= a_9 e^{-2\varepsilon_2} - 2a_8 e^{-2\varepsilon_2}\varepsilon_3 + 2a_3 e^{2\varepsilon_4}\varepsilon_8 \\ &\quad - 4a_4 e^{2\varepsilon_4}\varepsilon_3\varepsilon_8 + 2a_2\varepsilon_9. \end{aligned} \tag{2.11}$$

Remark 2.1. It is noteworthy to state clearly here that the existence of solution of system (2.10) with regards to real constants ε_i, $i = 1, 2, 3, \ldots, 9$ implies that the selected element that occasions the solution is an optimal system and so it is germane.

Next, we begin the optimal system computations proper in the subsequent part of the study by first contemplating the invariants earlier secured on the basis of its sign as presented in the algorithm adopted [64] in two stages as $a_2 = 1$, $a_4 = 1$, and $a_2 a_4 = 0$.

Case 1. $a_2 = 1$, $a_4 = 1$

We choose in this situation, the representative element $v = w_2 + w_4$. On inserting the parametric values $\tilde{a}_i = 0$, $i = 1, 3, 5, \ldots, 9$ alongside $\tilde{a}_i = 1$, $i = 2, 4$ into adjoint system (2.11), we achieve the solution

$$\begin{aligned} &\varepsilon_1 = \frac{1}{6}a_1, \quad \varepsilon_3 = \frac{1}{2}a_3, \quad \varepsilon_6 = -\frac{1}{4}a_6 e^{-2(\varepsilon_2+\varepsilon_4)}, \\ &\varepsilon_7 = \frac{1}{12}a_6 a_1 e^{\varepsilon_2+\varepsilon_4} + \frac{1}{2}a_7 e^{\varepsilon_2+\varepsilon_4}, \\ &\varepsilon_8 = \frac{1}{4}a_8 e^{-2(\varepsilon_2+\varepsilon_4)}, \ \varepsilon_9 = \frac{1}{2}a_8 e^{-2\varepsilon_2}a_3 - \frac{1}{2}a_9 e^{-2\varepsilon_2}. \end{aligned} \tag{2.12}$$

Case 2. $a_2a_4 = 0$

This case occasions three scenarios $a_2 = 0, a_4 \neq 0$; $a_2 \neq 0, a_4 = 0$; and $a_2 = 0, a_4 = 0$. Thus, without loss of generality, we contemplate $a_4 = a_2 = 1$ and investigate those cases shortly.

Case 2.1. $a_2 = 0, a_4 = 1$

We select in this subcase, the representative element $v = w_4$. On invoking the parameters $\tilde{a}_i = 0,\ i = 1, 2, 3, 5, \dots, 9$ together with $\tilde{a}_i = 1,\ i = 4$ into adjoint system (2.11), we gain the result

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{3}a_1,\\
\varepsilon_3 &= \frac{a_9}{2a_8},\\
\varepsilon_4 &= \ln\left(\left| -\frac{a_5}{\varepsilon_5} \right|\right) + \varepsilon_2,\\
\varepsilon_6 &= \frac{a_6\varepsilon_5^2}{2a_5^2e^{4\varepsilon_2}},\\
\varepsilon_7 &= \frac{1}{3\varepsilon_5 a_8}\{e^{2\varepsilon_2}a_5(a_1a_6a_8 + 3a_5a_9 - 3a_7a_8)\},\\
\varepsilon_8 &= \frac{a_8\varepsilon_5^2}{2a_5^2e^{4\varepsilon_2}},
\end{aligned} \tag{2.13}$$

Case 2.2. $a_2 = 1, a_4 = 0$

We choose the representative element $v = w_2$. Substituting the parametric values $\tilde{a}_i = 0,\ i = 1, 3, 4, 5, \dots, 9$ as well as $\tilde{a}_i = 1,\ i = 2$ into adjoint system (2.11), we secure the outcome

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{3}a_1,\\
\varepsilon_4 &= \ln\left(\frac{a_5}{\varepsilon_5}\right) + \varepsilon_2,\\
\varepsilon_6 &= \frac{a_6\varepsilon_5^2}{2a_5^2e^{4\varepsilon_2}},\\
\varepsilon_7 &= \frac{1}{3\varepsilon_5}\{e^{2\varepsilon_2}a_5(a_1a_6 + 6a_5\varepsilon_3 - 3a_7)\},\\
\varepsilon_8 &= -\frac{a_8\varepsilon_5^2}{2a_5^2e^{4\varepsilon_2}},\\
\varepsilon_9 &= a_8\varepsilon_3e^{-2\varepsilon_2} - \frac{1}{2}a_9e^{-2\varepsilon_2}.
\end{aligned} \tag{2.14}$$

Case 2.3. $a_2 = 0, a_4 = 0$

Here, we insert $a_2 = 0$ and $a_4 = 0$ into equation PDE system (2.9), solve the resultant equations, and thereby secure a new invariant with regards to $a_i,\ i = 1, 3, 5, \dots, 9$ as

$$\Phi(a_1, a_3, a_5, a_6, a_7, a_8, a_9) = G\left(\frac{a_3a_5}{\sqrt[3]{a_1}}, a_6\sqrt[3]{a_1^2}, a_8\sqrt[3]{a_1^2}\right). \tag{2.15}$$

In consequence, we treat this invariants accordingly in the succeeding part of this work by contemplating three basic cases where the invariants are 0, −1, and 1 [64].

Case 2.3.1. $\frac{a_3a_5}{\sqrt[3]{a_1}} = 1, a_6\sqrt[3]{a_1^2} = 1, a_8\sqrt[3]{a_1^2} = 0.$

In this situation, we have $a_3^3a_5^3 = a_6^{-3/2}$. Hence, by taking $a_8 = 0, a_1 \neq 0$, precisely $a_1 > 0$, we select the representative element $v = w_3 + w_5 - w_6$. As a result, we implore the parametric values $\tilde{a}_i = 0,\ i = 2, 4, \dots, 9,\ \tilde{a}_i = 1,\ i = 1, 3, 5$ with $a_6 = -1$ in adjoint system (2.11) and so gain the solution

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2}\ln\left(\frac{1}{a_5}\sqrt[3]{\frac{1}{a_1}}\right),\\
\varepsilon_4 &= \ln(a_5) + \frac{1}{2}\ln\left(\frac{1}{a_5}\sqrt[3]{\frac{1}{a_1}}\right),\\
\varepsilon_6 &= \frac{1}{a_1}\left\{2a_1a_5\varepsilon_3\sqrt[3]{\frac{1}{a_1}} - a_1a_7\sqrt[3]{\frac{1}{a_1}} - 2a_1\varepsilon_5 - \varepsilon_1\right\},\\
\varepsilon_8 &= -\frac{1}{2}a_1a_5a_9\sqrt[3]{\left(\frac{1}{a_1}\right)^2}.
\end{aligned} \tag{2.16}$$

Case 2.3.2. $\frac{a_3a_5}{\sqrt[3]{a_1}} = 1, a_6\sqrt[3]{a_1^2} = 0, a_8\sqrt[3]{a_1^2} = 1.$

Now, just as earlier demonstrated, here, we also have $a_3^3a_5^3 = a_8^{-3/2}$ and by assuming that $a_6 = 0, a_1 \neq 0$, we choose the representative element $v = w_3 + w_5 - w_8$. Thus, we insert the parameters $\tilde{a}_i = 0,\ i = 2, 4, 6, 7, 9$ together with $\tilde{a}_i = 1,\ i = 1, 3, 5$, and $a_8 = -1$ into adjoint system (2.11) and obtain the outcome

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2}\ln\left(\frac{1}{a_5}\sqrt[3]{\frac{1}{a_1}}\right),\\
\varepsilon_4 &= \ln(a_5) + \frac{1}{2}\ln\left(\frac{1}{a_5}\sqrt[3]{\frac{1}{a_1}}\right),\\
\varepsilon_6 &= 2a_5\sqrt[3]{\frac{1}{a_1}}\varepsilon_3 - a_7\sqrt[3]{\frac{1}{a_1}} - 2\varepsilon_5,\\
\varepsilon_8 &= -\frac{1}{2}a_5\sqrt[3]{\frac{1}{a_1}}\left[a_1a_9\left(\frac{1}{a_1}\right)^{1/3} + 2\varepsilon_3\right].
\end{aligned} \tag{2.17}$$

Case 2.3.3. $\frac{a_3a_5}{\sqrt[3]{a_1}} = 0, a_6\sqrt[3]{a_1^2} = 1, a_8\sqrt[3]{a_1^2} = 1$

In third subcase of *Case 2.3.*, we note that $a_6 = a_8$ and by supposing that $a_5 = 0, a_1 = a_3 \neq 0$, we select the representative element $v = w_6 + w_8$. Therefore, by engaging the parametric values $\tilde{a}_i = 0,\ i = 2, 4, 5, 7, 9$ together with $\tilde{a}_i = 1,\ i = 1, 3, 6, 8$ in adjoint system (2.11), we secure

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2}\ln(a_3a_8),\\
\varepsilon_3 &= \frac{1}{2a_8}(2a_3a_8\varepsilon_8 + a_9),\\
\varepsilon_4 &= \frac{1}{2}\ln\left(\frac{1}{a_3}\right),\\
\varepsilon_6 &= a_8\sqrt{a_3a_8}\left(\frac{1}{a_3}\right)^{1/3}\varepsilon_1 - a_7\sqrt{a_3a_8}\left(\frac{1}{a_3}\right)^{1/3} - 2\varepsilon_5.
\end{aligned} \tag{2.18}$$

Case 2.3.4. $\frac{a_3a_5}{\sqrt[3]{a_1}} = 1,\ a_6\sqrt[3]{a_1^2} = 1,\ a_8\sqrt[3]{a_1^2} = 1$

This scenario presents $a_1 = a_3^3a_5^3$ and also gives $a_6 = a_8$. Thus, we choose the representative element $v = w_1 + w_3 + w_5$. On inserting the parameters $\tilde{a}_i = 0,\ i = 2,4,7,9$ along with $\tilde{a}_i = 1,\ i = 1,3,5,6,8$ into adjoint system (2.11), we achieve the solution

$$\begin{aligned} \varepsilon_2 &= \frac{1}{2}\ln\left(\frac{1}{a_3a_5^2}\right),\\ \varepsilon_3 &= \frac{1}{2}a_3^2a_5^2a_9 + a_3\varepsilon_8,\\ \varepsilon_4 &= \ln(a_5) + \frac{1}{2}\ln\left(\frac{1}{a_3a_5^2}\right),\\ \varepsilon_6 &= \frac{1}{a_3^3a_5^3}\{a_3^4a_5^5a_9 - 2a_3^3a_5^3\varepsilon_5 + 2a_3^3a_5^3\varepsilon_8 - a_3^2a_5^2a_7 + \varepsilon_1\}. \end{aligned} \tag{2.19}$$

Case 2.3.5. $\frac{a_3a_5}{\sqrt[3]{a_1}} = -1,\ a_6\sqrt[3]{a_1^2} = -1,\ a_8\sqrt[3]{a_1^2} = -1$

Here, $a_1 = -a_3^3a_5^3$ and so for $a_1 = a_3 > 0, a_5 = a_6 = a_8 < 0$, we choose the representative element $v = w_1 + w_3 - w_5$. On substituting the parametric values $\tilde{a}_i = 0,\ i = 2,4,7,9$, $\tilde{a}_i = 1,\ i = 1,3$ with $\tilde{a}_i = -1,\ i = 5,6,8$ into adjoint system (2.11), we obtain the result

$$\begin{aligned} \varepsilon_2 &= \frac{1}{4}\ln\left(-\frac{a_8}{a_5^2}\right),\\ \varepsilon_3 &= \frac{1}{2a_8}\left(2\sqrt{-\frac{a_8}{a_5^2}}\,\varepsilon_8 + a_9\right),\\ \varepsilon_4 &= \ln(-a_5) + \frac{1}{4}\ln\left(-\frac{a_8}{a_5^2}\right),\\ \varepsilon_6 &= -\frac{1}{a_8}\left\{a_5a_8^2\sqrt{-\frac{a_8}{a_5^2}}\,\varepsilon_1 + a_5^2a_9\sqrt{-\frac{a_8}{a_5^2}} - a_5a_7a_8\sqrt{-\frac{a_8}{a_5^2}}\right.\\ &\qquad \left. + 2a_8\varepsilon_5 - 2a_8\varepsilon_8\right\}. \end{aligned} \tag{2.20}$$

Furthermore, for $a_1 = a_5 > 0, a_3 = a_6 = a_8 < 0$, we select the representative element $v = w_1 - w_3 + w_5$. Thus, invoking the relevant parameters of $\tilde{a}_i,\ i = 1,\dots,9$ into system (2.11), we obtain

$$\begin{aligned} \varepsilon_2 &= \frac{1}{2}\ln\left(-\frac{1}{a_3a_5^2}\right),\\ \varepsilon_3 &= -\frac{1}{2}a_3^2a_5^2a_9 - a_3\varepsilon_8,\\ \varepsilon_4 &= \ln(a_5) + \frac{1}{2}\ln\left(-\frac{1}{a_3a_5^2}\right),\\ \varepsilon_6 &= \frac{1}{a_3^3a_5^3}\{a_3^4a_5^5a_9 + 2a_3^3a_5^3\varepsilon_5 + 2a_3^3a_5^3\varepsilon_8 + a_3^2a_5^2a_7 + \varepsilon_1\}. \end{aligned} \tag{2.21}$$

Remark 2.2. We note that other possible representatives from *Case 2.3.4.* and *Case 2.3.5.* have been obtained earlier thereby contributing no additional subalgebra to the optimal system.

