Research Article

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Extracting solitary solutions of the nonlinear Kaup–Kupershmidt (KK) equation by analytical method

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Abstract: Finding analytical solutions for nonlinear partial differential equations is physically meaningful. The Kaup-Kupershmidt (KK) equation is studied in this article. The KK equation is of fifth order, such that several solitary solutions are obtained. In this article, however, the modified auxiliary function approach is applied to this model to find solitary solutions. These solutions are written in terms of Jacobi functions. Therefore, the obtained solutions can be implemented graphically to show different patterns for appropriate parameters.

Keywords: solitary waves, solitons, exact solutions, Jacobi functions, exponential solution

1 Introduction

Nonlinear partial differential equations (NPDEs) govern transport phenomena that manifest in various physics and engineering scenarios. These partial differential equations can be used to model, for instance, solitary waves in fluid problems such as earthquakes and the hydromagnetic flux of a dusty liquid through a porous medium [1] or fluid dynamics such as Bona–Mahony equation [2–5], Benney-Luke equation, the modified Kortewege–de-Varies equation [6,7], heat transfer in thermoelectric fluid [8] and Benjamin–Bona–Mohaony equation [9,10]. Great attention has already been taken to the Kaup–Kupershmidt (KK) equation [11,12] and also to Sawada–Kotera model [13] that are fifth-order NPDEs.

In literature, the latter two equations are well documented even though they are of the same order and different. The KK model is more complex than the Sawada–Kotera model regarding integrability and mathematical solutions. However, the KK equation is still an active system as far as we know of which one would seek further soliton solutions. The KK equation can be applied to various applications in physics, such as nonlinear optics, fluid dynamics, and plasma physics [14]. In 1980, the famous classical KK equation is introduced by Kaup [15] and modified by Kupershmidt in 1994 [16]. Most recently, numerical approaches to the fractional-order KK equation have been implemented to seek nonlinear dispersive waves and capillary gravity waves [17]. For fractional-order differential equations model, readers who express interest in the topic are advised to consult these references [18–23].

In the literature, Kaup found numerically solitary solutions to the KK equation by using inverse scattering theory. Herman and Nusier [24] have computed two and three solutions, but, they did not provide a further sequence of soliton solutions. This brought motivation to investigate the existence of exact solutions with the aid of Mathematica software and several analytical methods. For instance, the modified auxiliary equation (MAE) method [25,26], the exponential expansion method [27], the tanh-based expansion method [28], the Jacobi elliptic functions method [29], the modified exponential rational method [30], the G'/G-expansion method [31] with interested applications, and the Kudryashov method [32]. Moreover, one may refer to [33,34] for more analytical methods, for instance, the sub-equation method and more. Furthermore, several computational methods can be used to approximate soliton solutions, such as the Adomian decomposition approach [35], homotopy analyzis approach [36], and Laplace-homotopy perturbation method [37].

The objective of this present manuscript is to investigate new traveling wave solutions to the problem under investigation. This includes exponential periodic hyperbolic [2–4] and rational [4,5] analytical solutions different from those exposed in the literature. The analytical method MAE is utilized to construct different analytical solutions for the KK equation using the well-known Jacobi functions. Further solitons solution is determined and analyzed upon

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any constraint conditions if they exist. Furthermore, the solutions are displayed graphically and classified according to the Soliton profiles.

The current manuscript is organized as follows: an overview of the methodology used herein in Section 2; in Section 3, we use the proposed approach to the KK equation; Section 4 is devoted to discussing the obtained solutions; and at the end of Section 5, the conclusion and further ideas are presented.

2 The method of MAE

Here, we give an overview of the modified auxiliary method [25,26]. Thus, consider the general form of the NPDEs as follows:

$$\chi(\upsilon, \upsilon_x, \upsilon_t, \upsilon_{xt}, \upsilon_{xx}, \ldots) = 0, \quad (1)$$

where $\upsilon = \chi(x, t)$ is unknown function in space and time, respectively.

