Research Article

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Model of conversion of flow from confined to unconfined aquifers with stochastic approach

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Abstract: This work deals with the conversion of flow from confined to unconfined aquifers, a real-world problem that has attracted the attention of several authors. We have introduced a piecewise modified mathematical model where the first part deals with the flow within a confined system, and the second part deals with the flow within an unconfined system. In the unconfined part, we added the randomness to capture stochastic behaviours that could occur due to the geological formation. Moreover, we used a numerical method to solve the stochastic differential equations. The obtained model was evaluated numerically using some numerical scheme, and the stability analysis was performed using the von Neumann approach and the numerical simulations were presented.

Keywords: stochastic approach, stochastic and deterministic models, Lagrange polynomial method, confined aquifer, unconfined aquifer

1 Introduction

Real-world problems are generally modelled using two types of approaches: deterministic and stochastic models. Deterministic models have state variables that are distinctively specified by model parameters and sets of these variables' prior states [1]. For this, deterministic models behave the same for both a given set of parameters and initial conditions; however, their solution is different for a different set of initial conditions and parameter values [2,3]. However, deterministic models can be uncertain, implying that even the smallest changes in the parameters regulating the physical problem or the initial condition can have a significant impact on the solution [2,4,5]. The models allow us to precisely compute events that are yet to come without including randomness [6]. Hence, if a problem is deterministic, one has all the information needed to accurately predict the results with certainty [7]. It is presumed that all the given input parameters are known with certainty in time and space; as a result, a deterministic value of every parameter can be allocated [8,9]. The models have been used with great success to depict physical processes that show power-law, fading memory to power-law, and are good for capturing memory processes [10–13]. However, deterministic models are sometimes unstable, and this implies minor deviations caused by outside influences on the fundamental parameters governing the physical problem, which leads to weighty errors in the forecast, and thus, the intended goal of a numerical model cannot be reached [1]. It is indeed that these deterministic models fail to depict real-world problems, which show a kind of randomness [14,15]. On the other hand, stochastic models are the other way around. The models have been used in many physical problems with great success as they were introduced to deal with randomness [16–18]. Stochastic models are mathematical models that consist of parameters that include in their formulation random variables or distributions instead of single values [3,19]. Therefore, groups of probable solutions will result from using the same parameters and initial conditions, giving the researcher the task of analysing the underlying uncertainty of the physical problem being described [20]. Stochastic models have been used with great success to depict real-world problems that provide more than one possible outcome, hence making them useful for future predictions. Like any other model, stochastic models also have some limitations. Stochastic models can be more complex to carry out and may demand more thorough computational and statistical capacity than some simple deterministic models [18,21]. Therefore, making the results more difficult to describe than simple deterministic models [14].

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For this study, stochastic models are focused on [22]. One of the main weaknesses of deterministic models is that they do not give clear reasoning or explanation for uncertainty [23]. This limitation can cause problems due to nature being inherently heterogeneous and the system's just being computed at distinct (or sometimes few) places [1,24]. Early theories assumed that all media is homogeneous, but it was later found that in the real world, the assumption does not apply to natural formations. For instance, in the field of hydrogeology, these assumptions are false due to the heterogeneous nature of hydrogeological parameters that occur in aquifers [8,25]. To capture random behaviour, a stochastic approach is introduced. This will help us quantify and calculate the uncertainty and understand complex flow due to heterogeneity that exists in underground systems. The approach will make it easier to deal with hydrogeological parameters in the aquifer system and the prediction of uncertainties to increase confidence in making predictions in our generated mathematical models.

2 Equation solutions for confined and unconfined aquifers

The model under investigation describes the conversion of flow from confined to unconfined aquifers. Confined groundwater flow is considered the principal route for transporting water from recharge regions to wells and springs [26]. Theis [27] derived the basic equation of unsteady flow toward the well in 1935 using a comparison between groundwater flow and heat conduction. Therefore, flow in confined aquifers is captured by the following equation:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}, \tag{1}$$

where h is the hydraulic head, L; r is the radial distance from the well, L; S is the storage coefficient; T is the transmissivity, L^2/T; and t is the time since pumping started, T.

Boulton [28] extended the Theis transient confined theory to include the effect of the water table in unconfined aquifers due to the nonlinearity of unconfined aquifers, the integro-differential partial differential equation is given by:

$$\frac{S}{T}\frac{\partial h}{\partial t} = \frac{\partial h}{r\partial r} + \frac{\partial^2 h}{\partial r^2} - \alpha S_y \int_0^t \frac{\partial h}{\partial \tau} \exp[-\alpha(t-\tau)]\mathrm{d}\tau, \tag{2}$$

where S_y is the specific yield of the unconfined aquifer; α is the empirical constant, a reciprocal of the delay index, T^{-1}; and $e^{-\alpha(t-\tau)}$ is the exponential delay index.

The following system of partial differential equations is used to represent flow in confined and unconfined aquifers in this study, respectively:

$$\begin{cases} \dfrac{S_c}{T}\dfrac{\partial h}{\partial t} = \dfrac{1}{r}\dfrac{\partial h}{\partial r} + \dfrac{\partial^2 h}{\partial r^2}, h \geq b \\ \dfrac{S}{T}\dfrac{\partial h}{\partial t} = \dfrac{1}{r}\dfrac{\partial h}{\partial r} + \dfrac{\partial^2 h}{\partial r^2} - \alpha S_y \displaystyle\int_0^t \dfrac{\partial h}{\partial \tau} \exp[-\alpha(t-\tau)]\mathrm{d}\tau, \\ 0 < h \leq b. \end{cases} \tag{3}$$

While the above model has been used in several situations, it is worth noting that it does not consider randomness that could occur due to the complexity of the geological formations. Therefore, it cannot replicate random behaviours that could arise due to the complexities of these media. Hence, a model that considers these factors is needed and an attempt will be made in Section 3.

3 Conceptual model

Several complexities exist in nature that cannot be avoided; therefore, various mathematical concepts have been developed to understand and capture the complexities of the nature that we live in. To increase confidence in prediction, the idea of modelling in time and space was used, and advanced software was developed to capture the problems with high complexities [23]. Two common methods have been used to depict nature and its complexities, which are non-local operators and the stochastic approach [29–32]. Both methods are different and are used for different physical problems in modelling. Recently, there have been advancements in analytical and numerical solutions for non-local operators [33,34]. The concept of the stochastic approach depicts the heterogeneous nature of a closed system for a Markovian process [35], whereas the concept of non-local operators depicts non-Markovian processes, particularly when no local operators have any kind of index law properties [36]. The two concepts are different; however, recently Atangana and Bonyah [36] have combined the two to suggest a methodology that will be used in the future to model complex physical problems.