Case 2.3.6. $\frac{a_3a_5}{\sqrt[3]{a_1}} = 0,\ a_6\sqrt[3]{a_1^2} = 0,\ a_8\sqrt[3]{a_1^2} = 0$

In this situation, we take $a_3 = a_6 = a_8 = 0$ with $a_1 \neq 0$ and substitute it back into PDE system (2.9). Hence, we secure another new invariant dependent on (a_1, a_5, a_7, a_9) given as $\Phi(a_1, a_5, a_7, a_9) = G(a_5, a_9, \sqrt[3]{a_1})$. Thus, we explore the invariant to further scale down vector (2.5) shortly, bearing in mind the approach engaged in the study of Hu *et al.* [64] as demonstrated earlier.

Case 2.3.6.1. $a_1 \neq 0, a_5 \neq 0, a_9 \neq 0$

Now, we contemplate $a_1 > 0, a_5 > 0$, and $a_9 > 0$. Then, we select the optimal representative $v = w_1 + w_5 + w_9$. Substituting values of parameters, *viz*, $\tilde{a}_i = 0,\ i = 2,3,4,6,\dots,8$ alongside $\tilde{a}_i = 1,\ i = 1,5,9$ into system (2.11) gives the outcome

$$\begin{aligned} \varepsilon_2 &= \frac{1}{6}\ln\left(\frac{1}{a_1a_5^3}\right),\\ \varepsilon_3 &= \frac{1}{2}a_1a_5\left(\frac{1}{a_1a_5^3}\right)^{2/3}\varepsilon_6 + \frac{a_7}{2a_5},\\ \varepsilon_4 &= \frac{1}{6}\ln\left(\frac{a_5^3}{a_1}\right). \end{aligned} \tag{2.22}$$

Conversely, we consider $a_1 < 0, a_5 < 0$, and $a_9 < 0$. So, we choose the optimal representative $v = -w_1 - w_5 - w_9$. On invoking parameters $\tilde{a}_i = 0,\ i = 2,3,4,6,\dots,8$ alongside $\tilde{a}_i = -1,\ i = 1,5,9$ into adjoint Eq. (2.11), we have the result as

$$\begin{aligned} \varepsilon_2 &= \frac{1}{6}\ln\left(\frac{1}{a_1a_5^3}\right),\\ \varepsilon_3 &= \frac{1}{2}a_1a_5\left(\frac{1}{a_1a_5^3}\right)^{2/3}\varepsilon_6 + \frac{a_7}{2a_5},\\ \varepsilon_4 &= \frac{1}{6}\ln\left(\frac{a_5^3}{a_1}\right). \end{aligned} \tag{2.23}$$

Moreover, contemplating $a_1 > 0, a_5 > 0$ with $a_9 < 0$ which gives representative $v = w_1 + w_5 - w_9$ and also $a_1 < 0, a_5 < 0$ with $a_9 > 0$ that secures representative $v = -w_1 - w_5 + w_9$ and solving system (2.11) using appropriate values of $a_i,\ i = 1,2,3,\dots,9$ gives the same solution as earlier obtained. Furthermore, for $a_1 > 0, a_5 < 0$ with $a_9 < 0$ whose representative is $v = w_1 - w_5 - w_9$ as well as $a_1 < 0, a_5 > 0$ with $a_9 > 0$ which gives representative $v = -w_1 + w_5 + w_9$ together with $a_1 > 0, a_5 < 0$ with $a_9 > 0$ that occasions representative $v = w_1 - w_5 + w_9$, we solve system (2.11). On imploring adequate values $a_i,\ i = 1,2,3,\dots,9$, one obtains

$$\varepsilon_2 = \frac{1}{6}\ln\left(-\frac{1}{a_1a_5^3}\right),$$
$$\varepsilon_3 = \frac{1}{2}a_1a_5\left(-\frac{1}{a_1a_5^3}\right)^{2/3}\varepsilon_6 + \frac{a_7}{2a_5}, \tag{2.24}$$
$$\varepsilon_4 = \frac{1}{6}\ln\left(-\frac{a_5^3}{a_1}\right).$$

Next, we contemplate the case $a_5a_9\sqrt[3]{a_1} = 0$ whereby various other possibilities are explored.

Case 2.3.6.2. $a_1 = 0, a_5 \neq 0, a_9 \neq 0$

Here, we examine the case $a_5 > 0$ and $a_9 > 0$. Therefore, we select the representative element $v = w_5 + w_9$. In addition, for $a_5 < 0$ and $a_9 > 0$ whose representative is $v = -w_5 + w_9$, we substitute appropriate values of a_i, $i = 1, 2, 3, \ldots, 9$ in adjoint system (2.11) as earlier manifested and gain

$$\varepsilon_2 = \frac{1}{2}\ln(a_9),$$
$$\varepsilon_3 = \frac{a_7}{2a_5}, \tag{2.25}$$
$$\varepsilon_4 = \frac{1}{2}\ln(a_5^2a_9).$$

Moreover, we study the case when $a_5 < 0$ and $a_9 < 0$. We choose the representative element $v = -w_5 - w_9$. Besides, for $a_5 > 0$ and $a_9 < 0$ whose representative element is $v = w_5 - w_9$, we insert adequate parametric values of a_i, $i = 1, 2, 3, \ldots, 9$ into adjoint system (2.11) as exercised before and obtain

$$\varepsilon_2 = -\frac{1}{2}\ln\left(-\frac{1}{a_9}\right),$$
$$\varepsilon_3 = \frac{a_7}{2a_5}, \tag{2.26}$$
$$\varepsilon_4 = -\frac{1}{2}\ln\left(-\frac{1}{a_5^2a_9}\right), a_9 < 0.$$

Case 2.3.6.3. $a_5 = 0, a_1 \neq 0, a_9 \neq 0$

Reckoning $a_1 > 0$ and $a_9 > 0$ with $a_5 = 0$, we select the representative element $v = w_1 + w_9$. Not only that on making adequate choice of a_i, $i = 1, 2, 3, \ldots, 9$ and substituting same in adjoint system (2.11), we secure the solution

$$\varepsilon_2 = \frac{1}{3}\ln\left(\frac{1}{a_1}\right) - \frac{1}{3}\ln\left(\frac{1}{a_1\sqrt{a_9^3}}\right),$$
$$\varepsilon_4 = \frac{1}{3}\ln\left(\frac{1}{a_1\sqrt{a_9^3}}\right), \tag{2.27}$$
$$\varepsilon_6 = -\frac{a_7}{a_1a_9}(a_9\sqrt[3]{a_1^2}).$$

Besides, for $a_1 < 0$ and $a_9 < 0$ alongside $a_5 = 0$, we choose the representative element $v = -w_1 - w_9$. On invoking adequate parametric values of a_i, $i = 1, 2, 3, \ldots, 9$ in adjoint system (2.11) as revealed earlier, one obtains

$$\varepsilon_2 = \frac{1}{3}\ln\left(-\frac{1}{a_1}\right) - \frac{1}{3}\ln\left(\left|\frac{1}{a_1a_9\sqrt{-a_9}}\right|\right),$$
$$\varepsilon_4 = \frac{1}{3}\ln\left(\left|\frac{1}{a_1a_9\sqrt{-a_9}}\right|\right), \tag{2.28}$$
$$\varepsilon_6 = \frac{a_7}{a_1a_9}((a_1a_9)^{2/3}(-a_9)^3).$$

Considering $a_1 < 0$ and $a_9 > 0$ with $a_5 = 0$, we select the representative element $v = -w_1 + w_9$. Invoking adequate choice of a_i, $i = 1, 2, 3, \ldots, 9$ in adjoint system (2.11), we secure the solution

$$\varepsilon_2 = \frac{1}{3}\ln\left(-\frac{1}{a_1}\right) - \frac{1}{3}\ln\left(-\frac{1}{a_1\sqrt{a_9^3}}\right),$$
$$\varepsilon_4 = \frac{1}{3}\ln\left(-\frac{1}{a_1\sqrt{a_9^3}}\right), \tag{2.29}$$
$$\varepsilon_6 = \frac{a_7}{a_1}(\sqrt[3]{a_1^2}).$$

In the same vein, for $a_1 > 0$ and $a_9 < 0$ alongside $a_5 = 0$, we choose the representative element $v = w_1 - w_9$. On invoking adequate parametric values of a_i, $i = 1, 2, 3, \ldots, 9$ in adjoint system (2.11) as revealed earlier, one obtains

$$\varepsilon_2 = \frac{1}{3}\ln\left(\frac{1}{a_1}\right) - \frac{1}{3}\ln\left(-\frac{1}{a_1a_9\sqrt{-a_9}}\right),$$
$$\varepsilon_4 = \frac{1}{3}\ln\left(-\frac{1}{a_1a_9\sqrt{-a_9}}\right), \tag{2.30}$$
$$\varepsilon_6 = \frac{a_7}{a_1a_9}(-(a_1a_9)^{2/3}(-a_9)^3).$$

Case 2.3.6.4. $a_9 = 0, a_1 \neq 0, a_5 \neq 0$

Regarding the case of $a_1 > 0$ and $a_5 > 0$ together with $a_9 = 0$, we select representative element $v = w_1 + w_5$. In addition, for $a_1 < 0$, $a_5 < 0$ as well as $a_9 = 0$, we choose representative element $v = -w_1 - w_5$. On inserting relevant values of a_i, $i = 1, 2, 3, \ldots, 9$ in adjoint system (2.11), we achieve the result

$$\varepsilon_2 = \frac{1}{6}\ln\left(\frac{1}{a_1a_5^3}\right),$$
$$\varepsilon_3 = \frac{1}{2}a_1a_5\left(\frac{1}{a_1a_5^3}\right)^{2/3}\varepsilon_6 + \frac{a_7}{2a_5}, \tag{2.31}$$
$$\varepsilon_4 = \frac{1}{6}\ln\left(\frac{a_5^3}{a_1}\right).$$

On the contrary, if we reckon $a_1 > 0$, $a_5 < 0$ as well as $a_9 = 0$, we choose representative element $v = w_1 - w_5$.