The outline of the method is presented as follows as:

Step 1: we apply the following transformation:

$$\chi(x, t) = W(\zeta), \quad \zeta = x + c_1 t, \quad (2)$$

where c_1 is arbitrary constant. Then, employing the applied transformation in Eq. (2) to the partial equation in Eq. (1), it will be converted to an ordinary differential equation in the following nonlinear form:

$$Q(W, W', W'', W''', \ldots)| = 0. \quad (3)$$

Step 2: Assuming the solution of Eq. (3) would be

$$W(\zeta) = \sum_{j=-n}^{n} \eta_j \Psi^j(\zeta) \quad (4)$$

such that $n > 0$ is an integer and η_j are constants that will be specified. The function $\Psi^j(\zeta)$ satisfies:

$$\Psi'^2(\zeta) = \upsilon_0 + \upsilon_1 \Psi^2(\zeta) + \upsilon_2 \Psi^4(\zeta), \quad (5)$$

where υ_0, υ_1, and υ_2 are arbitrary constants. Therefore, Eq. (5) admits several solutions' cases as follows:

Case 1: If $\upsilon_0 = 1, \upsilon_1 = -(1 + k^2)$, and $\upsilon_2 = k^2$, then the solution to Eq. (5) is: $\Psi(\zeta) = sn(\zeta, k)$, where $sn(\zeta, k)$ is the Jacobi function and k refers to the elliptic modulus, where $0 < k < 1$.

Case 2: If $\upsilon_0 = 1 - k^2, \upsilon_1 = -2k^2 - 1$, and $\upsilon_2 = -k^2$, then Eq. (5) has a solution $\Psi(\zeta) = cn(\zeta, k)$, where $cn(\zeta, k)$ defines the Jacobi function and k as before.

Case 3: If $\upsilon_0 = k^2 - 1, \upsilon_1 = -2 - k^2$, and $\upsilon_2 = -1$, then Eq. (5) can be solved as $\Psi(\zeta) = dn(\zeta, k)$, where $dn(\zeta, k)$ defines the Jacobi function dn.

Case 4: If $\upsilon_0 = k^2, \upsilon_1 = -1 - k^2$, and $\upsilon_2 = 1$, then Eq. (5) can be solved as $\Psi(\zeta) = ns(\zeta, k)$, where $ns(\zeta, k)$ defines the Jacobi function ns.

Case 5: If $\upsilon_0 = 1 - k^2, \upsilon_1 = -2 - k^2$, and $\upsilon_2 = 1$, then Eq. (5) can be solved as $\Psi(\zeta) = cs(\zeta, k)$, where $cs(\zeta, k)$ defines the Jacobi function cs.

Case 6: If $\upsilon_0 = 1, \upsilon_1 = 2k^2 - 1$, and $\upsilon_2 = k^2(k^2 - 1)$, then Eq. (5) can be solved as $\Psi(\zeta) = sd(\zeta, k)$, where $sd(\zeta, k)$ defines the Jacobi function sd.

Step 3: In Eq. (4), n is determined *via* the application of homogenous balancing principle explained in the study by Hereman and Nuseir [24].

Step 4: Substituting Eqs (4) and (5) into Eq. (3) and vanishing those terms with the same exponent of $\Psi\zeta$, a set of equations in η_i can be determined. So, the solution Eq. (1) is well determined.

3 The solutions to the KK model

Here, the MAE will be used to solve the KK model. Thus, based on the study by Parker [11], the KK model is written as follows:

$$u_t + 45u^2u_x - \frac{75}{2}u_x u_{xx} - 15uu_{xxx} + u_{xxxxx} = 0. \quad (6)$$

Then, let us apply the following transformation:

$$u(x, t) = U(\xi), \quad \xi = x + ct, \quad (7)$$

where c is a constant. So that, Eq. (6) is transformed as follows:

$$cU' + 45U^2U' - \frac{75}{2}U'U'' - 15UU''' + U''''' = 0. \quad (8)$$

The principle of homogeneous balancing will be used in Eq. (8), we find $n = 2$. Hence, the solution to Eq. (8) can be written as follows:

$$U(\xi) = \eta_0 + \eta_1\psi(\xi) + \eta_2\psi^2(\xi) + \frac{\eta_{-1}}{\psi(\xi)} + \frac{\eta_{-2}}{\psi^2(\xi)}. \quad (9)$$

Therefore, by using Eq. (5) and substituting Eq. (9) into Eq. (8), one can vanish the coefficients η_0, η_1, η_{-1}, and η_{-2} in Eq. (9). Thereafter, an algebraic equations system is obtained. Then, solving the system for η_0, η_1, η_{-1}, and c, the following solutions are obtained:

