Research Article

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Spectral analysis of variable-order multi-terms fractional differential equations

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Abstract: In this work, a numerical scheme based on shifted Jacobi polynomials (SJPs) is deduced for variable-order fractional differential equations (FDEs). We find numerical solution of consider problem of fractional order. The proposed numerical scheme is based on operational matrices of variable-order differentiation and integration. To create the mentioned operational matrices for variable-order integration and differentiation, SJPs are used. Using the aforementioned operational matrices, we change the problem under consideration into matrix equation. The resultant matrix equation is solved by using Matlab, which executes the Gauss elimination method to provide the necessary numerical solution. The technique is effective and produced reliable outcomes. To determine the effectiveness of the suggested method, the results are compared to the outcomes of some other numerical procedure. Additional examples are included in this article to further clarify the process. For various scale levels and fractional-order values, absolute errors are also recorded.

Keywords: shifted Jacobi polynomials, matrix equation, numerical results, absolute errors

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1 Introduction

In the last several decades, it has been shown that calculus of non-integer-order differentiation and integration is a helpful tool for characterizing the characteristics of complex dynamic systems more efficiently than conventional integer-order derivatives and integrals. As a result, researchers are very interested in employing fractional differential equations (FDEs) to explore a variety of real-world issues and occurrences. These kind of equations can also more effectively capture the dynamics of practical problems. A fractional-order derivative expresses the entire spectrum or accumulation of a function applied to it and has a higher degree of freedom [1–3]. In this context, researchers have looked more closely at FDEs for existence and stability outcomes in recent years. In addition, numerous real-world occurrences have been modeled mathematically using the aforementioned field. Here, we cite a few articles that address the aforementioned topics (see [4–7]).

It is crucial to keep in mind that the effective methods that have been thoroughly studied practically for all kinds of FDEs connected to linear and nonlinear issues are qualitative theory and numerical analysis. A few well-known outcomes are listed in the sources such as [17–20]. Researchers have employed a variety of nonlinear analysis tools, including fixed point theory, Picard, monotone iterative methods, and topological degree ideas, to study existence theory. Schauder, Mawhin, and Sheafers have provided some well-known results that are primarily applicable to theoretical outcomes. The aforementioned findings and theories have been applied to the study of existence theory for solutions to a variety of FDE nonlinear issues (see [21–23]). Mathematical models of many real-world problems, including multiple fractional-order differential operators, are called multi-term FDEs. A system of mixed fractional and ordinary differential equations with more than one fractional differential operator is known as a multi-term FDE. FDEs have been resolved by a variety of numerical and analytical techniques (see [8–12]). Along the same vein, researchers have been particularly drawn to the numerical component. Khan *et al.* [13] studied numerically a disease model by using slide mode controller with fractional order. Some integro-partial FDE with weakly singular kernels have been solved in the study by Fuan *et al.* [14] using piecewise collocation techniques based on polynomials. The Jacobi collocation method was applied in the study by Amin *et al.* [15] to tackle nonlinear fractional partial problems. Murtaza *et al.* [16] studied a fractals-fractional order coupled stress nanofluid problem for numerical solution.

Here, we remark that in the aforesaid study, researchers have used constant fractional-order. Samko and Ross [24] in 1993 originated the idea to investigate variable-order FDEs. But this area has not received proper attention later on, like its real-order counterpart. After 2010, the area has received much attention, and various theoretical and numerical results have been published up to date [25–28]. The said FDEs with variable-order have more degree of freedom and further globalize the dynamics of a problem. Keeping the importance of variable-order, here we investigate a class of multi-term variable-order FDEs by using spectral method based on shifted Jacobi polynomials (SJPs) operational matrices method. The concerned method has been used for dealing traditional fractional-order problems very well, see [28–30]. Spectral methods based on SJPs have been used in various linear problems of fractional-orders. Hence, a huge literature exists on the mentioned area, see, *e.g.*, [31–34]. Recently, some more efficient numerical schemes have been deduced like [35–37] for non-variable order problems in fractional calculus. Recently, some remarkable work on variable order have been done. Some reputed results are refereed too, see [38].