Besides, for $a_1 < 0$, $a_5 > 0$ as well as $a_9 = 0$, we choose representative element $v = -w_1 + w_5$. On invoking pertinent values of a_i, $i = 1, 2, 3, \dots, 9$ in adjoint system (2.11), we secure the outcome

$$\begin{aligned} \varepsilon_2 &= \frac{1}{6}\ln\left(-\frac{1}{a_1a_5^3}\right), \\ \varepsilon_3 &= \frac{1}{2}a_1a_5\left(-\frac{1}{a_1a_5^3}\right)^{2/3}\varepsilon_6 + \frac{a_7}{2a_5}, \\ \varepsilon_4 &= \frac{1}{6}\ln\left(-\frac{a_5^3}{a_1}\right). \end{aligned} \tag{2.32}$$

Remark 2.3. It is noteworthy to state that some of the representative elements obtained with both positive and negative signs are equivalent. For instance, $w_1 + w_5 + w_9$ and $-w_1 - w_5 - w_9$ are equivalent. The study reveals that their solutions ((2.22) and (2.23)) are exactly the same. Moreover, we observe the same occurrence for representative elements $w_1 + w_5$ and $-w_1 - w_5$.

Finally, in view of the detailed calculations and analysis presented alongside remarks (2.2) and (2.3), we reduce the list of the representatives slightly by admitting that the discrete symmetry $(t, x, y, z, u) \longmapsto (-t, -x, -y, -z, u)$ not in the connected component of the identity of the full symmetry group maps $w_1 + w_5 + w_9$ to $w_1 - w_5 - w_9$, $w_5 + w_9$ to $-w_5 + w_9$, $w_1 + w_9$ to $w_1 - w_9$, and $w_1 + w_5$ to $w_1 - w_5$, and so on, thereby minimizing the number of inequivalent subalgebras [17]. Thus, we arrive at the theorem.

Theorem 2.4. *The optimal system of one-dimensional Lie subalgebra of HD-SOLeqn* (1.6) *comprises the list*: $w_2 + w_4$; w_4; w_2; $w_3 + w_5 - w_6$; $w_6 + w_8$; $w_3 + w_5 - w_8$; $w_1 + w_3 + w_5$; $w_1 + w_5 + w_9$; $w_5 + w_9$; $w_1 + w_9$; $w_1 + w_5$; $w_1 - w_5 + w_9$.

We note that various invariant solutions associated with the subalgebras presented in Theorem 2.4 have been copiously explored [61,62]. Thus, we compute the conserved vectors related to the vectors in the subsequent part of the research paper.

3 Conserved currents of system (1.6) with applications

This part of the article reveals the calculation of conserved vectors related to (1.6) by invoking the Ibragimov's theorem [65,66] for determining conserved quantities. Hence, some salient information are divulged in order to understand the technique.

3.1 Preliminaries

A new theorem in [65] was introduced by Ibragimov for the computations of conserved vectors associated with a given DE. In addition, availability of classical Lagrangian is not demanded for the theorem to function. Ibragimov's technique suggests fundamentally that infinitesimal generators are uniquely associated with their conserved current. Furthermore, the concept stands on theavailability of adjoint equation related to nonlinear DEs. Thus, a detailed outline of the theorem is furnished shortly.

Formal Lagrangian and adjoint equation

Theorem 3.1. [65] *The system of adjoint relations given as*

$$\begin{aligned} &\Xi^*_\sigma(x, \Psi, \Omega, \dots, \Psi_{(s)}, \Omega_{(s)}) \equiv \frac{\delta(\Omega^\beta \Xi_\beta)}{\delta \Psi^\sigma} = 0, \\ &\sigma = 1, \dots, \alpha, \Omega = (\Omega^1, \dots, \Omega^\alpha), \end{aligned} \tag{3.1}$$

which exists for a known system of α sets of relations presented as

$$\Xi_\sigma(x, \Psi, \Psi_{(1)}, \dots, \Psi_{(s)}) = 0, \quad \sigma = 1, \dots, \alpha, \tag{3.2}$$

where $\Omega = \Omega(x)$ *with* κ *independent together with* α *dependent variables accordingly explicated as* $x = (x^1, x^2, \dots, x^\kappa)$ *and* $\Psi = (\Psi^1, \Psi^2, \dots, \Psi^\alpha)$, *alongside the variational derivative designated as Euler–Lagrange operator, expressed for each* σ, *by the formal sum defined as*

$$\begin{aligned} \frac{\delta}{\delta \Psi^\sigma} = \frac{\partial}{\partial \Psi^\sigma} &+ \sum_{s=1}^{\infty} (-1)^s D_{i_1} \cdots D_{i_s} \\ &\times \frac{\delta}{\delta \Psi^\sigma_{i_1, i_2, \dots, i_s}}, \quad i = 1, \dots, \kappa, \end{aligned} \tag{3.3}$$

owns the symmetries inherited by set of relations (3.2). *The complete derivative* D_i *explicates as*

$$\begin{aligned} D_i = \frac{\partial}{\partial x^i} &+ \Psi^\sigma_i \frac{\partial}{\partial \Psi^\sigma} + \Psi^\sigma_{ij} \\ &\times \frac{\partial}{\partial \Psi^\sigma_j} + \cdots, \ i = 1, \dots, \kappa, \ j = 1, \dots, \kappa. \end{aligned} \tag{3.4}$$

Noteworthy, it is to declare that suppose set of relations (3.2) admit a point transformation group, having a generator convey as

$$\begin{aligned} &\mathcal{R} = \xi^i \frac{\partial}{\partial x^i} + \varphi^\sigma \frac{\partial}{\partial \Psi^\sigma}, \\ &\xi^i = \xi^i(x, \Psi), \varphi^\sigma = \varphi^\sigma(x, \Psi), \end{aligned} \tag{3.5}$$

then the system of adjoint relations (3.1) admit the operator (3.5) whose extension to a variable Ω^σ is given by

$$\mathcal{S} = \xi^i \frac{\partial}{\partial x^i} + \varphi^\sigma \frac{\partial}{\partial \Psi^\sigma} + \varphi_*^\sigma \frac{\partial}{\partial \Omega^\sigma} \tag{3.6}$$

with coefficient $\varphi_*^\sigma = \varphi_*^\sigma(x, \Psi, \Omega, \ldots)$ defined as

$$\varphi_*^\sigma = -[\lambda_\beta^\sigma \Omega^\beta + \Omega^\sigma D_i(\xi^i)], \tag{3.7}$$

which is selected approximately and the included variable $\Omega^\sigma = \Omega^\sigma(x)$. λ_β^σ is achievable by the formula:

$$\lambda_\beta^\sigma \Xi_\beta = X(\Xi_\sigma). \tag{3.8}$$

Hence, the formal Lagrangian is explicated as

$$\mathcal{L} = \Omega^\sigma \Xi_\sigma(x, \Psi, \Psi_{(1)}, \ldots, \Psi_{(s)}), \tag{3.9}$$

with the adjoint relation to (3.2) stated as

$$\Xi_\sigma^*(x, \Psi, \Omega, \ldots, \Psi_{(s)}, \Omega_{(s)}) = 0, \tag{3.10}$$

together with criteria (3.1) holding.

Theorem 3.2. *Every nonlocal symmetry* (3.5), *Lie–Bäcklund, as well as Lie point, admitted by the system of* (3.2) *produce a conserved current for relations* (3.2) *alongside the adjoint* (3.10), *with the conserved current $T = (T^1, \ldots, T^\kappa)$ having components T^i and decided by*

$$\begin{aligned} T^i = \xi^i \mathcal{L} &+ \Pi^\sigma \left[\frac{\partial \mathcal{L}}{\partial \Psi_i^\sigma} - D_j \frac{\partial \mathcal{L}}{\partial \Psi_{ij}^\sigma} + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial \Psi_{ijk}^\sigma} \right) + \cdots \right] \\ &+ D_j(\Pi^\sigma) \left[\frac{\partial \mathcal{L}}{\partial \Psi_{ij}^\sigma} - D_k \frac{\partial \mathcal{L}}{\partial \Psi_{ijk}^\sigma} + \cdots \right] \\ &+ D_j D_k(\Pi^\sigma) \frac{\partial \mathcal{L}}{\partial \Psi_{ijk}} + \cdots, \quad i, j, k = 1, \ldots, \kappa, \end{aligned} \tag{3.11}$$

where Π^σ is the involved Lie characteristic function is engendered as

$$\begin{aligned} &\Pi^\sigma = \varphi^\sigma - \xi^j \Psi_j^\sigma, \\ &\sigma = 1, \ldots, \alpha, \ j = 1, \ldots, \kappa. \end{aligned} \tag{3.12}$$

Remark 3.1. We assert here that a given system of (3.2) is said to be self-adjoint if after replacing $\Omega = \Psi$ in the system of adjoint relations (3.10) gives that same system. A more detailed knowledge about the proof and various other copious information on the results made available here, can be accessed by the reader in the studies of Ibragimov [65,66].

3.2 Derivation of conservation laws *via* Ibragimov's theorem

We give the Ibragimov's conservation theorem [65] in this part of the article to find the conserved vectors of the HD-SOLeqn (1.6). Utilizing the highlighted information earlier presented, one achieves the theorem:

Theorem 3.3. *The HD-SOLeqn* (1.6) *as well as its adjoint equation is accordingly explicated as*

$$\begin{aligned} &G_{\alpha_1} \equiv 3u_{xz} - 2u_{ty} + 2u_x u_y + 2u u_{xy} + 2v u_{xx} + 2v_x u_x - u_{xxxy}, \\ &G_{\alpha_2} \equiv u_y - v_x, \\ &G_{\alpha_1}^* \equiv 3p_{xz} - 2p_{ty} + 2p_x v_x + 2u p_{xy} + 2p_{xx} v - q_y - p_{xxxy} = 0, \\ &G_{\alpha_2}^* \equiv q_x - 2p_x u_x = 0, \end{aligned} \tag{3.13}$$

with a second-order formal Lagrangian given as

$$\begin{aligned} \mathcal{L} = {} & p(2u_{xx} v + 2u u_{xy} - 2u_{ty} + 2u_x v_x + 2u_x u_y + 3u_{xz}) \\ & + q(u_y - v_x) - p_{xx} u_{xy}. \end{aligned} \tag{3.14}$$

Obviously, one can frankly state that from the adjoint relation (3.13) and remark (3.1), HD-SOLeqn (1.6) is not self-adjoint. By applying of the earlier highlighted facts in Theorem 3.2, we calculate the conserved currents associated with the 12 elements of the achieved optimal system of one-dimensional subalgebras as well as other Lie point symmetries attained for (1.6). These conserved currents are

$$\begin{aligned} T_1^t = {} & 2u_x v_x p + 2u_{xx} p v + 2u_x u_y p + 3u_{xz} p \\ & + 2u_{xy} p u - u_{ty} p + u_y q - v_x q - p_y u_t - p_{xx} u_{xy}, \\ T_1^x = {} & 2p_x u_t v - 2v_t u_x p - 2u_{tx} p v + p_y u_t u \\ & - u_t u_y p - \frac{3}{2} u_{tz} p - u_{ty} p u + v_t q - p_t u_{xxy} \\ & - \frac{1}{2} u_t p_{xxy} + \frac{1}{2} p_{xx} u_{ty} + p_{tx} u_{xy} + \frac{3}{2} p_z u_t, \\ T_1^y = {} & p_x u_t u - u_t u_x p - u_{tx} p u + u_{tt} p - u_t q \\ & - \frac{1}{2} p_{xxx} u_t + \frac{1}{2} p_{xx} u_{tx} - p_t u_t, \\ T_1^z = {} & \frac{3}{2} p_x u_t - \frac{3}{2} u_{tx} p; \end{aligned}$$