Set 1.

$$\eta_0 = \frac{\upsilon_1}{3}, \eta_1 = 0, \eta_2 = 0, \eta_{-1} = 0, \eta_{-2} = \upsilon_0, c = 3\upsilon_0\upsilon_2 - \upsilon_1^2. \quad (10)$$

After replacing these values into Eq. (9), several cases of analytical solutions are constructed as follows:

Case 1. If $\upsilon_0 = 1$, $\upsilon_1 = -(1 + k^2)$, and $\upsilon_2 = k^2$, then the solution to KK equation in Eq. (8) is given as follows:

$$u(\xi) = -\frac{1}{3}(k^2 + 1) + \frac{1}{\mathrm{sn}(\xi, k)^2} \quad (11)$$

such that $\xi = x + ct$.

This leads to the following equation:

$$u(x, t) = \frac{-2}{3} + \coth^2(t - x), \quad (12)$$

$$u(x, t) = \frac{-1}{3} + \csc^2(t - x) \quad (13)$$

when $k \to 1$ and $k \to 0$, respectively. The given solutions in Eqs (12) and (13) are plotted in Figure 1, respectively.

Case 2. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2k^2 - 1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{1}{3}(2k^2 - 1) + \frac{1 - k^2}{\mathrm{cn}(\xi, k)^2}. \quad (14)$$

Furthermore, Eq. (14) leads to the following solution:

$$u(x, t) = \frac{-1}{3} + \sec^2(t - x) \quad (15)$$

when $k \to 0$.

Case 3. If $\upsilon_0 = k^2 - 1$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = -1$, then Eq. (6) leads to the Jacobi solution as follows:

$$u(\xi) = \frac{1}{3}(2 - k^2) + \frac{k^2 - 1}{\mathrm{dn}(\xi, k)^2}. \quad (16)$$

Case 4. If $\upsilon_0 = k^2$, $\upsilon_1 = -1 - k^2$, and $\upsilon_2 = 1$, then, Eq. (6) satisfies the solution as follows:

$$u(\xi) = -\frac{1}{3}(k^2 + 1) + \frac{k^2}{\mathrm{ns}(\xi, k)^2}. \quad (17)$$

Thus, the solution in Eq. (17) leads to the following equation:

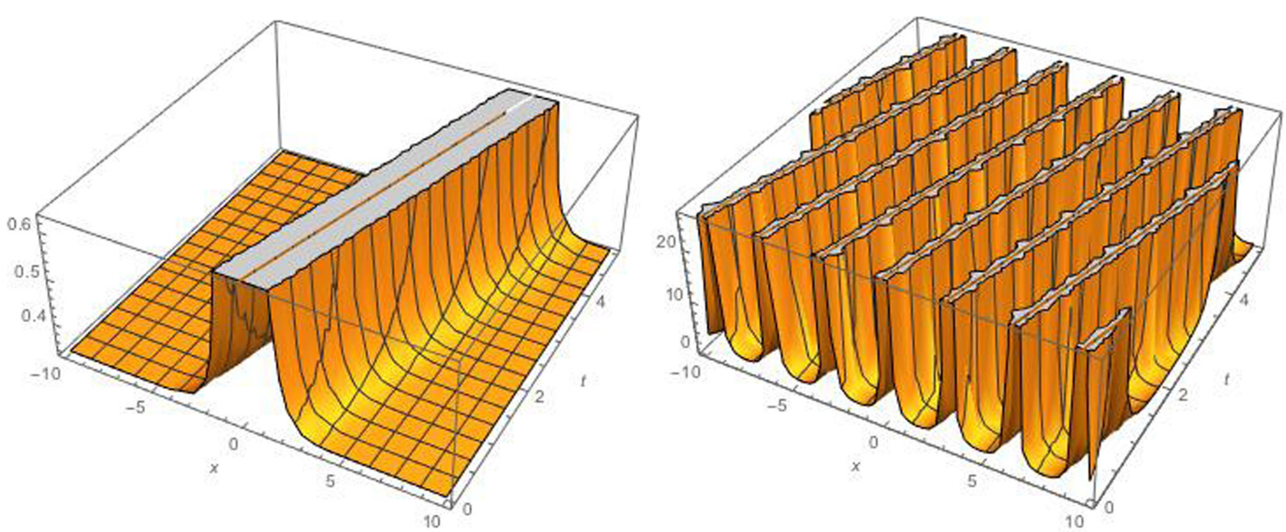
$$u(x, t) = \frac{-2}{3} + \tanh^2(t - x), \quad (18)$$

when $k \to 1$. The given solution to Eq. (18) will be represented in Figure 2.

