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Research Article

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On the localized and periodic solutions to the time-fractional Klein-Gordan equations: Optimal additive function method and new iterative method

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Abstract: This investigation explores two numerical approaches: the optimal auxiliary function method (OAFM) and the new iterative method (NIM). These techniques address the physical fractional-order Klein-Gordon equations (FOKGEs), a class of partial differential equations (PDEs) that model various physical phenomena in engineering and diverse plasma models. The OAFM is a recently introduced method capable of efficiently solving several nonlinear differential equations (DEs), whereas the NIM is a well-established method specifically designed for solving fractional DEs. Both approaches are utilized to analyze different variations in FOKGE. By conducting numerous numerical experiments on the FOKGE, we compare the accuracy, efficiency, and convergence of these two proposed methods. This study is expected to yield significant findings that will help researchers study various nonlinear phenomena in fluids and plasma physics.

Keywords: time-fractional Klein-Gordan equations, fractional calculus, new iterative method, optimal auxiliary function method

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1 Introduction

Fractional calculus (FC) has become a significant branch of practical mathematics and theoretical physics. Modeling real-world occurrences with fractional derivatives and integrals are more accurate than using the classical derivative. Some physical processes, such as signal processing, electronics, chemistry, viscoelasticity, dynamical systems, economics, biology, and nonlinear phenomena in plasma physics, can be modeled correctly and accurately using fractional derivatives. Many authors focused on significant advances and additions to FC [1–8]. FC is an essential area of study for most researchers and experts due to its fascinating applications. The analysis of fractional differential equations (DEs) is essential to many professions. Comparatively, fractional derivatives can represent a wide range of broad problems. A fascinating area of study in the study of wave movements in the real world is traveling waveforms. Mathematics and physics are engaged in wave dispersion and breaking on coasts, ship waves on water, storm-caused river flood waves, and free movements of confined water like lakes and ports in addition to nonlinear waves in plasmas [9–11].

On the other hand, the wave propagation equation explains how waves travel through dispersive media, such as liquid flow containing microscopic air bubbles or flow features in elastic channels, such as streams, rivers, and seas, as well as gravity waves in a nearby regions and waves in plasmas. This kind of dynamical system may be helpful in research on fluid movement, ocean wave dynamics, and the mechanism of propagating nonlinear waves in plasmas. Therefore, finding precise answers to fractional DEs is challenging [12–14]. Therefore, it can be handled using analytical and estimation techniques. Many practical approaches have been investigated and proposed for the solution of fractional DEs, including the fractional differential transform method [15–17], the Adomian decomposition method [18–20], the residual power series method [21–23], the variational iteration method [24–26], and the HAM method [27–29].

The time-fractional Klein-Gordon equation (TFKGE) has been considered in this research [30,31]. The Klein-Gordon (KG) equation plays a vital role in mathematical physics and numerous other scientific disciplines like solid-state physics, quantum field theory, nonlinear optics, and nonlinear waves in plasmas [32–36]. On the other hand, by exchanging the time order derivative with the fractional derivative of order, the classical KG equation is transformed into the fractional-order KG equation. The KG equation in fractional order reads

$$D_t^{\rho}\psi - D_x^2\psi + a_1\psi + a_2G(\psi) = f(x,t), \tag{1}$$

with initial condition (IC),

$$\psi(x,0) = f_1(x),\ \psi_t(x,0) = f_2(x),$$

where D_t^{ρ} represents the Caputo fractional time derivative, a_1 and a_2 are real constants, $f(x,t)$, $f_1(x)$, and $f_2(x)$ are known as analytical functions, whereas $G(\psi)$ is a non-linear, and $\psi \equiv \psi(x,t)$ is an unknown function of x and t.

Due to its importance in several scientific and technical fields, the investigation of novel methods for solving fractional partial differential equations (PDEs) has attracted considerable interest. In this context, the Optimal auxiliary function method (OAFM) and the new iterative method (NIM) have distinguished themselves as potential approaches to deal with the difficulties in fractional PDEs. These techniques provide fresh ways to deal with the problems present in these equations, including fractional derivatives and nonlinear dynamics. By using auxiliary functions to convert fractional PDEs into systems of ordinary DEs, the OAFM makes it possible to use tried-and-true solution methods. The NIM, on the other hand, uses iterative refinement to increase the precision of approximation solutions for these equations. Both approaches have great promise in developing mathematical analysis and their use in various industries. This study examines the NIM and the OAFM in-depth to illuminate their ability to solve complex fractional PDEs.

The main purpose of the current study is to implement the OAFM and NIM for analyzing and solving the time-fractional Klein-Gordon equation (TFKGE).

2 Preliminaries

Some basic definitions of the Caputo fractional derivative are introduced and discussed here.

Definition 1. The formula for the Riemann fractional integral reads [32]

$$J_t^{\rho}\psi(x,t) = \frac{1}{\Gamma(\rho)}\int_0^t (t-r)^{\rho-1}\psi(x,r)\mathrm{d}r, \tag{2}$$

Definition 2. The fractional derivative of f, according to the Caputo formula, is defined as [33]

$$ {}^{C}D_t^{\rho}\psi(x,t) = \frac{1}{\Gamma(m-\rho)}\int_0^t (t-r)^{m-\rho-1} \psi(x,r)\mathrm{d}r,\ m-1<\rho\le m,\ t>0, \tag{3}$$

Lemma 1. *For $n-1<\rho\le n$, $p>-1$, $t\ge 0$ and $\lambda\in R$, we have*

1. $D_t^{\rho}t^{p} = \dfrac{\Gamma(\rho+1)}{\Gamma(p-\rho+1)}t^{p-\rho}$,
2. $D_t^{\rho}\lambda = 0$,
3. $D_t^{\rho}I_t^{\rho}\psi(x,t) = \psi(x,t)$,
4. $I_t^{\rho} = \psi(x,t) - \sum_{i=0}^{n-1}\partial^{i}\psi(x,0)\dfrac{t^{i}}{i!}$.