$$\begin{aligned} T_2^t = {} & u_{yz} p - p_y u_z, \\ T_2^x = {} & 2p_x u_z v - 2u_x v_z p - 2u_{xz} p v \\ & + p_y u_z u - u_y u_z p - \frac{3}{2} u_{zz} p - u_{yz} p u + v_z q \\ & - \frac{1}{2} u_z p_{xxy} + p_{xz} u_{xy} \\ & + \frac{1}{2} p_{xx} u_{yz} - p_z u_{xxy} + \frac{3}{2} p_z u_z, \\ T_2^y = {} & p_x u_z u - u_x u_z p - u_{xz} p u \\ & + u_{tz} p - u_z q - p_t u_z - \frac{1}{2} p_{xxx} u_z + \frac{1}{2} p_{xx} u_{xz}, \\ T_2^z = {} & 2u_x v_x p + 2u_{xx} p v + 2u_x u_y p + \frac{3}{2} u_{xz} p \\ & + 2u_{xy} p u - 2u_{ty} p + u_y q - v_x q \\ & - p_{xx} u_{xy} + \frac{3}{2} p_x u_z; \end{aligned}$$

$$
\begin{aligned}
T_3^t = {} & 2yu_y p_y - 2up_y - xu_x p_y - 3tu_t p_y + 3tqu_y - 2ypu_{yy} \\
& + 6tpu_y u_x - 3tqv_x + 6tpu_x v_x + 9tpu_{xz} + xpu_{xy} \\
& + 6tpuu_{xy} - 3tu_{xy} p_{xx} + 6tpvu_{xx} - 3tpu_{ty},
\end{aligned}
$$

$$
\begin{aligned}
T_3^x = {} & 2p_y u^2 + 3p_z u - 2pu_y u - 2yp_y u_y u \\
& + 2ypu_{yy} u + 4vp_x u + xp_y u_x u + xpu_{xy} u - p_{xxy} u + 3tp_y u_t u \\
& - 3tpu_{ty} u + 2ypu_y^2 - qv - 3pu_z + xqu_y - 3yp_z u_y \\
& - 2yqv_y + 3ypu_{yz} - 4yvu_y p_x - 4pvu_x + \frac{3}{2}xp_z u_x + xpu_y u_x \\
& + 4ypv_y u_x + 2xvp_x u_x + \frac{3}{2}xpu_{xz} + 4ypvu_{xy} + p_x u_{xy} \\
& - 2yp_{xy} u_{xy} - yu_{yy} p_{xx} + \frac{9}{2}tp_z u_t + \frac{1}{2}xu_{xy} p_{xx} \\
& + yu_y p_{xxy} - \frac{1}{2}xu_x p_{xxy} + 2yp_y u_{xxy} - xp_x u_{xxy} - 3tu_{xxy} p_t \\
& - 3tpu_y u_t + 6tvp_x u_t - \frac{3}{2}tp_{xxy} u_t + 3tqv_t - 6tpu_x v_t \\
& - \frac{9}{2}tpu_{tz} - 2xpu_{ty} + \frac{3}{2}tp_{xx} u_{ty} + 3tu_{xy} p_{tx} - 6tpvu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
T_3^y = {} & 2p_x u^2 - 2qu - 2yu_y p_x u - 5pu_x u + xp_x u_x u - 2ypu_{xy} u \\
& - xpu_{xx} u - p_{xxx} u - 2p_t u + 3tp_x u_t u - 3tpu_{tx} u \\
& - xpu_x^2 - xqu_x - 2ypu_y u_x + 2yqv_x \\
& - 4ypu_x v_x - 6ypu_{xz} + \frac{3}{2}u_x p_{xx} + yu_{xy} p_{xx} - 4ypvu_{xx} \\
& + \frac{1}{2}xp_{xx} u_{xx} + yu_y p_{xxx} - \frac{1}{2}xu_x p_{xxx} + 2yu_y p_t - xu_x p_t \\
& + 5pu_t - 3tqu_t - 3tpu_x u_t - \frac{3}{2}tp_{xxx} u_t - 3tp_t u_t \\
& + 2ypu_{ty} + xpu_{tx} + \frac{3}{2}tp_{xx} u_{tx} + 3tpu_{tt},
\end{aligned}
$$

$$
\begin{aligned}
T_3^z = {} & 3up_x - 3yu_y p_x + \frac{3}{2}xu_x p_x + \frac{9}{2}tu_t p_x \\
& - \frac{9}{2}pu_x + 3ypu_{xy} - \frac{3}{2}xpu_{xx} - \frac{9}{2}tpu_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_4^t = {} & 2pu_y - 2up_y - 2zu_z p_y - xu_x p_y - 3tu_t p_y + 3tqu_y \\
& + 2zpu_{yz} + xpu_{xy} + 6tpu_y u_x - 3tqv_x + 6tpu_x v_x + 9tpu_{xz} \\
& + 6tpuu_{xy} - 3tu_{xy} p_{xx} + 6tpvu_{xx} - 3tpu_{ty},
\end{aligned}
$$

$$
\begin{aligned}
T_4^x = {} & 2p_y u^2 + 3p_z u + 2zu_z p_y u - 4pu_y u - 2zpu_{yz} u \\
& + 4vp_x u + xp_y u_x u + xpu_{xy} u - p_{xxy} u + 3tp_y u_t u - 3tpu_{ty} u \\
& + qv - 6pu_z + 3zp_z u_z + 2zqv_z - 3zpu_{zz} + xqu_y \\
& - 2zpu_z u_y + 4zvu_z p_x - 8pvu_x + \frac{3}{2}xp_z u_x - 4zpv_z u_x \\
& + xpu_y u_x + 2xvp_x u_x + \frac{3}{2}xpu_{xz} - 4zpvu_{xz} + p_x u_{xy} + 2zp_{xz} u_{xy} \\
& + u_y p_{xx} + zu_{yz} p_{xx} + \frac{1}{2}xu_{xy} p_{xx} - zu_z p_{xxy} - \frac{1}{2}xu_x p_{xxy} \\
& - 2zp_z u_{xxy} - xp_x u_{xxy} - 3tu_{xxy} p_t + \frac{9}{2}tp_z u_t - 3tpu_y u_t \\
& + 6tvp_x u_t - \frac{3}{2}tp_{xxy} u_t + 3tqv_t - 6tpu_x v_t - \frac{9}{2}tpu_{tz} \\
& - 2xpu_{ty} + \frac{3}{2}tp_{xx} u_{ty} + 3tu_{xy} p_{tx} - 6tpvu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
T_4^y = {} & 2p_x u^2 - 2qu + 2zu_z p_x u - 5pu_x u + xp_x u_x u \\
& - 2zpu_{xz} u - xpu_{xx} u - p_{xxx} u - 2p_t u + 3tp_x u_t u - 3tpu_{tx} u \\
& - xpu_x^2 - 2zqu_z - xqu_x - 2zpu_z u_x + \frac{3}{2}u_x p_{xx} \\
& + zu_{xz} p_{xx} + \frac{1}{2}xp_{xx} u_{xx} - zu_z p_{xxx} - \frac{1}{2}xu_x p_{xxx} \\
& - 2zu_z p_t - xu_x p_t + 5pu_t - 3tqu_t - 3tpu_x u_t - \frac{3}{2}tp_{xxx} u_t \\
& - 3tp_t u_t + 2zpu_{tz} + xpu_{tx} + \frac{3}{2}tp_{xx} u_{tx} + 3tpu_{tt},
\end{aligned}
$$

$$
\begin{aligned}
T_4^z = {} & 2zqu_y + 4zpu_x u_y + 3up_x + 3zu_z p_x - \frac{9}{2}pu_x + \frac{3}{2}xp_x u_x \\
& - 2zqv_x + 4zpu_x v_x + 3zpu_{xz} + 4zpuu_{xy} - 2zu_{xy} p_{xx} \\
& - \frac{3}{2}xpu_{xx} + 4zpvu_{xx} + \frac{9}{2}tp_x u_t - 4zpu_{ty} - \frac{9}{2}tpu_{tx};
\end{aligned}
$$

$$
\begin{aligned}
T_5^t = {} & 2zu_{xy} p - 2zp_y u_x, \\
T_5^x = {} & 4zp_x u_x v + 3u_x p + 2zp_y u_x u \\
& + 2zu_x u_y p + 3zu_{xz} p + 2zu_{xy} pu - \frac{1}{2}zu_{xxxy} p \\
& - 4zu_{ty} p - 2zu_y q + 3q + \frac{1}{2}zp_{xx} u_{xy} \\
& + zu_{xx} p_{xy} - \frac{3}{2}zu_x p_{xxy} - zp_x u_{xxy} \\
& - \frac{1}{2}zp_y u_{xxx} + 3zp_z u_x, \\
T_5^y = {} & 2zp_x u_x u - 2zu_x^2 p - 2zu_{xx} pu + \frac{1}{2}zu_{xxxx} p \\
& + 2zu_{tx} p + 2zu_x q - 2zp_t u_x - \frac{1}{2}zp_{xxx} u_x \\
& + \frac{1}{2}zp_{xx} u_{xx} - \frac{1}{2}zp_x u_{xxx}, \\
T_5^z = {} & 3zp_x u_x - 3zu_{xx} p; \\
T_6^t = {} & tu_{xy} p - tp_y u_x - p_y, \\
T_6^x = {} & 2tp_x u_x v + p_y u - u_y p + tp_y u_x u + tu_x u_y p \\
& + \frac{3}{2}tu_{xz} p + tu_{xy} pu - \frac{1}{4}tu_{xxxy} p - 2tu_{ty} p \\
& + 2p_x v - tu_y q + \frac{1}{4}tp_{xx} u_{xy} + \frac{1}{2}tu_{xx} p_{xy} \\
& - \frac{3}{4}tu_x p_{xxy} - \frac{1}{2}tp_x u_{xxy} - \frac{1}{4}tp_y u_{xxx} \\
& + \frac{3}{2}tp_z u_x - \frac{3}{4}p_{xxy} + \frac{3}{2}p_z,
\end{aligned}
$$

$$T_6^y = p_x u - t u_x^2 p + t p_x u_x u - t u_{xx} p u + \frac{1}{4} t u_{xxxx} p + t u_{tx} p + t u_x q + q - \frac{1}{4} t p_{xxx} u_x - t p_t u_x + \frac{1}{4} t p_{xx} u_{xx} - \frac{1}{4} t p_x u_{xxx} - p_t - \frac{1}{4} p_{xxx},$$

$$T_6^z = \frac{3}{2} t p_x u_x - \frac{3}{2} t u_{xx} p + \frac{3}{2} p_x;$$

$$T_7^t = u_{xy} p - p_y u_x,$$

$$T_7^x = 2 p_x u_x v + u_x u_y p + p_y u_x u + \frac{3}{2} u_{xz} p + u_{xy} p u - \frac{1}{4} u_{xxxy} p - 2 u_{ty} p - u_y q + \frac{1}{4} p_{xx} u_{xy} + \frac{1}{2} u_{xx} p_{xy} - \frac{3}{4} u_x p_{xxy} - \frac{1}{2} p_x u_{xxy} - \frac{1}{4} p_y u_{xxx} + \frac{3}{2} p_z u_x,$$

$$T_7^y = p_x u_x u - u_x^2 p - u_{xx} p u + \frac{1}{4} u_{xxxx} p + u_{tx} p + u_x q - p_t u_x - \frac{1}{4} p_{xxx} u_x + \frac{1}{4} p_{xx} u_{xx} - \frac{1}{4} p_x u_{xxx},$$

$$T_7^z = \frac{3}{2} p_x u_x - \frac{3}{2} u_{xx} p;$$

$$T_8^t = -2 z u_y p_y + 3 p_y + 2 z p u_{yy},$$

$$T_8^x = 3 z p_z u_y - 2 z p u_y^2 + 2 z u p_y u_y + 4 z v p_x u_y - \frac{3}{2} z p_{xxy} u_y - \frac{9}{2} p_z - 3 u p_y - 2 z q v_y - 3 z p u_{yz} - 2 z p u u_{yy} - 6 v p_x - 4 z p v_y u_x - 4 z p v u_{xy} + z p_{xy} u_{xy} - z p_x u_{xyy} + \frac{1}{2} z u_{yy} p_{xx} + \frac{9}{4} p_{xxy} - \frac{1}{2} z p_y u_{xxy} + \frac{3}{2} z p u_{xxyy},$$

