Research Article

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Numerical simulation and analysis of Airy's-type equation

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Abstract: In this article, we propose a novel new iteration method and homotopy perturbation method (HPM) along with the Elzaki transform to compute the analytical and semi-analytical approximations of fractional Airy's-type partial differential equations (FAPDEs) subjected to specific initial conditions. A convergent series solution form with easily commutable coefficients is used to examine and compare the performance of the suggested methods. Using Maple graphical method analysis, the behavior of the estimated series results at various fractional orders ς and its modeling in two-dimensional (2D) and three-dimensional (3D) spaces are compared with actual results. Also, detailed descriptions of the physical and geometric implications of the calculated graphs in 2D and 3D spaces are provided. As a result, the obtained solutions of FAPDEs that are subject to particular initial values quite closely match the exact solutions. In this way, to solve FAPDEs quickly, the proposed approaches are considered to be more accurate and efficient.

Keywords: Elzaki transform, new iterative method, Caputo derivative, homotopy perturbation method, fractional order Airy's-type equation

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1 Introduction

Recent years have seen a significant increase in interest in fractional calculus (FC) among scholars in various study topics. The integration and differentiation are generalized to any fractional order using FC [1]. The ideas of n-fold integration and differentiation having integer order are unified and generalized by the idea of integrals and derivatives of arbitrary order. As the response of the fractional-order system eventually converges to the integer-order equations, the theory of fractional differentiation is currently receiving a lot more attention.

Global features, which are not present in classical-order models, are the most iterative aspect of these models. Numerous areas of science, including financial mathematics, fluid dynamics, ecology, solid mechanics, biological diseases, and other fields, have given considerable attention to FC [2–8].

FC and fractional differential equations (FDEs) have made numerous remarkable developments in recent years, which have been studied. By creating models employing the FC theory, some genuine phenomena that appear in engineering and scientific areas can be convincingly demonstrated in the studies by Li *et al.* [9] and and Jin and Wang [10]. These include, but are not limited to, time-fractional wave equations, time-fractional telegraphic equations, time-fractional heat-like equations, fractional Airy's-type partial differential equations (FAPDEs), and others. Solving such fractional differential equations is quite easy because these equations are described by linear and nonlinear differential equations and have so many scientific applications [11,12]. The fundamental benefit of FDEs is that they generate accurate and reliable solutions because they are global operators [13]. Due to the significance of this class of differential equations, we encourage readers to review the work in [14–21] for some recent results. Fractional partial differential equations (FPDEs) are the most efficient kind of partial differential equations (FDEs) for simulating a range of complex processes in applied sciences. Two important models that are represented by FPDEs are the groundwater float and the El

Nino-Southern oscillation mannequin. The mathematical models of extended FPDE are essential for comprehending natural processes. In order to determine the precise dynamics of the described events, researchers have tried to numerically or analytically solve these models [22–25].

George Biddell Airy, interested in optics, was the first to use the name "Airy differential equation" [26]. Calculating the amount of light present near a caustic surface also caught his attention. Many researchers agree that the Airy equation, which is a traditional mathematical physics equation, plays a crucial role in many different scientific domains. Numerous applications of the Airy equation can be found in many areas, especially in describing physical phenomena. Although its use is not just restricted to optics, modeling the diffraction of light and other problems is one of its applications. One of the first models of water waves, Airy's partial differential equation, depicts a modest wave traveling in "wave trains" in deep water. It is one of the linear partial differential equations utilized in numerous real applications [27]. A typical example of a linear dispersive equation is Airy's partial differential equation [28]. In the early days of mathematical modeling of water waves, it was thought that the wave height was small as compared to the water. Such equations are somewhat satisfactory in this sense because their solutions resemble waves moving along the water's surface at a steady speed and with a fixed profile, as one observes in nature [29].

Several authors have described the analysis of the solution of Airy's-type differential equations (ADEs) using various methods over the last few years, including combining the knowledge of the mean and the variance and the principle of maximum entropy [30], classical and nonclassical Lie symmetry analysis and some technical calculations [31], the variational iteration method (VIM) and the steepest descent method [26]. The result of the solution to ADEs equation based on the energy estimations utilizing weighted Sobolev norms was also demonstrated [32], as well as its existence, uniqueness, and regularity. All of these techniques, though, have their own drawbacks. The analytical and semi-analytical approximate solutions of the FAPDEs, were found using two different techniques to obtain around these problems:

(i) The FAPDEs of the form:

$$\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}} = \beta\frac{\partial^{3}\mathbb{P}(\phi,\lambda)}{\partial\phi^{3}} \quad \phi\in\mathbb{R}, \lambda>0, 0<\varsigma\leq 1, \quad (1)$$

where $\beta = \pm 1$, having initial guess $\mathbb{P}(\phi, 0) = \xi(\phi)$.

(ii) The FAPDEs of the form:

$$\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}} = \mathbb{P}(\phi,\lambda)\frac{\partial^{3}\mathbb{P}(\phi,\lambda)}{\partial\phi^{3}} \quad \phi\in\mathbb{R}, \lambda>0, \quad 0<\varsigma\leq 1, \quad (2)$$

having initial guess $\mathbb{P}(\phi, 0) = \xi_1(\phi)$. Here, $\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}}$ denoted the Caputo fractional derivative having order ς. $\mathbb{P}(\phi, \lambda)$ is a function of space and time that vanishes at values of $\phi < 0$ and $\lambda < 0$.

In the current study, we use both the innovative iterative approach offered by Daftardar-Gejji and Jafari [33] and the homotopy perturbation transform method introduced by Madani *et al.* [34] and Khan and Wu [35]. Daftardar-Gejji and Jafari introduced a new iterative method for finding numerical solutions to nonlinear functional equations in 2006 [33,36]. Many nonlinear differential equations of integer and fractional order [37] and fractional boundary value problems [38] have been solved using the iterative method. We combine the iterative method with Elzaki transform (ET) [39] in the first method. The traditional Fourier integral is the source of the ET. Tarig Elzaki developed the ET to make it easier to solve ordinary and partial differential equations in the time domain. In a simple manner, the second strategy combines the Elzaki transformation, the homotopy perturbation method (HPM), and He's polynomials. He [40,41] introduced the HPM, which is a series expansion approach for solving nonlinear partial differential equations. To ensure convergence of the approximation series over a certain interval of physical parameters, the HPM employs a so-called convergence-control parameter. The importance of this research is finding an approximate solution to the Airy's-type equation having fractional order using two novel methods that are comparably new, as well as effective in comparing the accurate solution of the proposed models to fourth-order approximations for a range of fractional derivative values. Researchers can use this study as a fundamental reference to examine these strategies and employ it in many applications to obtain accurate and approximative results in a few easy steps. The unique aspect of this work is the description of two novel techniques for fractional Airy's-type equation with minimal and consecutive steps.

The following is our article's outline: Section 2 provides the fundamental definitions of FC; Section 3 presents the fundamental concept of NITM, whereas Section 4 provides the general approach of HPTM; In Section 5, we discussed the uniqueness and convergence results; to show the applicability of the technique under consideration, numerical problems are given in Section 6; in Section 7, conclusions of the main results are discussed in brief.

2 Basic definitions

In this part, we show FC basic results related to our study.