Groundwater systems are open and complex systems that are influenced by factors such as hydrological conditions, geological structure, and topography, to name a few [37–39]. Therefore, several aquifer system characteristics cannot be monitored directly; hence, they are measured indirectly by evaluating the input and output measurements [40]. Considering the model suggested in this study, due to the random nature of the aquifer system, randomness can occur in space or time. In addition to this, the flow from confined to unconfined aquifers becomes a stochastic change over time. The flow in the aquifer during the process may take long periods, but the inherent uncertainty in time cannot be removed. Therefore, an accurate time series of the process cannot be acquired. Even though a hydrological parameter such as conductivity is low in the aquitard, it is considered uncertain in space. This is due to the difficulty in measuring hydrogeological parameters at every point of a model. Hence, the stochastic approach is introduced to depict the random setting of nature in a larger time and space. This will help develop a predictive model and get reliable results for the conversion of flow. Several researchers over the years have used mathematical models that have been proposed in the past to model the conversion of flow [41–46]. The models, which used differential and integral operators based on the concept of rate of change, were developed to capture the conversion based on one type of aquitard setting. Although various fields have successfully used these operators, researchers found that they could only be used to express classical mechanical problems with no memory [47]. Hence, in this study, a stochastic approach is introduced to capture the complex nature and uncertainties of the aquifer systems, in particular randomness that may occur. This approach will help in giving a better representation of the conversion of flow that occurs in the real world.

The model under investigation is in comparison with the existing model, the Moench and Prickett model [46] (Figure 1), with similar assumptions. The conceptual model consists of a confined aquifer with a horizontal initial piezometric head, h. A pumping well fully penetrates the aquifer and discharges at a constant rate, Q. When the groundwater level falls below the upper border of the confined aquifer, the piezometric surface continues to fall below the overlaying aquitard, forming an unconfined zone near the well. With continuous pumping, the hydraulic head in the pumping well, h_x drops below the top of the confined aquifer, $0 < h_x \leq b$ the confined unit's thickness is given by $[L]$. This gives rise to an unconfined flow with a radial distance of $0 < h_x \leq R$, where r $[L]$ is the radial distance from the pumping well and R $[L]$ is the radial distance between the pumping well and the transient conversion contact [48].

The addition of a random element to a deterministic differential equation results in a change from an ordinary differential equation (ODE) to a stochastic differential equation (SDE) [49,50], and SDEs generalize ODEs by introducing random noise into the dynamics [51]. SDEs were also employed in geological investigations to derive accounts of particle size distributions [52] and were later used to investigate flow in heterogeneous porous media [53]. The stochastic approach proposed by Freeze [54] on the field of flow in porous media opened the door for stochastic modelling in hydrological studies. In this study, the randomness will represent the inflow of water due to recharge or water trap that is being released due to force induced during abstraction. However, analytical solutions for these equations are not always available; thus, researchers rely on numerical approaches to approximate the solution.

Considering the general form of an SDE given by

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \tag{4}$$

where B_t denotes the standard Brownian motion. The drift term is represented by $\mu(X_t)\mathrm{d}t$, this explains the deterministic portion of the equation, and in the case where this is the only term, a canonical ODE is acquired [55]. The diffusion term is denoted by $\sigma(X_t)\mathrm{d}B_t$, this explains random movement proportional to a Brownian motion; in the case of small times, the diffusion term makes the probability to disperse diffusively with a diffusivity exclusively proportional to σ^2 [55]. Thus, to include in mathematical formulation the randomness effects that could occur due to complexities of geological formation, in particular in the unconfined part, the existing model is converted to

$$\begin{cases} \dfrac{S_c}{T}\dfrac{\partial h_2}{\partial t} = \dfrac{1}{r}\dfrac{\partial h_2}{\partial r} + \dfrac{\partial^2 h_2}{\partial r^2}, h_x \geq b, \\ \dfrac{S}{T}\dfrac{\partial h_1}{\partial t} = \dfrac{1}{r}\dfrac{\partial h_1}{\partial r} + \dfrac{\partial^2 h_1}{\partial r^2} - \alpha S_y \displaystyle\int_0^t \dfrac{\partial h_1}{\partial \tau}\exp[-\alpha(t-\tau)\mathrm{d}\tau] \\ \quad + \sigma h B(t), 0 < h_x \leq b, \\ h_2(r\to\infty, t) = h_x, \ \lim\limits_{r\to 0}\dfrac{\partial h_2(r,t)}{\partial r} = -\dfrac{Q}{2\pi r}, \\ \dfrac{\partial h_1}{\partial r}\Big|_{r=R} = \dfrac{\partial h_2}{\partial r}\Big|_{r=R}, \\ h_1 = h_2 = b, \end{cases} \tag{5}$$

where b is the aquifers thickness, L; h_x is theinitial head, L; h_2 is the elevation of the piezometric surface in the confined unit, L; h_1 is the elevation of the piezometric surface in the unconfined unit, L; $B(t)$ is th the environmental noise, L; σ is the fluctuations in the water level, L; S_c is the storage coefficient for confined zone; S_y is the storage coefficient for unconfined zone; and Q is the discharge rate.

The above equation represents the transient confined to the unconfined flow of the conceptual model, respectively. It is assumed that the process within the confined part obeys the Theis conditions, which can be found in the study by Kruseman and de Ridder [56]. In this study, we will not stress finding the exact solution of the confined

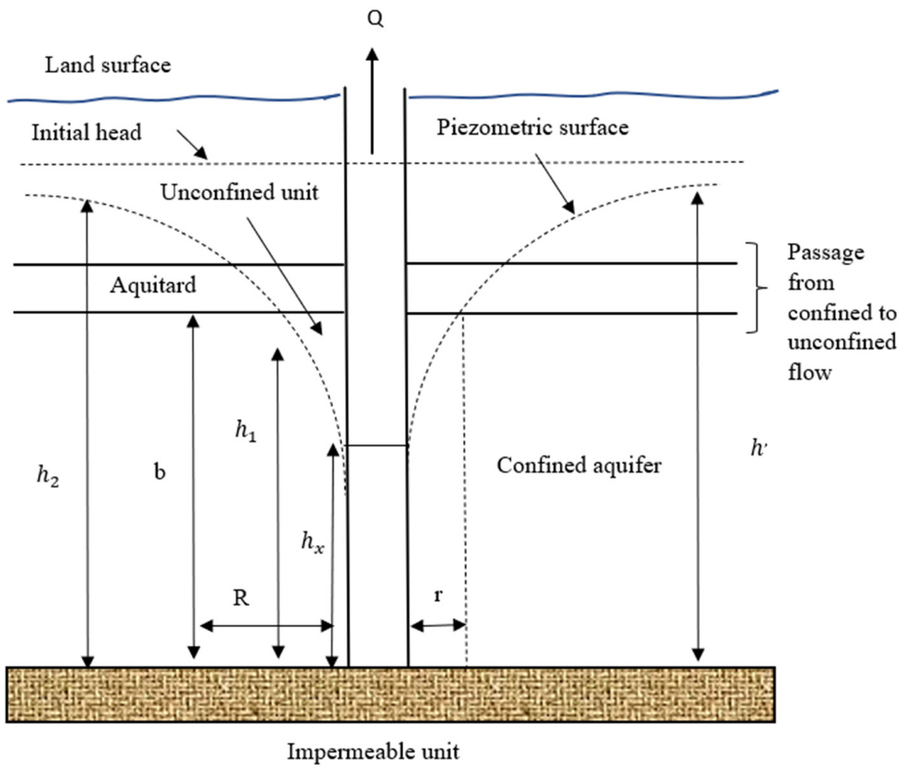

Figure 1: Schematic illustration of confined to unconfined flow towards a completely penetrated well passing through an overlaying aquitard (sandy clay) in a confined aquifer [48].

aquifer part since this solution has been already obtained in the literature as

$$h_2(r, t) = \frac{Q}{4\pi T}\int_u^\infty \frac{\exp(-t)}{t}\mathrm{d}t, u = \frac{Sr^2}{4Tt}.$$

Therefore, we will only focus on the derivation of the numerical scheme of the stochastic part.

