Research Article

Yong Tang and Zhao Li*

# Bifurcation, chaotic behavior, and traveling wave solution of stochastic coupled Konno–Oono equation with multiplicative noise in the Stratonovich sense

https://doi.org/10.1515/phys-2023-0130
received March 12, 2023; accepted October 12, 2023

**Abstract:** The main aim of this article is to focus on the dynamics and traveling wave solution of stochastic coupled Konno–Oono equation with multiplicative noise in the Stratonovich sense. First, the considered model is converted to the nonlinear ordinary differential equations by using traveling wave transformation. Secondly, two-dimensional phase portrait of the nonlinear ordinary differential equation and its periodic perturbation system are drawn by using Maple software. Finally, the traveling wave solutions of the investigated equation are obtained via the planar dynamic system method. Moreover, three-dimensional graphs of some obtained solutions are drawn.

**Keywords:** stochastic coupled Konno–Oono equation, phase portrait, traveling wave solution, planar dynamic system method

## 1 Introduction

Stochastic partial differential equation (SPDE) [1,2] plays a very important role in the fields of physics, life science, nonlinear optics, engineering technology, and control science. More and more models from natural and social sciences need to be simulated by SPDE. Therefore, the study of SPDE is particularly important. At present, the main problems of SPDE include the existence of solutions, the uniqueness of solutions, the stability of solutions, martingale representation theory, numerical solutions, and exact solutions [3–8]. However, in real life and scientific calculation, it is more likely to use numerical or analytical solutions of SPDE. However, due to the complexity of stochastic problems, there are still many challenging problems in finding the exact solutions of SPDE. As is well known, the problem of traveling wave solutions for nonlinear partial differential equation (NLPDE) has received widespread attention [9–19], and many classic methods have been proposed for constructing traveling wave solutions for NLPDE [20–32]. In some recent studies, Mohammed and his collaborators [33,34] can simplify some special SPDE into nonlinear ordinary differential equations by means of traveling wave transformation and mathematical analysis, which is very helpful for us to study SPDE.

In this article, we consider the stochastic coupled Konno–Oono (K–O) equation [35]:

$$\begin{cases} v_{xt} - 2vu = \sigma F(v), \\ u_t + 2vv_x = 0, \end{cases} \tag{1.1}$$

where $v = v(t, x), u = u(t, x)$. $\sigma$ is the noise strength. $F(v) = v_x \circ \beta_t$ represents the noise term in the Stratonovich sense. $\beta(t)$ stands for the standard Wiener process. When $\sigma = 0$, Eq. (1.1) describes a current-fed string interacting with an external magnetic fed [36]. Stratonovich integral and Itô integral are as follows [37]:

$$\rho\int_0^t \varphi \circ \mathrm{d}\beta(s) = \rho\int_0^t \varphi \mathrm{d}\beta(s) + \frac{\rho^2}{2}\int_0^t \varphi \mathrm{d}s. \tag{1.2}$$

Mohammed *et al.* [35] studied the exact solutions of Eq. (1.1) by using the generalized $\frac{G'}{G}$-expansion method. However, Mohammed *et al.* [35] did not obtain a Jacobian function solution and did not discuss the dynamic and chaotic behavior of Eq. (1.1). The main purpose of this article is to discuss the dynamic behavior and exact traveling wave solutions of Eq. (1.1).

* **Corresponding author: Zhao Li,** College of Computer Science, Chengdu University, Chengdu, 610106, China, e-mail: lizhao10.26@163.com
**Yong Tang:** College of Computer Science, Chengdu University, Chengdu, 610106, China

The rest of this article is arranged as follows. In Section 2, two-dimensional phase portrait of the nonlinear ordinary differential equation and its periodic perturbation system are drawn. Moreover, the traveling wave solutions of systems (1.1) are obtained. Finally, in Section 3, a conclusion is given.

