Research Article

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A homotopy perturbation method with Elzaki transformation for solving the fractional Biswas–Milovic model

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Abstract: In this research, we use the homotopy perturbation method (HPM) combined with the Elzaki transform to investigate the fractional Biswas–Milovic equation (BME) within the framework of the Caputo operator. The fractional BME is a significant mathematical model with applications in various scientific and engineering fields, including physics, biology, and chemistry. However, its fractional nature introduces analytical complexities. By integrating the HPM with the Elzaki transform, we aim to provide an effective approach for obtaining accurate solutions to this equation. The combination of these mathematical techniques allows us to explore the behavior of the fractional BME in a comprehensive manner. The research outcomes are supported by numerical results and comparisons, demonstrating the reliability and efficiency of the proposed methodology. This study contributes to advancing the tools for solving fractional equations and enhances our understanding of the intricate dynamics described by the fractional BME.

Keywords: homotopy perturbation method, Elzaki transform, fractional Biswas–Milovic equation, Caputo operator

1 Introduction

Fractional partial differential equations (FPDEs) are mathematical problems that extend the traditional concept of partial differential equations (PDEs) by incorporating fractional derivatives and integrals. FPDEs have been used to model a broad range of phenomena in fields such as engineering, physics, and finance [1–6]. The use of fractional derivatives allows for the description of memory and hereditary effects, which are not captured by traditional PDEs. Applications of FPDEs include modeling of viscoelastic materials, PDEs and fractional financial models. The study of FPDEs is an active area of research, and new developments are continually being made in both the theoretical and numerical aspects of these equations [7–9].

The Biswas–Milovic equation (BME), also known as the Biswas–Milovic model, is a mathematical equation that describes the behavior of two-phase flow in porous media. The equation was first proposed by Biswas and Milovic in 1989 and has been widely used in the field of subsurface hydrology and petroleum engineering. The BME is a nonlinear differential equation that describes the dynamic behavior of two-phase flow in porous media [10]. It takes into account the effects of capillarity, viscous forces, and gravity on the flow of fluids in porous media [11–13]. The equation is based on the assumption that the fluids are incompressible and the porous medium is isotropic and homogeneous [19–23].

The fractional-order BME is a type of fractional differential equation that describes the dynamics of certain physical systems. It is a generalization of the standard BME, which is a second-order differential equation. The fractional-order BME involves derivatives of fractional order, rather than integer order derivatives. These types of equations are used to model phenomena such as viscoelasticity, anomalous diffusion, and chaotic systems. The solution of these equations can be challenging, and various numerical and analytical methods have been developed to solve them.

The novelty of our contribution in this study is deeply rooted in the innovative approach we have taken to address the complex dynamics described by the BME within the framework of the Caputo operator. Fractional differential equations have become increasingly relevant in modeling various physical and engineering phenomena, owing to their ability to capture non-local and memory-dependent effects. However,

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solving these equations analytically remains a formidable challenge. Our research breaks new ground by introducing and successfully applying the homotopy perturbation transform method (HPTM) to this specific fractional equation. First, our methodology stands out as a novel and powerful approach for solving the BME. While HPTM has been used in diverse contexts, its application to this equation, characterized by its complexity and importance in modeling real-world processes, is a novel contribution. This highlights the versatility of HPTM as an analytical tool for addressing fractional differential equations, especially those involving the Caputo operator. Furthermore, our research goes beyond numerical approximations by providing analytical solutions to the BME. These solutions offer profound insights into the intricate dynamics governed by this fractional equation [24,25]. This analytical depth is crucial for understanding the underlying physics and behavior of systems described by such equations, setting our contribution apart from purely numerical approaches [26,27]. Ultimately, our article not only advances the toolbox of analytical techniques available for tackling fractional differential equations but also enhances our understanding of the BME's dynamics. This research has broader implications, as it contributes to the field of fractional calculus, offering researchers and practitioners a valuable methodology for addressing complex, memory-dependent systems in various scientific and engineering domains [28–31].

The HPM is a numerical technique for solving nonlinear differential equations. It is based on the concept of homotopy, which is a continuous deformation of one problem into another. The HPM is used to solve a nonlinear problem by introducing a small parameter, called the perturbation parameter, and then using it to continuously deform the original problem into a simpler problem. The Elzaki transformation is a modification of the HPM that was proposed by Elzaki [32]. The Elzaki transformation is used to improve the convergence of the HPM by introducing an additional term in the homotopy equation. This additional term is based on the gradient of the solution, and it helps to guide the deformation of the problem toward a solution [32–34].