To the best of our information, multi-term variable-order FDEs have not been testified by the use of said procedure. In order to close this gap, we, therefore, take into consideration the subsequent linear multi-term FDE under variable-order as

$$\begin{cases} u''(t) + {}_0^C\mathbb{D}_t^{\varepsilon_1(t)}u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_2(t)}u(t) + u(t) = \mathrm{f}(t), \\ 0 \le t \le 1, \\ u(t)|_{t=0} = \mu_1, \quad u'(t)|_{t=0} = \mu_2, \end{cases} \tag{1}$$

where ${}_0^C\mathbb{D}_t^{\varepsilon_1(t)}$ and ${}_0^C\mathbb{D}_t^{\varepsilon_2(t)}$ represent variable-order operators of order $\varepsilon_1(t)$ and $\varepsilon_2(t)$, where $\mu_1, \mu_2 \in \mathbb{R}$. In addition, ε_1 and ε_2 are continuous functions such that $\varepsilon_1 : \mathbb{R} \to (0, 1]$ and $\varepsilon_2 : \mathbb{R} \to (1, 2]$. Furthermore, $\mathrm{f} \in C[0, 1]$ is a given continuous function. We use the aforesaid method to compute numerical solution. The concerned spectral method has been proved stable and convergent when applied to any linear problem. Furthermore, the computational cost is low as compared to wavelet and finite element methods. No need of prior discretization or collocation. In addition, like perturbation method, this technique needs no axillary parameter to control the method. Also, the suggested method has some rawbacks. We also establish some existence results to the considered problems by using fixed point theory. A comparison between our results and that computed by Haar wavelet method (HWM) has given to illustrate the efficiency of the procedure. Numerous examples are given along with graphical presentation and error analysis.

This article is organized as follows: in Section 1, we give detail introduction. Some needful results are given in Section 2. Further, the required operational matrices are given in Section 3. Moreover, the numerical scheme has established in Section 4. Illustrative examples are given in Section 5. Finally conclusion and discussion are fixed in Section 6.

2 Auxiliary results

Some fundamental materials are recollected as follows:

Definition 2.1. [24] If $\varepsilon(t) > 0$, then variable-order integration of $\mathrm{h} \in L[0, 1]$ is given as follows:

$${}_0\mathbb{I}_t^{\varepsilon(t)}\mathrm{h}(t) = \frac{1}{\Gamma(\varepsilon(t))}\int_0^t (t - \varrho)^{\varepsilon(t)-1}\mathrm{h}(\varrho)\mathrm{d}\varrho. \tag{2}$$

Definition 2.2. [24] The variable-order derivative of $\mathrm{h} \in C[0, 1]$ is given as follows:

$${}_0^C\mathbb{D}_t^{\varepsilon(t)}\mathrm{h}(t) = \begin{cases} \dfrac{1}{\Gamma(n - \varepsilon(t))}\displaystyle\int_0^t (t - \varrho)^{n-\varepsilon(t)-1}\mathrm{h}^{(n)}(\varrho)\mathrm{d}\varrho, \\ n - 1 < \varepsilon(t) < n, \\ {}_0\mathbb{D}_t\mathrm{h}(t), \quad \varepsilon(t) = n. \end{cases} \tag{3}$$

Lemma 2.3. [24] *If $\varepsilon(t) > 0$, then for variable-order problem*

$${}_0^C\mathbb{D}_t^{\varepsilon(t)}\mathrm{h}(t) = \mathrm{g}(t), \tag{4}$$

the solution is

$$\mathrm{h}(t) = \mathrm{d}_0 + \mathrm{d}_1 t + \mathrm{d}_2 t^2 + \ldots + \mathrm{d}_{n-1}t^{n-1} + {}_0\mathbb{I}_t^{\varepsilon(t)}\mathrm{g}(t).$$

3 SJPs and operational matrices

The analytical form of the SJPs [31] over $[0, \tau]$ can be expressed as follows:

$$\mathcal{Q}_{\tau,i}^{(\theta,\vartheta)}(t) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\Gamma(i+k+\theta+\vartheta+1)}{\Gamma(k+b+1)\Gamma(i+\theta+b+1)\Gamma(i-k+1)\Gamma(k+1)\tau^k} t^k, \quad \text{with}$$
$$\mathcal{Q}_{\tau,i}^{(\theta,\vartheta)}(0) = \frac{\Gamma(i+\vartheta+1)}{\Gamma(\vartheta+1)\Gamma(i+1)}(-1)^i.$$