$$T_8^y = 2 z v_x q - 3 q - 3 u p_x + 2 z u u_y p_x + 3 p u_x + 2 z p u_y u_x + 4 z p u_x v_x + 6 z p u_{xz} + 2 z p u u_{xy} + \frac{1}{2} z u_{xy} p_{xx} + 4 z p v u_{xx} - \frac{1}{2} z p_x u_{xxy} - \frac{1}{2} z u_y p_{xxx} + \frac{3}{4} p_{xxx} - \frac{3}{2} z p u_{xxxy} - 2 z u_y p_t + 3 p_t - 2 z p u_{ty},$$

$$T_8^z = 3 z u_y p_x - \frac{9}{2} p_x - 3 z p u_{xy};$$

$$T_9^t = u_{yy} p - p_y u_y,$$

$$T_9^x = 2 p_x u_y v - 2 u_x v_y p - 2 u_{xy} p v - u_y^2 p + p_y u_y u - \frac{3}{2} u_{yz} p - u_{yy} p u + \frac{3}{4} u_{xxyy} p - v_y q - \frac{3}{4} u_y p_{xxy} + \frac{1}{2} p_{xy} u_{xy} - \frac{1}{2} p_x u_{xyy} + \frac{1}{4} p_{xx} u_{yy} - \frac{1}{4} p_y u_{xxy} + \frac{3}{2} p_z u_y,$$

$$T_9^y = 2 u_x v_x p + 2 u_{xx} p v + p_x u_y u + u_x u_y p + 3 u_{xz} p + u_{xy} p u - \frac{3}{4} u_{xxxy} p - u_{ty} p + v_x q - p_t u_y - \frac{1}{4} p_x u_{xxy} + \frac{1}{4} p_{xx} u_{xy} - \frac{1}{4} p_{xxx} u_y,$$

$$T_9^z = \frac{3}{2} p_x u_y - \frac{3}{2} u_{xy} p;$$

$$T_{10}^t = 2 y u_y p_y - 4 u p_y - 2 z u_z p_y - 2 x u_x p_y - 6 t u_t p_y + 2 p u_y + 6 t q u_y + 2 z p u_{yz} - 2 y p u_{yy} + 12 t p u_y u_x - 6 t q v_x + 12 t p u_x v_x + 18 t p u_{xz} + 2 x p u_{xy} + 12 t p u u_{xy} - 6 t u_{xy} p_{xx} + 12 t p v u_{xx} - 6 t p u_{ty},$$

$$T_{10}^x = 4 p_y u^2 + 6 p_z u + 2 z u_z p_y u - 6 p u_y u - 2 y p_y u_y u - 2 z p u_{yz} u + 2 y p u_{yy} u + 8 v p_x u + 2 x p_y u_x u + 2 x p u_{xy} u - 2 p_{xxy} u + 6 t p_y u_t u - 6 t p u_{ty} u + 2 y p u_y^2 - 9 p u_z + 3 z p_z u_z + 2 z q v_z - 3 z p u_{zz} + 2 x q u_y - 3 y p_z u_y - 2 z p u_z u_y - 2 y q v_y + 3 y p u_{yz} + 4 z v u_z p_x - 4 y v u_y p_x - 12 p v u_x + 3 x p_z u_x - 4 z p v_z u_x + 2 x p u_y u_x + 4 y p v_y u_x + 4 x v p_x u_x + 3 x p u_{xz} - 4 z p v u_{xz} + 4 y p v u_{xy} + 2 p_x u_{xy} + 2 z p_{xz} u_{xy} - 2 y p_{xy} u_{xy} + u_y p_{xx} + z u_{yz} p_{xx} - y u_{yy} p_{xx} + x u_{xy} p_{xx} - z u_z p_{xxy} + y u_y p_{xxy} - x u_x p_{xxy} - 2 z p_z u_{xxy} + 2 y p_y u_{xxy} - 2 x p_x u_{xxy} - 6 t u_{xxy} p_t + 9 t p_z u_t - 6 t p u_y u_t + 6 t q v_t + 12 t v p_x u_t - 3 t p_{xxy} u_t - 12 t p u_x v_t - 9 t p u_{tz} - 4 x p u_{ty} + 3 t p_{xx} u_{ty} + 6 t u_{xy} p_{tx} - 12 t p v u_{tx},$$

$$T_{10}^y = 4 p_x u^2 - 4 q u + 2 z u_z p_x u - 2 y u_y p_x u - 10 p u_x u + 2 x p_x u_x u - 2 z p u_{xz} u - 2 y p u_{xy} u - 2 x p u_{xx} u - 2 p_{xxx} u - 4 p_t u + 6 t p_x u_t u - 6 t p u_{tx} u - 2 x p u_x^2 - 2 z q u_z - 2 x q u_x - 2 z p u_z u_x - 2 y p u_y u_x + 2 y q v_x - 4 y p u_x v_x - 6 y p u_{xz} + 3 u_x p_{xx} + z u_{xz} p_{xx} + y u_{xy} p_{xx} - 4 y p v u_{xx} + x p_{xx} u_{xx} - z u_z p_{xxx} + y u_y p_{xxx} - x u_x p_{xxx} - 2 z u_z p_t + 2 y u_y p_t - 2 x u_x p_t + 10 p u_t - 6 t q u_t - 6 t p u_x u_t - 3 t p_{xxx} u_t - 6 t p_t u_t + 2 z p u_{tz} + 2 y p u_{ty} + 2 x p u_{tx} + 3 t p_{xx} u_{tx} + 6 t p u_{tt},$$

$$T_{10}^z = 2 z q u_y - 3 y p_x u_y + 4 z p u_x u_y + 6 u p_x + 3 z u_z p_x - 9 p u_x + 3 x p_x u_x - 2 z q v_x + 4 z p u_x v_x + 3 z p u_{xz} + 3 y p u_{xy} + 4 z p u u_{xy} - 2 z u_{xy} p_{xx} - 3 x p u_{xx} + 4 z p v u_{xx} + 9 t p_x u_t - 4 z p u_{ty} - 9 t p u_{tx};$$

$$T_{11}^t = q u_y + 2 p u_x u_y - 2 z p_y u_x - q v_x + 2 p u_x v_x + 3 p u_{xz} + 2 z p u_{xy} + 2 p u u_{xy} - p_y u_t + 2 p v u_{xx} - u_{xy} p_{xx} - p u_{ty},$$

$$
\begin{aligned}
T_{11}^{x} &= 2zu_yq + v_tq - 3q + 3pu_x + 3zp_zu_x \\
&\quad + 2zup_yu_x + 2zpu_yu_x + 4zvp_xu_x \\
&\quad + 3zpu_{xz} + 2zpuu_{xy} + zu_{xy}p_{xx} - zu_xp_{xxy} \\
&\quad - 2zp_xu_{xxy} - u_{xxy}p_t + \frac{3}{2}p_zu_t + up_yu_t \\
&\quad - pu_yu_t + 2vp_xu_t - \frac{1}{2}p_{xxy}u_t - 2pu_xv_t \\
&\quad - \frac{3}{2}pu_{tz} - 4zpu_{ty} - puu_{ty} + \frac{1}{2}p_{xx}u_{ty} \\
&\quad + u_{xy}p_{tx} - 2pvu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
T_{11}^{y} &= -2zpu_x^2 - 2zqu_x + 2zup_xu_x - zp_{xxx}u_x \\
&\quad - 2zp_tu_x - pu_tu_x - 2zpuu_{xx} + zp_{xx}u_{xx} \\
&\quad - qu_t + up_xu_t - \frac{1}{2}p_{xxx}u_t - p_tu_t \\
&\quad + 2zpu_{tx} - puu_{tx} + \frac{1}{2}p_{xx}u_{tx} + pu_{tt},
\end{aligned}
$$

$$
T_{11}^{z} = 3zp_xu_x - 3zpu_{xx} + \frac{3}{2}p_xu_t - \frac{3}{2}pu_{tx};
$$

$$
\begin{aligned}
T_{12}^{t} &= tu_xp_y - u_zp_y - 2zu_xp_y + p_y \\
&\quad + pu_{yz} - tpu_{xy} + 2zpu_{xy},
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^{x} &= v_zq - tu_yq + 2zu_yq - 3q - \frac{3}{2}p_z \\
&\quad + \frac{3}{2}p_zu_z - \frac{3}{2}pu_{zz} - up_y + uu_zp_y + pu_y \\
&\quad - puu_{yz} + 2vu_zp_x + 3pu_x - \frac{3}{2}tp_zu_x \\
&\quad + 3zp_zu_x - 2pv_zu_x - tup_yu_x + 2zup_yu_x \\
&\quad - 2vp_x - tpu_yu_x + 2zpu_yu_x - 2tvp_xu_x \\
&\quad + 4zvp_xu_x - \frac{3}{2}tpu_{xz} + 3zpu_{xz} - 2pvu_{xz} \\
&\quad - tpuu_{xy} + 2zpuu_{xy} + p_{xz}u_{xy} + \frac{1}{2}u_{yz}p_{xx} \\
&\quad - \frac{1}{2}tu_{xy}p_{xx} + zu_{xy}p_{xx} - \frac{1}{2}u_zp_{xxy} + \frac{1}{2}tu_xp_{xxy} \\
&\quad - zu_xp_{xxy} + \frac{1}{2}p_{xxy} - p_zu_{xxy} + tp_xu_{xxy} \\
&\quad - 2zp_xu_{xxy} + 2tpu_{ty} - 4zpu_{ty} - pu_zu_y,
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^{y} &= tpu_x^2 - 2zpu_x^2 + tqu_x - 2zqu_x - pu_zu_x \\
&\quad - tup_xu_x + 2zup_xu_x + \frac{1}{2}tp_{xxx}u_x - zp_{xxx}u_x + tp_tu_x \\
&\quad - 2zp_tu_x + q - qu_z - up_x + uu_zp_x - puu_{xz} + \frac{1}{2}u_{xz}p_{xx} \\
&\quad + tpuu_{xx} - 2zpuu_{xx} - \frac{1}{2}tp_{xx}u_{xx} + zp_{xx}u_{xx} - \frac{1}{2}u_zp_{xxx} \\
&\quad + \frac{1}{2}p_{xxx} - u_zp_t + p_t + pu_{tz} - tpu_{tx} + 2zpu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^{z} &= qu_y + 2pu_xu_y + \frac{3}{2}u_zp_x - \frac{3}{2}p_x - \frac{3}{2}tp_xu_x + 3zp_xu_x \\
&\quad - qv_x + 2pu_xv_x + \frac{3}{2}pu_{xz} + 2puu_{xy} - u_{xy}p_{xx} + \frac{3}{2}tpu_{xx} \\
&\quad - 3zpu_{xx} + 2pvu_{xx} - 2pu_{ty};
\end{aligned}
$$

$$
T_{13}^{t} = tpu_{xy} - 2zu_yp_y - tu_xp_y + 2p_y + 2zpu_{yy},
$$

$$
\begin{aligned}
T_{13}^{x} &= 2zup_yu_y - 2zpu_y^2 - pu_y + tqu_y + 3zp_zu_y + 4zvp_xu_y \\
&\quad + tpu_xu_y - zp_{xxy}u_y - 3p_z - 2up_y + 2zqv_y - 3zpu_{yz} \\
&\quad - 2zpuu_{yy} - 4vp_x + \frac{3}{2}tp_zu_x + tup_yu_x \\
&\quad - 4zpv_yu_x + 2tvp_xu_x + \frac{3}{2}tpu_{xz} + tpuu_{xy} - 4zpvu_{xy} \\
&\quad + 2zp_{xy}u_{xy} + zu_{yy}p_{xx} + \frac{1}{2}tu_{xy}p_{xx} - \frac{1}{2}tu_xp_{xxy} \\
&\quad + p_{xxy} - 2zp_yu_{xxy} - tp_xu_{xxy} - 2tpu_{ty},
\end{aligned}
$$