2 Basic definitions

2.1 Definition

The fractional derivative $D^{ß}$ in Abel-Riemann sense having order ß is given as [35,36]:

$$D^{ß}\theta(\mu) = \begin{cases} \dfrac{d^{\kappa}}{d\mu^{\kappa}}\theta(\mu), & ß = \kappa, \\ \dfrac{1}{\Gamma(\kappa - ß)}\dfrac{d}{d\mu^{\kappa}}\displaystyle\int_0^{\mu}\dfrac{\theta(\mu)}{(\mu-\phi)^{ß-\kappa+1}}d\phi, \\ \kappa - 1 < ß < \kappa, \end{cases}$$

where $\kappa \in Z^{+}$, $ß \in R^{+}$ and

$$D^{-ß}\theta(\mu) = \frac{1}{\Gamma(ß)}\int_0^{\mu}(\mu-\phi)^{ß-1}\theta(\phi)d\phi, \quad 0 < ß \le 1.$$

2.2 Definition

In the Abel-Riemann sense, the fractional integration operator κ^{ϕ} is expressed as [35,36]:

$$\kappa^{ß}\theta(\mu) = \frac{1}{\Gamma(ß)}\int_0^{\mu}(\mu-\phi)^{ß-1}\theta(\mu)d\mu, \quad \mu > 0, \quad ß > 0,$$

having properties:

$$\kappa^{ß}\mu^{\kappa} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa+ß+1)}\mu^{\kappa+\phi},$$
$$D^{ß}\mu^{\kappa} = \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-ß+1)}\mu^{\kappa-\phi}.$$

2.3 Definition

The Caputo derivative $D^{ß}$ of fractional-order ß is defined as [35,36]:

$$^{C}D^{ß}\theta(\mu) = \begin{cases} \dfrac{1}{\Gamma(\kappa-ß)}\displaystyle\int_0^{\mu}\dfrac{\theta^{\kappa}(\phi)}{(\mu-\phi)^{ß-\kappa+1}}d\phi, \\ \kappa - 1 < ß < \kappa, \\ \dfrac{d^{\kappa}}{d\mu^{\kappa}}\theta(\mu), & \kappa = ß. \end{cases} \tag{1}$$

2.4 Definition

$$\kappa_{\mu}^{ß}D_{\mu}^{ß}\theta(\mu) = \theta(\mu) - \sum_{k=0}^{m}\theta^{k}(0^{+})\frac{\mu^{k}}{k!}, \text{ for } \mu > 0,$$
$$\text{and } \kappa - 1 < ß \le \kappa, \quad \kappa \in N. \tag{2}$$
$$D_{\mu}^{ß}\kappa_{\mu}^{ß}\theta(\mu) = \theta(\mu).$$

2.5 Definition

The Elzaki transform in the sense of Caputo operator is given as [35,36]:

$$E[D_\mu^ß \theta(\mu)] = s^{-ß}E[\theta(\mu)] - \sum_{k=0}^{\kappa-1} s^{2-ß+k}\theta^{(k)}(0),$$
where $\kappa - 1 < ß < \kappa$.

3 General discussion of the proposed method

We consider the following differential equation as the basis for implementing the HPTM.

$$D_\varepsilon^ß \theta(\mu, \varepsilon) = \mathcal{P}_1[\mu]\theta(\mu, \varepsilon) + Q_1[\mu]\theta(\mu, \varepsilon), \quad 0 < ß \le 2, \tag{3}$$

with the initial conditions (ICs):

$$\theta(\mu, 0) = \xi(\mu), \quad \frac{\partial}{\partial \varepsilon}\theta(\mu, 0) = \zeta(\mu),$$

where $D_\varepsilon^ß = \frac{\partial^ß}{\partial \varepsilon^ß}$ stands for the Caputo fractional derivative and $\mathcal{P}_1[\mu]$ and $Q_1[\mu]$ denote the linear and nonlinear functions.

On using the Elzaki transform, we obtain

$$E[D_\varepsilon^ß \theta(\mu, \varepsilon)] = E[\mathcal{P}_1[\mu]\theta(\mu, \varepsilon) + Q_1[\mu]\theta(\mu, \varepsilon)], \tag{4}$$

$$\begin{aligned}&\frac{1}{u^ß}\{M(u) - u^2\theta(0) - u^3\theta'(0)\}\\ &= E[\mathcal{P}_1[\mu]\theta(\mu, \varepsilon) + Q_1[\mu]\theta(\mu, \varepsilon)].\end{aligned} \tag{5}$$

Using the inverse Elzaki transformation, we obtain

$$\begin{aligned}\theta(\mu, \varepsilon) = \theta(0) + \theta'(0) + E^{-1}[u^ß E[\mathcal{P}_1[\mu]\theta(\mu, \varepsilon)\\ + Q_1[\mu]\theta(\mu, \varepsilon)]].\end{aligned} \tag{6}$$

On applying the HPM,

$$\theta(\mu, \varepsilon) = \sum_{k=0}^{\infty} \varrho^k \theta_k(\mu, \varepsilon). \tag{7}$$

The perturbation parameter $\varrho \in [0, 1]$:

$$Q_1[\mu]\theta(\mu, \varepsilon) = \sum_{k=0}^{\infty} \varrho^k H_n(\theta), \tag{8}$$

and He's polynomials represent $H_k(\theta)$ as:

$$H_n(\theta_0, \theta_1, \ldots, \theta_n) = \frac{1}{\Gamma(n+1)} D_\varrho^k \left[Q_1\left(\sum_{k=0}^{\infty} \varrho^i \theta_i\right)\right]_{\varrho=0}, \tag{9}$$

where $D_\varrho^k = \frac{\partial^k}{\partial \varrho^k}$.