Here, for SJPs, the orthogonal relation is given as follows:

$$\int_0^\tau \mathcal{Q}_{\tau,j}^{(\theta,\vartheta)}(t)\mathcal{Q}_{\tau,i}^{(\theta,\vartheta)}(t)W_\tau^{(\theta,\vartheta)}(t)\mathrm{d}t = \Omega_{\tau,j}^{(\theta,\vartheta)}\delta_{j,i}$$

such that

$$\delta_{j,i} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{other wise}. \end{cases}$$

Furthermore, $\mathbf{W}_\tau^{(\theta,\vartheta)}(t) = (\tau - t)^\theta t^\vartheta$ is weight function and

$$\Omega_{\tau,j}^{(\theta,\vartheta)}(t) = \frac{\tau^{\theta+\vartheta+1}\Gamma(j+\theta+1)\Gamma(j+\vartheta+1)}{(2j+\theta+\vartheta+1)\Gamma(j+1)\Gamma(j+\theta+\vartheta+1)}. \tag{5}$$

Any function $\mathrm{f} \in L^2[0,\tau]$ can be estimated using the aforementioned polynomials as follows:

$$\mathrm{f}(t) \simeq \sum_{k=0}^{m} \mathbf{b}_j \mathcal{Q}_{\tau,k}^{(\theta,\vartheta)}(t) = \mathbf{K}_M^T \mathbf{F}_M, \tag{6}$$

where the notion $\mathbf{K}_M$ and $\mathbf{F}_M$ are M terms coefficient and function vectors, respectively. Here, $M = m + 1$ and $\mathbf{b}_j$ is derived from Eqs (3)–(6) as follows

$$\mathbf{b}_j = \frac{1}{\Omega_{\tau,j}^{(\theta,\vartheta)}} \int_0^\tau \mathbf{W}_\tau^{(\theta,\vartheta)}(t)\mathrm{f}(t)\mathcal{Q}_{\tau,i}^{(\theta,\vartheta)}(t)\mathrm{d}t, \quad i = 0, 1, \ldots. \tag{7}$$

Here, we create a few operational matrices for fractional integration and differentiation of variable-order.

Theorem 3.1. *For vector* $\mathbf{F}_M$, *one has*

$${}_0^C\mathbb{D}_t^{\varepsilon(t)}[\mathbf{F}_M] \simeq \mathbf{Q}_{M\times M}^{(\varepsilon(t))}\mathbf{F}_M, \tag{8}$$

where $\mathbf{Q}_{M\times M}^{(\varepsilon(t))}$ *is provided as follows*:

$$\mathbf{Q}_{M\times M}^{(\varepsilon(t))} = \begin{bmatrix} \mathbf{a}_{0,0,k} & \mathbf{a}_{0,1,k} & \cdots & \mathbf{a}_{0,j,k} & \cdots & \mathbf{a}_{0,m,k} \\ \mathbf{a}_{2,1,k} & \mathbf{a}_{2,2,k} & \cdots & \mathbf{a}_{2,r,k} & \cdots & \mathbf{a}_{1,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_{i,0,k} & \mathbf{a}_{i,1,k} & \cdots & \mathbf{a}_{i,j,k} & \cdots & \mathbf{a}_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{a}_{m,0,k} & \mathbf{a}_{m,1,k} & \cdots & \mathbf{a}_{m,j,k} & \cdots & \mathbf{a}_{m,m,k} \end{bmatrix}, \tag{9}$$

where

$$\begin{aligned} \mathbf{a}_{i,j,k} = &\sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\vartheta+1)\xi^{\varepsilon(t)-k}\Gamma(i+k+\theta+\vartheta+1)\Gamma(k+1)}{\Gamma(k+\vartheta+1)\Gamma(i+\theta+\vartheta+1)\Gamma(i-k+1)\Gamma(k+1)\Gamma(k+1-\varepsilon(t))} \\ &\times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+l+\theta+\vartheta+1)(2j+\theta+\vartheta+1)\Gamma(j+1)\Gamma(k-\varepsilon(t)+l+\vartheta+1)\Gamma(\theta+1)}{\Gamma(j+\theta+1)\Gamma(l+\vartheta+1)\Gamma(j-l+1)\Gamma(l+1)\Gamma(k-\varepsilon(t)+l+\vartheta+\theta+2)}, \\ &\text{else } \mathbf{a}_{i,j,k} = 0, \quad \text{with } i < \varepsilon(t). \end{aligned} \tag{10}$$