# 2 Phase portraits and traveling wave solutions of Eq. (1.1)

## 2.1 Traveling wave transformation

First, the following transformation is considered

$$v(t,x) = V(\xi)e^{\sigma\beta(t)-\sigma^2 t},\ u(t,x) = U(\xi),\ \xi = x - kt, \tag{2.1}$$

where $V(\xi)$ and $U(\xi)$ are deterministic real function, $\sigma$ stands for the noise strength, and $k$ is the nonzero constant.

Substituting Eq. (2.1) into Eq. (1.1), we have

$$-kV'' + \sigma V'\beta(t) - \sigma^2 V' - 2VU = \sigma V' \circ \beta(t), \tag{2.2}$$

$$-kU' + 2VV'e^{2\sigma\beta(t)-2\sigma^2 t} = 0. \tag{2.3}$$

Applying (1.2) to Eq. (2.2), we can obtain

$$kV'' + 2VU = 0. \tag{2.4}$$

Next, taking mathematical expectation on both sides of Eq. (2.3) at the same time, we can obtain

$$-kU' + 2VV' = 0. \tag{2.5}$$

Integrating both sides of Eq. (2.5) with respect to $\xi$, we have

$$U = \frac{1}{k}(V^2 + c), \tag{2.6}$$

where $c$ is the integration constant.

Substituting Eq. (2.6) into Eq. (2.4), we have

$$V'' + \frac{2}{k^2}V^3 + \frac{2c}{k^2}V = 0. \tag{2.7}$$

## 2.2 Phase portraits of Eq. (2.7)

Suppose that $\frac{dV(\xi)}{d\xi} = y$, then we obtain the following two-dimensional plane system:

$$\begin{cases} \dfrac{dV(\xi)}{d\xi} = y, \\ \dfrac{dy}{d\xi} = -\Upsilon_1 V^3 + \Upsilon_2 V, \end{cases} \tag{2.8}$$

which has the first integral

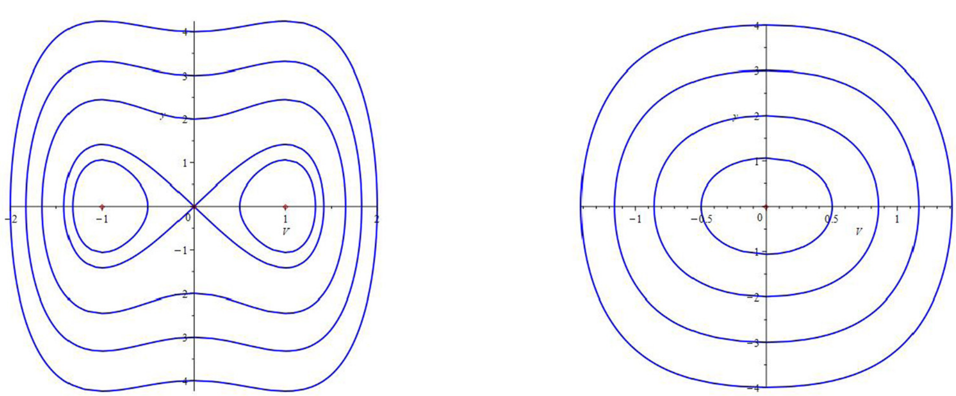
$$H(V,y) = \frac{1}{2}y^2 + \frac{\Upsilon_1}{4}V^4 - \frac{\Upsilon_2}{2}V^2 = h, \tag{2.9}$$

where $\Upsilon_1 = \frac{2}{k^2}$ and $\Upsilon_2 = -\frac{2c}{k^2}$.

When $k = \frac{\sqrt{2}}{2}$ and $c = -1$, system (2.8) has three equilibrium points, namely $(0,0)$, $(0,1)$, and $(0,-1)$. Its phase portrait is shown in Figure 1a. When $k = \frac{\sqrt{2}}{2}$ and $c = 1$, system (2.8) has only one equilibrium point, namely $(0,0)$. Its phase portrait [38] is shown in Figure 1b.