4 Numerical scheme for a general stochastic partial differential equation

Partial differential equations replicate processes as a function of space and time. These types of equations can be classified into two major classes, as have been recorded in the literature, deterministic and stochastics, which have been discussed earlier in this study. Stochastic equations are used to capture processes that show some randomness as a function of time and space. They are used in applications for several real-world problems; for instance, the conversion of flow from confined to unconfined aquifers. Several of these equations are nonlinear [57], thus they cannot be solved easily using analytical methods. Therefore, researchers used numerical schemes to provide a numerical solution for future predictions. In this section, we shall consider a general nonlinear stochastic equation and present an application of a numerical scheme based on the Lagrange interpolation formula [33].

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = f(x,t,u(x,t)) + \sigma u(x,t)B(t), \\ u(x,0) = g(x), \\ u(0,t) = l(t). \end{cases} \tag{6}$$

Assuming, $g(x,)$ and $l(t)$ are conditions are initial and bounded conditions, respectively, the function $f(x,t,u(x,t))$ is bounded and twice differentiable with respect to the variable t and n-times differentiable with respect to the variable x. The function $B(t)$ may not be differentiable.

We assume that $t_0 < t < T$ and $x_0 < x < L$.

$\forall i \in [1, ...M],\ x_i - x_{i-1} = \Delta x$.

$\forall j \in [1, ...N],\ t_j - t_{j-1} = \Delta t$.

To solve our equation, we first apply, as a routine, the integral on both sides, and obtain the following equation. This is achieved due to the fundamental theorem of calculus, and the general aim is to obtain an integral equation that will be further discretized using polynomial interpolation.

$$u(x,t)-u(x,0)=\int_0^t f(x,\tau,u(x,\tau))\mathrm{d}\tau + \sigma\int_0^t u(x,\tau)\mathrm{d}B(\tau). \tag{7}$$

We consider that $\Delta x = x_{i+1} - x_i$ and $\Delta t = t_{n+1} - t_n$.
And fixing $x = x_i$.

To discretize, we consider the coupling point (x_i, t_{n+1}), then the following expression is obtained:

$$u(x_i,t_{n+1})-u(x_i,t_0)=\int_{t_0}^{t_{n+1}} f(x_i,\tau,u(x_i,\tau))\mathrm{d}\tau + \sigma\int_{t_0}^{t_{n+1}} u(x_i,\tau)\mathrm{d}B(\tau). \tag{8}$$

As a routine, we consider again the couple points (x_i, t_n), then the following expression is obtained:

$$u(x_i,t_n)-u(x_i,t_0)=\int_{t_0}^{t_n} f(x_i,\tau,u(x_i,\tau))\mathrm{d}\tau + \sigma\int_{t_0}^{t_n} u(x_i,\tau)\mathrm{d}B(\tau). \tag{9}$$

Now to proceed, we subtract as in the standard derivation of the well-known Adams–Bashforth approach for ODEs Eq. (9) from Eq. (8) to obtain

$$u(x_i,t_{n+1})=u(x_i,t_n)+\int_{t_n}^{t_{n+1}} f(x_i,\tau,u(x_i,\tau))\mathrm{d}\tau + \sigma\int_{t_n}^{t_{n+1}} u(x_i,\tau)\mathrm{d}B(\tau). \tag{10}$$

At this point, since the function f is nonlinear, we are using the Lagrange interpolation approach to approximate the function within the interval $\tau \in [t_{n-1}, t_n]$ for the deterministic part to have

$$f(x_i,\tau,u(x_i,\tau)) \approx P_j(\tau) = \frac{\tau-t_{n-1}}{\Delta t} f(x_i,t_n,u(x_i,t_n)) - \frac{\tau-t_n}{\Delta t} f(x_i,t_{n-1},u(x_i,t_{n-1})). \tag{11}$$

Thus, replacing the above in Eq. (10) and integrating on both sides and rearranging gives

$$u(x_i,t_{n+1})=u(x_i,t_n)+\frac{3}{2}\Delta t f(x_i,t_n,u(x_i,t_n)) - \frac{\Delta t}{2} f(x_i,t_{n-1},u(x_i,t_{n-1})) + \sigma\int_{t_n}^{t_{n+1}} u(x_i,\tau)\mathrm{d}B(\tau). \tag{12}$$

However, the discretization of the stochastic part is obtained according to the properties of the function $B(t)$. Thus, if this function $B(t)$ could be differentiable, we have

$$\int_{t_n}^{t_{n+1}} u(x_i,\tau)\mathrm{d}B(\tau) \approx u(x_i,t_n)(B(t_{n+1})-B(t_n)). \tag{13}$$

Replacing the above in Eq. (12), we obtain

$$u(x_i,t_{n+1})=u(x_i,t_n)+\frac{3}{2}\Delta t f(x_i,t_n,u(x_i,t_n)) - \frac{\Delta t}{2} f(x_i,t_{n-1},u(x_i,t_{n-1})) + \sigma u(x_i,c_n)(B(t_{n+1})-B(t_n)). \tag{14}$$

Important note: To start the above scheme, $u(x_i, 0)$ is obtained *via* the Euler forward method.

$$u(x_i,t_1)=u(x_i,t_0)+\Delta t f(x_i,0,u(x_i,0)). \tag{15}$$

If $B(t)$ is not differentiable, then,

$$\int_{t_n}^{t_{n+1}} u(x_i,\tau)\mathrm{d}B(\tau) \approx u(x_i,c_n)[B(t_{n+1})-B(t_n)], \tag{16}$$

where $c_n \in [t_n,t_{n+1}]$. Thus, the scheme is given by

$$u(x_i,t_{n+1})=u(x_i,t_n)+\frac{3}{2}\Delta t f(x_i,t_n,u(x_i,t_n)) - \frac{\Delta t}{2} f(x_i,t_{n-1},u(x_i,t_{n-1})) + \sigma u(x_i,c_n)[B(t_{n+1})-B(t_n)]. \tag{17}$$

Therefore, the numerical solution obtained from this approach yields

$$\begin{cases} u_i^{n+1}=u_i^n+\frac{3}{2}\Delta t f(x_i,t_n,u(x_i,t_n)) \\ -\frac{\Delta t}{2} f(x_i,t_{n-1},u(x_i,t_{n-1})) + \sigma u(x_i,t_n)(B(t_{n+1})-B(t_n)) \\ u(x_i,0)=g(x_i) \\ u(0,t_n)=l(t_n). \end{cases} \tag{18}$$

If $B(t)$ is differentiable,

$$\begin{cases} u_i^{n+1}=u_i^n+\frac{3}{2}\Delta t f(x_i,t_n,u(x_i,t_n)) \\ -\frac{\Delta t}{2} f(x_i,t_{n-1},u(x_i,t_{n-1})) + \sigma u(x_i,c_n)(B(t_{n+1}) \\ \quad - B(t_n));\ c_n \in [t_n,t_{n+1}] \\ u(x_i,t_1)=u(x_i,0)+\Delta t f(x_i,0,u(x_i,0)) \\ u(x_i,0)=g(x_i). \end{cases} \tag{19}$$

We stress that we need two components to start the process, the first component is obtained *via* initial condition, and the second component can be obtained using the simple Euler approach, therefore $n \geq 1$.