2.1 Definition

The fractional Abel–Riemann derivative is as follows [42–44]:

$$D^{\varsigma}\nu(\psi) = \begin{cases} \dfrac{\mathrm{d}^{\ell}}{\mathrm{d}\psi^{\ell}}\nu(\psi), \quad \varsigma = \ell, \\ \dfrac{1}{\Gamma(\ell - \varsigma)}\dfrac{\mathrm{d}}{\mathrm{d}\psi^{\ell}}\displaystyle\int_0^{\psi}\dfrac{\nu(\psi)}{(\psi - \phi)^{\varsigma - \ell + 1}}\mathrm{d}\phi, \\ \ell - 1 < \varsigma < \ell, \end{cases} \tag{3}$$

with $\ell \in Z^+$, $\varsigma \in R^+$ and

$$D^{-\varsigma}\nu(\psi) = \frac{1}{\Gamma(\varsigma)}\int_0^{\psi}(\psi - \phi)^{\varsigma-1}\nu(\phi)\mathrm{d}\phi, \ \ 0 < \varsigma \leq 1. \tag{4}$$

2.2 Definition

The fractional integration operator in Abel–Riemann sense is given as follows [42–44]:

$$J^{\varsigma}\nu(\psi) = \frac{1}{\Gamma(\varsigma)}\int_0^{\psi}(\psi - \phi)^{\varsigma-1}\nu(\psi)\mathrm{d}\psi, \ \ \psi > 0, \ \ \varsigma > 0, \tag{5}$$

having the following properties:

$$J^{\varsigma}\psi^{\ell} = \frac{\Gamma(\ell + 1)}{\Gamma(\ell + \varsigma + 1)}\psi^{\ell+\phi},$$
$$D^{\varsigma}\psi^{\ell} = \frac{\Gamma(\ell + 1)}{\Gamma(\ell - \varsigma + 1)}\psi^{\ell-\psi}.$$

2.3 Definition

The Caputo fractional derivative is given as follows [45]:

$$^{C}D^{\varsigma}\nu(\psi) = \begin{cases} \dfrac{1}{\Gamma(\ell - \varsigma)}\displaystyle\int_0^{\psi}\dfrac{\nu^{\ell}(\phi)}{(\psi - \phi)^{\varsigma - \ell + 1}}\mathrm{d}\phi, \ \ \ell - 1 < \varsigma < \ell, \\ \dfrac{\mathrm{d}^{\ell}}{\mathrm{d}\psi^{\ell}}\nu(\psi), \ \ \ell = \varsigma, \end{cases} \tag{6}$$

with the properties $\ell_{\psi}^{\varsigma}D_{\psi}^{\varsigma}g(\psi) = g(\psi) - \sum_{k=0}^{m}g^{k}(0^+)\frac{\psi^k}{k!}$, for $\psi > 0$, and $\ell - 1 < \varsigma \leq \ell, \ell \in N$. $D_{\psi}^{\varsigma}\ell_{\psi}^{\varsigma}g(\psi) = g(\psi)$.

2.4 Definition

The ET of a function is given as follows [42,43]:

$$E[g(\psi)] = G(r) = r\int_0^{\infty}h(\psi)e^{\frac{-\psi}{r}}\mathrm{d}\psi, \quad r > 0. \tag{7}$$

2.5 Definition

The Caputo operator ET is defined by the following equation [42,43]:

$$\mathrm{E}[D_{\psi}^{\varsigma}g(\psi)] = s^{-\varsigma}\mathrm{E}[g(\psi)] - \sum_{k=0}^{\ell-1}s^{2-\varsigma+k}g^{(k)}(0),$$
$$\text{where } \ell - 1 < \varsigma < \ell.$$

3 Methodology of NITM

Here, the general solution of the FPDEs that are subjected to a certain initial guess reads

$$D_{\lambda}^{\varsigma}\mathbb{P}(\phi, \lambda) + N\mathbb{P}(\phi, \lambda) + M\mathbb{P}(\phi, \lambda) = h(\phi, \lambda), \ \ \lambda > 0, \ \ 1 < \varsigma \leq 0, \tag{8}$$

subjected to the initial condition

$$\mathbb{P}^{k}(\phi, 0) = g_k(\phi), \ \ k = 0, 1, 2, \ldots, n - 1, \tag{9}$$

where N and M are, respectively, the linear and nonlinear components.

On employing the ET to Eq. (8), we obtain

$$E[D_{\lambda}^{\varsigma}\mathbb{P}(\phi, \lambda)] + E[N\mathbb{P}(\phi, \lambda) + M\mathbb{P}(\phi, \lambda)] = E[h(\phi, \lambda)]. \tag{10}$$

Applying the ET, we obtain

$$E[\mathbb{P}(\phi, \lambda)] = \sum_{k=0}^{m}s^{2-\varsigma+k}u^{(k)}(\phi, 0) + s^{\varsigma}E[h(\phi, \lambda)] - s^{\varsigma}E[N\mathbb{P}(\phi, \lambda) + M\mathbb{P}(\phi, \lambda)]. \tag{11}$$

By employing the inverse of the ET to Eq. (11), we obtain

$$\mathbb{P}(\phi, \lambda) = E^{-1}\left[\left\{\sum_{k=0}^{m}s^{2-\varsigma+k}u^{k}(\phi, 0) + s^{\varsigma}E[h(\phi, \lambda)]\right\}\right] - E^{-1}[s^{\varsigma}E[N\mathbb{P}(\phi, \lambda) + M\mathbb{P}(\phi, \lambda)]]. \tag{12}$$

By iterative technique, we have

$$\mathbb{P}(\phi, \lambda) = \sum_{m=0}^{\infty}\mathbb{P}_m(\phi, \lambda), \tag{13}$$

$$N\left(\sum_{m=0}^{\infty}\mathbb{P}_m(\phi, \lambda)\right) = \sum_{m=0}^{\infty}N[\mathbb{P}_m(\phi, \lambda)], \tag{14}$$

and the nonlinear term N is decomposed as follows:

$$N\left(\sum_{m=0}^{\infty}\mathbb{P}_m(\phi,\lambda)\right) = \mathbb{P}_0(\phi,\lambda) + N\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right) - M\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right). \tag{15}$$

Inserting Eqs (13)–(15) into Eq. (12), we have

$$\sum_{m=0}^{\infty}\mathbb{P}_m(\phi,\lambda) = E^{-1}\left[s^{\varsigma}\left(\sum_{k=0}^{m}s^{2-\phi+k}u^k(\phi,0) + E[h(\phi,\lambda)]\right)\right] - E^{-1}\left[s^{\varsigma}E\left[N\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right) - M\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right)\right]\right]. \tag{16}$$

By means of an iterative formula, we have

$$\mathbb{P}_0(\phi,\lambda) = E^{-1}\left[s^{\varsigma}\left(\sum_{k=0}^{m}s^{2-\phi+k}u^k(\phi,0) + s^{\varsigma}E(g(\phi,\lambda))\right)\right], \tag{17}$$

$$\mathbb{P}_1(\phi,\lambda) = -E^{-1}[s^{\varsigma}E[N\mathbb{P}_0(\phi,\lambda)] + M[\mathbb{P}_0(\phi,\lambda)]], \tag{18}$$

$$\mathbb{P}_{m+1}(\phi,\lambda) = -E^{-1}\left[s^{\varsigma}E\left[-N\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right) - M\left(\sum_{k=0}^{m}\mathbb{P}_k(\phi,\lambda)\right)\right]\right], \quad m \geq 1. \tag{19}$$

Finally, the series form solution to Eq. (8) for m-term is given as

$$\mathbb{P}(\phi,\lambda) \cong \mathbb{P}_0(\phi,\lambda) + \mathbb{P}_1(\phi,\lambda) + \mathbb{P}_2(\phi,\lambda) + \cdots, \quad m = 1, 2, \ldots. \tag{20}$$

4 Methodology of HPTM

Here, the general solution of the FPDEs that are subjected to a certain initial guess reads

$$D_{\lambda}^{\varsigma}\mathbb{P}(\phi,\lambda) + M\mathbb{P}(\phi,\lambda) + N\mathbb{P}(\phi,\lambda) = h(\phi,\lambda), \quad \lambda > 0, \quad 0 < \varsigma \leq 1, \quad \mathbb{P}(\phi,0) = g(\phi), \quad \nu \in \mathfrak{R}. \tag{21}$$

By applying the ET to Eq. (21), we obtain

$$E[D_{\lambda}^{\varsigma}\mathbb{P}(\phi,\lambda) + M\mathbb{P}(\phi,\lambda) + N\mathbb{P}(\phi,\lambda)] = E[h(\phi,\lambda)],$$
$$\mathbb{P}(\phi,\lambda) = s^2 g(\phi) + s^{\varsigma}E[h(\phi,\lambda)] - s^{\varsigma}E[M\mathbb{P}(\phi,\lambda) + N\mathbb{P}(\phi,\lambda)]. \tag{22}$$

By applying the inverse ET, we obtain

$$\mathbb{P}(\phi,\lambda) = F(x,\lambda) - E^{-1}[s^{\varsigma}E\{M\mathbb{P}(\phi,\lambda) + N\mathbb{P}(\phi,\lambda)\}], \tag{23}$$

where

$$F(\phi,\lambda) = E^{-1}[s^2 g(\phi) + s^{\varsigma}E[h(\phi,\lambda)]] = g(\nu) + E^{-1}[s^{\varsigma}E[h(\phi,\lambda)]]. \tag{24}$$

Now in series form, the perturbation technique having parameter ε is stated as follows:

$$\mathbb{P}(\phi,\lambda) = \sum_{k=0}^{\infty}\varepsilon^k\mathbb{P}_k(\phi,\lambda), \tag{25}$$

with $\varepsilon \in [0,1]$ representing the perturbation parameter.