$$
\begin{aligned}
T_{13}^{y} &= 3pu_x - tpu_x^2 - tqu_x + 2zpu_yu_x + tup_xu_x + 4zpv_xu_x \\
&\quad - \frac{1}{2}tp_{xxx}u_x - tp_tu_x + 2q - 2up_x + 2zuu_yp_x \\
&\quad - 2zqv_x + 6zpu_{xz} + 2zpuu_{xy} - zu_{xy}p_{xx} - tpuu_{xx} \\
&\quad + 4zpvu_{xx} + \frac{1}{2}tp_{xx}u_{xx} - zu_yp_{xxx} + p_{xxx} - 2zu_yp_t + 2p_t \\
&\quad - 2zpu_{ty} + tpu_{tx},
\end{aligned}
$$

$$
T_{13}^{z} = 3zu_yp_x + \frac{3}{2}tu_xp_x - 3p_x - 3zpu_{xy} - \frac{3}{2}tpu_{xx};
$$

$$
\begin{aligned}
T_{14}^{t} &= -u_zp_y + 2zu_yp_y - 2zu_xp_y - 3p_y + pu_{yz} - 2zpu_{yy} \\
&\quad + 2zpu_{xy},
\end{aligned}
$$

$$
\begin{aligned}
T_{14}^{x} &= 2zpu_y^2 + 2zqu_y - 3zp_zu_y - pu_zu_y - 2zup_yu_y \\
&\quad - 4zvp_xu_y + 2zpu_xu_y + zp_{xxy}u_y - 3q + \frac{9}{2}p_z \\
&\quad + \frac{3}{2}p_zu_z + qv_z - \frac{3}{2}pu_{zz} + 3up_y + uu_zp_y - 2zqv_y \\
&\quad + 3zpu_{yz} - puu_{yz} + 2zpuu_{yy} + 6vp_x + 2vu_zp_x \\
&\quad + 3pu_x + 3zp_zu_x - 2pv_zu_x + 2zup_yu_x + 4zpv_yu_x \\
&\quad + 4zvp_xu_x + 3zpu_{xz} - 2pvu_{xz} + 2zpuu_{xy} + 4zpvu_{xy} \\
&\quad + p_{xz}u_{xy} - 2zp_{xy}u_{xy} + \frac{1}{2}u_{yz}p_{xx} - zu_{yy}p_{xx} + zu_{xy}p_{xx} \\
&\quad - \frac{1}{2}u_zp_{xxy} - zu_xp_{xxy} - \frac{3}{2}p_{xxy} - p_zu_{xxy} \\
&\quad + 2zp_yu_{xxy} - 2zp_xu_{xxy} - 4zpu_{ty},
\end{aligned}
$$

$$
\begin{aligned}
T_{14}^{y} &= 2zup_xu_x - 2zpu_x^2 - 3pu_x - 2zqu_x - pu_zu_x \\
&\quad - 2zpu_yu_x - 4zpv_xu_x - zp_{xxx}u_x - 2zp_tu_x - 3q - qu_z \\
&\quad + 3up_x + uu_zp_x - 2zuu_yp_x + 2zqv_x - 6zpu_{xz} \\
&\quad - puu_{xz} - 2zpuu_{xy} + \frac{1}{2}u_{xz}p_{xx} + zu_{xy}p_{xx} - 2zpuu_{xx} \\
&\quad - 4zpvu_{xx} + zp_{xx}u_{xx} - \frac{1}{2}u_zp_{xxx} + zu_yp_{xxx} - \frac{3}{2}p_{xxx} \\
&\quad - u_zp_t + 2zu_yp_t - 3p_t + pu_{tz} + 2zpu_{ty} + 2zpu_{tx},
\end{aligned}
$$

$$
\begin{aligned}
T_{14}^{z} &= qu_y - 3zp_xu_y + 2pu_xu_y + \frac{3}{2}u_zp_x + \frac{9}{2}p_x + 3zp_xu_x \\
&\quad - qv_x + 2pu_xv_x + \frac{3}{2}pu_{xz} + 3zpu_{xy} + 2puu_{xy} - u_{xy}p_{xx} \\
&\quad - 3zpu_{xx} + 2pvu_{xx} - 2pu_{ty};
\end{aligned}
$$

$$T^t_{15} = qu_y - u_z p_y - 2zu_x p_y - u_t p_y + pu_{yz} + 2pu_y u_x - qv_x + 2pu_x v_x + 3pu_{xz} + 2zpu_{xy} + 2puu_{xy} - u_{xy} p_{xx} + 2pvu_{xx} - pu_{ty},$$

$$T^x_{15} = v_z q + 2zu_y q + v_t q - 3q + \frac{3}{2}p_z u_z - \frac{3}{2}pu_{zz} + uu_z p_y - pu_z u_y - puu_{yz} + 2vu_z p_x + 3pu_x + 3zp_z u_x - 2pv_z u_x + 2zup_y u_x + 2zpu_y u_x + 4zvp_x u_x + 3zpu_{xz} - 2pvu_{xz} + 2zpuu_{xy} + p_{xz} u_{xy} + \frac{1}{2}u_{yz} p_{xx} + zu_{xy} p_{xx} - \frac{1}{2}u_z p_{xxy} - zu_x p_{xxy} - p_z u_{xxy} - 2zp_x u_{xxy} - u_{xxy} p_t + \frac{3}{2}p_z u_t + up_y u_t - pu_y u_t + 2vp_x u_t - \frac{1}{2}p_{xxy} u_t - 2pu_x v_t - \frac{3}{2}pu_{tz} - 4zpu_{ty} - puu_{ty} + \frac{1}{2}p_{xx} u_{ty} + u_{xy} p_{tx} - 2pvu_{tx},$$

$$T^y_{15} = uu_z p_x - 2zpu_x^2 - 2zqu_x - pu_z u_x + 2zup_x u_x - zp_{xxx} u_x - 2zp_t u_x - pu_t u_x - qu_z - puu_{xz} + \frac{1}{2}u_{xz} p_{xx} - 2zpuu_{xx} + zp_{xx} u_{xx} - \frac{1}{2}u_z p_{xxx} - u_z p_t - qu_t + up_x u_t - \frac{1}{2}p_{xxx} u_t - p_t u_t + pu_{tz} + 2zpu_{tx} - puu_{tx} + \frac{1}{2}p_{xx} u_{tx} + pu_{tt},$$

$$T^z_{15} = qu_y + 2pu_x u_y + \frac{3}{2}u_z p_x + 3zp_x u_x - qv_x + 2pu_x v_x + \frac{3}{2}pu_{xz} + 2puu_{xy} - u_{xy} p_{xx} - 3zpu_{xx} + 2pvu_{xx} + \frac{3}{2}p_x u_t - 2pu_{ty} - \frac{3}{2}pu_{tx};$$

$$T^t_{16} = qu_y - p_y u_y + 2pu_x u_y + pu_{yy} - 2zp_y u_x + 3pu_{xz} + 2zpu_{xy} + 2puu_{xy} - u_{xy} p_{xx} + 2pvu_{xx} - p_y u_t - pu_{ty},$$

$$T^x_{16} = 2zqu_y - pu_y^2 + \frac{3}{2}p_z u_y + up_y u_y + 2vp_x u_y + 2zpu_x u_y - \frac{1}{2}p_{xxy} u_y - pu_t u_y - 3q + qv_y - \frac{3}{2}pu_{yz} - puu_{yy} + 3pu_x + 3zp_z u_x + 2zup_y u_x - 2pv_y u_x + 4zvp_x u_x + 3zpu_{xz} + 2zpuu_{xy} - 2pvu_{xy} + p_{xy} u_{xy} + \frac{1}{2}u_{yy} p_{xx} + zu_{xy} p_{xx} - zu_x p_{xxy} - p_y u_{xxy} - 2zp_x u_{xxy} - u_{xxy} p_t + \frac{3}{2}p_z u_t + up_y u_t + 2vp_x u_t - \frac{1}{2}p_{xxy} u_t + qv_t - 2pu_x v_t - \frac{3}{2}pu_{tz} - 4zpu_{ty} - puu_{ty} + \frac{1}{2}p_{xx} u_{ty} + u_{xy} p_{tx} - 2pvu_{tx},$$

$$T^y_{16} = pu_y u_x - 2zpu_x^2 - 2zqu_x + 2zup_x u_x + 2pv_x u_x - zp_{xxx} u_x - 2zp_t u_x - pu_t u_x + uu_y p_x - qv_x + 3pu_{xz} + puu_{xy} - \frac{1}{2}u_{xy} p_{xx} - 2zpuu_{xx} + 2pvu_{xx} + zp_{xx} u_{xx} - \frac{1}{2}u_y p_{xxx} - u_y p_t - qu_t + up_x u_t - \frac{1}{2}p_{xxx} u_t - p_t u_t - pu_{ty} + 2zpu_{tx} - puu_{tx} + \frac{1}{2}p_{xx} u_{tx} + pu_{tt},$$

$$T^z_{16} = \frac{3}{2}u_y p_x + 3zu_x p_x + \frac{3}{2}u_t p_x - \frac{3}{2}pu_{xy} - 3zpu_{xx} - \frac{3}{2}pu_{tx};$$

$$T^t_{17} = pu_{yy} - p_y u_y - 2zp_y u_x + 2zpu_{xy},$$

$$T^x_{17} = 2zqu_y - pu_y^2 + \frac{3}{2}p_z u_y + up_y u_y + 2vp_x u_y + 2zpu_x u_y - \frac{1}{2}p_{xxy} u_y - 3q + qv_y - \frac{3}{2}pu_{yz} - puu_{yy} + 3pu_x + 3zp_z u_x + 2zup_y u_x - 2pv_y u_x + 4zvp_x u_x + 3zpu_{xz} + 2zpuu_{xy} - 2pvu_{xy} + p_{xy} u_{xy} + \frac{1}{2}u_{yy} p_{xx} + zu_{xy} p_{xx} - zu_x p_{xxy} - p_y u_{xxy} - 2zp_x u_{xxy} - 4zpu_{ty},$$

$$T^y_{17} = pu_y u_x - 2zpu_x^2 - 2zqu_x + 2zup_x u_x + 2pv_x u_x - zp_{xxx} u_x - 2zp_t u_x + uu_y p_x - qv_x + 3pu_{xz} + puu_{xy} - \frac{1}{2}u_{xy} p_{xx} - 2zpuu_{xx} + 2pvu_{xx} + zp_{xx} u_{xx} - \frac{1}{2}u_y p_{xxx} - u_y p_t - pu_{ty} + 2zpu_{tx},$$

$$T^z_{17} = \frac{3}{2}u_y p_x + 3zu_x p_x - \frac{3}{2}pu_{xy} - 3zpu_{xx};$$

$$T^t_{18} = qu_y - p_y u_y + 2pu_x u_y + pu_{yy} - qv_x + 2pu_x v_x + 3pu_{xz} + 2puu_{xy} - u_{xy} p_{xx} + 2pvu_{xx} - p_y u_t - pu_{ty},$$

$$T^x_{18} = -pu_y^2 + \frac{3}{2}p_z u_y + up_y u_y + 2vp_x u_y - \frac{1}{2}p_{xxy} u_y - pu_t u_y + qv_y - \frac{3}{2}pu_{yz} - puu_{yy} - 2pv_y u_x - 2pvu_{xy} + p_{xy} u_{xy} + \frac{1}{2}u_{yy} p_{xx} - p_y u_{xxy} - u_{xxy} p_t + \frac{3}{2}p_z u_t + up_y u_t - \frac{1}{2}p_{xxy} u_t + qv_t - 2pu_x v_t - \frac{3}{2}pu_{tz} - puu_{ty} + \frac{1}{2}p_{xx} u_{ty} + u_{xy} p_{tx} + 2vp_x u_t - 2pvu_{tx},$$

$$T^y_{18} = uu_y p_x + uu_t p_x + pu_y u_x - qv_x + 2pu_x v_x + 3pu_{xz} + puu_{xy} - \frac{1}{2}u_{xy} p_{xx} + 2pvu_{xx} - \frac{1}{2}u_y p_{xxx} - u_y p_t - qu_t - pu_x u_t - \frac{1}{2}p_{xxx} u_t - p_t u_t - pu_{ty} - puu_{tx} + \frac{1}{2}p_{xx} u_{tx} + pu_{tt},$$