5 Application to convert from confined to unconfined model

The system of equations that governs the conversion from confined to unconfined flow has a nonlinearity in the second equation. While a derivation of the exact solution could be achieved using some integral transform like Laplace and Sumudu, however, we foresee some complications in obtaining the inverse Laplace or Sumudu transform. To avoid such a situation, we employed the presented numerical scheme above to derive a numerical solution. However, for simplicity, we let

$$f_1(r,t,h(r,t)) = \frac{T}{S_C}\left(\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2}\right), h > b, \tag{20}$$

$$\begin{aligned} f_2(r,t,h(r,t)) = \frac{T}{S}\Bigg\{&\frac{1}{r}\frac{\partial h}{\partial r} + \frac{\partial^2 h}{\partial r^2} \\ &- \alpha S_y \int_0^t \frac{\partial h}{\partial \tau}\exp[-\alpha(t-\tau)]d\tau\Bigg\}. \end{aligned} \tag{21}$$

According to the suggested approach, we have the following system of iterative formula,

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{3}{2}\Delta t\frac{T}{S_C}\left\{\frac{1}{r_i}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} + \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2}\right\} \\ &- \frac{1}{2}\Delta t\frac{T}{S_C}\left\{\frac{1}{r_i}\frac{h(r_{i+1},t_{n-1}) - h(r_{i-1},t_{n-1})}{2\Delta r} + \frac{h(r_{i+1},t_{n-1}) - 2h(r_i,t_{n-1}) + h(r_{i-1},t_{n-1})}{\Delta r^2}\right\}, \\ h > b. \end{aligned} \tag{22}$$

In Eq. (22), one can see the presence of $n-1$; this could confuse especially those who are not used to multiple steps approaches. We then stress that the process to be activated first needs two steps. The first step is of course the initial condition, and the second step is presented below, which is obtained from the Euler forward method between 0 and t_1 as presented in Eq. (23).

$$\begin{aligned} h(r_i,t_1) = h(r_i,0) + \Delta t\frac{T}{S_C}&\left\{\frac{1}{r_i}\frac{h(r_{i+1},0) - h(r_{i-1},0)}{2\Delta r} + \frac{h(r_{i+1},0) - 2h(r_i,0) + h(r_{i-1},0)}{\Delta r^2}\right\}, \\ h(r_i,0) = g(r_i), \end{aligned} \tag{23}$$

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{3}{2}\Delta t + f_2(r_i,t_n,h(r_i,t_n)) \\ &- \frac{\Delta t}{2}f_2(r_i,t_{n-1},h(r_i,t_{n-1})) \\ &+ \sigma h(r_i,t_n)(B(t_{n+1}) - B(t_n)), \end{aligned} \tag{24}$$

if $B(t)$ is differentiable.

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{3}{2}\Delta t f_2(r_i,t_n,h(r_i,t_n)) \\ &- \frac{\Delta t}{2}f_2(r_i,t_{n-1},h(r_i,t_{n-1})) \\ &+ \sigma h(r_i,c_n)(B(t_{n+1}) - B(t_n)), \end{aligned} \tag{25}$$

if $B(t)$ is not differentiable.

However, noting that

$$\begin{aligned} f_2(r_i,t_n,h(r_i,t_n)) = \frac{T}{S}\Bigg[&\frac{1}{r_i}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} \\ &+ \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2} \\ &- \alpha S_y\int_0^{t_{n+1}}\frac{\partial h}{\partial \tau}\exp[-\alpha(t_{n+1}-\tau)]d\tau\Bigg], \end{aligned} \tag{26}$$

where

$$\int_0^{t_n}\frac{\partial h}{\partial \tau}\exp[-\alpha(t_n-\tau)]d\tau = \sum_{j=1}^{n-2}\frac{h(r_i,t_j) - h(r_i,t_{j-1})}{\Delta t}\delta_{n,j}, \tag{27}$$

$$\begin{aligned} &\int_0^{t_{n+1}}\frac{\partial h}{\partial \tau}\exp[-\alpha(t_{n+1}-\tau)]d\tau \\ &= \sum_{j=1}^{n-1}\frac{h(r_i,t_j) - h(r_i,t_{j-1})}{\Delta t}\delta_{n+1,j}. \end{aligned} \tag{28}$$

Thus,

If $B(t)$ is differentiable while we have

$$\begin{aligned} h_i^{n+1} = h_i^n &+ \frac{3}{2}\Delta t\frac{T}{S}\Bigg\{\frac{1}{r_i}\frac{h(r_{i+1},t_n) - h(r_{i-1},t_n)}{2\Delta r} \\ &+ \frac{h(r_{i+1},t_n) - 2h(r_i,t_n) + h(r_{i-1},t_n)}{\Delta r^2} \\ &- \alpha S_y\sum_{j=1}^{n-1}\frac{h(r_i,t_{j+1}) - h(r_i,t_j)}{\Delta t}\delta_{n,j}\Bigg\} \\ &+ \sigma h(r_i,t_n)(B(t_{n+1}) - B(t_n)) \\ &- \Delta t\frac{T}{2S}\Bigg\{\frac{1}{r_i}\frac{h(r_{i+1},t_{n-1}) - h(r_{i-1},t_{n-1})}{\Delta t} \\ &+ \frac{h(r_{i+1},t_{n-1}) - 2h(r_i,t_{n-1}) + h(r_{i-1},t_{n-1})}{\Delta r^2} \\ &- \alpha S_y\sum_{j=1}^{n-2}h(r_i,t_{j+1}) - h(r_i,t_j)\delta_{n,j}\Bigg\}. \end{aligned} \tag{29}$$

If $B(t)$ is not differentiable while we have

$$h_i^{n+1} = h_i^n + \frac{3}{2}\Delta t \frac{T}{S}\left\{\frac{1}{r_i}\frac{h_{i+1}^n - h_{i-1}^n}{2\Delta r} + \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\Delta r^2} - aS_y \sum_{j=1}^{n-1} \frac{h_i^{j+1} - h_i^j}{\Delta t}\delta_{n,j}\right\} - \frac{\Delta t}{2}\frac{T}{S}\left\{\frac{1}{r_i}\frac{h_{i+1}^{n-1} - h_{i-1}^{n-1}}{2\Delta r} + \frac{h_{i+1}^{n-1} - 2h_i^{n-1} + h_{i-1}^{n-1}}{\Delta r^2} - aS_y \sum_{j=0}^{n-2} \frac{h_i^{j+1} - h_i^j}{\Delta t}\delta_{n,j}^2\right\} + \sigma h(r_i, C_n)(B(t_{n+1}) - B(t_n)). \quad (30)$$