The calculation of nonlinear terms is as follows:

$$N\mathbb{P}(\phi,\lambda) = \sum_{k=0}^{\infty}\varepsilon^k H_k(\mathbb{P}_k), \tag{26}$$

where H_n denotes He's polynomials in terms of $\mathbb{P}_0, \mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n$ and is expressed as follows:

$$H_n(\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_n) = \frac{1}{\varsigma(n+1)}D_{\varepsilon}^{k}\left[N\left(\sum_{k=0}^{\infty}\varepsilon^k\mathbb{P}_k\right)\right]_{\varepsilon=0}, \tag{27}$$

where $D_{\varepsilon}^{k} = \frac{\partial^k}{\partial\varepsilon^k}$.

Inserting Eqs (26) and (27) into Eq. (23), we obtain

$$\sum_{k=0}^{\infty}\varepsilon^k\mathbb{P}_k(\phi,\lambda) = F(\phi,\lambda) - \varepsilon\left[E^{-1}\left\{s^{\varsigma}E\left\{M\sum_{k=0}^{\infty}p^k\mathbb{P}_k(\phi,\lambda) + \sum_{k=0}^{\infty}\varepsilon^k H_k(\mathbb{P}_k)\right\}\right\}\right]. \tag{28}$$

By comparing the coefficient of ε on both sides, we have

$$\begin{aligned}
\varepsilon^0 &: \mathbb{P}_0(\phi,\lambda) = F(\phi,\lambda),\\
\varepsilon^1 &: \mathbb{P}_1(\phi,\lambda) = E^{-1}[s^{\varsigma}E(M\mathbb{P}_0(\phi,\lambda) + H_0(\mathbb{P}))],\\
\varepsilon^2 &: \mathbb{P}_2(\phi,\lambda) = E^{-1}[s^{\varsigma}E(M\mathbb{P}_1(\phi,\lambda) + H_1(\mathbb{P}))],\\
&\vdots\\
\varepsilon^k &: \mathbb{P}_k(\phi,\lambda) = E^{-1}[s^{\varsigma}E(M\mathbb{P}_{k-1}(\phi,\lambda) + H_{k-1}(\mathbb{P}))].
\end{aligned} \tag{29}$$

Thus, $\mathbb{P}_k(\phi,\lambda)$ are easily calculated showing that they easily lead to a convergent series. For $\varepsilon \to 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = \lim_{M\to\infty}\sum_{k=1}^{M}\mathbb{P}_k(\phi,\lambda). \tag{30}$$

5 Applications

5.1 Example

Let us assume the 1D FAPDEs for $\beta = 1$:

$$\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}} = \frac{\partial^{3}\mathbb{P}(\phi,\lambda)}{\partial\phi^{3}}, \tag{31}$$

having initial guess

$$\mathbb{P}(\phi,0) = \cos(\pi\phi) + e^{\pi\phi}. \tag{32}$$

On employing the ET to Eq. (31), we have

$$E[\mathbb{P}(\phi,\lambda)] = s^{2}(\cos(\pi\phi) + e^{\pi\phi}) + s^{\varsigma}E\left[\frac{\partial^{3}\mathbb{P}(\phi,\lambda)}{\partial\phi^{3}}\right]. \tag{33}$$

By applying the inverse of the ET, we obtain

$$\mathbb{P}(\phi,\lambda) = \cos(\pi\phi) + e^{\pi\phi} + E^{-1}\left[s^{\varsigma}E\left[\frac{\partial^{3}\mathbb{P}(\phi,\lambda)}{\partial\phi^{3}}\right]\right]. \tag{34}$$

By NITM, we obtain

$$\begin{aligned}
\mathbb{P}_0(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi},\\
\mathbb{P}_1(\phi,\lambda) &= E^{-1}\left[s^{\varsigma}E\left\{\frac{\partial^{3}\mathbb{P}_0(\phi,\lambda)}{\partial\phi^{3}}\right\}\right]\\
&= \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda^{\varsigma}}{\Gamma(\varsigma+1)},\\
\mathbb{P}_2(\phi,\lambda) &= E^{-1}\left[s^{\varsigma}E\left\{\frac{\partial^{3}\mathbb{P}_1(\phi,\lambda)}{\partial\phi^{3}}\right\}\right]\\
&= \frac{-\pi^{6}(\cos(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{2}}{\Gamma(2\varsigma+1)},\\
\mathbb{P}_3(\phi,\lambda) &= E^{-1}\left[s^{\varsigma}E\left\{\frac{\partial^{3}\mathbb{P}_2(\phi,\lambda)}{\partial\phi^{3}}\right\}\right]\\
&= \frac{-\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{3}}{\Gamma(3\varsigma+1)},\\
&\vdots
\end{aligned}$$

Thus, the series solution takes the form

$$\begin{aligned}
\mathbb{P}(\phi,\lambda) &= \mathbb{P}_0(\phi,\lambda) + \mathbb{P}_1(\phi,\lambda) + \mathbb{P}_2(\phi,\lambda)\\
&\quad + \mathbb{P}_3(\phi,\lambda) + \cdots \mathbb{P}_n(\phi,\lambda),\\
\mathbb{P}(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi} + \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda^{\varsigma}}{\Gamma(\varsigma+1)}\\
&\quad - \frac{\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{3}}{\Gamma(3\varsigma+1)} + \cdots.
\end{aligned} \tag{35}$$

For $\varsigma = 1$, we obtain

$$\begin{aligned}
\mathbb{P}(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi} + \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda}{1!}\\
&\quad - \frac{\pi^{6}(\cos(\pi\phi) - e^{\pi\phi})\lambda^{2}}{2!}\\
&\quad - \frac{\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})\lambda^{3}}{3!} + \cdots.
\end{aligned} \tag{36}$$

By implementing the HPTM, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty}\varepsilon^{n}w_n(\phi,\lambda) &= (\cos(\pi\phi) + e^{\pi\phi})\\
&\quad + \varepsilon\left[E^{-1}\left(s^{\varsigma}E\left[\sum_{n=0}^{\infty}\varepsilon^{n}H_n(w)\right]\right)\right].
\end{aligned} \tag{37}$$

Thus, by equating equivalent powers of ε yields

$$\begin{aligned}
\varepsilon^{0}: w_0(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi},\\
\varepsilon^{1}: w_1(\phi,\lambda) &= [E^{-1}\{s^{\varsigma}E(H_0(w))\}]\\
&= \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda^{\varsigma}}{\Gamma(\varsigma+1)},\\
\varepsilon^{2}: w_2(\phi,\lambda) &= [E^{-1}\{s^{\varsigma}E(H_1(w))\}]\\
&= \frac{-\pi^{6}(\cos(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{2}}{\Gamma(2\varsigma+1)},\\
\varepsilon^{3}: w_3(\phi,\lambda) &= [E^{-1}\{s^{\varsigma}E(H_2(w))\}]\\
&= \frac{-\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{3}}{\Gamma(3\varsigma+1)},\\
&\vdots
\end{aligned} \tag{38}$$

So, the series solution takes the following form:

$$\mathbb{P}(\phi,\lambda) = \sum_{n=0}^{\infty}\varepsilon^{n}w_n(\phi,\lambda). \tag{39}$$

Hence,

$$\begin{aligned}
\mathbb{P}(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi} + \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda^{\varsigma}}{\Gamma(\varsigma+1)}\\
&\quad - \frac{\pi^{6}(\cos(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{2}}{\Gamma(2\varsigma+1)}\\
&\quad - \frac{\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})(\lambda^{\varsigma})^{3}}{\Gamma(3\varsigma+1)} + \cdots.
\end{aligned} \tag{40}$$