Case 5. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) leads to the following solution:

$$u(\xi) = \frac{1}{3}(2 - k^2) + \frac{1 - k^2}{\mathrm{cs}(\xi, k)^2}, \quad (19)$$

which can be reduced to



(a) (b)

Figure 1: The 3D positive solutions (12) and (13) are plotted in (a) and (b), respectively.

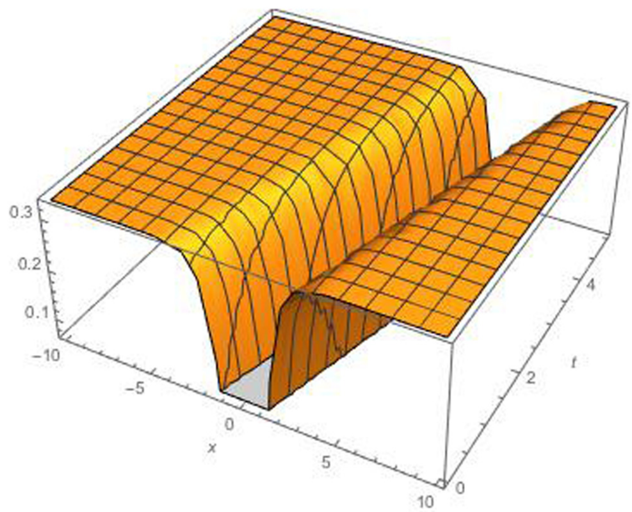


Figure 2: The 3D plot of the solutions (18) when $k = 1$.

$$u(x,t) = \frac{2}{3} + \tan^2(t - x) \tag{20}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1$, $\upsilon_1 = 2k^2 - 1$, and $\upsilon_2 = k^2(k^2 - 1)$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{1}{3}(2k^2 - 1) + \frac{1}{\mathrm{sd}(\xi,k)^2}. \tag{21}$$

Furthermore, the determined solution above converts to:

$$u(x,t) = \frac{1}{3} + \mathrm{csch}^2(t - x) \tag{22}$$

as $k \to 1$.

Set 2.

$$\eta_0 = \frac{8\upsilon_1}{3}, \eta_1 = 0, \eta_2 = 0, \eta_{-1} = 0, \eta_{-2} = 8\upsilon_0, \quad c = 176(3\upsilon_0\upsilon_2 - \upsilon_1^2). \tag{23}$$

By substituting previous values into Eq. (9), several cases of analytical solutions are constructed as follows:

Case 1. If $\upsilon_0 = 1$, $\upsilon_1 = -(1 + k^2)$, and $\upsilon_2 = k^2$, then the KK equation in Eq. (8) can be solved as follows:

$$u(\xi) = -\frac{8}{3}(k^2 + 1) + \frac{8}{\mathrm{sn}(\xi,k)^2}. \tag{24}$$

Furthermore, Eq. (24) leads to the following solution:

$$u(x,t) = \frac{-16}{3} + 8\coth^2(176t - x), \tag{25}$$

$$u(x,t) = \frac{-8}{3} + 8\csc^2(176t - x) \tag{26}$$

when $k \to 1$ and $k \to 0$, respectively.

Case 2. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2k^2 - 1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the solution as follows:

$$u(\xi) = \frac{8}{3}(2k^2 - 1) + \frac{8(1 - k^2)}{\mathrm{cn}(\xi,k)^2}. \tag{27}$$

Furthermore, Eq. (14) leads to the following solution:

$$u(x,t) = \frac{-8}{3} + 8\sec^2(176t - x) \tag{28}$$

when $k \to 0$.