We present the stability analysis of this method using the von Neumann method for the first part of the equation.

$$\varepsilon_i^{n+1} = \varepsilon_i^n + a_1(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n) + a_2(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) - a_3(\varepsilon_{i+1}^{n-1} - \varepsilon_{i-1}^{n-1}) + a_4(\varepsilon_{i+1}^{n-1} - 2\varepsilon_i^{n-1} + \varepsilon_{i-1}^{n-1}). \quad (31)$$

We replace
$\varepsilon_j^n = \delta_m \exp(i\Re_m x)$,
$\varepsilon_j^{n+1} = \delta_{m+1} \exp(i\Re_m x)$,
$\varepsilon_{j+1}^{n+1} = \delta_{m+1} \exp[i\Re_m(x + \Delta x)]$.
Replacing yields

$$\begin{aligned}\delta_{m+1}\exp(i\Re_m x) = {} & \delta_m \exp(i\Re_m x) \\ & + a_1\delta_m \exp(i\Re_m(x+\Delta x)) \\ & - a_1\delta_m \exp(i\Re_m(x-\Delta x)) \\ & + a_2\delta_m \exp(i\Re_m(x+\Delta x)) \\ & - 2a_2\delta_m \exp(i\Re_m x) \\ & + \delta_m a_2 \exp(i\Re_m(x-\Delta x)) \\ & - a_3\delta_{m-1}\exp(i\Re_m(x+\Delta x)) \\ & - a_3\delta_{m-1}\exp(i\Re_m(x-\Delta x)) \\ & + a_4\delta_{m-1}\exp(i\Re_m(x+\Delta x)) \\ & - 2a_4\delta_{m-1}\exp(i\Re_m x) \\ & + \delta_{m-1}a_4\exp(i\Re_m(x-\Delta x)).\end{aligned} \quad (32)$$

We can proceed with the simplification to have

$$\begin{aligned}\delta_{m+1} = {} & \delta_m(1 + a_1(\exp(i\Re_m\Delta x)) - a_1\exp(-i\Re_m\Delta x)) \\ & + a_2(\exp(i\Re_m\Delta x) - 2 + \exp(-i\Delta x\Re_m)) \\ & - a_3\delta_{m-1}\{(\exp(i\Re_m\Delta x) + \exp(-i\Re_m\Delta x)) \\ & + a_4(\exp(i\Re_m\Delta x) - 2 + \exp(-i\Re_m\Delta x))\}.\end{aligned} \quad (33)$$

We have that $\theta = \Re_m\Delta x \in [-\pi, \pi]$, now using,

$$\sin\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2i},$$

$$\cos\theta = \frac{\exp(i\theta) - \exp(-i\theta)}{2},$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \exp(i\theta) - \exp(-i\theta)}{2},$$

$$\delta_{m+1} = \delta_m\left(1 + a_1 2i\sin\theta - 4a_2\sin\left(\frac{\theta}{2}\right)^2\right) - \delta_{m-1}\left(-a_3 2i\sin\theta + 4a_4\sin\left(\frac{\theta}{2}\right)^2\right). \quad (34)$$

We can now use the Euler approximation on the first step to have

$$\frac{u_i^1 - u_i^0}{\Delta t} = \left(\frac{1}{r_i}\frac{u_{i+1}^0 - u_{i-1}^0}{2\Delta r} + \frac{u_{i+1}^0 - 2u_i^0 + u_{i-1}^0}{\Delta r^2}\right)\frac{T}{S}. \quad (35)$$

$$u_i^1 = u_i^0 + \frac{\Delta tT}{2Sr_i\Delta t}(u_{i+1}^0 - u_{i-1}^0) + \frac{\Delta tT}{2\Delta r^2}(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0), \quad (36)$$

$$= u_i^0 + a_1(u_{i+1}^0 - u_{i-1}^0) + a_2(u_{i+1}^0 - 2u_i^0 + u_{i-1}^0). \quad (37)$$

By the von Neumann approach, we have

$$\delta_1 = \delta_0 + 2ia_1\sin\theta\delta_0 - 4a_2\sin\left(\frac{\theta}{2}\right)^2\delta_0, \quad (38)$$

$$= \delta_0\left(1 + 2ia_1\sin\theta - 4a_2\sin\left(\frac{\theta}{2}\right)^2\right), \quad (39)$$

$$\frac{\delta_1}{\delta_0} = 1 + 2ia_1\sin\theta - 4a_2\sin\left(\frac{\theta}{2}\right)^2, \quad (40)$$

$$\left|\frac{\delta_1}{\delta_0}\right| < 1 \implies \left|1 + 2ia_1\sin\theta - 4a_2\sin\left(\frac{\theta}{2}\right)^2\right| < 1, \quad (41)$$

$$\implies \left(1 - 4a_2\sin\left(\frac{\theta}{2}\right)^2\right)^2 + 4a_1^2\sin(\theta)^2 < 1, \quad (42)$$

$$\implies 1 - 8a_2\sin\left(\frac{\theta}{2}\right)^2 + 16a_2^2\sin\left(\frac{\theta}{2}\right)^4 + 4a_1^2\sin(\theta)^2 < 1, \quad (43)$$

$$\implies 4a_2^2\sin\left(\frac{\theta}{2}\right)^4 + a_1^2\sin(\theta)^2 < 2a_2\sin\left(\frac{\theta}{2}\right)^2. \quad (44)$$

The above can further be simplified; however, under the above condition, we have that

$$\left|\frac{\delta_1}{\delta_0}\right| < 1.$$

We now assume that for a fixed $m, c > 0$, such that when $c < 1$, then $\left|\frac{\delta_m}{\delta_0}\right| < 1$.