Proof. Following the same procedure as derived in the study by Youssri and Atta [31] and Shah *et al.* [32], we can obtain the above matrix of variable fractional-order derivative. □

Theorem 3.2. *For vector of functions* $\mathbf{F}_M$, *we compute matrix corresponding to variable-order integral as follows*:

$$ {}_0\mathbb{I}_t^{\varepsilon(t)}[\mathbf{F}_M] \simeq \mathbf{H}_{M\times M}^{(\varepsilon(t))}\mathbf{F}_M \tag{11}$$

such that

$$\mathbf{H}_{M\times M}^{(\varepsilon(t))} = \begin{bmatrix} \mathbf{b}_{0,0,k} & \mathbf{b}_{0,1,k} & \cdots & \mathbf{b}_{0,j,k} & \cdots & \mathbf{b}_{0,m,k} \\ \mathbf{b}_{2,1,k} & \mathbf{b}_{2,2,k} & \cdots & \mathbf{b}_{2,r,k} & \cdots & \mathbf{b}_{1,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{b}_{i,0,k} & \mathbf{b}_{i,1,k} & \cdots & \mathbf{b}_{i,j,k} & \cdots & \mathbf{b}_{i,m,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{b}_{m,0,k} & \mathbf{b}_{m,1,k} & \cdots & \mathbf{b}_{m,j,k} & \cdots & \mathbf{b}_{m,m,k} \end{bmatrix}, \tag{12}$$

where

$$\begin{aligned} \mathbf{b}_{i,j,k} = & \sum_{k=0}^{i} \frac{\Gamma(i+\vartheta+1)\xi^{\varepsilon(t)-k}(-1)^{i-k}\Gamma(i+k+\theta+\vartheta+1)\Gamma(k+1)}{\Gamma(k+\vartheta+1)\Gamma(i+\theta+\vartheta+1)\Gamma(i-k+1)\Gamma(k+1)\Gamma(k+1+\varepsilon(t))} \\ & \times \sum_{l=0}^{j} \frac{(-1)^{j-l}\Gamma(j+1)(2j+\theta+\vartheta+1)\Gamma(j+l+\theta+\vartheta+1)\Gamma(k+\varepsilon(t)+l+\vartheta+1)\Gamma(\theta+1)}{\Gamma(j+\theta+1)\Gamma(l+\vartheta+1)\Gamma(j-l+1)\Gamma(k+\varepsilon(t)+l+\vartheta+\theta+2)\Gamma(l+1)}. \end{aligned} \tag{13}$$

Proof. The above matrix can be obtained by following the same procedure as done by Youssri and Atta [31] and Shah *et al.* [32]. □

4 Numerical algorithm

Here, by using Theorems 3.1 and 3.2, we establish the required algorithm for the considered problem.

Assume that

$$ {}_0^C\mathbb{D}_t^2 u(t) = \mathbf{K}_M^T\mathbf{F}_M. \tag{14}$$

From Lemma 2.3 and Eq. (14), we have

$$u(t) = \mathrm{d}_0 + \mathrm{d}_1 t + {}_0\mathbb{D}_t^2[\mathbf{K}_M^T\mathbf{F}_M], \tag{15}$$

and using initial condition $u(0) = \mu_1$ and $u'(0) = \mu_2$, Eq. (15) yields

$$u(t) = \mu_1 + \mu_2 t + \mathbf{K}_M^T\mathbf{H}_{M\times M}^{(2)}\mathbf{F}_M. \tag{16}$$

Table 1: Maximum absolute errors and computer processing unite (CPU) time at two different scale levels and various values of (θ, ϑ) and t for Example 1 by taking $\varepsilon_1(t) = 1 - \exp(-t)$ and $\varepsilon_2(t) = 1 + 0.5t$