For $\varsigma = 1$, we obtain

$$\begin{aligned}
\mathbb{P}(\phi,\lambda) &= \cos(\pi\phi) + e^{\pi\phi} + \frac{\pi^{3}(\sin(\pi\phi) + e^{\pi\phi})\lambda}{1!}\\
&\quad - \frac{\pi^{6}(\cos(\pi\phi) - e^{\pi\phi})\lambda^{2}}{2!}\\
&\quad - \frac{\pi^{9}(\sin(\pi\phi) - e^{\pi\phi})\lambda^{3}}{3!} + \cdots.
\end{aligned} \tag{41}$$

(Figure 1)

5.2 Example

Let us assume the 1D FAPDEs for $\beta = -1$:

$$\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}} = -\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}, \tag{42}$$

having initial guess

$$\mathbb{P}(\phi,0) = \sin(\phi). \tag{43}$$

On employing the ET to Eq. (31), we have

$$E[\nu(\phi,\lambda)] = s^2(\sin(\phi)) + s^{\varsigma}E\left[-\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]. \tag{44}$$

By applying the inverse of the ET, we obtain

$$\nu(\phi,\lambda) = \sin(\phi) + E^{-1}\left[s^{\varsigma}E\left[-\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]\right]. \tag{45}$$

By applying the NITM, we obtain

$$\mathbb{P}_0(\phi,\lambda) = \sin(\phi),$$

$$\mathbb{P}_1(\phi,\lambda) = E^{-1}\left[s^{\varsigma}E\left\{-\frac{\partial^3\mathbb{P}_0(\phi,\lambda)}{\partial\phi^3}\right\}\right] = \frac{\cos(\phi)\lambda^{\varsigma}}{\Gamma(\varsigma+1)},$$

$$\mathbb{P}_2(\phi,\lambda) = E^{-1}\left[s^{\varsigma}E\left\{-\frac{\partial^3\mathbb{P}_1(\phi,\lambda)}{\partial\phi^3}\right\}\right] = \frac{-\sin(\phi)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)},$$

$$\mathbb{P}_3(\phi,\lambda) = E^{-1}\left[s^{\varsigma}E\left\{-\frac{\partial^3\mathbb{P}_2(\phi,\lambda)}{\partial\phi^3}\right\}\right] = \frac{-\cos(\phi)(\lambda^{\varsigma})^3}{\Gamma(3\varsigma+1)},$$

$$\vdots$$

Thus, the series solution takes the following form:

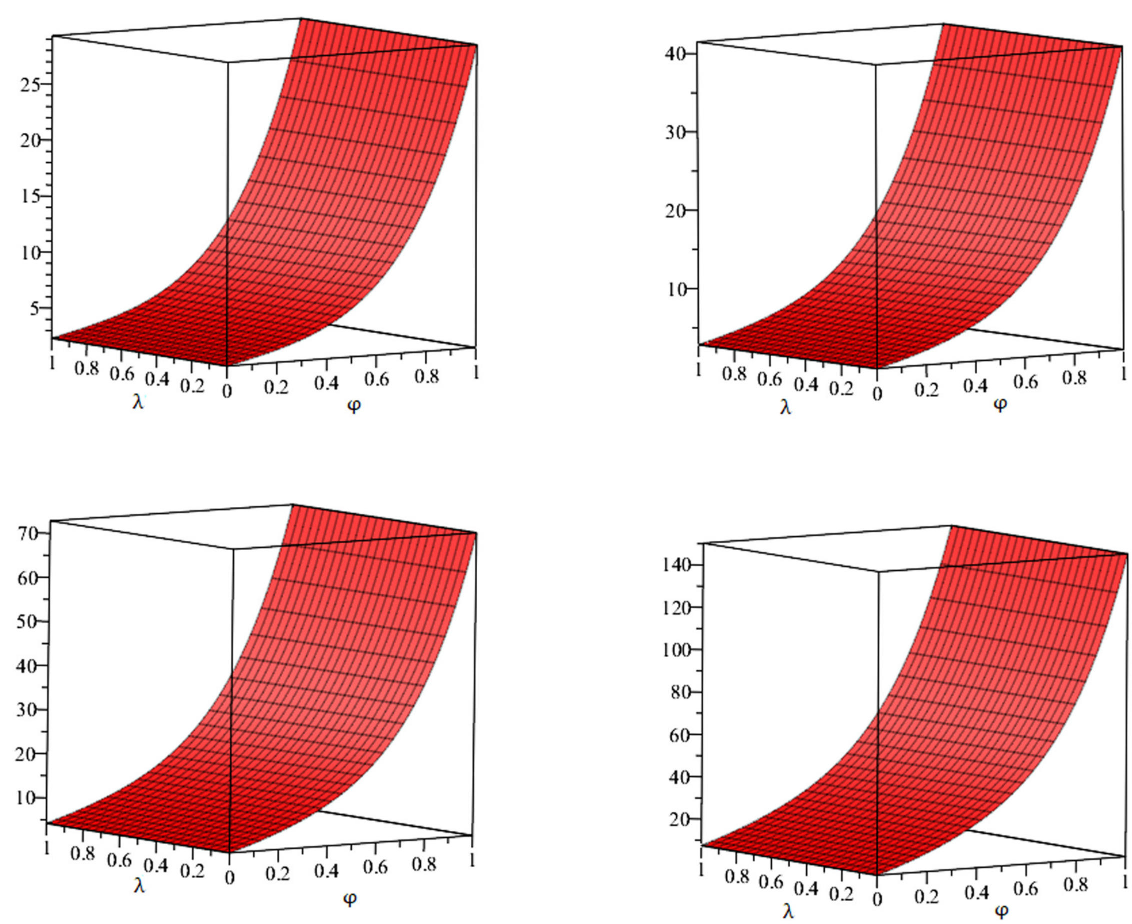

Figure 1: Plot of numerical solution *vs* ϕ *vs* λ using some different values of ς and caption of subfigure can be (a) $\varsigma = 1$, (b) $\varsigma = 0.75$, (c) $\varsigma = 0.50$, and (d) $\varsigma = 0.25$.

$$\mathbb{P}(\phi,\lambda) = \mathbb{P}_0(\phi,\lambda) + \mathbb{P}_1(\phi,\lambda) + \mathbb{P}_2(\phi,\lambda) + \mathbb{P}_3(\phi,\lambda) + \cdots \mathbb{P}_n(\phi,\lambda),$$
$$\mathbb{P}(\phi,\lambda) = \sin(\phi) + \frac{\cos(\phi)\lambda^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{\sin(\phi)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)} - \frac{\cos(\phi)(\lambda^{\varsigma})^3}{\Gamma(3\varsigma+1)} + \cdots . \tag{46}$$

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = \sin(\phi)\left[1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right] + \cos(\phi)\left[\lambda - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right]. \tag{47}$$

By implementing the HPTM, we obtain

$$\sum_{n=0}^{\infty} \varepsilon^n w_n(\phi,\lambda) = (\sin(\phi)) + \varepsilon\left\{E^{-1}\left\{s^{\varsigma}E\left[\sum_{n=0}^{\infty}\varepsilon^n H_n(w)\right]\right\}\right\}. \tag{48}$$

Thus, by equating equivalent powers of ε yields

$$\begin{aligned}
\varepsilon^0 &: w_0(\phi,\lambda) = \sin(\phi),\\
\varepsilon^1 &: w_1(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_0(w))\}] = \frac{\cos(\phi)\lambda^{\varsigma}}{\Gamma(\varsigma+1)},\\
\varepsilon^2 &: w_2(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_1(w))\}] = \frac{-\sin(\phi)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)},\\
\varepsilon^3 &: w_3(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_2(w))\}] = \frac{-\cos(\phi)(\lambda^{\varsigma})^3}{\Gamma(3\varsigma+1)},\\
&\vdots
\end{aligned} \tag{49}$$

So, the series solution takes the following form:

$$\mathbb{P}(\phi,\lambda) = \sum_{n=0}^{\infty}\varepsilon^n w_n(\phi,\lambda). \tag{50}$$

Hence,

$$\mathbb{P}(\phi,\lambda) = \sin(\phi) + \frac{\cos(\phi)\lambda^{\varsigma}}{\Gamma(\varsigma+1)} - \frac{\sin(\phi)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)} - \frac{\cos(\phi)(\lambda^{\varsigma})^3}{\Gamma(3\varsigma+1)} + \cdots . \tag{51}$$