The periodic perturbation of system (2.8) is expressed as follows:

$$\begin{cases} \dfrac{dV(\xi)}{d\xi} = y, \\ \dfrac{dy}{d\xi} = -\Upsilon_1 V^3 + \Upsilon_2 V + A\cos(B\xi), \end{cases} \tag{2.10}$$



(a) (b)

**Figure 1:** The phase portraits of system (2.8). (a) $\Upsilon_1 > 0$, $\Upsilon_2 > 0$ and (b) $\Upsilon_1 > 0$, $\Upsilon_2 < 0$.

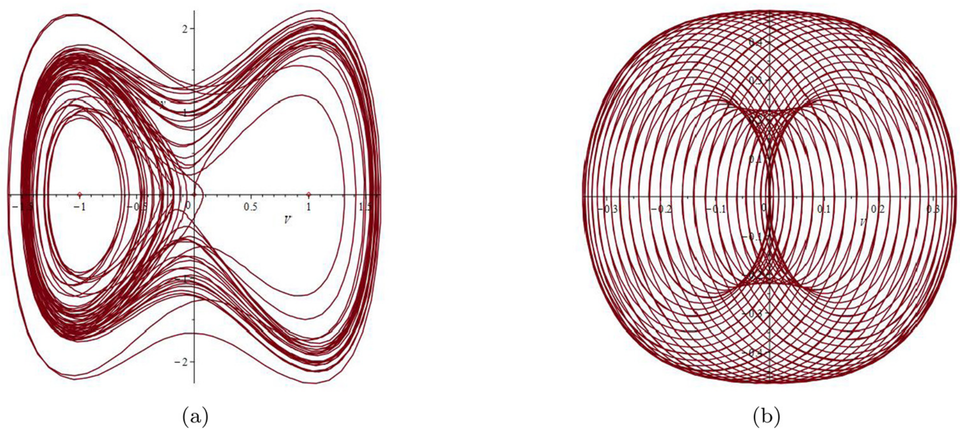


**Figure 2:** The phase portraits of system (2.10). (a) $\Upsilon_1 > 0$, $\Upsilon_2 > 0$, $A = 0.6$, $B = 0.8$ and (b) $\Upsilon_1 > 0$, $\Upsilon_2 < 0$, $A = 0.6$, $B = 0.8$.

where $A$ stands for the periodic perturbation and $B$ represents the frequency. It can be seen from the comparison between Figures 1 and 2 that when a disturbance term is added to the system, the system will produce chaotic behavior [39].

## 2.3 Traveling wave solutions of Eq. (1.1)

Let $h_1 = H(\pm\sqrt{\frac{\Upsilon_1}{\Upsilon_2}}, 0) = -\frac{\Upsilon_1^2}{4\Upsilon_2}$.

(i) When $h \in (h_1, 0)$, system (2.6) can be rewritten as

$$\begin{aligned} y^2 &= \frac{\Upsilon_2}{2}\left(-V^4 + \frac{2\Upsilon_1}{\Upsilon_2}V^2 + \frac{4h}{\Upsilon_2}\right) \\ &= \frac{\Upsilon_2}{2}(\delta_1^2 - V^2)(V^2 - \delta_2^2), \end{aligned} \tag{2.11}$$

where $\delta_1^2 = \frac{1}{\Upsilon_2}(\Upsilon_1 + \sqrt{\Upsilon_1^2 + 4\Upsilon_2 h})$ and $\delta_2^2 = \frac{1}{\Upsilon_2}(\Upsilon_1 - \sqrt{\Upsilon_1^2 + 4\Upsilon_2 h})$.