(θ, ϑ)	t	$\lVert u_{ex} - u_{app}\rVert_X$, at $M = 4$	CPU time (s)	$\lVert u_{ex} - u_{app}\rVert_X$, at $M = 6$	CPU time (s)
(0, 0)	0.2	0.005330×10^{-14}	15	0.001206×10^{-16}	20
(0.1, 0)	0.2	0.005016×10^{-14}	17	0.002660×10^{-16}	23
(0, 0.1)	0.2	0.002152×10^{-14}	18	0.001125×10^{-16}	28
(0.2, 0.2)	0.4	8.456253×10^{-14}	25	4.407010×10^{-16}	30
(0.5, 0.5)	0.6	3.182218×10^{-14}	30	1.658655×10^{-16}	35
(–0.5, 0.5)	0.8	1.169592×10^{-14}	35	6.104875×10^{-16}	40
(0.5, –0.5)	0.8	4.239774×10^{-14}	30	2.216698×10^{-16}	40
(1, 1)	1.0	1.523946×10^{-14}	30	7.979700×10^{-16}	45

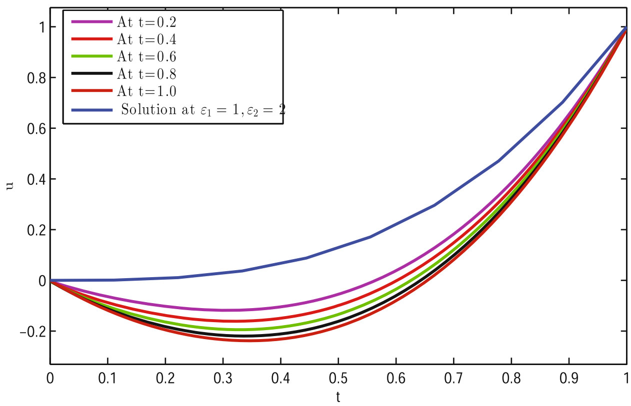

Figure 1: Graphical presentation of variable-order and integer-order solution of Example 1 for $M = 6$, $\varepsilon_1 = 1 \ \exp(t)$, $\varepsilon_2(t) = 2 + 0.5t$, and taking $(\theta, \vartheta) = (0, 0)$.

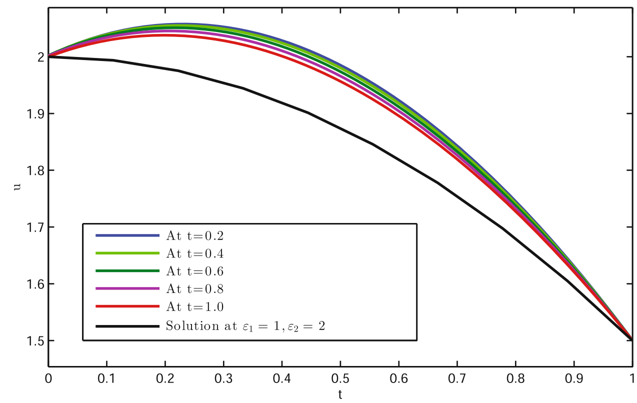

Figure 2: Graphical presentation of variable-order and integer-order solutions of Example 1 for $M = 6$, $\varepsilon_1 = \frac{1}{2}[t + 1]$, $\varepsilon_2(t) = 1 + 0.5t$, and taking $(\theta, \vartheta) = (0, 0)$.

By approximating $\mu_1 + \mu_2 t \approx \mathbf{G}_M^T \mathbf{F}_M$, Eq. (16) reduces to

$$u(t) \approx \mathbf{G}_M^T \mathbf{F}_M + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} \mathbf{F}_M. \tag{17}$$

Now taking $\varepsilon_1(t)$ and $\varepsilon_2(t)$ order differentiations of Eq. (17), we have

$$\begin{aligned} {}_0^C\mathbb{D}_t^{\varepsilon_1(t)}[u(t)] &= {}_0^C\mathbb{D}_t^{\varepsilon_1(t)}[\mathbf{G}_M^T \mathbf{F}_M + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} \mathbf{F}_M] \\ &= \mathbf{G}_M^T \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} \mathbf{F}_M + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} \mathbf{F}_M \\ &= [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} \mathbf{F}_M. \end{aligned} \tag{18}$$