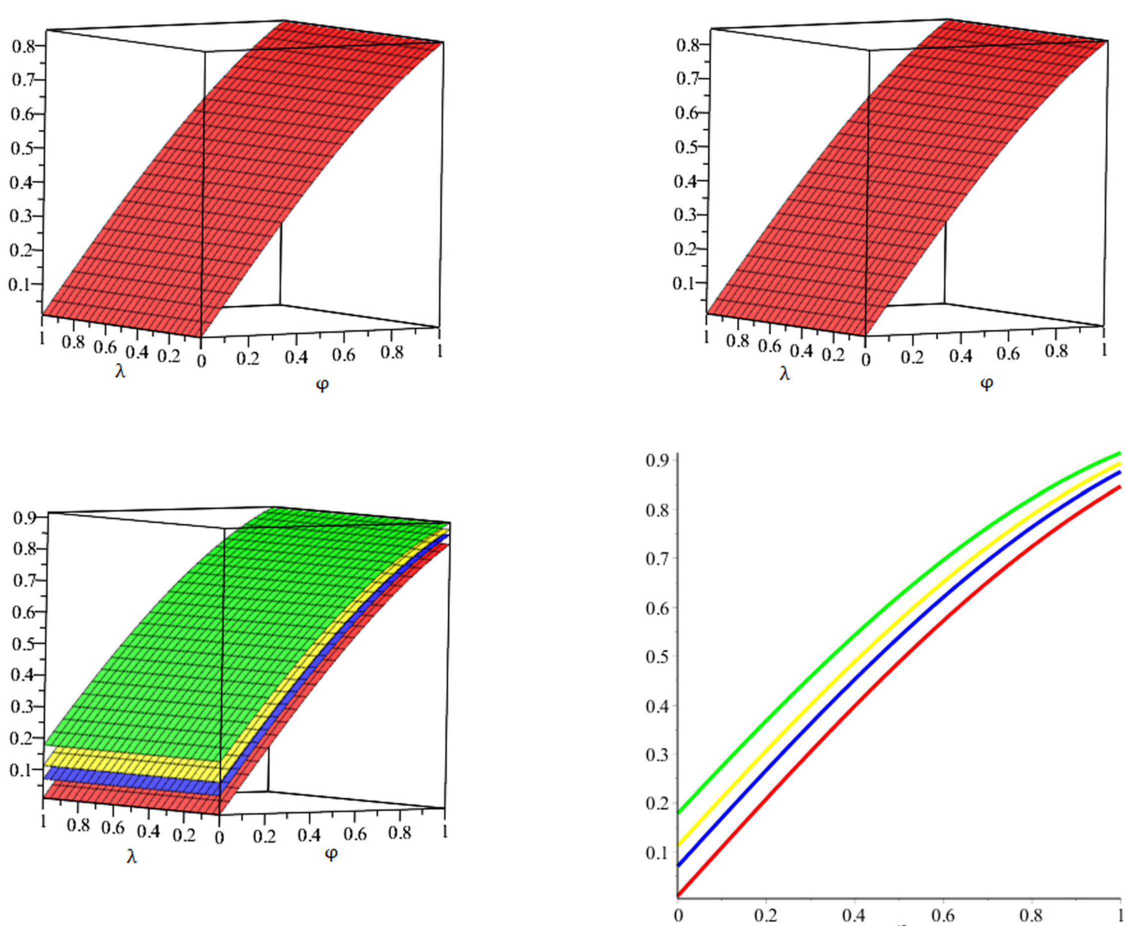

Figure 2: Plot of numerical solution *vs* ϕ *vs* λ using some different values of ς and caption of subfigure can be (a) exact solution, (b) proposed method solution, (c) 3D behavior of the proposed method solution when $\varsigma = 1, 0.8, 0.6, 0.4$, and (d) 2D behavior of the proposed method solution when $\varsigma = 1, 0.8, 0.6, 0.4$.

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = \sin(\phi)\left[1 - \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots\right] + \cos(\phi)\left[\lambda - \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots\right]. \tag{52}$$

The exact solution reads

$$\mathbb{P}(\phi,\lambda) = \sin(\phi)\cos(\lambda) + \cos(\phi)\sin(\lambda) \tag{53}$$

(Figure 2 and Table 1, Table 2)

5.3 Example

Let us assume the 1D FAPDEs as follows:

$$\frac{\partial^\varsigma \mathbb{P}(\phi,\lambda)}{\partial\lambda^\varsigma} = \mathbb{P}(\phi,\lambda)\frac{\partial^3 \mathbb{P}(\phi,\lambda)}{\partial\phi^3}, \tag{54}$$

having initial guess

$$\mathbb{P}(\phi,0) = (1-\phi)^{\left(\frac{1}{2}\right)}. \tag{55}$$

By employing the ET to Eq. (54), we have

$$E[\mathbb{P}(\phi,\lambda)] = s^2((1-\phi)^{\left(\frac{1}{2}\right)}) + s^\varsigma E\left[\mathbb{P}(\phi,\lambda)\frac{\partial^3 \mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]. \tag{56}$$

By applying the inverse of the ET, we obtain

$$\mathbb{P}(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)} + E^{-1}\left[s^\varsigma E\left[\mathbb{P}(\phi,\lambda)\frac{\partial^3 \mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]\right]. \tag{57}$$

By applying the NITM, we obtain

Table 1: Error comparison of HPTM and NITM solution for Example 2

ϕ	λ	|Exact–NITM|	|Exact–NITM|	|Exact–HPTM|	|Exact–HPTM|
	0.5	$1.45796785 \times 10^{-02}$	$2.0000000 \times 10^{-10}$	$2.86844810 \times 10^{-03}$	$2.0000000 \times 10^{-10}$
	1	$8.85201540 \times 10^{-03}$	$1.0000000 \times 10^{-10}$	$1.76023770 \times 10^{-03}$	$1.0000000 \times 10^{-10}$
	1.5	$9.57070100 \times 10^{-04}$	$2.0000000 \times 10^{-10}$	$2.21059600 \times 10^{-04}$	$2.0000000 \times 10^{-10}$
	2	$7.17219920 \times 10^{-03}$	$2.0000000 \times 10^{-10}$	$1.37224160 \times 10^{-03}$	$2.0000000 \times 10^{-10}$
	2.5	$1.35454641 \times 10^{-02}$	$1.0000000 \times 10^{-10}$	$2.62957010 \times 10^{-03}$	$1.0000000 \times 10^{-10}$
0.1	3	$1.66023269 \times 10^{-02}$	$2.0000000 \times 10^{-10}$	$3.24308820 \times 10^{-03}$	$2.0000000 \times 10^{-10}$
	3.5	$1.55943611 \times 10^{-02}$	$2.0000000 \times 10^{-10}$	$3.06258530 \times 10^{-03}$	$2.0000000 \times 10^{-10}$
	4	$1.07683518 \times 10^{-02}$	$1.0000000 \times 10^{-10}$	$2.13225460 \times 10^{-03}$	$1.0000000 \times 10^{-10}$
	4.5	$3.30587450 \times 10^{-03}$	$2.0000000 \times 10^{-10}$	$6.79873600 \times 10^{-04}$	$2.0000000 \times 10^{-10}$
	5	$4.96599630 \times 10^{-03}$	$1.0000000 \times 10^{-10}$	$9.38964100 \times 10^{-04}$	$1.0000000 \times 10^{-10}$
	0.5	$2.15888208 \times 10^{-02}$	1.100000×10^{-09}	$4.76073760 \times 10^{-03}$	$1.1000000 \times 10^{-09}$
	1	$1.30065763 \times 10^{-02}$	$7.0000000 \times 10^{-10}$	$2.91378210 \times 10^{-03}$	$7.0000000 \times 10^{-10}$
	1.5	$1.23986850 \times 10^{-03}$	$1.0000000 \times 10^{-10}$	$3.53431300 \times 10^{-04}$	$1.0000000 \times 10^{-10}$
	2	$1.08304025 \times 10^{-02}$	$6.0000000 \times 10^{-10}$	$2.29345200 \times 10^{-03}$	$6.0000000 \times 10^{-10}$
	2.5	$2.02490132 \times 10^{-02}$	$1.0000000 \times 10^{-09}$	$4.37881820 \times 10^{-03}$	$1.0000000 \times 10^{-09}$
0.2	3	$2.47099594 \times 10^{-02}$	$1.3000000 \times 10^{-09}$	$5.39209700 \times 10^{-03}$	$1.3000000 \times 10^{-09}$
	3.5	$2.31210456 \times 10^{-02}$	$1.3000000 \times 10^{-09}$	$5.08520240 \times 10^{-03}$	$1.3000000 \times 10^{-09}$
	4	$1.58712935 \times 10^{-02}$	$8.0000000 \times 10^{-10}$	$3.53327290 \times 10^{-03}$	$8.0000000 \times 10^{-10}$
	4.5	$4.73569530 \times 10^{-03}$	$3.0000000 \times 10^{-10}$	$1.11627500 \times 10^{-03}$	$3.0000000 \times 10^{-10}$
	5	$7.55936640 \times 10^{-03}$	$4.0000000 \times 10^{-10}$	$1.57402600 \times 10^{-03}$	$4.0000000 \times 10^{-10}$
	0.5	$2.70534724 \times 10^{-02}$	$4.0000000 \times 10^{-09}$	$6.37209510 \times 10^{-03}$	$4.0000000 \times 10^{-09}$
	1	$1.61930500 \times 10^{-02}$	$2.4000000 \times 10^{-09}$	$3.89044970 \times 10^{-03}$	$2.4000000 \times 10^{-09}$
	1.5	$1.36800430 \times 10^{-03}$	$3.0000000 \times 10^{-10}$	$4.56286500 \times 10^{-04}$	$3.0000000 \times 10^{-10}$
	2	$1.37919766 \times 10^{-02}$	$1.9000000 \times 10^{-09}$	$3.08959160 \times 10^{-03}$	$1.9000000 \times 10^{-09}$
	2.5	$2.55752007 \times 10^{-02}$	$3.6000000 \times 10^{-09}$	$5.87902990 \times 10^{-03}$	$3.6000000 \times 10^{-09}$
0.3	3	$3.10967236 \times 10^{-02}$	$4.5000000 \times 10^{-09}$	$7.22907660 \times 10^{-03}$	$4.5000000 \times 10^{-09}$
	3.5	$2.90046840 \times 10^{-02}$	$4.3000000 \times 10^{-09}$	$6.80919330 \times 10^{-03}$	$4.3000000 \times 10^{-09}$
	4	$1.98112863 \times 10^{-02}$	$3.0000000 \times 10^{-09}$	$4.72218190 \times 10^{-03}$	$3.0000000 \times 10^{-09}$
	4.5	$5.76739480 \times 10^{-03}$	$1.0000000 \times 10^{-09}$	$1.47901590 \times 10^{-03}$	$1.0000000 \times 10^{-09}$
	5	$9.68855610 \times 10^{-03}$	$1.3000000 \times 10^{-09}$	$2.12626490 \times 10^{-03}$	$1.30000000 \times 10^{-10}$