By substituting (7) and (3) into (6), we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \varrho^k \theta_k(\mu, \varepsilon) &= \theta(0) + \theta'(0) + \varrho\\ &\times \left(E^{-1}\left[u^ß E\{\mathcal{P}_1 \sum_{k=0}^{\infty} \varrho^k \theta_k(\mu, \varepsilon) + \sum_{k=0}^{\infty} \varrho^k H_k(\theta)\}\right]\right).\end{aligned} \tag{10}$$

By comparing the coefficients, ϱ, we achieved

$$\begin{aligned}&\varrho^0 : \theta_0(\mu, \varepsilon) = \theta(0) + \theta'(0),\\ &\varrho^1 : \theta_1(\mu, \varepsilon) = E^{-1}[u^ß E(\mathcal{P}_1[\mu]\theta_0(\mu, \varepsilon) + H_0(\theta))],\\ &\varrho^2 : \theta_2(\mu, \varepsilon) = E^{-1}[u^ß E(\mathcal{P}_1[\mu]\theta_1(\mu, \varepsilon) + H_1(\theta))],\\ &\quad\vdots\\ &\varrho^k : \theta_k(\mu, \varepsilon) = E^{-1}[u^ß E(\mathcal{P}_1[\mu]\theta_{k-1}(\mu, \varepsilon) + H_{k-1}(\theta))],\\ &\quad k > 0, k \in N.\end{aligned} \tag{11}$$

Thus, the analytic result $\theta_k(\mu, \varepsilon)$ is achieved by applying the truncate series:

$$\theta(\mu, \varepsilon) = \lim_{M\to\infty} \sum_{k=1}^{M} \theta_k(\mu, \varepsilon). \tag{12}$$

4 Numerical results

4.1 Problem

Consider the fractional Biswas–Milovic model is given as:

$$\iota\frac{\partial^ß \theta}{\partial \varepsilon^ß} + \ell\frac{\partial^2 \theta}{\partial \varepsilon^2} + \hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon) = 0, \quad 0 < ß \le 1, \tag{13}$$

with the IC:

$$\theta(\mu, 0) = \exp(\iota\mu).$$

On using the Elzaki transform, we have

$$E\left(\iota\frac{\partial^ß \theta}{\partial \varepsilon^ß}\right) = -E\left[\ell\frac{\partial^2 \theta}{\partial \varepsilon^2} + \hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon)\right], \tag{14}$$

After calculation, we obtain

$$\frac{1}{u^ß}\{M(u) - u^2\theta(0)\} = -E\left[\ell\iota\frac{\partial^2 \theta}{\partial \varepsilon^2} + \iota\hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon)\right], \tag{15}$$

$$M(u) = u^2\theta(0) - u^ß E\left[\ell\iota\frac{\partial^2 \theta}{\partial \varepsilon^2} + \iota\hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon)\right]. \tag{16}$$

By applying the inverse Elzaki transform, we obtain

$$\begin{aligned}\theta(\mu, \varepsilon) &= \theta(0) - E^{-1}\left[u^ß\left\{E\left[\ell\iota\frac{\partial^2 \theta}{\partial \varepsilon^2} + \iota\hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon)\right]\right\}\right],\\ \theta(\mu, \varepsilon) &= \exp(\iota\mu) - E^{-1}\left[u^ß\left\{E\left[\ell\iota\frac{\partial^2 \theta}{\partial \varepsilon^2}\right.\right.\right.\\ &\quad \left.\left.\left. + \iota\hbar|\theta(\mu, \varepsilon)|^2\theta(\mu, \varepsilon)\right]\right\}\right].\end{aligned} \tag{17}$$

On applying the HPM,

$$\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon) = \exp(\iota\mu) + \varrho\left(E^{-1}\left[u^{ß}E\left[\ell\iota\left(\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon)\right)_{\mu\mu} - \iota\hbar\left(\sum_{k=0}^{\infty}\varrho^k H_k(\theta)\right)\right]\right]\right). \tag{18}$$