In same line from Eq. (18), we have

$${}_0^C\mathbb{D}_t^{\varepsilon_2(t)}[u(t)] = [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_2(t))} \mathbf{F}_M. \tag{19}$$

We further approximate $\mathrm{f}(t)$ as follows:

$$\mathrm{f}(t) \approx \mathbf{L}_M^T \mathbf{F}_M. \tag{20}$$

Using Eqs (14) and (17)–(20) in Eq. (1), we obtain

$$\begin{aligned} &\mathbf{K}_M^T \mathbf{F}_M + [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} \mathbf{F}_M \\ &\quad + [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_2(t))} \mathbf{F}_M \\ &\quad + \mathbf{G}_M^T \mathbf{F}_M + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} \mathbf{F}_M - \mathbf{L}_M^T \mathbf{F}_M = 0 \\ &\mathbf{K}_M^T + [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} \\ &\quad + [\mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)}] \mathbf{Q}_{M\times M}^{(\varepsilon_2(t))} \\ &\quad + \mathbf{G}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} - \mathbf{L}_M^T = 0. \end{aligned} \tag{21}$$

We can also write as

$$\mathbf{P}_{M\times M}^{(\varepsilon_1(t), \varepsilon_2(t))} = \mathbf{Q}_{M\times M}^{(\varepsilon_1(t))} + \mathbf{Q}_{M\times M}^{(\varepsilon_2(t))},$$
$$\mathbf{R}_M^T = \mathbf{G}_M^T - \mathbf{L}_M^T.$$

Eq. (22) yields that

$$\begin{aligned} &\mathbf{K}_M^T + \mathbf{K}_M^T \mathbf{H}_{M\times M}^{(2)} (\mathbf{P}_{M\times M}^{(\varepsilon_1(t), \varepsilon_2(t))} + \mathbf{I}_{M\times M}) \\ &\quad + \mathbf{G}_M^T \mathbf{P}_{M\times M}^{(\varepsilon_1(t), \varepsilon_2(t))} + \mathbf{R}_M^T = 0. \end{aligned} \tag{22}$$

Table 2: Maximum absolute errors and CPU time at two different scale levels and various values of (θ, ϑ) and t for Example 2 by taking $\varepsilon_1(t) = \frac{t+1}{2}$ and $\varepsilon_2(t) = 2 - \frac{t+1}{2}$

(θ, ϑ)	t	$\|u_{ex} - u_{app}\|_X$, at $M = 4$	CPU time (s)	$\|u_{ex} - u_{app}\|_X$, at $M = 6$	CPU time (s)
(0, 0)	0.2	3.9000×10^{-13}	16	3.0000×10^{-16}	21
(0.1, 0)	0.2	2.5000×10^{-13}	23	3.1000×10^{-16}	35
(0, 0.1)	0.2	2.4000×10^{-13}	28	3.2000×10^{-16}	36
(0.2, 0.2)	0.4	1.3203×10^{-13}	32	9.1234×10^{-16}	36
(0.5, 0.5)	0.6	1.8907×10^{-13}	37	1.0000×10^{-16}	38
(-0.5, 0.5)	0.8	1.3400×10^{-13}	37	1.4001×10^{-16}	42
(0.5, -0.5)	0.8	1.8000×10^{-14}	38	1.4000×10^{-16}	42
(1, 1)	1.0	1.3000×10^{-14}	40	1.3000×10^{-16}	48

Table 3: Maximum absolute errors and CPU time at two different scale levels and various values of (θ, ϑ) and t for Example 3 by taking $\varepsilon_1(t) = \frac{t+1}{2}$ and $\varepsilon_2(t) = 2 - \frac{t+1}{2}$