At $m + 1$, we have,

$$\delta_{m+1} = \delta_m\bar{a}_1 + \delta_{m-1}\bar{a}_2,$$

$$|\delta_{m+1}| \le |\delta_m||\bar{a}_1| + |\delta_{m-1}||\bar{a}_2|.$$

By induction, we have that

$$|\delta_{m+1}| \le |\delta_0|\{|\bar{a}_1| + |\bar{a}_2|\},$$

$$\frac{|\delta_{m+1}|}{|\delta_0|} \le \{|\bar{a}_1| + |\bar{a}_2|\} < 1.$$

Therefore, we will need

$$Y = \max\left\{\left|1 + 2ia_1 \sin\theta - 4a_2 \sin\left(\frac{\theta}{2}\right)^2\right|; c, |\bar{a}_1| + |\bar{a}_2|\right\} < 1. \tag{45}$$

To reach the stability,
we shall now present the analyses for the second part.
For the second part, we have

$$\begin{aligned} h_i^{n+1} &= h_i^n + a_1 h_{i+1}^n - a_2 h_{i-1}^n + a_2 h_{i+1}^n - 2a_2 h_i^n + a_2 h_{i-1}^n \\ &\quad + a_3 \sum_{j=1}^{n-1} \{h_i^{j+1} - h_i^j\}\delta_{n,j} + \sigma h_i^n \widetilde{B}_{n+1} - \bar{a}_1 h_{i+1}^{n-1} \\ &\quad + \bar{a}_1 h_{i-1}^{n-1} - \bar{a}_2 h_{i+1}^{n-1} + 2\bar{a}_2 h_i^{n-1} - \bar{a}_2 h_{i-1}^{n-1} \\ &\quad + \bar{a}_3 \sum_{j=1}^{n-2} \{h_i^{j+1} - h_i^j\}\delta_{n,j}. \end{aligned} \tag{46}$$

Replacing by the error yields

$$\begin{aligned} \delta_{n+1} &= \delta_n + a_1\delta_n \exp(+\text{Ʀ}_m\Delta x) - a_1\delta_n \exp(-\text{Ʀ}_m\Delta x) \\ &\quad + a_2\delta_n \exp(\text{Ʀ}_m\Delta x) - 2a_2\delta_n + a_2 \exp(-\text{Ʀ}_m\Delta x)\delta_n \\ &\quad + a_3 \sum_{j=1}^{n-1} (\delta_{j+1} - \delta_j)\delta_{n,j} + 6\delta_n \widetilde{B}_{n+1} \\ &\quad - \bar{a}_1\delta_{n-1} \exp(+\text{Ʀ}_m\Delta x) + \bar{a}_1\delta_{n-1} \exp(-\text{Ʀ}_m\Delta x) \\ &\quad - \bar{a}_2\delta_{n-1} \exp(-\text{Ʀ}_m\Delta x) + 2\bar{a}_2\delta_{n-1} \\ &\quad - a_2\delta_{n-1} \exp(-\text{Ʀ}_m\Delta x) + \bar{a}_3 \sum_{j=1}^{n-2} \{\delta_{j+1} - \delta_j\}\delta_{n,j}. \end{aligned} \tag{47}$$

But, noting that to start the process, we need two components.

$\left|\frac{\delta_1}{\delta_0}\right| < 1.$

If we have the following

$$|1 + \bar{a}_1 \exp(\text{Ʀ}_m\Delta x) + \bar{a}_1 \exp(-\text{Ʀ}_m\Delta x) - 2\bar{a}_1| < 1,$$

$$|1 + \bar{a}_1(2\cos(\text{Ʀ}_m\Delta x) - 2)| < 1,$$

$$\left|1 - 4a_1 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right)\right| < 1,$$

$$4a_1 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right) < 2,$$

$$4a_1 < 2,$$

$$a_1 < \tfrac{1}{2}.$$

We shall assume that $\forall\ n \ge 1$, $\left|\frac{\delta_n}{\delta_0}\right| < 1$. Now showing that $\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1$, under some conditions. We have that δ_1 and δ_0. These can be obtained using a simple approximation. But we shall note that the original state of this phase corresponds to the last state of the first phase; thus, if we can take $\bar{t}_0 = t_{n+1}$ of the last stage, then we shall have,

$$h_i^0 = \frac{Q}{4\pi T} W(\bar{r}_i, \bar{t}_0) = \bar{\Omega}, \tag{48}$$

$$h_i^1 - h_i^0 = \frac{\Delta t}{r^2}\frac{T}{S}\{h_{i+1}^0 - 2h_i^0 + h_{i-1}^0\} - \frac{T}{Sr_i}\left\{\frac{h_{i+1}^0 - h_{i-1}^0}{2\Delta r}\right\}, \tag{49}$$

$$\bar{\delta}_1 = \bar{\delta}_0 + a_1\bar{\delta}_0 \exp(\text{Ʀ}_m\Delta x) - 2a_1\bar{\delta}_0 + \bar{a}_1\bar{\delta}_0 \exp(-\text{Ʀ}_m\Delta x), \tag{50}$$

$$= \bar{\delta}_0\{1 + \bar{a}_1 \exp(\text{Ʀ}_m\Delta x) + \bar{a}_1 \exp(-\text{Ʀ}_m\Delta x) - 2\bar{a}_1\}, \tag{51}$$

$$\begin{aligned} \delta_{n+1} &= \delta_n + \left(2a_1 i \sin(\text{Ʀ}_m\Delta x) - 4a_1 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right)\right)\delta_n \\ &\quad + \sigma\widetilde{B}_{n+1}\delta_n + a_3 \sum_{j=1}^{n-2} (\delta_{j+1} - \delta_j)\delta_{n,j} \\ &\quad - 2\bar{a}_1 i \sin(\text{Ʀ}_m\Delta x) - 4\bar{a}_2 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right) \\ &\quad - \bar{a}_3 \sum_{j=1}^{n-2} (\delta_{j+1} - \delta_j)\delta_{n,j}, \end{aligned} \tag{52}$$

$$\begin{aligned} |\delta_{n+1}| &\le |\delta_n| + \left|2a_1 i \sin(\text{Ʀ}_m\Delta x) - 4\bar{a}_2 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right) \right. \\ &\quad \left. + \sigma \max_{n\ge 0}\{\widetilde{B}_{n+1}\}\right| + \left|2\bar{a}_1 i \sin(\text{Ʀ}_m\Delta x) \right. \\ &\quad \left. - 4\bar{a}_2 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right)\right| + |a_3| \sum_{j=1}^{n-1} |\delta_{j+1} - \delta_j||\delta_{n,j}| \\ &\quad + \bar{a}_3 \left|\sum_{j=1}^{n-2} |\delta_{j+1} - \delta_j||\delta_{n,j}|\right|. \end{aligned} \tag{53}$$

By hypothesis, we have $\forall_n \ge 1 \left|\frac{\delta_n}{\delta_0}\right| < 1$. Therefore, $\left|\frac{\delta_{n+1}}{\delta_0}\right| < 1$ if,

$$\begin{aligned} 1 &+ \left|2a_1 i \sin(\text{Ʀ}_m\Delta x) - 4a_2 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right)\right| \\ &+ \sigma \max_{n\ge 1}\{\widetilde{B}_{n+1}\} + a_3 \sum_{j=1}^{n-1} \delta_{n,j} + \bar{a}_3 \sum_{j=1}^{n-1} \delta_{n,j} \\ &+ \left|2\bar{a}_1 i \sin(\text{Ʀ}_m\Delta x) - 4\bar{a}_2 \sin^2\left(\frac{\text{Ʀ}_m\Delta x}{2}\right)\right| < 1. \end{aligned} \tag{54}$$

6 Numerical simulations

The numerical simulations are presented in Figures 2–4. To obtain these figures, the following theoretical parameters were used: $r = 0.01$ m which is the radial distance from the well, the transmissivity $T = 1{,}000$ m^3/day, the storativity of the confined aquifer $S_c = 0.009$, the storativity of the unconfined aquifer $S_y = 0.0009$, the thickness of the aquifer $b = 30$ m, the time steps size d$t = 0.00000001$ m, and the space step size d$r = 0.1$ m. These figures were achieved for various values of fractional order alpha, the part that was introduced to represent the passage from confined to unconfined. We shall note that the used fractional orders are chosen above and below 0.5. Above 0.5 especially near 1, we have a slow but not very slow flow while when the fractional order is less than 0.5, we have a fast flow that represents the flow within a fracture.