The nonlinear term find by He's polynomial $H_k(\theta)$ is defined as:

$$\sum_{k=0}^{\infty}\varrho^k H_k(\theta) = |\theta(\mu,\varepsilon)|^2\theta(\mu,\varepsilon). \tag{19}$$

Some He's polynomials term are calculated as:

$$\begin{aligned}
H_0(\theta) &= |\theta_0(\mu,\varepsilon)|^2\theta_0(\mu,\varepsilon),\\
H_1(\theta) &= \frac{1}{1!}\frac{\partial}{\partial\varrho}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon))|^2(\theta_0(\mu,\varepsilon) + \varrho\theta_0(\mu,\varepsilon))]_{\varrho=0},\\
H_2(\theta) &= \frac{1}{2!}\frac{\partial^2}{\partial\varrho^2}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon)|^2)(\theta_0(\mu,\varepsilon),\\
&\quad + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon))]_{\varrho=0}
\end{aligned}$$

By comparing the coefficients ϱ, we obtain

$$\begin{aligned}
\varrho^0 &: \theta_0(\mu,\varepsilon) = \exp(\iota\mu),\\
\varrho^1 &: \theta_1(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_0(\theta)\right]\right)\\
&= \iota(\hbar - \ell)\exp(\iota\mu)\frac{\varepsilon^{ß}}{\Gamma(ß+1)},\\
\varrho^2 &: \theta_2(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_1(\theta)\right]\right)\\
&= \frac{1}{2}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu),\\
\varrho^3 &: \theta_3(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_2(\theta)\right]\right)\\
&= \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3\left\{\frac{-\iota\ell}{2}(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu)\right.\\
&\quad\left. + \frac{3}{2}\hbar\iota(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu) + \hbar\iota(\hbar - \ell)^2\exp(\iota\mu)\right\}. \vdots
\end{aligned}$$

Thus, the analytic result is achieved by applying the truncate series as:

$$\begin{aligned}
\theta(\mu,\varepsilon) &= \exp(\iota\mu) + \iota(\hbar - \ell)\exp(\iota\mu)\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\\
&\quad + \frac{1}{2}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu)\\
&\quad + \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3\left\{\frac{-\iota\ell}{2}(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu)\right.\\
&\quad\left. + \frac{3}{2}\hbar\iota(\hbar - \ell)(\ell - \hbar)\exp(\iota\mu) + \hbar\iota(\hbar - \ell)^2\exp(\iota\mu)\right\}\\
&\quad + \cdots .
\end{aligned}$$

4.2 Problem

Consider the fractional nonlinear BME

$$\iota\frac{\partial^{ß}\theta}{\partial\varepsilon^{ß}} + \ell\frac{\partial^2\theta}{\partial\varepsilon^2} + \hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon) = 0, \quad 0 < ß \le 1, \tag{20}$$

with the IC:

$$\theta(\mu, 0) = \exp(\iota\mu).$$

On using the Elzaki transform, we obtain

$$E\left(\iota\frac{\partial^{ß}\theta}{\partial\varepsilon^{ß}}\right) = -E\left[\ell\frac{\partial^2\theta}{\partial\varepsilon^2} + \hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon)\right]. \tag{21}$$

After calculation, we obtain

$$\frac{1}{u^{ß}}\{M(u) - u^2\theta(0)\} = -E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon)\right], \tag{22}$$

$$M(u) = u^2\theta(0) - u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon)\right]. \tag{23}$$

By applying the inverse Elzaki transformation, we obtain

$$\begin{aligned}
\theta(\mu,\varepsilon) &= \theta(0) - E^{-1}\left[u^{ß}\left\{E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon)\right]\right\}\right],\\
\theta(\mu,\varepsilon) &= \exp(\iota\mu) - E^{-1}\left[u^{ß}\left\{E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar|\theta(\mu,\varepsilon)|^4\theta(\mu,\varepsilon)\right]\right\}\right].
\end{aligned} \tag{24}$$

On applying the HPM,

$$\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon) = \exp(\iota\mu) + \varrho\left(E^{-1}\left[u^{ß}E\left[\ell\iota\left(\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon)\right)_{\mu\mu} - \iota\hbar\left(\sum_{k=0}^{\infty}\varrho^k H_k(\theta)\right)\right]\right]\right). \tag{25}$$

The nonlinear term find by He's polynomials $H_k(\theta)$ is defined as:

$$\sum_{k=0}^{\infty}\varrho^k H_k(\theta) = |\theta(\mu,\varepsilon)|^2\theta(\mu,\varepsilon). \tag{26}$$