(θ, ϑ)	t	$\|u_{ex} - u_{app}\|_X$, at $M = 4$	CPU time (Sec.)	$\|u_{ex} - u_{app}\|_X$, at $M = 6$	CPU time (Sec.)
(0, 0)	0.2	9.9100×10^{-14}	18	6.1000×10^{-16}	25
(0.1, 0)	0.2	9.9102×10^{-14}	22	4.5432×10^{-16}	30
(0, 0.1)	0.2	9.9876×10^{-14}	26	6.1234×10^{-16}	30
(0.2, 0.2)	0.4	4.4523×10^{-14}	33	7.9821×10^{-15}	32
(0.5, 0.5)	0.6	7.5678×10^{-14}	34	4.2340×10^{-15}	35
(–0.5, 0.5)	0.8	9.9231×10^{-14}	37	6.2341×10^{-15}	44
(0.5, –0.5)	0.8	9.9123×10^{-14}	39	5.6432×10^{-15}	45
(1, 1)	1.0	1.5991×10^{-14}	41	6.9000×10^{-16}	50

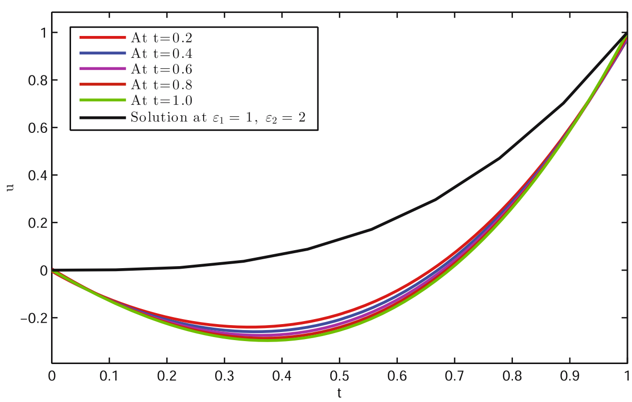

Figure 3: Graphical presentation of variable-order and integer-order solutions of Example 3 for $M = 6$, $\varepsilon_1 = \frac{t+1}{2}$, $\varepsilon_2(t) = 2 \ \frac{t+1}{2}$, and taking $(\theta, \vartheta) = (0, 0)$.

By using Matlab and exercising the Gauss elimination method, we solve Eq. (22) to compute matrix $\mathbf{K}_M^T$, which, on putting in Eq. (17), we obtain numerical solution of Eq. (1).

Theorem 4.1. [33] **Convergence result:** *If* $\mathrm{w} : [t_0, \tau] \to \mathbb{R}$ *differentiated m times and all derivatives are continuous such that* $\mathrm{w} \in C^m[0, \tau]$ *and let*

$$\mathbf{S}^{m,\theta,\vartheta} = \langle \mathcal{Q}_0^{(\theta,\vartheta)}(t), \mathcal{Q}_1^{(\theta,\vartheta)}(t), \ldots, \mathcal{Q}_m^{(\theta,\vartheta)}(t) \rangle.$$

Let $\mathrm{w}(t) \approx \mathbf{K}_M^T \mathbf{F}_M$ *be the best approximation of* w *in terms of* $\mathbf{S}^{m,\theta,\vartheta}$, *then the error bounds are given as follows:*

$$\|\mathrm{w}(t) - \mathbf{K}_M^T \mathbf{F}_M\| \leq \max_{t \in [t_0, \tau]} \frac{|\mathrm{w}^m(t)| S^M}{\Gamma(m+1)} \sqrt{\mathscr{B}(\theta+1, \vartheta+1)},$$

where $S = \max\{1 - t_0, t_0\}$. *Here, the point to be noted is that at* $m \to \infty$, *the numerical result converges to the exact value of the function.*

5 Experimental problems

Here, we present some test problems to testify our algorithm.

Problem 1. Consider the given problem

$$\begin{cases} {}_0^C\mathbb{D}_t^2 u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_1(t)} u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_2(t)} u(t) + u(t) = \mathrm{f}(t), \\ u(0) = u'(0) = 0, \end{cases} \tag{23}$$

where

$$\mathrm{f}(t) = 6t + t^3 + t^3 + 6t + \frac{6t^{3-\varepsilon_1(t)}}{\Gamma(4 - \varepsilon_1(t))} + \frac{6t^{3-\varepsilon_2(t)}}{\Gamma(4 - \varepsilon_2(t))}.$$

The true solution is $u(t) = t^3$.