Case 3. If $\upsilon_0 = k^2 - 1$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = -1$, then Eq. (6) admits a solution that can be written as follows:

$$u(\xi) = \frac{8}{3}(2 - k^2) + \frac{8(k^2 - 1)}{\mathrm{dn}(\xi,k)^2}. \tag{29}$$

Case 4. If $\upsilon_0 = k^2$, $\upsilon_1 = -1 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = -\frac{8}{3}(k^2 + 1) + \frac{8k^2}{\mathrm{ns}(\xi,k)^2}. \tag{30}$$

Furthermore, Eq. (30) leads to the following form:

$$u(x,t) = \frac{-16}{3} + 8\tanh^2(176t - x) \tag{31}$$

when $k \to 1$.

Case 5. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = \frac{8}{3}(2 - k^2) + \frac{8(1 - k^2)}{\mathrm{cs}(\xi,k)^2}, \tag{32}$$

which can be reduced to the following equation:

$$u(x,t) = \frac{16}{3} + 8\tan^2(176t - x) \tag{33}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1$, $\upsilon_1 = 2k^2 - 1$, and $\upsilon_2 = k^2(k^2 - 1)$, then, Eq. (6) satisfies the following form:

$$U(\xi) = \frac{8}{3}(2k^2 - 1) + \frac{8}{\mathrm{sd}(\xi,k)^2}. \tag{34}$$

Furthermore, the solution determined above transforms to the following equation:

$$u(x,t) = \frac{8}{3} + 8\,\mathrm{csch}^2(176t - x) \tag{35}$$

as $k \to 1$.

Set 3.

$$\eta_0 = \frac{\upsilon_1}{3}, \eta_1 = 0, \eta_2 = \upsilon_2, \eta_{-1} = 0, \eta_{-2} = 0, \quad c = 3\upsilon_0\upsilon_2 - \upsilon_1^2. \tag{36}$$

By substituting previous values into Eq. (9), several cases of analytical solutions are constructed in the following form:

Case 1. If $\upsilon_0 = 1, \upsilon_1 = -(1+k^2)$, and $\upsilon_2 = k^2$, then the KK equation in Eq. (8) satisfies a solution that can be written as follows:

$$u(\xi) = -\frac{1}{3}(k^2+1) + k^2\,\mathrm{sn}(\xi,k)^2. \tag{37}$$

Case 2. If $\upsilon_0 = 1-k^2, \upsilon_1 = -2k^2-1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the solution as follows:

$$u(\xi) = \frac{1}{3}(2k^2-1) - k^2\,\mathrm{cn}(\xi,k)^2. \tag{38}$$

Furthermore, the form in Eq. (38) leads to the following solution:

$$u(x,t) = \frac{1}{3} - \mathrm{sech}^2(t-x), \tag{39}$$

when $k \to 1$.

Case 3. If $\upsilon_0 = k^2-1, \upsilon_1 = -2-k^2$, and $\upsilon_2 = -1$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = \frac{1}{3}(2-k^2) - \mathrm{dn}(\xi,k)^2. \tag{40}$$

Case 4. If $\upsilon_0 = k^2, \upsilon_1 = -1-k^2$, and $\upsilon_2 = 1$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = -\frac{1}{3}(k^2+1) + \mathrm{ns}(\xi,k)^2. \tag{41}$$

Case 5. If $\upsilon_0 = 1-k^2, \upsilon_1 = -2-k^2$, and $\upsilon_2 = 1$, then Eq. (6) admits a solution that can be written as follows:

$$u(\xi) = \frac{1}{3}(2-k^2) + \mathrm{cs}(\xi,k)^2, \tag{42}$$

which can be reduced to the following form:

$$u(x,t) = \frac{2}{3} + \cot^2(t-x) \tag{43}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1, \upsilon_1 = 2k^2-1$, and $\upsilon_2 = k^2(k^2-1)$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = \frac{1}{3}(2k^2-1) + k^2(k^2-1)\,\mathrm{sd}(\xi,k)^2. \tag{44}$$

Set 4.

$$\eta_0 = \frac{\upsilon_1}{3}, \eta_1 = 0, \eta_2 = \upsilon_2, \eta_{-1} = 0, \eta_{-2} = \upsilon_0, \quad c = -12\upsilon_0\upsilon_2 - \upsilon_1^2. \tag{45}$$

By substituting previous values into Eq. (9), several cases of exact solutions are constructed in the following forms:

Case 1. If $\upsilon_0 = 1, \upsilon_1 = -(1+k^2)$, and $\upsilon_2 = k^2$, then the KK equation in Eq. (8) satisfies a solution that can be written as follows:

$$u(\xi) = -\frac{1}{3}(k^2+1) + \frac{1}{\mathrm{sn}(\xi,k)^2} + k^2\,\mathrm{sn}(\xi,k)^2. \tag{46}$$

Furthermore, Eq. (11) leads to following form:

$$u(x,t) = \frac{-2}{3} + \coth^2(16t-x) + \tanh^2(16t-x) \tag{47}$$

when $k \to 1$.