To incorporate in the mathematical model the result of random nature, randomness is added to the classical model and the numerical simulations are performed using sigma = 0.009. We presented the simulations for different values of fractional order, where the passage from confined to unconfined was made up of an exponential decay kernel.

One of the commonly asked questions is how to determine a fractional order in practice, as this is a mathematical parameter. To provide an answer to this question, one will recall the primary aim of a mathematical model. Indeed, if the aim is to replicate observed facts using a mathematical model, then if there is a good agreement between observed facts and the solution of the mathematical model, prediction can proceed. In our case, one aims to determine the aquifer's parameters, including storativity and transmissivity, and then the crossover time.

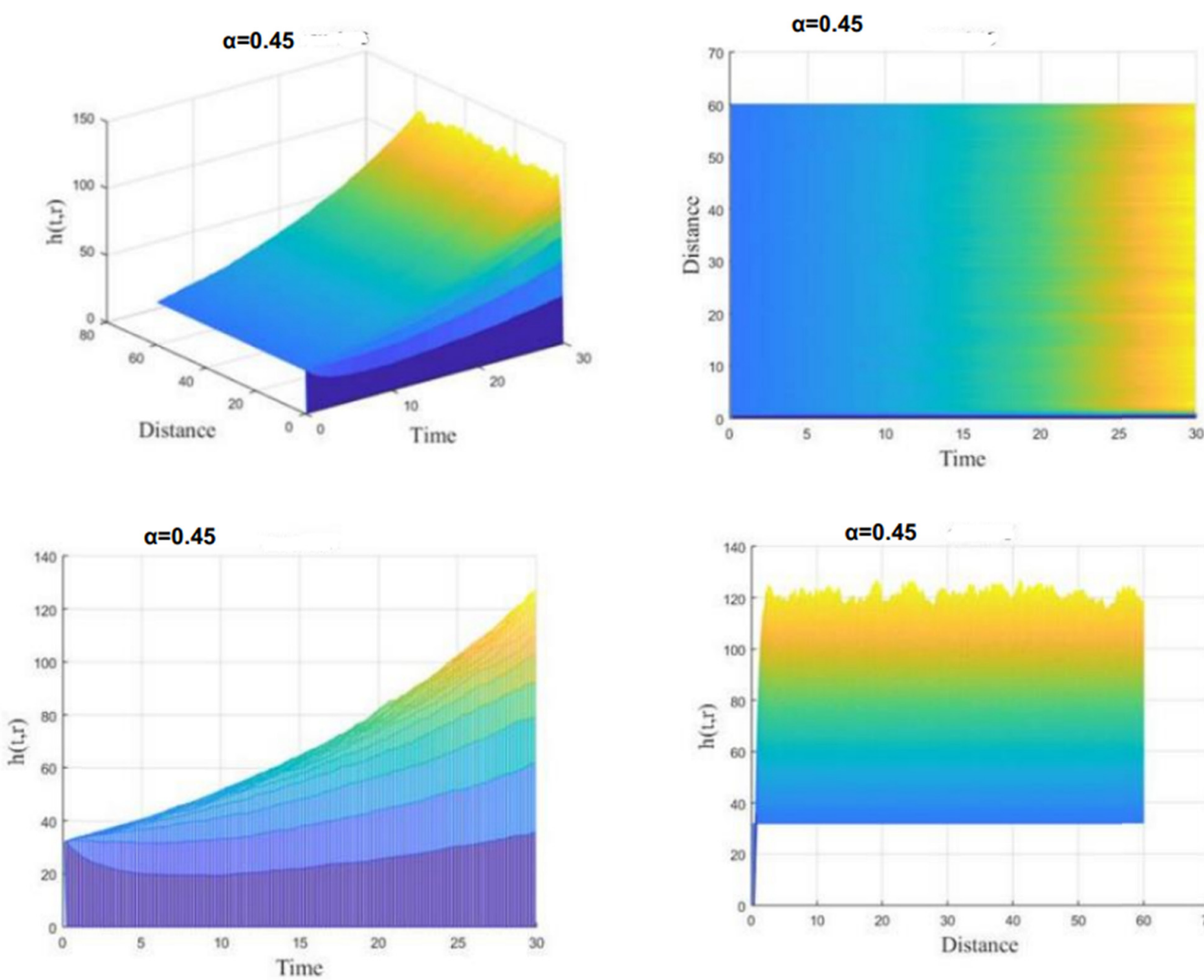

Figure 2: Numerical solution of the confined to unconfined groundwater flow by introducing a stochastic approach with $\alpha = 0.45$.