Substituting (2.11) into $\frac{\mathrm{d}V}{\mathrm{d}\xi} = y$ and integrating it, we have

$$\int_V^{\delta_2} \frac{\mathrm{d}\phi}{\sqrt{(\delta_1^2 - \phi^2)(\phi^2 - \delta_2^2)}} = \mp\sqrt{\frac{\Upsilon_2}{2}}(\xi - \xi_0). \tag{2.12}$$

$$\int_{-\delta_2}^{V} \frac{\mathrm{d}\phi}{\sqrt{(\delta_1^2 - \phi^2)(\phi^2 - \delta_2^2)}} = \pm\sqrt{\frac{\Upsilon_2}{2}}(\xi - \xi_0). \tag{2.13}$$

Integrating Eqs (2.12) and (2.13), we obtain

$$\begin{aligned} v_1(t,x) = \pm\delta_2 \mathbf{dn}\left(\delta_2\sqrt{\frac{\Upsilon_1}{2}}(x - kt\right. \\ \left. - \xi_0), \frac{\sqrt{\delta_1^2 - \delta_2^2}}{\delta_1}\right)\mathbf{e}^{\sigma\beta(t) - \sigma^2 t}. \end{aligned} \tag{2.14}$$

$$\begin{aligned} u_1(t,x) = \frac{1}{k}\left[c + \delta_2^2\mathbf{dn}^2\left(\delta_2\sqrt{\frac{\Upsilon_1}{2}}(x - kt\right.\right. \\ \left.\left. - \xi_0), \frac{\sqrt{\delta_{11}^2 - \delta_{21}^2}}{\delta_1}\right)\right]\mathbf{e}^{\sigma\beta(t) - \sigma^2 t}. \end{aligned} \tag{2.15}$$

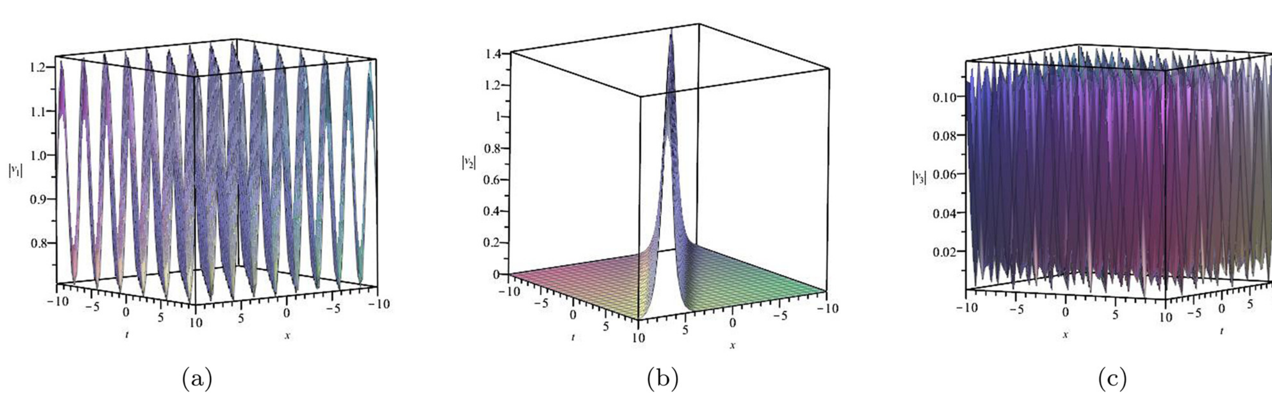


**Figure 3:** The traveling wave solution of system (1.1) for $k = \frac{\sqrt{2}}{2}$, $c = 1$, $\sigma = 0$. (a) $h = \frac{3}{4}$, (b) $h = 0$, and (c) $h = \frac{1}{4}$.