Table 2: Error comparison of reduced differential transform method (RDTM) with proposed methods for Example 2

ϕ	λ	RDTM $\varsigma = 1$	NITM $\varsigma = 1$	HPTM $\varsigma = 1$
	0.5	$1.6125789000 \times 10^{-08}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	1	$6.1317061000 \times 10^{-08}$	$1.0000000 \times 10^{-10}$	$1.0000000 \times 10^{-10}$
	1.5	$4.3621745000 \times 10^{-07}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	2	$1.6588950200 \times 10^{-07}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	2.5	$4.8456311000 \times 10^{-07}$	$1.0000000 \times 10^{-10}$	$1.0000000 \times 10^{-10}$
0.1	3	$2.0470878500 \times 10^{-07}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	3.5	$1.7918592000 \times 10^{-08}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	4	$7.5713342000 \times 10^{-08}$	$1.0000000 \times 10^{-10}$	$1.0000000 \times 10^{-10}$
	4.5	$2.9603994048 \times 10^{-08}$	$2.0000000 \times 10^{-10}$	$2.0000000 \times 10^{-10}$
	5	$2.3733505759 \times 10^{-07}$	$1.0000000 \times 10^{-10}$	$1.0000000 \times 10^{-10}$
	0.5	$3.8791346860 \times 10^{-09}$	1.100000×10^{-09}	$1.1000000 \times 10^{-09}$
	1	$7.8076524100 \times 10^{-09}$	$7.0000000 \times 10^{-10}$	$7.0000000 \times 10^{-10}$
	1.5	$2.7814445650 \times 10^{-07}$	$1.0000000 \times 10^{-10}$	$1.0000000 \times 10^{-10}$
	2	$3.64653597841 \times 10^{-07}$	$6.0000000 \times 10^{-10}$	$6.0000000 \times 10^{-10}$
	2.5	$5.5982968930 \times 10^{-07}$	$1.0000000 \times 10^{-09}$	$1.0000000 \times 10^{-09}$
0.2	3	$2.7371653580 \times 10^{-07}$	$1.3000000 \times 10^{-09}$	$1.3000000 \times 10^{-09}$
	3.5	$5.4189576320 \times 10^{-07}$	$1.3000000 \times 10^{-09}$	$1.3000000 \times 10^{-09}$
	4	$3.8173772900 \times 10^{-09}$	$8.0000000 \times 10^{-10}$	$8.0000000 \times 10^{-10}$
	4.5	$7.5575220640 \times 10^{-09}$	$3.0000000 \times 10^{-10}$	$3.0000000 \times 10^{-10}$
	5	$5.5426847543 \times 10^{-07}$	$4.0000000 \times 10^{-10}$	$4.0000000 \times 10^{-10}$
	0.5	$2.1862474647 \times 10^{-08}$	$4.0000000 \times 10^{-09}$	$4.0000000 \times 10^{-09}$
	1	$1.6487275658 \times 10^{-07}$	$2.4000000 \times 10^{-09}$	$2.4000000 \times 10^{-09}$
	1.5	$2.8831254647 \times 10^{-06}$	$3.0000000 \times 10^{-10}$	$3.0000000 \times 10^{-10}$
	2	$2.9831257369 \times 10^{-06}$	$1.9000000 \times 10^{-09}$	$1.9000000 \times 10^{-09}$
	2.5	$2.0962494632 \times 10^{-08}$	$3.6000000 \times 10^{-09}$	$3.6000000 \times 10^{-09}$
0.3	3	$1.7287638302 \times 10^{-07}$	$4.5000000 \times 10^{-09}$	$4.5000000 \times 10^{-09}$
	3.5	$2.990390826 \times 10^{-08}$	$4.3000000 \times 10^{-09}$	$4.3000000 \times 10^{-09}$
	4	$2.3333502823 \times 10^{-07}$	$3.0000000 \times 10^{-09}$	$3.0000000 \times 10^{-09}$
	4.5	$3.9659274738 \times 10^{-06}$	$1.0000000 \times 10^{-09}$	$1.0000000 \times 10^{-09}$
	5	$3.9759208942 \times 10^{-06}$	$1.3000000 \times 10^{-09}$	$1.30000000 \times 10^{-10}$

$$\mathbb{P}_0(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)},$$

$$\mathbb{P}_1(\phi,\lambda) = E^{-1}\left[s^{\varsigma}E\left\{\mathbb{P}_0(\phi,\lambda)\frac{\partial^3\mathbb{P}_0(\phi,\lambda)}{\partial\phi^3}\right\}\right] = \frac{\left(\frac{-3}{8}\right)\lambda^{\varsigma}}{\Gamma(\varsigma+1)(1-\phi)^2},$$

$$\mathbb{P}_2(\phi,\lambda) = E^{-1}\left[s^{\varsigma}E\left\{\mathbb{P}_1(\phi,\lambda)\frac{\partial^3\mathbb{P}_1(\phi,\lambda)}{\partial\phi^3}\right\}\right] = \frac{\left(\frac{-3}{8}\right)^2(63)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)(1-\phi)^{\left(\frac{9}{2}\right)}},$$

$$\vdots$$

Thus, the series solution takes the following form:

$$\mathbb{P}(\phi,\lambda) = \mathbb{P}_0(\phi,\lambda) + \mathbb{P}_1(\phi,\lambda) + \mathbb{P}_2(\phi,\lambda) + \mathbb{P}_3(\phi,\lambda) + \cdots \mathbb{P}_n(\phi,\lambda),$$

$$\mathbb{P}(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)} + \frac{\left(\frac{-3}{8}\right)\lambda^{\varsigma}}{\Gamma(\varsigma+1)(1-\phi)^2} + \frac{\left(\frac{-3}{8}\right)^2(63)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)(1-\phi)^{\left(\frac{9}{2}\right)}} + \cdots. \tag{58}$$