$$T^z_{18} = \frac{3}{2}u_y p_x + \frac{3}{2}u_t p_x - \frac{3}{2}pu_{xy} - \frac{3}{2}pu_{tx};$$

$$T_{19}^{t} = qu_y - p_y u_y + 2pu_x u_y + pu_{yy} + 2zp_y u_x - qv_x + 2pu_x v_x + 3pu_{xz} - 2zpu_{xy} + 2puu_{xy} - u_{xy}p_{xx} + 2pvu_{xx} - p_y u_t - pu_{ty},$$

$$T_{19}^{x} = -pu_y^2 - 2zqu_y + \frac{3}{2}p_z u_y + up_y u_y + 2vp_x u_y - 2zpu_x u_y - \frac{1}{2}p_{xxy}u_y - pu_t u_y + qv_y - \frac{3}{2}pu_{yz} - puu_{yy} - 3pu_x - 3zp_z u_x - 2zup_y u_x - 2pv_y u_x - 4zvp_x u_x - 3zpu_{xz} - 2zpuu_{xy} - 2pvu_{xy} + p_{xy}u_{xy} + \frac{1}{2}u_{yy}p_{xx} - zu_{xy}p_{xx} + zu_x p_{xxy} - p_y u_{xxy} + 2zp_x u_{xxy} - u_{xxy}p_t + \frac{3}{2}p_z u_t + up_y u_t + 2vp_x u_t + 3q - \frac{1}{2}p_{xxy}u_t + qv_t - 2pu_x v_t - \frac{3}{2}pu_{tz} + 4zpu_{ty} - puu_{ty} + \frac{1}{2}p_{xx}u_{ty} + u_{xy}p_{tx} - 2pvu_{tx},$$

$$T_{19}^{y} = 2zpu_x^2 + 2zqu_x + pu_y u_x - 2zup_x u_x + 2pv_x u_x + zp_{xxx}u_x + 2zp_t u_x - pu_t u_x + uu_y p_x - qv_x + 3pu_{xz} + puu_{xy} - \frac{1}{2}u_{xy}p_{xx} + 2zpuu_{xx} + 2pvu_{xx} - zp_{xx}u_{xx} - \frac{1}{2}u_y p_{xxx} - u_y p_t - qu_t + up_x u_t - \frac{1}{2}p_{xxx}u_t - p_t u_t - pu_{ty} - 2zpu_{tx} - puu_{tx} + \frac{1}{2}p_{xx}u_{tx} + pu_{tt},$$

$$T_{19}^{z} = \frac{3}{2}u_y p_x - 3zu_x p_x + \frac{3}{2}u_t p_x - \frac{3}{2}pu_{xy} + 3zpu_{xx} - \frac{3}{2}pu_{tx}.$$

3.2.1 Physical meaning of some of the obtained conservation laws in physical sciences and engineering

Local conservation laws for the higher dimensional integrodifferential soliton Eq. (1.6) observe a divergence criterion on the whole space ε solution of Eq. (1.6)

$$(D_z T^z + D_y T^y + D_x T^x + D_t T^t)|_{\varepsilon} = 0, \tag{3.15}$$

where (D_z, D_y, D_x, D_t) are total differential operators with conserved density T as well as T^z, T^y, T^x, which are spatial fluxes are functions depending on (z, y, x, t). In physical sense, we observe that every conservation law generates a related conserved integral [67,68]

$$\mathcal{P}[u, v] = \int_{\Gamma} C^t \mathrm{d}x, \tag{3.16}$$

as can be obtained in this case, where Γ denotes the solution domain $u(t, x, y, z)$. It has been established that conservation laws with discontinuous coefficients, such as fluxes, densities as well as source terms, emerge in quite a large number of problems in the fields of physics and engineering. In this research, the resultant conservation laws have been observed to have relevance in physical sciences. Physically speaking, part of the obtained symmetries that delineate conserved quantities, *viz*, conservation of energy, momentum, and so on. More precisely, it is observed that time translation symmetries purvey energy, whereas in the case of space translation alongside boost symmetries, momenta are attained.

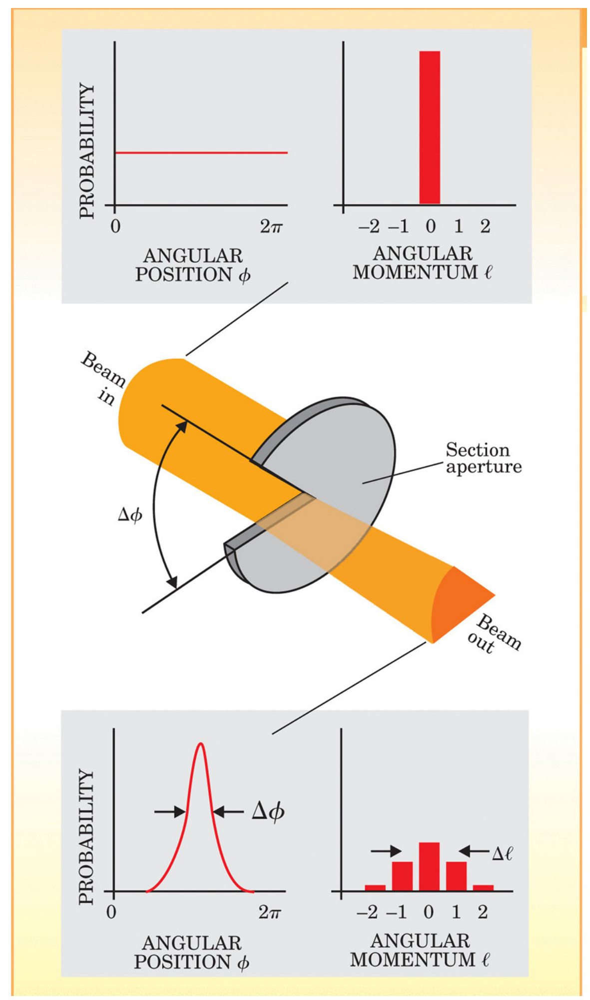

Figure 1: In an experiment recently conducted at Glasgow University, Heisenberg's uncertainty principle is manifested in which a light beam that initially possesses no orbital angular momentum was observed. We define $\Delta\phi$ as the range of azimuthal angular positions appearing for a photon in a cross-sectional part of the beam through a sector aperture. Moreover, at the upstream of the said aperture, the beam is observed to be in an $\ell = 0$ eigen state of orbital angular momentum (*i.e.*, upper panel). Nonetheless, the uncertainty principle states that the limitation in ϕ causes a spread in orbital angular momentum represented as $L = \ell\hbar$ for every photon. Additionally, for strait apertures, the connection is portrayed as $\Delta\phi\Delta L \geq \hbar/2$ [74].

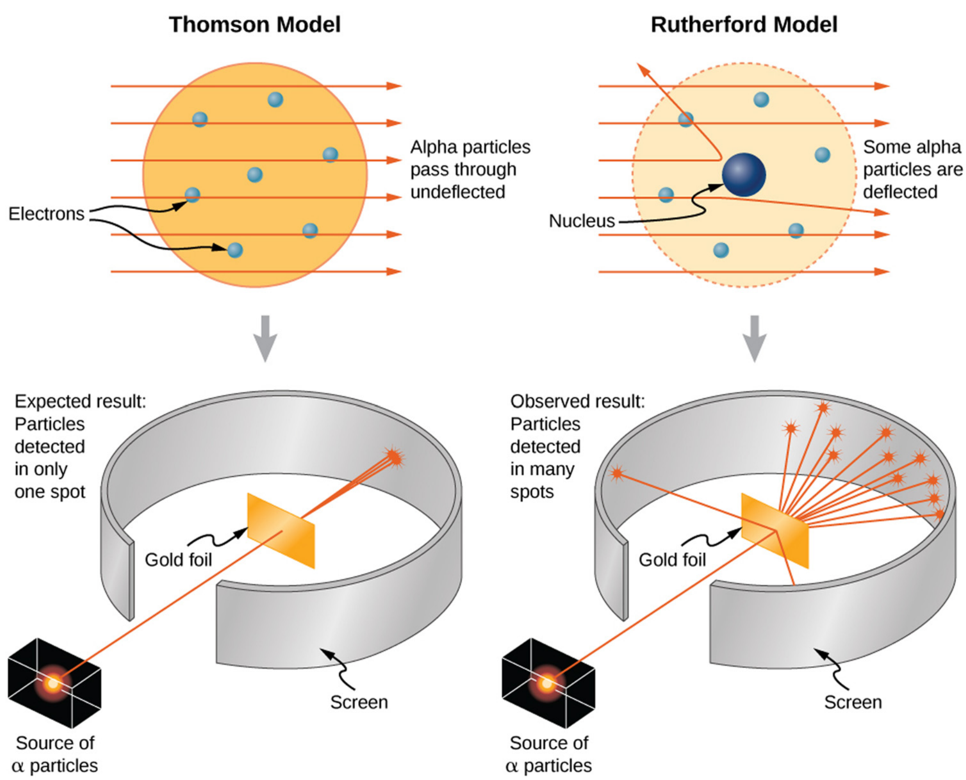

Figure 2: In the models of atom introduced by Rutherford and Thomson. The later postulated a model that predicted that virtually at small angles, all of the incident alpha particles would be dispersed. Geiger and Rutherford discovered that almost none of the alpha particles were dispersed. However, *via* a very large angle, a few were found to deflect. Thus, the outcome of the Thomson model was in disagreement with Rutherford's experiments. Rutherford engaged conservation of energy alongside momentum in the development of a new, and much better model of the atom, and that is the nuclear model [75].

Theorem 3.4. *The set of 19 non-trivial local conservation laws generated by HD-SOLeqn* (1.6) *via Ibragimov's theorem comprise components* $(T^t_j, T^x_j, T^y_j, T^z_j, j = 1, 2, \ldots, 19)$, *with density as the first component and fluxes constituting the other components.*

Consequently, in the case of HD-SOLeqn (1.6), for example, conserved vectors $(T^t_1, T^x_1, T^y_1, T^z_1)$ yield conservation of energy obtained as

$$\mathcal{P}_1[u,v] = \int_\Gamma (2u_x v_x p + 2u_{xx} pv + 2u_x u_y p + 3u_{xz} p + 2u_{xy} pu - u_{ty} p + u_y q - v_x q - p_y u_t - p_{xx} u_{xy})\mathrm{d}x, \tag{3.17}$$

for solution $u(t,x,y,z)$, corresponding to time translation symmetry. Besides, conserved vectors $(T^t_2, T^x_2, T^y_2, T^z_2)$, accordingly give the conserved quantity of momentum explicated as

$$\mathcal{P}_2[u,v] = \int_\Gamma (u_{yz} p - p_y u_z)\mathrm{d}x, \tag{3.18}$$

which relates to the corresponding space translation symmetry. In addition, one obtains conserved quantity of dilation energy as

$$\mathcal{P}_3[u,v] = \int_\Gamma (2yu_y p_y - 2up_y - xu_x p_y - 3tu_t p_y + 3tqu_y - 2ypu_{yy} + 6tpu_y u_x - 3tqv_x + 6tpu_x v_x + 9tpu_{xz} + xpu_{xy} + 6tpuu_{xy} - 3tu_{xy} p_{xx} + 6tpvu_{xx} - 3tpu_{ty})\mathrm{d}x, \tag{3.19}$$

which correspond to $(T^t_3, T^x_3, T^y_3, T^z_3)$, which furnishes conservation laws associated with scaling symmetry. Moreover, another conserved quantity of dilation energy is obtainable as

$$\mathcal{P}_4[u,v] = \int_\Gamma (2pu_y - 2up_y - 2zu_z p_y - xu_x p_y - 3tu_t p_y + 3tqu_y + 2zpu_{yz} + xpu_{xy} + 6tpu_y u_x - 3tqv_x + 6tpu_x v_x + 9tpu_{xz} + 6tpuu_{xy} - 3tu_{xy} p_{xx} + 6tpvu_{xx} - 3tpu_{ty})\mathrm{d}x,$$