Some He's polynomials term are calculated as:

$$\begin{aligned}
H_0(\theta) &= |\theta_0(\mu,\varepsilon)|^4\theta_0(\mu,\varepsilon),\\
H_1(\theta) &= \frac{1}{1!}\frac{\partial}{\partial\varrho}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon))|^4(\theta_0(\mu,\varepsilon) + \varrho\theta_0(\mu,\varepsilon))]_{\varrho=0},\\
H_2(\theta) &= \frac{1}{2!}\frac{\partial^2}{\partial\varrho^2}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon)|^4)(\theta_0(\mu,\varepsilon).\\
&\quad + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon))]_{\varrho=0}
\end{aligned}$$

By comparing the coefficients ϱ, we obtain

$$\varrho^0 : \theta_0(\mu,\varepsilon) = \exp(\iota\mu),$$

$$\varrho^1 : \theta_1(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_0(\theta)\right]\right) = \iota(\hbar-\ell)\exp(\iota\mu)\frac{\varepsilon^{ß}}{\Gamma(ß+1)},$$

$$\varrho^2 : \theta_2(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_1(\theta)\right]\right) = \frac{1}{2}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2(\hbar-\ell)(\hbar-\ell-4\hbar\exp(2\iota\mu))\exp(\iota\mu),$$

$$\varrho^3 : \theta_3(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_2(\theta)\right]\right) = \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3(\hbar-\ell)\{\iota\ell(-\ell\exp(\iota\mu)+16\hbar\exp(5\iota\mu)) + \hbar\iota[\exp(5\iota\mu)(2+\ell-\hbar-\exp(4\iota\mu))]\} + \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3 (\hbar-\ell)\hbar\iota\left\{\frac{3}{2}\exp(3\iota\mu)[(\ell-\hbar)\exp(\iota\mu)+\ell-\hbar\exp(4\iota\mu)]\right\},$$

$$\vdots$$

Thus, the analytic result is achieved by applying the truncate series as:

$$\theta(\mu,\varepsilon) = \exp(\iota\mu) + \iota(\hbar-\ell)\exp(\iota\mu)\frac{\varepsilon^{ß}}{\Gamma(ß+1)} + \frac{1}{2}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2(\hbar-\ell)(\hbar-\ell-4\hbar\exp(2\iota\mu))\exp(\iota\mu) + \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3(\hbar-\ell)\{\iota\ell(-\ell\exp(\iota\mu) + 16\hbar\exp(5\iota\mu)) + \hbar\iota[\exp(5\iota\mu)(2+\ell-\hbar - \exp(4\iota\mu))]\} + \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3 (\hbar-\ell)\hbar\iota\left\{\frac{3}{2}\exp(3\iota\mu)[(\ell-\hbar)\exp(\iota\mu)+\ell - \hbar\exp(4\iota\mu)]\right\} + \cdots.$$

For $\ell = 1$ and $\hbar = 1$, we achieved the exact result as:

$$\theta(\mu,\varepsilon) = \exp(\iota\mu).$$

4.3 Problem

Consider the fractional nonlinear BME:

$$\iota\frac{\partial^{ß}\theta}{\partial\varepsilon^{ß}} + \ell\frac{\partial^2\theta}{\partial\varepsilon^2} + \hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon) = 0, \quad 0 < ß \le 1, \tag{27}$$

with the IC:

$$\theta(\mu,0) = \exp^{-\frac{\iota\mu}{2}}.$$

On using the Elzaki transform, we obtain

$$E\left(\iota\frac{\partial^{ß}\theta}{\partial\varepsilon^{ß}}\right) = -E\left[\ell\frac{\partial^2\theta}{\partial\varepsilon^2} + \hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon)\right]. \tag{28}$$

After calculation, we obtain

$$\frac{1}{u^{ß}}\{M(u) - u^2\theta(0)\} = -E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon)\right], \tag{29}$$

$$M(u) = u^2\theta(0) - u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon)\right]. \tag{30}$$

By applying the inverse Elzaki transform, we obtain

$$\theta(\mu,\varepsilon) = \theta(0) - E^{-1}\left[u^{ß}\left\{E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon)\right]\right\}\right],$$
$$\theta(\mu,\varepsilon) = \exp^{-\frac{\iota\mu}{2}} - E^{-1}\left[u^{ß}\left\{E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar(|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon)\right]\right\}\right]. \tag{31}$$

On applying the HPM,

$$\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon) = \exp^{-\frac{\iota\mu}{2}} + \varrho\left(E^{-1}\left[u^{ß}E\left[\ell\iota\left(\sum_{k=0}^{\infty}\varrho^k\theta_k(\mu,\varepsilon)\right)_{\mu\mu} - \iota\hbar\left(\sum_{k=0}^{\infty}\varrho^k H_k(\theta)\right)\right]\right]\right). \tag{32}$$

The nonlinear term find by He's polynomial $H_k(\theta)$ is defined as:

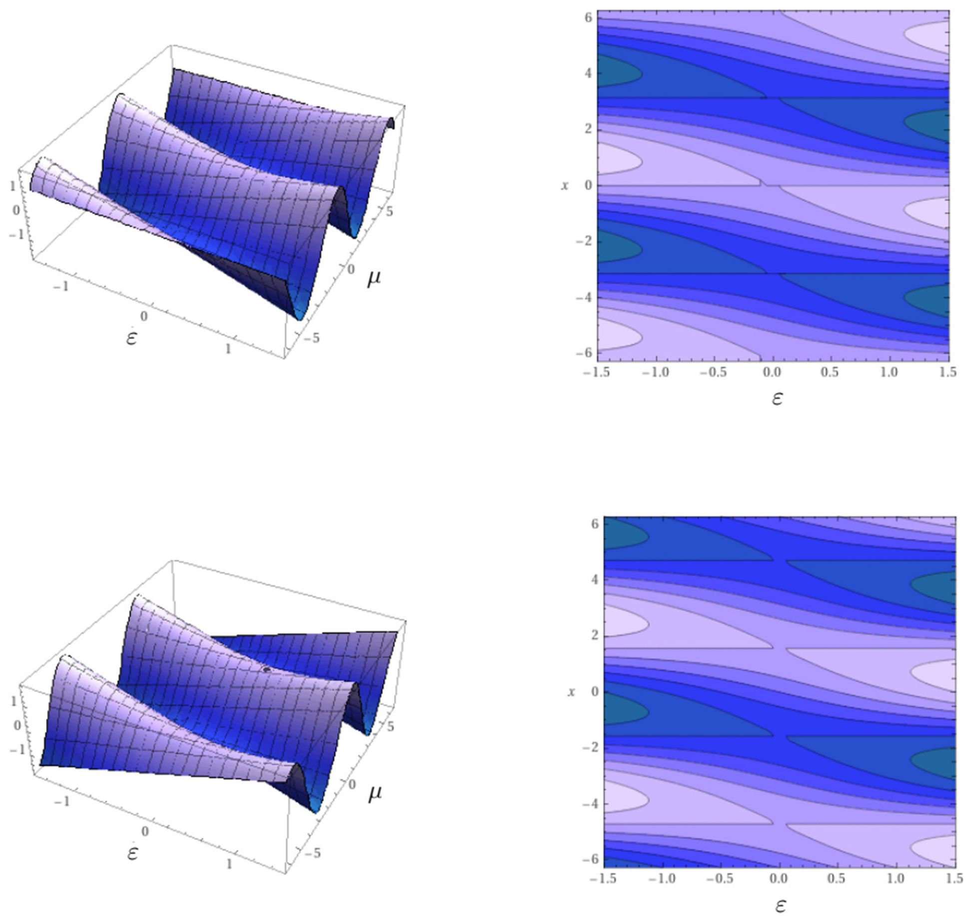

Figure 1: Analytical solution of three-dimensional graphes of Problem 1.

$$\sum_{k=0}^{\infty} \varrho^k H_k(\theta) = (|\theta(\mu,\varepsilon)|^2 + |\theta(\mu,\varepsilon)|^4)\theta(\mu,\varepsilon). \tag{33}$$

Some He's polynomial terms are calculated as:

$$H_0(\theta) = |\theta_0(\mu,\varepsilon)|^2\theta_0(\mu,\varepsilon) + |\theta_0(\mu,\varepsilon)|^4\theta_0(\mu,\varepsilon),$$

$$H_1(\theta) = \frac{1}{1!}\frac{\partial}{\partial\varrho}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon)|^2 + |\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon)|^4)(\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon))]_{\varrho=0},$$

$$H_2(\theta) = \frac{1}{2!}\frac{\partial^2}{\partial\varrho^2}[(|\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon)|^2 + |\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon)|^4)(\theta_0(\mu,\varepsilon) + \varrho\theta_1(\mu,\varepsilon) + \varrho^2\theta_2(\mu,\varepsilon))]_{\varrho=0}$$

By comparing the coefficients ϱ, we obtain

$$\varrho^0 : \theta_0(\mu,\varepsilon) = \exp^{-\frac{\iota\mu}{2}},$$

$$\varrho^1 : \theta_1(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_0(\theta)\right]\right)$$

$$= \iota\left(2\hbar - \frac{\ell}{4}\right)\exp^{-\frac{\iota\mu}{2}}\frac{\varepsilon^{ß}}{\Gamma(ß+1)},$$

$$\varrho^2 : \theta_2(\mu,\varepsilon) = E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_1(\theta)\right]\right)$$

$$= \frac{1}{2}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2\left(2\hbar - \frac{\ell}{4}\right)\left(\frac{\ell}{4} - 2\hbar - 3\iota + 3\iota\exp(-2\iota\mu)\right)\exp^{-\frac{\iota\mu}{2}},$$