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)} + \left(\frac{\left(-\frac{3}{8}\right)}{1!}\right)\left(\frac{1}{(1-\phi)^2}\right)\lambda + \left(\frac{\left(-\frac{3}{8}\right)^2}{2!}\right)\left(\frac{63}{(1-\phi)^{\left(\frac{9}{2}\right)}}\right)\lambda^2 + \cdots . \tag{59}$$

By implementing the HPTM, we obtain

$$\sum_{n=0}^{\infty}\varepsilon^n w_n(\phi,\lambda) = \left((1-\phi)^{\left(\frac{1}{2}\right)}\right) + \varepsilon\left\{E^{-1}\left\{s^{\varsigma}E\left[\sum_{n=0}^{\infty}\varepsilon^n H_n(w)\right]\right\}\right\}. \tag{60}$$

The polynomials $H_n(w)$ serve as a representation for the nonlinear terms. For instance, the recursive relation $H_n(w) = \mathbb{P}(\phi,\lambda)\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}$ $\forall n \in N$ is used to obtain the elements of He's polynomials. Thus, equating the equivalent power of ε yields us the following result:

$$\begin{aligned}
\varepsilon^0 &: w_0(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)},\\
\varepsilon^1 &: w_1(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_0(w))\}]\\
&= \frac{\left(\frac{-3}{8}\right)\lambda^{\varsigma}}{\Gamma(\varsigma+1)(1-\phi)^2},\\
\varepsilon^2 &: w_2(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_1(w))\}]\\
&= \frac{\left(\frac{-3}{8}\right)^2(63)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)(1-\phi)^{\left(\frac{9}{2}\right)}},\\
&\ \ \vdots
\end{aligned} \tag{61}$$

Hence, we obtain a solution in series form as follows:

$$\mathbb{P}(\phi,\lambda) = \sum_{n=0}^{\infty}\varepsilon^n w_n(\phi,\lambda). \tag{62}$$

Hence,

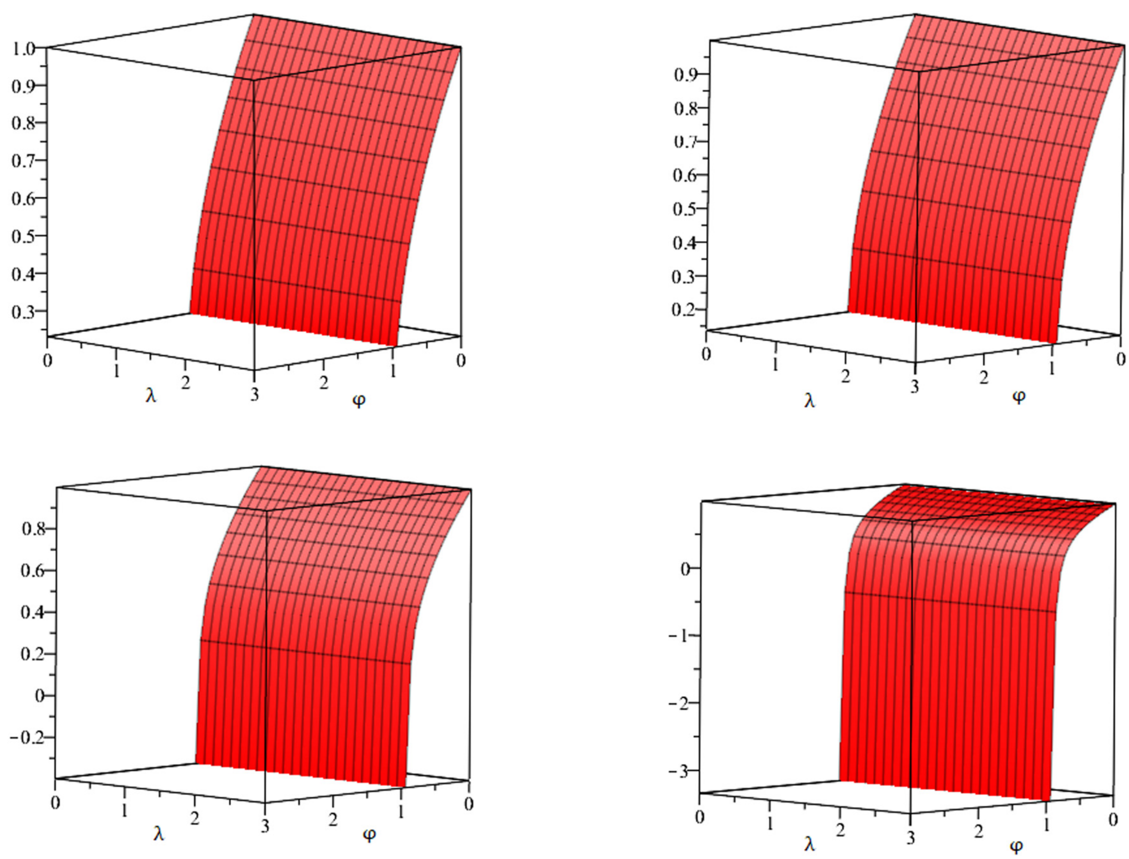

Figure 3: Plot of numerical solution *vs* ϕ *vs* λ using some different values of ς and caption of subfigure can be (a) $\varsigma = 1$, (b) $\varsigma = 0.75$, (c) $\varsigma = 0.50$, and (d) $\varsigma = 0.25$.

$$\mathbb{P}(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)} + \frac{\left(\frac{-3}{8}\right)\lambda^{\varsigma}}{\Gamma(\varsigma+1)(1-\phi)^2} + \frac{\left(\frac{-3}{8}\right)^2 (63)(\lambda^{\varsigma})^2}{\Gamma(2\varsigma+1)(1-\phi)^{\left(\frac{9}{2}\right)}} + \cdots . \tag{63}$$

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = (1-\phi)^{\left(\frac{1}{2}\right)} + \left(\frac{\left(-\frac{3}{8}\right)}{1!}\right)\left(\frac{1}{(1-\phi)^2}\right)\lambda + \left(\frac{\left(-\frac{3}{8}\right)^2}{2!}\right)\left(\frac{63}{(1-\phi)^{\left(\frac{9}{2}\right)}}\right)\lambda^2 + \cdots . \tag{64}$$

(Figure 3)

5.4 Example

Let us assume the 1D FAPDEs as:

$$\frac{\partial^{\varsigma}\mathbb{P}(\phi,\lambda)}{\partial\lambda^{\varsigma}} = \mathbb{P}(\phi,\lambda)\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}, \tag{65}$$

having initial guess

$$\mathbb{P}(\phi,0) = e^{\left(\frac{-\phi}{3}\right)}. \tag{66}$$

On employing the ET to Eq. (65), we have

$$E[\mathbb{P}(\phi,\lambda)] = s^2\left(e^{\left(\frac{-\phi}{3}\right)}\right) + s^{\varsigma}E\left[\mathbb{P}(\phi,\lambda)\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]. \tag{67}$$

By applying the inverse of the ET, we obtain

$$\mathbb{P}(\phi,\lambda) = e^{\left(\frac{-\phi}{3}\right)} + E^{-1}\left[s^{\varsigma}E\left[\mathbb{P}(\phi,\lambda)\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}\right]\right]. \tag{68}$$