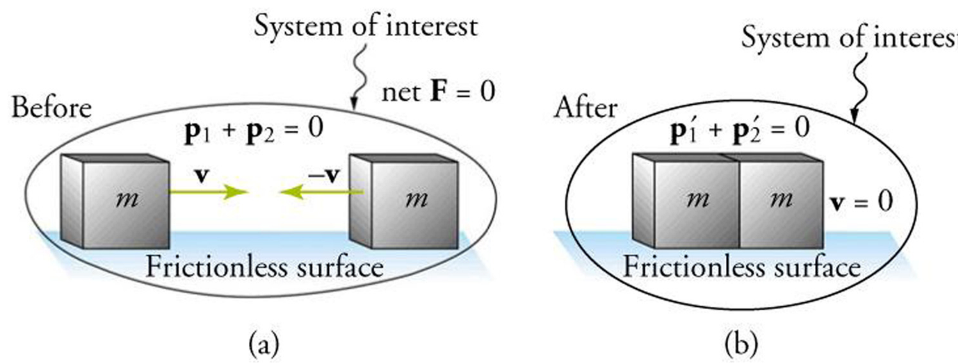

Figure 3: We have a uni-dimensional inextensible collision in play between two objects. Momentum is conserved, whereas kinetic energy is found not to be conserved. In (a), two objects that are initially having equal mass, move directly toward each other at a uniform speed. In the case of (b), the objects cohere, making way for a perfectly inductile collision. Therefore, the consolidated objects stop and that is the instance revealed in this figure. However, this is untrue for all unyielding collisions [76].

corresponding to $(T_4^t, T_4^x, T_4^y, T_4^z)$. The rest could also be studied in the same way. For more interesting insights, see the study of Adeyemo and Khalique [68]. In particle physics, various other conservation laws apply to subatomic particles' properties which are invariant in the course of interactions. In addition, an essential function of conserved vectors is the fact that they occasion the possibility of making prediction for the macroscopic character of a system without taking cognizance of the microscopic details in the course of a chemical reaction or physical process [69].

3.3 Application of conserved quantities in physical sciences and engineering

Frankly speaking, conservation law, also referred to as the law of conservation, in physics, depicts a principle that states that a certain physical property (*i.e.*, a measurable quantity) remains unchanged with time within an isolated physical system. Thus, in physical sciences and engineering mathematics, conservation laws state that a particular measurable property associated with an isolated physical system remains constant (that is does not change) as the system evolves over time [70].

Conservation laws [71–73] alternatively, are an area of applicable engrossment in engineering and physics, with the inclusion of theoretical *vis-à-vis* quantum-mechanics. This part of our research scrutinizes the conserved quantities of model HD-SOLeqn (1.6) with an observable trait that showcases results unveiling the availability of conservation of momentum alongside that of energy. Physical quantities resident in isolated systems, namely, mass, charge, angular momentum, energy, along with linear momentum are conserved. Meanwhile, it is unveiled that conserved quantities invoke an advantageous feature of DE ntegrability check. Further to that, conserved quantities significantly engender

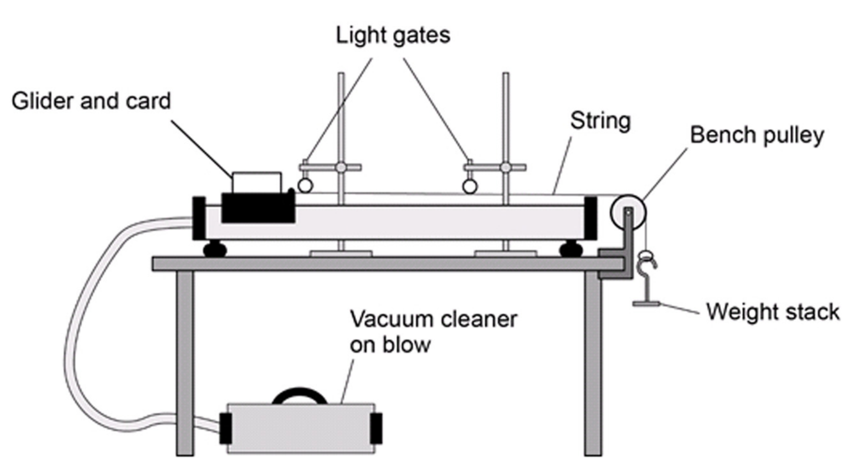

Figure 4: A graphical depiction of experimental relationship between force and acceleration – Newton's law [77].

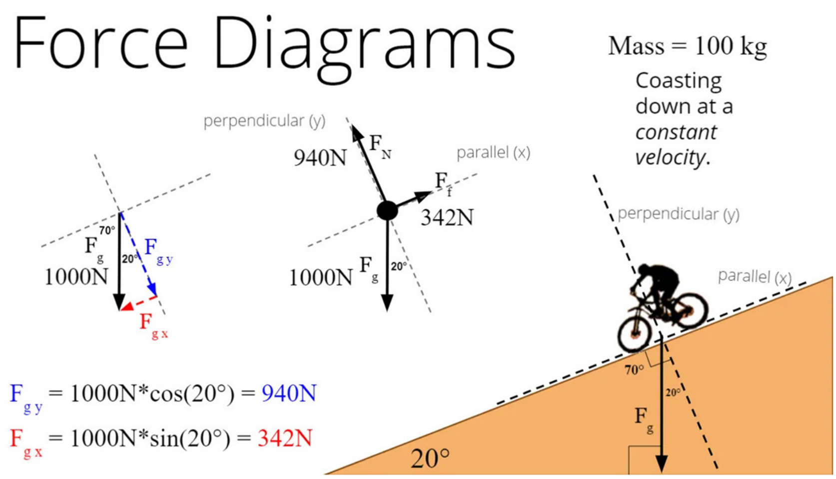

Figure 5: A diagrammatic representation of relationship between force, weight, and mass [78].

the establishment of existence-uniqueness characteristic for linearization mappings and solutions. Besides, they bring to play stability analysis and to ascertain global behavior of solutions.

In addition, conserved quantities play a leading role in the evolution of numerical techniques. They also furnish a crucial starting point in securing non-locally related systems and potential variables. In particular, a conserved quantity is fundamental in the investigation of a given DE, which implies that it holds for any posed data whether initial conditions and/or boundary conditions. Furthermore, the conservation law structure is such that it is not depending on co-ordinates since it involves a contact transformation mapping one to the other. Momentum is simply referred to as the resistance of a given object with regards to an alteration in the object's velocity. Engineers employ this concept to make lives safer for people through the design of products in lengthening the time over which a deceleration happens.

Moreover, momentum is crucial in Physics due to the fact that it recounts the connection that is existing between mass, speed, as well as direction. Besides, it also delineates the force required to put an object to a halt and to further keep them in motion. It is to be noted that an apparently small object can deploy a large amount of force, provided it possesses sufficient momentum. Figures 1 and 2 present some practical applications of momentum.

Consequently, understanding momentum makes it possible for engineers to have an understanding of various kinds of collisions (Figure 3). As a result, having the knowledge can assist in ensuring that cars are much safer, predict the results observed when two objects bang into each other, or investigate the proof of a traffic accident. One typical example is the utilization of air bags in automobiles. Using air bags in automobiles makes it possible for the effect of the force exerted on an object that is involved in a collision to be minimized. Air

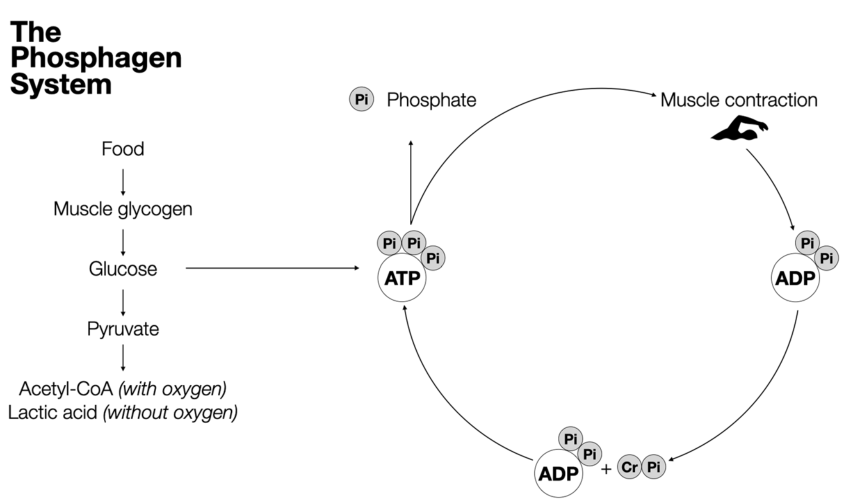

Figure 6: A diagrammatic representation revealing the Phosphagen system [80].

bags attain this by lengthening the required time in stopping the momentum of the driver as well as passenger. It is to be acknowledged that whether it is achieved or otherwise, situations involving momentum and impulse can be experienced everywhere.

Mass can be defined as a measure of the quantum of material present in an object, being directly associated with the type and number of atoms that is found in the object. Mass remains constant with regards to the position of the involved body, its movement or alteration of its shape, except material is removed or added.

In engineering science, mass is used to signal the size of something. Mass which is measured in grams or kilograms likewise referred to as drive is a measure of anxiety, which makes it important in human lives.

Moreover, weight and mass are essential in engineering due to the fact that the greater the mass of any given object is, the greater the force required in accomplishing the same change in motion needed. Thus, for a given object, a larger force initiates a larger change in motion. Figure 4 shows a set-up relaying the relationship between force and acceleration. Besides, Figure 5 portrays some connections among force, weight, and mass.

Besides, mass is crucial in science as a result of two major factors affecting the movement of things in space: gravity and inertia. The more mass, an object possesses, the more experience it has regarding both properties. That is the reason heavy things (things with large mass) are difficult to move. Next, energy quantity implies energy quantum found in a certain volume of natural gas expressed in kilowatt hour (kWh).

People walk, ride bicycles, move cars along roads as well as boats through water, cook food on stoves, make ice in freezers, light our homes together with offices, manufacture products, and also send astronauts into space using energy. There are various different forms of energy, inclusive of heat, mechanical, electrical, and so on.

Therefore, energy systems are utilized on daily basis by humans to make life easy. Some of these ways include washing clothes, watching television, taking a shower, heating and lighting the home, working from home on desktop computers or laptop, running appliances, cooking, and so on. Residential uses of energy on a global scale account for almost 40% of total energy utilized.

More applications of energy is found in the phosphagen system (Figure 6) that is active during all-out exercise lasting for about 5–10 s including a 100-m dash, diving, lifting a heavy weight, jumping, dashing up a flight of stairs, or any other scheme that engages a maximal, short burst of power [79].

4 Conclusion

In this article, we clearly purveyed a conspectus investigation carried out on the HD-SOLeqn (1.6). We engage the universal technique, namely, Lie group analysis to which when engaged to solve a DE, it occasions the methodical procedures of generating Lie point symmetries of such equation. In the study, a demonstration of the robust usefulness of the aforementioned technique assisted us in performing a detailed and comprehensive construction of a one-dimensional optimal system of the Lie subalgebras for the nine-dimensional Lie algebra obtained for the equation, which affords us the chance to obtain various more general and robust combinations of the symmetries. Moreover, owing to the relevance of conserved quantities in the study of DEs, we attained diverse associated conservation laws to the HD-SOLeqn using the Lie subalgebras, where various quantities such as conservation of energy are derived. In consequence, this study clearly highlighted the importance of soliton solutions of higher-dimensional NLNPDEs in physics and engineering mathematics and the robust application of the Lie group theory of DEs in proffering solutions to them. Therefore, this research can be beneficial in various fields and in particular in the research area of physical sciences and engineering. Particularly, in an area where further analysis of the result could be of immense usefulness.

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