Case 2. If $\upsilon_0 = 1-k^2, \upsilon_1 = -2k^2-1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{1}{3}(2k^2-1) + \frac{1-k^2}{\mathrm{cn}(\xi,k)^2} + k^2\,\mathrm{cn}(\xi,k)^2. \tag{48}$$

Case 3. If $\upsilon_0 = k^2-1, \upsilon_1 = -2-k^2$, and $\upsilon_2 = -1$, then Eq. (6) admits the following form:

$$u(\xi) = \frac{1}{3}(2-k^2) + \frac{k^2-1}{\mathrm{dn}(\xi,k)^2} - \mathrm{dn}(\xi,k)^2. \tag{49}$$

Case 4. If $\upsilon_0 = k^2, \upsilon_1 = -1-k^2$, and $\upsilon_2 = 1$, then Eq. (6) admits the following form:

$$u(\xi) = -\frac{1}{3}(k^2+1) + \frac{k^2}{\mathrm{ns}(\xi,k)^2} + \mathrm{ns}(\xi,k)^2. \tag{50}$$

Case 5. If $\upsilon_0 = 1-k^2, \upsilon_1 = -2-k^2$, and $\upsilon_2 = 1$, then Eq. (6) admits the following form:

$$u(\xi) = \frac{1}{3}(2-k^2) + \frac{1-k^2}{\mathrm{cs}(\xi,k)^2} + \mathrm{cs}(\xi,k)^2, \tag{51}$$

which can be reduced to the following form:

$$u(x,t) = \frac{2}{3} + \cot^2(16t-x) + \tan^2(16t-x) \tag{52}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1, \upsilon_1 = 2k^2-1$, and $\upsilon_2 = k^2(k^2-1)$, then Eq. (6) satisfies the following form:

$$u(\xi) = \frac{1}{3}(2k^2-1) + \frac{1}{\mathrm{sd}(\xi,k)^2} + k^2(k^2-1)\,\mathrm{sd}(\xi,k)^2. \tag{53}$$

Set 5.

$$\eta_0 = \frac{8\upsilon_1}{3}, \eta_1 = 0, \eta_2 = 8\upsilon_2, \eta_{-1} = 0, \eta_{-2} = 0, \quad c = 176(3\upsilon_0\upsilon_2 - \upsilon_1^2). \tag{54}$$

By substituting previous values into Eq. (9), several cases of the following exact solutions can be constructed:

Case 1. If $\upsilon_0 = 1, \upsilon_1 = -(1+k^2)$, and $\upsilon_2 = k^2$, then the KK equation in Eq. (8) admits the following form:

$$u(\xi) = -\frac{8}{3}(k^2+1) + 8k^2\,\mathrm{sn}(\xi,k)^2. \tag{55}$$

Case 2. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2k^2 - 1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{8}{3}(2k^2 - 1) - 8k^2\,\mathrm{cn}(\xi, k)^2. \tag{56}$$

Furthermore, Eq. (56) leads to the following form:

$$u(x,t) = \frac{8}{3} - 8\,\mathrm{sech}^2(176t - x) \tag{57}$$

when $k \to 1$.

Case 3. If $\upsilon_0 = k^2 - 1$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = -1$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{8}{3}(2 - k^2) - 8\,\mathrm{dn}(\xi, k)^2. \tag{58}$$

Case 4. If $\upsilon_0 = k^2$, $\upsilon_1 = -1 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) satisfies a solution that can be written as follows:

$$u(\xi) = -\frac{8}{3}(k^2 + 1) + 8\,\mathrm{ns}(\xi, k)^2. \tag{59}$$

Case 5. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) has a solution of the following form:

$$u(\xi) = \frac{8}{3}(2 - k^2) + 8\,\mathrm{cs}(\xi, k)^2, \tag{60}$$

that can be reduced to the following equation:

$$u(x,t) = \frac{16}{3} + 8\cot^2(176t - x), \tag{61}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1$, $\upsilon_1 = 2k^2 - 1$, and $\upsilon_2 = k^2(k^2 - 1)$, then Eq. (6) satisfies the following form of solution:

$$U(\xi) = \frac{8}{3}(2k^2 - 1) + 8k^2(k^2 - 1)\,\mathrm{sd}(\xi, k)^2. \tag{62}$$