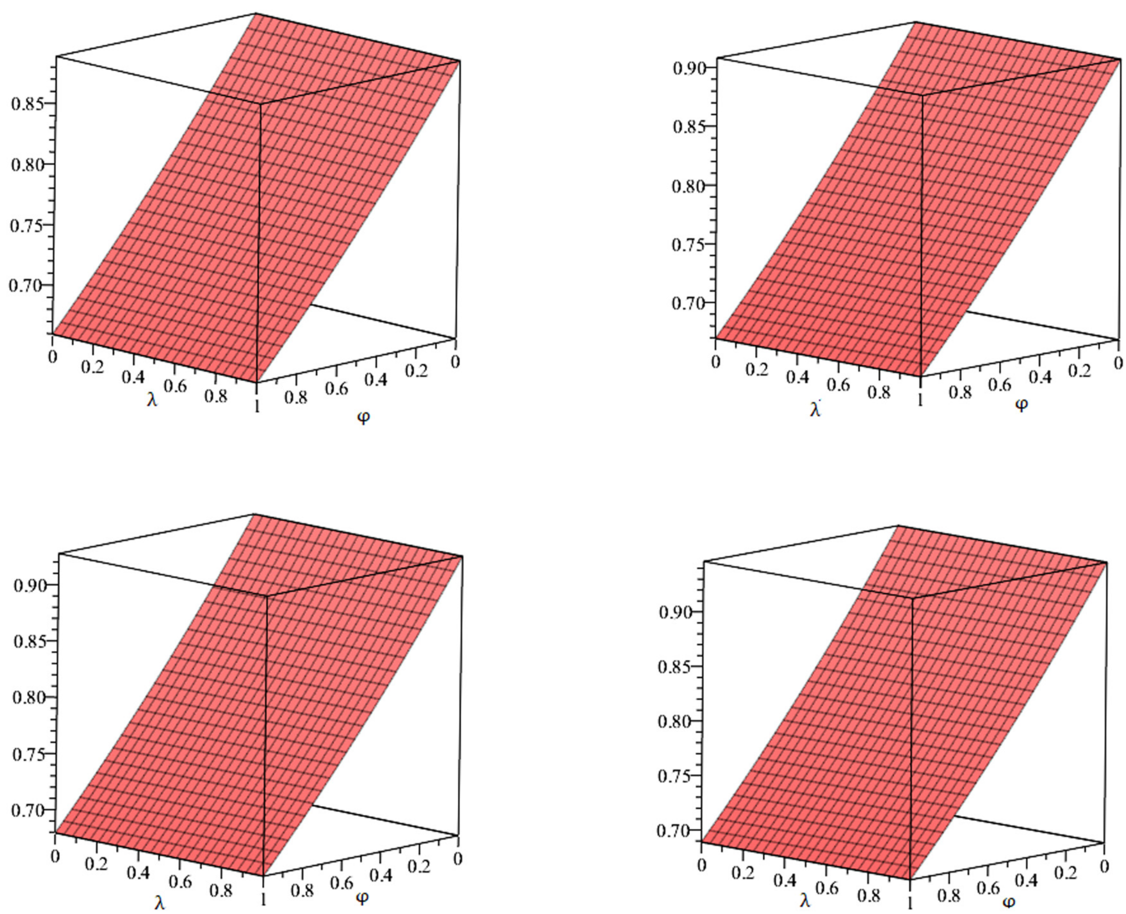

Figure 4: Plot of numerical solution *vs* ϕ *vs* λ using some different values of ς and caption of subfigure can be (a) $\varsigma = 1$, (b) $\varsigma = 0.75$, (c) $\varsigma = 0.50$, and (d) $\varsigma = 0.25$.

By applying the NITM, we obtain

$$\begin{aligned}
\mathbb{P}_0(\phi,\lambda) &= e^{\left(\frac{-\phi}{3}\right)},\\
\mathbb{P}_1(\phi,\lambda) &= E^{-1}\left[s^{\varsigma}E\left\{\mathbb{P}_0(\phi,\lambda)\frac{\partial^3\mathbb{P}_0(\phi,\lambda)}{\partial\phi^3}\right\}\right]\\
&= -\frac{(e^{\left(\frac{-2\phi}{3}\right)})\lambda^{\varsigma}}{27\Gamma(\varsigma+1)},\\
\mathbb{P}_2(\phi,\lambda) &= E^{-1}\left[s^{\varsigma}E\left\{\mathbb{P}_1(\phi,\lambda)\frac{\partial^3\mathbb{P}_1(\phi,\lambda)}{\partial\phi^3}\right\}\right]\\
&= \frac{e^{-\phi}(\lambda^{\varsigma})^2}{81\Gamma(2\varsigma+1)},\\
&\vdots
\end{aligned}$$

Thus, the series solution takes the form

$$\begin{aligned}
\mathbb{P}(\phi,\lambda) &= \mathbb{P}_0(\phi,\lambda)+\mathbb{P}_1(\phi,\lambda)+\mathbb{P}_2(\phi,\lambda)+\mathbb{P}_3(\phi,\lambda)\\
&\quad+\cdots\mathbb{P}_n(\phi,\lambda),\\
\mathbb{P}(\phi,\lambda) &= e^{\left(\frac{-\phi}{3}\right)}-\frac{(e^{\left(\frac{-2\phi}{3}\right)})\lambda^{\varsigma}}{27\Gamma(\varsigma+1)}+\frac{e^{-\phi}(\lambda^{\varsigma})^2}{81\Gamma(2\varsigma+1)}+\cdots.
\end{aligned}\tag{69}$$

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = \frac{1}{e^{\left(\frac{\phi}{3}\right)}}-\frac{\left(\frac{1}{27}\right)}{e^{\left(\frac{2\phi}{3}\right)}}\lambda+\frac{\left(\frac{1}{81}\right)}{2!e^{\phi}}\lambda^2+\cdots.\tag{70}$$

By implementing the HPTM, we obtain

$$\sum_{n=0}^{\infty}\varepsilon^n w_n(\phi,\lambda) = \left(e^{\left(\frac{-\phi}{3}\right)}\right)+\varepsilon\left\{E^{-1}\left(s^{\varsigma}E\left[\sum_{n=0}^{\infty}\varepsilon^n H_n(w)\right]\right)\right\}.\tag{71}$$

The polynomials $H_n(w)$ serve as a representation for the nonlinear terms. For instance, the recursive relation $H_n(w) = \mathbb{P}(\phi,\lambda)\frac{\partial^3\mathbb{P}(\phi,\lambda)}{\partial\phi^3}$ $\forall n \in N$ is used to obtain the elements of He's polynomials. Thus, equating the equivalent power of ε yields us the following result:

$$\begin{aligned}
&\varepsilon^0: w_0(\phi,\lambda) = e^{\left(\frac{-\phi}{3}\right)},\\
&\varepsilon^1: w_1(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_0(w))\}] = -\frac{(e^{\left(\frac{-2\phi}{3}\right)})\lambda^{\varsigma}}{27\Gamma(\varsigma+1)},\\
&\varepsilon^2: w_2(\phi,\lambda) = [E^{-1}\{s^{\varsigma}E(H_1(w))\}] = \frac{e^{-\phi}(\lambda^{\varsigma})^2}{81\Gamma(2\varsigma+1)},\\
&\vdots
\end{aligned}\tag{72}$$

So, the series solution takes the following form:

$$\mathbb{P}(\phi,\lambda) = \sum_{n=0}^{\infty}\varepsilon^n w_n(\phi,\lambda).\tag{73}$$

Hence,

$$\mathbb{P}(\phi,\lambda) = e^{\left(\frac{-\phi}{3}\right)}-\frac{(e^{\left(\frac{-2\phi}{3}\right)})\lambda^{\varsigma}}{27\Gamma(\varsigma+1)}+\frac{e^{-\phi}(\lambda^{\varsigma})^2}{81\Gamma(2\varsigma+1)}+\cdots.\tag{74}$$

For $\varsigma = 1$, we obtain

$$\mathbb{P}(\phi,\lambda) = \frac{1}{e^{\left(\frac{\phi}{3}\right)}}-\frac{\left(\frac{1}{27}\right)}{e^{\left(\frac{2\phi}{3}\right)}}\lambda+\frac{\left(\frac{1}{81}\right)}{2!e^{\phi}}\lambda^2+\cdots.\tag{75}$$

(Figure 4)

6 Conclusion

In this study, both the NITM and HPTM have been applied to examine both the approximate analytical and semi-analytical solutions to the FAPDEs. These results demonstrate that the complexity in evaluating some particular integrals while solving FAPDEs is resolved by using the suggested procedures. Furthermore, it points out that no symbolic computing is necessary, which can be challenging, particularly in nonlinear situations. When implemented in FAPDEs, both convergence approaches were demonstrated analytically and visually, demonstrating the method's dependability and effectiveness. By using specific instances, the physical and geometrical illustrations have been shown, and their graphs indicate the accurate solutions within certain approximation errors. As a result, the suggested techniques are strong, trustworthy, and effective to determine the analytical approximations of solutions for fractional order differential equations of Airy's-type. The suggested methods can be applied for analyzing many evolution equations that are used in describing several nonlinear structures in a plasma.

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