(ii) When $h = 0$, so $\delta_1^2 = \frac{2\Upsilon_1}{\Upsilon_2}$ and $\delta_2^2 = 0$. Thus, the solution of Eq. (1.1) is presented as

$$v_2(t,x) = \pm\sqrt{\frac{2\Upsilon_2}{\Upsilon_1}}\,\mathbf{sech}(\sqrt{\Upsilon_2}(x - kt - \xi_0))\mathbf{e}^{\sigma\beta(t)-\sigma^2 t}. \quad (2.16)$$

$$u_2(t,x) = \frac{1}{k}\left[c + \frac{2\Upsilon_2}{\Upsilon_1}\mathbf{sech}^2(\sqrt{\Upsilon_2}(x - kt - \xi_0))\right]\mathbf{e}^{\sigma\beta(t)-\sigma^2 t}. \quad (2.17)$$

(iii) When $h \in (0, +\infty)$, system (2.7) is recorded as

$$y^2 = \frac{\Upsilon_2}{2}(V^2 + \delta_3^2)(\delta_4^2 - V^2), \quad (2.18)$$

where $\delta_3^2 = \frac{1}{\Upsilon_2}(-\Upsilon_1 + \sqrt{\Upsilon_1^2 + 4\Upsilon_2 h})$ and $\delta_4^2 = \frac{1}{\Upsilon_2}(\Upsilon_1 + \sqrt{\Upsilon_1^2 + 4\Upsilon_2 h})$.

Plugging (2.18) into $\frac{dV}{d\xi} = y$ and integrating it along the periodic orbits, then, we have

$$\int_0^V \frac{d\phi}{\sqrt{(\phi^2 + \delta_3^2)(\delta_4^2 - \phi^2)}} = \pm\sqrt{\frac{\Upsilon_2}{2}}(\xi - \xi_0). \quad (2.19)$$

So, we have

$$v_3(t,x) = \pm\delta_3^2\mathbf{cn}\left(\sqrt{\frac{\Upsilon_1(\delta_3^2 + \delta_4^2)}{2}}(x - kt - \xi_0), \frac{\delta_4}{\sqrt{\delta_3^2 + \delta_4^2}}\right)\mathbf{e}^{\sigma\beta(t)-\sigma^2 t}. \quad (2.20)$$

$$u_3(t,x) = \frac{1}{k}\left[c + \delta_3^2\mathbf{cn}^2\left(\sqrt{\frac{\Upsilon_1(\delta_3^2 + \delta_4^2)}{2}}(x - kt - \xi_0), \frac{\delta_4}{\sqrt{\delta_3^2 + \delta_4^2}}\right)\right]\mathbf{e}^{\sigma\beta(t)-\sigma^2 t}. \quad (2.21)$$

**Remark 2.1.** Figure 3(a) and (c) shows solutions to a Jacobian function, which are periodic solutions. Figure 3(b) shows a bell shaped solitary wave.

## 3 Conclusion

In this article, the dynamics and traveling wave solution of stochastic coupled K–O equation with multiplicative noise is studied. Two-dimensional phase portrait of stochastic coupled K–O equation with multiplicative noise and its periodic perturbation system are drawn by using the Maple software. Compared with the existing literature [35], this article also obtains Jacobian elliptic function solutions, which are new solutions. In the future, our research will still focus on the dynamic behavior and traveling wave solutions of SPDEs.