Set 6.

$$\begin{aligned}&\eta_0 = \frac{8\upsilon_1}{3}, \eta_1 = 0, \eta_2 = 8\upsilon_2, \eta_{-1} = 0, \eta_{-2} = 8\upsilon_0,\\ &c = -176(12\upsilon_0\upsilon_2 + \upsilon_1^2).\end{aligned} \tag{63}$$

By substituting previous values into Eq. (9), several cases of the following exact solutions are constructed:

Case 1. If $\upsilon_0 = 1$, $\upsilon_1 = -(1 + k^2)$, and $\upsilon_2 = k^2$, then the KK equation in Eq. (8) satisfies the form of solution as follows:

$$u(\xi) = -\frac{8}{3}(k^2 + 1) + \frac{8}{\mathrm{sn}(\xi, k)^2} + 8k^2\,\mathrm{sn}(\xi, k)^2. \tag{64}$$

The solution in Eq. (24) leads to

$$\begin{aligned}u(x,t) = \frac{-16}{3} &+ 8\coth^2(2816t - x)\\ &+ 8\tanh^2(2816t - x),\end{aligned} \tag{65}$$

when $k \to 1$.

Case 2. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2k^2 - 1$, and $\upsilon_2 = -k^2$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{8}{3}(2k^2 - 1) + \frac{8(1 - k^2)}{\mathrm{cn}(\xi, k)^2} - 8k^2\,\mathrm{cn}(\xi, k)^2. \tag{66}$$

Case 3. If $\upsilon_0 = k^2 - 1$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = -1$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{8}{3}(2 - k^2) + \frac{8(k^2 - 1)}{\mathrm{dn}(\xi, k)^2} - 8\,\mathrm{dn}(\xi, k)^2. \tag{67}$$

Case 4. If $\upsilon_0 = k^2$, $\upsilon_1 = -1 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) admits the following solution:

$$u(\xi) = -\frac{8}{3}(k^2 + 1) + \frac{8k^2}{\mathrm{ns}(\xi, k)^2} + 8\,\mathrm{ns}(\xi, k)^2. \tag{68}$$

Case 5. If $\upsilon_0 = 1 - k^2$, $\upsilon_1 = -2 - k^2$, and $\upsilon_2 = 1$, then Eq. (6) admits the following solution:

$$u(\xi) = \frac{8}{3}(2 - k^2) + \frac{8(1 - k^2)}{\mathrm{cs}(\xi, k)^2} + 8\,\mathrm{cs}(\xi, k)^2, \tag{69}$$

which can be reduced to:

$$u(x,t) = \frac{16}{3} + 8\tan^2(2816t - x) + 8\cot^2(2816t - x), \tag{70}$$

when $k \to 0$.

Case 6. If $\upsilon_0 = 1$, $\upsilon_1 = 2k^2 - 1$, and $\upsilon_2 = k^2(k^2 - 1)$, then Eq. (6) admits the following form of solution:

$$\begin{aligned}U(\xi) = \frac{8}{3}(2k^2 - 1) &+ \frac{8}{\mathrm{sd}(\xi, k)^2}\\ &+ 8k^2(k^2 - 1)\,\mathrm{sd}(\xi, k)^2.\end{aligned} \tag{71}$$

4 Conclusion

In this manuscript, the KK equation has been studied to find new analytical solitary solutions. The MAE method has been utilized to obtain a variety of solutions to the problem under investigation. As far as we know, our solutions are new and different from those in the literature. The obtained solutions are written in terms of Jacobi functions that can be reduced to hyperbolic and trigonometric forms. All solutions obtained in the article have been verified through insertion into the primary equation. Some of the obtained solutions are plotted based on appropriate parameter values. In future work, the analytical methods MAE used in this article can be applied to other types of nonlinear equations for determining the solutions of a suitable model.

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