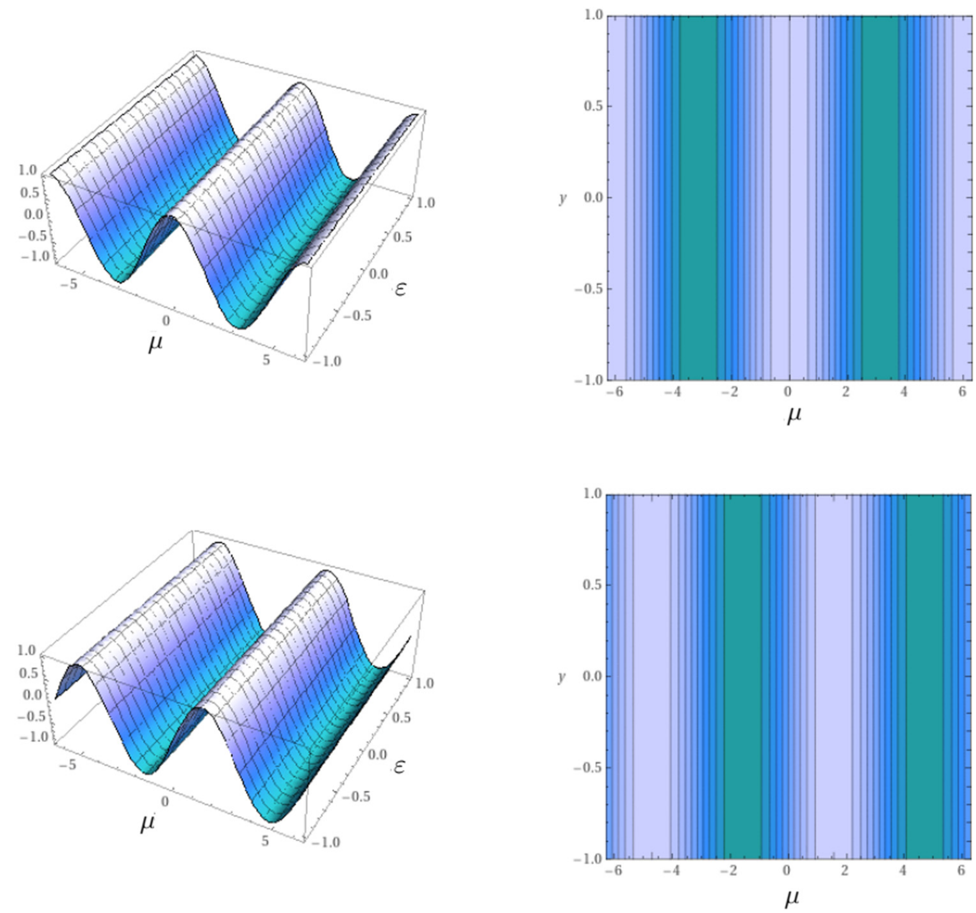

Figure 2: Analytical solution of three-dimensional graphes of Problem 2.

$$
\begin{aligned}
\varrho^3: \theta_3(\mu,\varepsilon) &= E^{-1}\left(u^{ß}E\left[\ell\iota\frac{\partial^2\theta}{\partial\varepsilon^2} + \iota\hbar H_2(\theta)\right]\right)\\
&= \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3\left(2\hbar - \frac{\ell}{4}\right)\iota\ell(-\ell\exp(\iota\mu))\\
&\quad + \{\iota\ell(-\ell\exp(\iota\mu) + 16\hbar\exp(5\iota\mu))\\
&\quad + \hbar\iota\exp(\iota\mu)(\hbar-\ell)^2(2+\ell-\hbar-\exp(4\iota\mu))\},\\
&\vdots
\end{aligned}
$$

Thus, the analytic result is achieved by applying the truncate series as:

$$
\begin{aligned}
\theta(\mu,\varepsilon) = {} & \exp^{-\frac{\iota\mu}{2}} + \iota\left(2\hbar - \frac{\ell}{4}\right)\exp^{-\frac{\iota\mu}{2}}\frac{\varepsilon^{ß}}{\Gamma(ß+1)} + \frac{1}{2}\\
& \left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^2\left(2\hbar - \frac{\ell}{4}\right)\left(\frac{\ell}{4} - 2\hbar - 3\iota + 3\iota\exp(-2\iota\mu)\right)\\
& \exp^{-\frac{\iota\mu}{2}} + \frac{1}{3}\left(\frac{\varepsilon^{ß}}{\Gamma(ß+1)}\right)^3\left(2\hbar - \frac{\ell}{4}\right)\\
& \{\iota\ell(-\ell\exp(\iota\mu) + 16\hbar\exp(5\iota\mu))\\
& \quad + \hbar\iota\exp(\iota\mu)(\hbar-\ell)^2(2+\ell-\hbar-\exp(4\iota\mu))\}\\
& \quad + \cdots.
\end{aligned}
$$