**Funding information:** The authors state no funding involved.

**Author contributions:** All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

**Conflict of interest**: The authors state no conflict of interest.

## References

[1] Zhang ZQ, Karniadakis GE. Numerical methods for stochastic partial differential equations with White Noise. Cham: Springer; 2010.
[2] Han TY, Li Z, Zhang K. Exact solutions of the stochastic fractional long-short wave interaction system with multiplicative noise in generalized elastic medium. Results Phys. 2023;44:106174.
[3] Ulutas E. Traveling wave and optical soliton solutions of the Wick-type stochastic NLSE with conformable derivatives. Chaos Solitons Fract. 2021;148:111052.
[4] Gess B, Tölle JM. Stability of solutions to stochastic partial differential equations. J Differ Equations. 2016;260:4973–5025.
[5] Gao DD, Li JL. Existence and mean-square exponential stability of mild solutions for impulsive stochastic partial differential equations with noncompact semigroup. J Math Anal Appl. 2020;484:123717.
[6] Yang M, Chen GG. Stability and constructions of the Poincaré maps for a class of stochastic partial differential equations. Physica D. 2023;446:133659.
[7] Zayed EME, Shohib RMA, Alngar MEM, Gepreel K, Nofal TA, Yıldırım Y. Optical solitons for Biswas-Arshed equation with multiplicative noise via Itô calculus using three integration algorithms. Optik. 2022;258:168847.
[8] Li Z, Hu HL. Chaotic pattern, bifurcation, sensitivity and traveling wave solution of the coupled Kundu-Mukherjee-Naskar equation. Results Phys. 2023;48:106441.
[9] Li Z. Bifurcation and traveling wave solution to fractional Biswas-Arshed equation with the beta time derivative. Chaos Solitons Fract. 2022;160:112249.
[10] Karakoc SBG, Asit Saha, Sucu D. A novel implementation of Petrov-Galerkin method to shallow water solitary wave pattern and superperiodic traveling wave and its multistability generalized Korteweg-de Vries equation. Chinese J Phys. 2020;68:605–17.
[11] Ali KK, Karakoc BG, Rezazadeh H. Optical soliton solutions of the fractional perturbed nonlinear Schrödinger equation. TWMS J Appl Eng Math. 2020;10:930–9.
[12] Saha A, Karakoc BG, Ali KK. New exact soliton solutions, bifurcation and multistability behaviors of traveling waves for the (3.1)-dimensional modified Zakharov-Kuznetsov equation with higher order dispersion. Math Sci Appl. 2023;11:226–40.
[13] Wu WJ, Manafian J, Ali KK, Karakoc SBG, Taqi AH, Mahmoud MA. Numerical and analytical results of the 1D BBM equation and 2D coupled BBM-system by finite element method. Int J Mod Phys B. 2022;36:2250201.
[14] Karakoc SBG, Saha A, Bhowmik SK, Sucu D. Numerical and dynamical behaviors of nonlinear traveling wave solutions of the