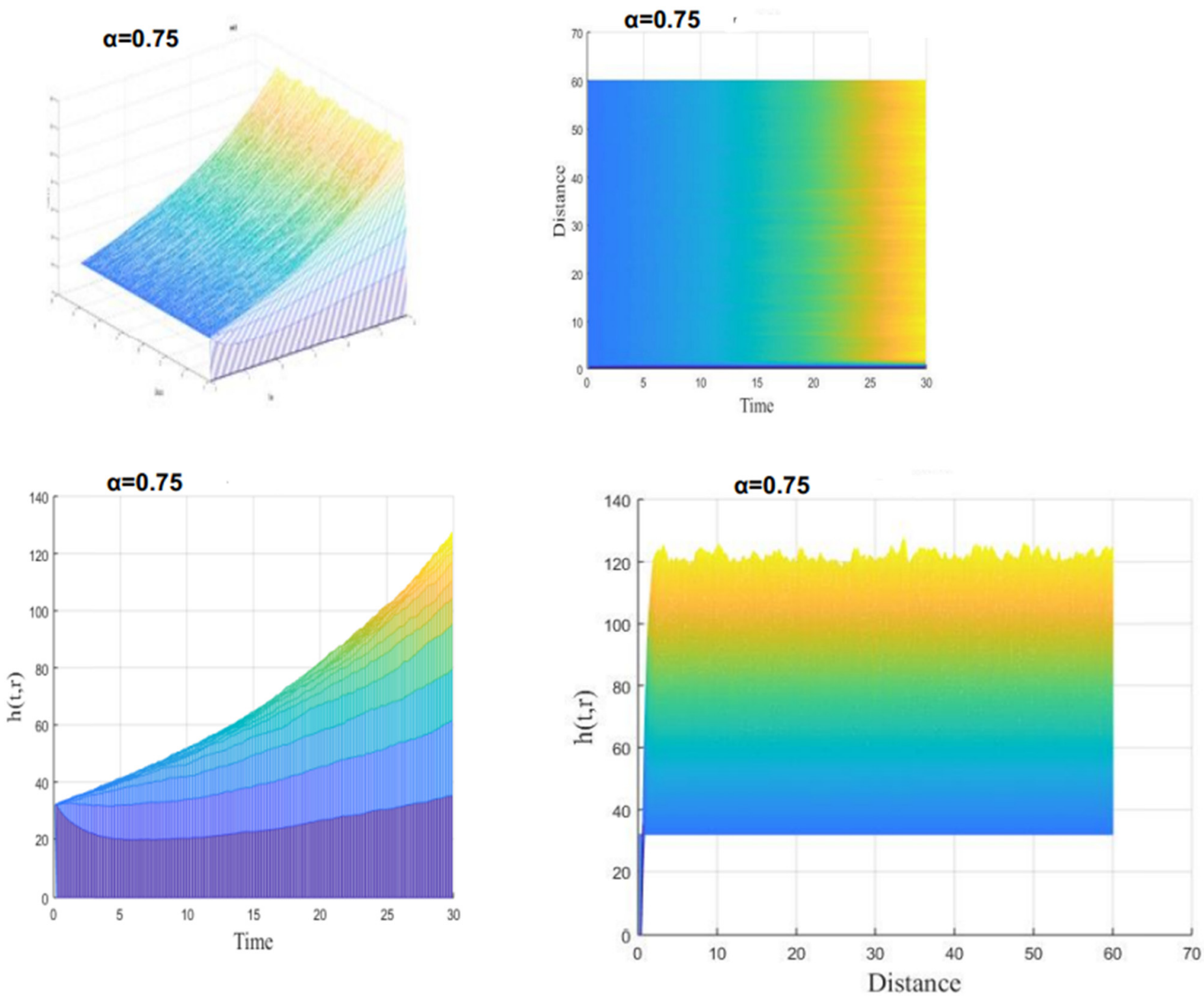

Figure 3: Numerical solution of the confined to unconfined groundwater flow by introducing a stochastic approach with $\alpha = 0.75$.

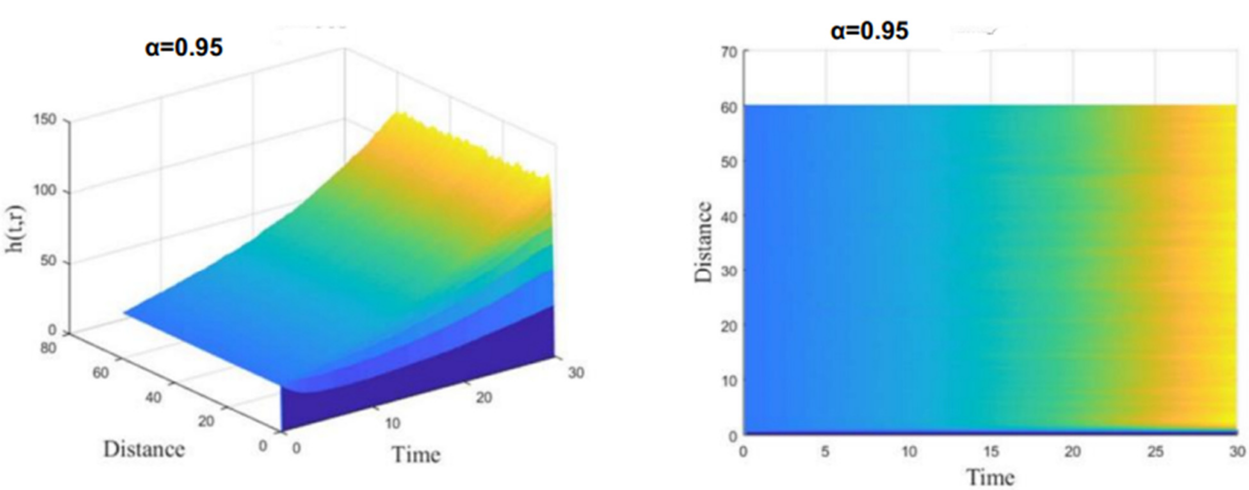

Figure 4: Numerical solution of the confined to unconfined groundwater flow by introducing a stochastic approach with $\alpha = 0.95$.

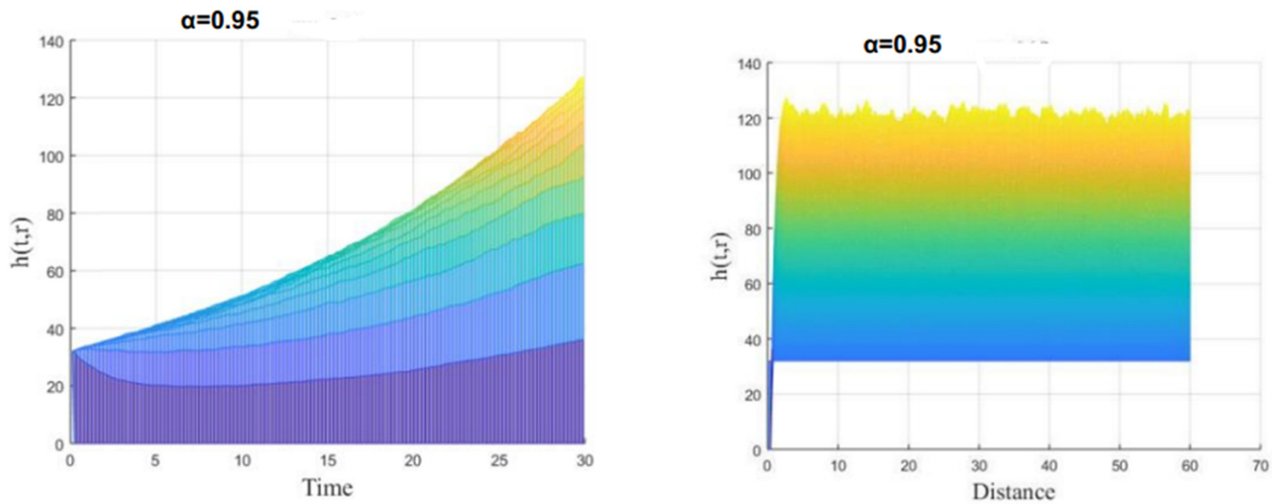

Figure 4: *Continued.*

The fractional order alpha will be determined by comparison of mathematical solution and observed data; the alpha that provides the best fit will be the suitable alpha for prediction.

7 Conclusion

The dynamic process underlying the conversion of groundwater flow from confined aquifers to unconfined has been a centre of interest for several researchers in the last decades. This conversion often occurs due to over-abstraction of the subsurface water and can sometimes lead to depletion, a situation that should be avoided through groundwater management. However, good management used prediction to make sound decisions; indeed, this could be achieved through monitoring and modelling using differential equations. In this work, we added a stochastic component to an existing part that was constructed to replicate this conversion. We aimed to include in the mathematical formulas randomness that could occur due to recharge or water trapping that is released due to the force induced during abstraction. A simple numerical scheme was used to solve numerically the system of equations. Numerical simulations were performed for different densities of randomness.

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