Table 4: Comparison between Haar wave let method and our proposed method for $\varepsilon_1 = \frac{3}{4}$, $(\theta, \vartheta) = (0, 0)$ for Example 3

J	$N = 2^{J+1}$	$\|u_{ex} - u_{app}\|_X$ at HWM	M	$\|u_{ex} - u_{app}\|_X$ at SJPs
1	4	0.007115	4	9.9100×10^{-16}
2	8	0.002810	4	9.9000×10^{-16}
3	16	0.001071	4	9.4536×10^{-16}
4	32	3.964817×10^{-4}	4	9.93456×10^{-16}
5	64	1.443621×10^{-4}	6	6.5674×10^{-16}
6	128	5.204125×10^{-5}	6	6.1234×10^{-16}
7	256	1.864213×10^{-5}	6	5.2341×10^{-16}
8	512	6.650365×10^{-6}	6	5.067807×10^{-16}

From Table 1, one can see that the absolute error is much more smaller at reasonable scale level. Figure 1 also shows this comparison graphically.

Problem 2. Take another test problem as

$$\begin{cases} {}_0^C\mathbb{D}_t^2 u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_1(t)} u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_2(t)} u(t) + u(t) = \mathrm{f}(t), \\ u(0) = 2, \quad u'(0) = 0, \end{cases} \tag{24}$$

where

$$\mathrm{f}(t) = 1 - \frac{t^2}{2} - \frac{t^{2-\varepsilon_1(t)}}{\Gamma(3-\varepsilon_1(t))} - \frac{t^{2-\varepsilon_2(t)}}{\Gamma(3-\varepsilon_2(t))}.$$

Here, the exact solution at $\varepsilon_1(t) = 1$ and $\varepsilon_2(t) = 2$ is given as follows:

$$u(t) = 2 - \frac{1}{2}t^2.$$

We approximate the solution for different scale levels and consider the variable order as follows:

$$\varepsilon_1(t) = \frac{1}{2}[t+1], \quad \varepsilon_2(t) = 2 - \frac{1}{2}[t+1].$$

From Table 2, at a tolerable scale level, we observe that the absolute inaccuracy is significantly reduced. Further in Figure 2, we have shown this comparison graphically.

Table 2 shows absolute errors and CPU times at various values of the parameters of SJPs, t and scale level M.

Problem 3. Consider the following multi-term FDEs [33]

$$\begin{cases} {}_0^C\mathbb{D}_t^2 u(t) + {}_0^C\mathbb{D}_t^{\varepsilon_1(t)} u(t) + u(t) \\ \quad = t^3 + 6t + \dfrac{8.53333333}{\Gamma\left(\frac{1}{4}\right)} t^{\frac{9}{4}}, \\ u(0) = u'(0) = 0. \end{cases} \tag{25}$$

At $\varepsilon_1(t) = 1$, the exact solution is given by $u(t) = t^3$. From Table 3, at a tolerable scale level, we observe that the absolute inaccuracy is significantly reduced. Further in Figure 3, we have shown this comparison graphically.

Table 3 shows absolute errors and CPU times at different values of the parameters of SJPs, t and scale level M.

Furthermore, we compare our results with that of HWM used in the study by Kazem (Table 4, [33]).

6 Conclusion and discussion

For the purpose of computing approximate solutions to a class of multi-term variable-order FDEs, we have developed a proper numerical approach. We have created two operational matrices for integration and fractional variable-order derivative using SJPs. We have translated the problem under consideration into the matching algebraic equations using these matrices. We have solved the received matrix equations using Matlab and the Gauss elimination process to obtain numerical answers. To confirm that the suggested numerical technique performs better for multi-term FDEs, certain numerical experiments have been conducted. The tables also include a list of the CPU's duration. Figures also show a comparison between exact and numerical solutions. We may observe from the table and graph that the suggested method performs better. The analysis demonstrates a quick convergence. In addition, the suggested approach's comparison to the HWM method already in use revealed that the latter is likewise a strong and effective numerical procedure with the highest accuracy. Furthermore, for any linear issue, the current spectral technique is stable. Extending the scale level can lead to even greater accuracy. The aforesaid method is a scale-oriented procedure. Greater the scale the greater the accuracy, and vice versa. In addition, the accuracy can also be enhanced by fixing the order in suitable manes. In the future, we will extend this scheme to fractals-fractional derivatives.

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