5 Graphical discussion

The results obtained in this research provide a deeper understanding of the dynamics governed by the fractional BME and demonstrate the efficacy of the HPTM in solving such equations. The methodology's accuracy and reliability were affirmed through comparisons with existing approaches and numerical simulations. To visually represent our findings, we present three-dimensional graphs of the analytical solutions for three different problems in Figures 1–3. These graphs illustrate the behavior of the solutions and provide insights into the dynamics of the BME. Overall, this study contributes to the growing body of research on fractional differential equations and their applications. It highlights the potential of HPTM as a valuable tool for addressing complex mathematical models involving the Caputo operator, paving the way for further advancements in the analysis of fractional systems in various scientific and engineering domains.

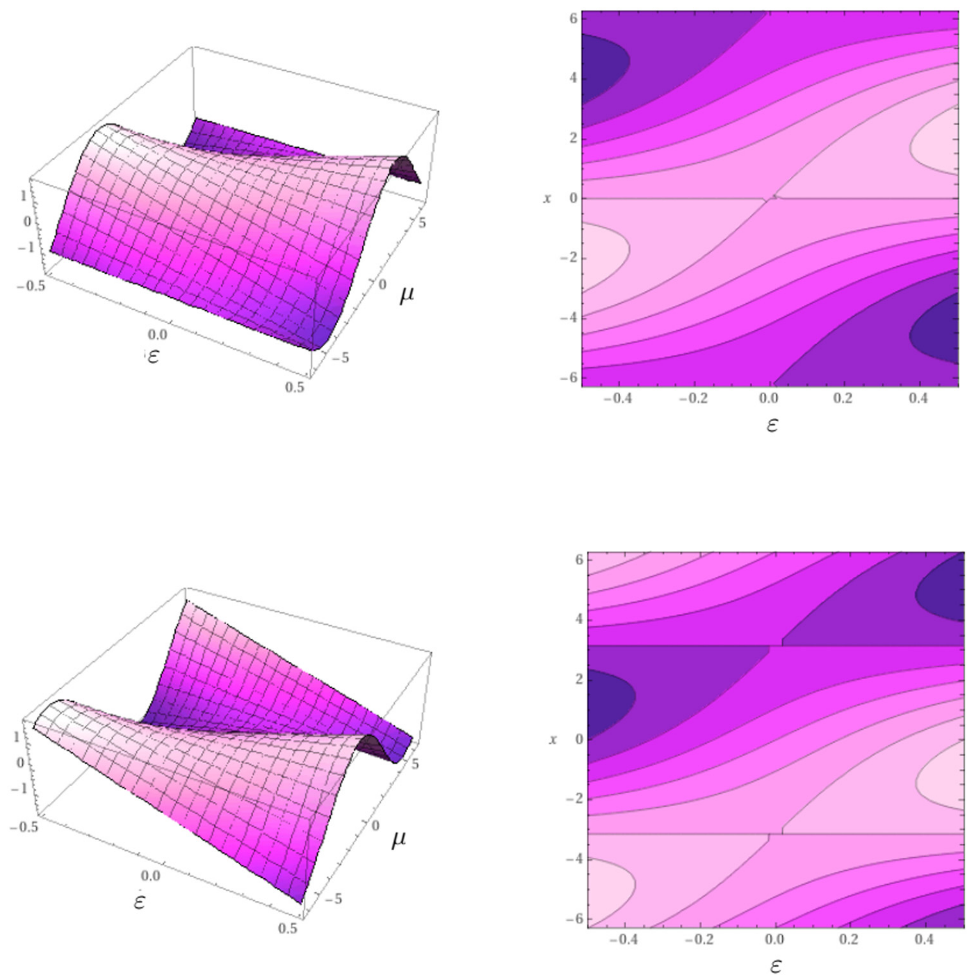

Figure 3: Analytical solution of three-dimensional graphes of Problem 3.

6 Conclusion

In this study, we applied the HPTM to rigorously investigate the fractional BME, considering the Caputo operator. The utilization of HPTM allowed us to tackle this complex fractional equation, which is known for its significance in modeling various physical and engineering phenomena. Through systematic mathematical transformations and perturbation techniques, we successfully derived analytical solutions and gained valuable insights into the behavior of the BME. The results obtained in this research provide a deeper understanding of the dynamics governed by the fractional BME and demonstrate the efficacy of the HPTM in solving such equations. The methodology's accuracy and reliability were affirmed through comparisons with existing approaches and numerical simulations. Overall, this study contributes to the growing body of research on fractional differential equations and their applications. It highlights the potential of HPTM as a valuable tool for addressing complex mathematical models involving the Caputo operator, paving the way for further advancements in the analysis of fractional systems in various scientific and engineering domains.

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