Kudryashov-Sinelshchikov Equation. Wave Motion. 2023;118:103121.
[15] Selima ES, Abu-Nab AK, Morad AM. Dynamics of the (2+1)-dimensional coupled cubic-quintic complex Ginzburg-Landau model for convecting binary fluid: analytical investigations and multiple-soliton solutions. Math Method Appl Sci. 2023;2023:1–25.
[16] Morad AM, Abu-Nab AK. Bubbles interactions in fluidized granular medium for the van der Waals hydrodynamic regime. Eur Phys J Plus. 2021;136:306.
[17] Abourabia AM, Morad AM. Exact traveling wave solutions of the van der Waals normal form for fluidized granular matter. Physica A. 2015;437:330–50.
[18] Morad AM, Maize SMA, Nowaya A A, Rammah YS. Stability analysis of magnetohydrodynamics waves in compressible turbulent plasma. J Nanofluids. 2020;9:1740.
[19] Morad AM, Maize SMA, Nowaya AA, Rammah YS. A new derivation of exact solutions for incompressible magnetohydrodynamic plasma turbulence. J Nanofluids. 2021;10:1765.
[20] Wang J, Shehzad K, Seadawy AR, Arshad M, Asmat F. Dynamic study of multi-peak solitons and other wave solutions of new coupled KdV and new coupled Zakharov-Kuznetsov systems with their stability. J Taibah Univ Sci. 2023;17:2163872.
[21] Seadawy AR, Rizvi STR, Ahmad S, Younis M, Baleanu D. Lump, lump-one stripe, multiwave and breather solutions for the Hunter-Saxton equation. Open Phys. 2021;19:1–10.
[22] Seadawy AR, Kumar D, Chakrabarty AK. Dispersive optical soliton solutions for the hyperbolic and cubic-quintic nonlinear Schrödinger equations via the extended sinh-Gordon equation expansion method. Eur Phys J Plus. 2018;133:182.
[23] Seadawy AR, Cheemaa N. Some new families of spiky solitary waves of one-dimensional higher-order K-dV equation with power law nonlinearity in plasma physics. Indian J Phys. 2019;94:117–26.
[24] Younas U, Seadawy AR, Younis M, Rizvi STR. Optical solitons and closed form solutions to the (3+1)-dimensional resonant Schrödinger dynamical wave equation. Int J Mod Phys B. 2020;34:2050291.
[25] Rizvi STR, Seadawy AR, Ahmed S, Younis M, Ali K. Study of multiple lump and rogue waves to the generalized unstable space time fractional nonlinear Schrödinger equation. Chaos Solitons Fractals. 2021;151:111251.
[26] Li Z, Huang C, Wang BJ. Phase portrait, bifurcation, chaotic pattern and optical soliton solutions of the Fokas-Lenells equation with cubic-quartic dispersion in optical fibers. Phys Lett A. 2023;465:128714.
[27] Hafez MG, Iqbal SA, Asaduzzaman N, Hammouch Z. Dynamical behaviors and oblique resonant nonlinear waves with dual-power law nonlinearity and conformable temporal evolution. Discrete Cont Dyn-A. 2021;14:2245–60.
[28] Hafez MG. Nonlinear ion acoustic solitary waves with dynamical behaviours in the relativistic plasmas. Astrophys Space Sci. 2020;365:78.
[29] Uddin MF, Hafez MG, Iqbal SA. Dynamical plane wave solutions for the Heisenberg model of ferromagnetic spin chains with beta derivative evolution and obliqueness. Helyon. 2022;8:E09199.
[30] Iqbal SA, Hafez MG, Uddin MF. Bifurcation features, chaos, and coherent structures for one-dimensional nonlinear electrical transmission line. Com Appl Math. 2022;41:50.
[31] Iqbal SA, Hafez MG, Karim SAA. Bifurcation analysis with chaotic motion of oblique plane wave for describing a discrete nonlinear electrical transmission line with conformable derivative. Results Phys. 2020;18:103309.
[32] Iqbal SA, Hafez MG, Chu YM, Parik C. Dynamical analysis of non-autonomous RLC circuit with the absence and presence of Atangana-Baleanu fractional derivative. J Appl Anal Comput. 2022;12:770–89.
[33] Mohammed WW, Qahiti R, Ahmad H, Baili J, Mansour FE, EL-Morshedy E. Exact solutions for the system of stochastic equations for the ion sound and Langmuir waves. Results Phys. 2021;30:104841.
[34] Mohammed WW, Ahmad H, Hamza AE, Aly ES, El-Morshedy M, Elabbasy EM. The exact solutions of the stochastic Ginzburg-Landau equation. Results Phys. 2021;23:103988.
[35] Mohammed WW, Iqbal N, Ali A, El-Morshedy M. Exact solutions of the stochastic new coupled Konno-Oono equation. Results Phys. 2021;21:103830.
[36] Elbrolosy ME, Elmandouh AA. Dynamical behaviour of conformable time-fractional coupled Konno-Oono equation in magnetic field. Math Probl Eng. 2022;2022:3157217.
[37] Mackevičius V. Introduction to stochastic analysis: integrals and differential equations. Hoboken: John Wiley & Sons ltd; 2011.
[38] Li Z, Peng C. Bifurcation, phase portrait and traveling wave solution of time-fractional thin-film ferroelectric material equation with beta fractional derivative. Phys Lett A. 2023;484:129080.
[39] Li Z, Huang C. Bifurcation, phase portrait, chaotic pattern and optical soliton solutions of the conformable Fokas-Lenells model in optical fibers. Chaos Solitons Fract. 2023;169:113237.