3 General procedure for the proposed methods

3.1 General procedure of OAFM

This section describes the OAFM approach for solving general fractional-order PDEs. Let us introduce the following general form for time fractional order PDEs:

$$\frac{\partial^{\rho}\psi(x,t)}{\partial t^{\rho}} = g(x,t) + N(\psi(x,t)), \tag{4}$$

which is subjected to the ICs,

$$\left.\begin{array}{c} D_t^{\rho-r}(x,0) = \varphi_r(x),\ r=0,1\ldots,s-1 \\ D_t^{\rho-s}(x,0) = 0,\ s=[\rho], \\ D_t^{r}(x,0) = \varsigma_r(x),\ r=0,1\ldots,s \\ D_t^{s}(x,0) = 0,\ s=[\rho] \end{array}\right\}. \tag{5}$$

Here $\frac{\partial^\rho}{\partial t^\rho}$ represents the Caputo operator, $\psi(x, t)$ is an unknown function, and $g(x, t)$ is a known analytic function. The approach algorithm is summarized in the following brief steps:

Step 1: We utilize a two-component approximate solution to address Eq. (4), which is as follows:

$$\psi(x, t) = \psi_0(x, t) + \psi_1(x, t, C_i), i = 1, 2, 3, \ldots, p \tag{6}$$

Step 2: To obtain the solutions for the zeroth- and first-order, we insert Eq. (6) in Eq. (4), to get

$$\begin{aligned} &\frac{\partial^\rho \psi_0(x, t)}{\partial t^\rho} + \frac{\partial^\rho \psi_1(x, t)}{\partial t^\rho} + g(x, t) \\ &+ N\left[\left(\frac{\partial^\rho \psi_0(x, t)}{\partial t^\rho}\right) + \left(\frac{\partial^\rho \psi_1(x, t, C_i)}{\partial t^\rho}\right)\right] = 0, \end{aligned} \tag{7}$$

Step 3: For the purpose of determining the first approximation $\psi_0(x, t)$ based on the linear equation

$$\frac{\partial^\rho \psi_0(x, t)}{\partial t^\rho} + g(x, t) = 0, \tag{8}$$

by utilizing the inverse operator, we can obtain the expression for$\psi_0(x, t)$, which can be stated as follows:

$$\psi_0(x, t) = g(x, t). \tag{9}$$

Step 4: The nonlinear term that appeared in Eq. (7) reads

$$\begin{aligned} &N\left[\frac{\partial^\rho \psi_0(x, t)}{\partial t^\rho} + \frac{\partial^\rho \psi_1(x, t, C_i)}{\partial t^\rho}\right] \\ &= N[\psi_0(x, t)] + \sum_{k=1}^{\infty} \frac{\psi_1^k}{k!} N^{(k)}[\psi_0(x, t)]. \end{aligned} \tag{10}$$

Step 5: To enhance the convergence of the first-order approximation $\psi(x, t)$ and effectively solve Eq. (10), we propose an alternative equation which can be expressed as follows:

$$\begin{aligned} &\frac{\partial^\rho \psi_1(x, t, C_i)}{\partial t^\rho} \\ &= -\nabla_1[\psi_0(x, t)] N[\psi_0(x, t)] - \nabla_2[\psi_0(x, t), C_j]. \end{aligned} \tag{11}$$

Step 6: We can calculate a first-order solution, $\psi_0(x, t)$ by using the inverse operator and substituting the auxiliary function in Eq. (11) as per the method mentioned.

Step 7: There are various methods for determining the numerical values of convergence control parameters C_i, including but not limited to least squares, Galerkin's, Ritz, and collocation. In our case, we have opted to utilize the least squares approach as it helps minimize errors.

$$J(C_i, C_j) = \int_0^t \int_\Omega^q R^2(x, s, C_i, C_j) \mathrm{d}x \mathrm{d}s, \tag{12}$$

where $R \equiv R(x, t, C_i, C_j)$ denotes the residual,

$$\begin{aligned} R = &\frac{\partial \psi \quad (x, t, C_i, C_j)}{\partial t} + g(x, t) \\ &+ N[\psi(x, t, C_i, C_j)], \quad i = 1, 2, \ldots, s, j = s + 1, \\ &s + 2, \ldots, p. \end{aligned} \tag{13}$$

3.2 Analysis of the NIM

To explain the basic idea of the NIM, the following general functional equation is considered

$$\psi(x) = f(x) + N(\psi(x)), \tag{14}$$

where N is the nonlinear operator from a Banach space B to B and f is an unknown function. Now, we are looking for a solution to Eq. (14) having the series form

$$\psi(x) = \sum_{i=0}^{\infty} \psi_i(x). \tag{15}$$

The nonlinear term can be decomposed as

$$N\left(\sum_{i=0}^{\infty} \psi_i(x)\right) = N(\psi_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} \psi_j(x)\right) \quad N\left(\sum_{j=0}^{i-1} \psi_j(x)\right)\right]. \tag{16}$$

Inserting Eqs (15) and (16) in Eq. (14), we get

$$\sum_{i=0}^{\infty} \psi_i(x) = f + N(\psi_0) + \sum_{i=0}^{\infty}\left[N\left(\sum_{j=0}^{i} \psi_j(x)\right) - N\left(\sum_{j=0}^{i-1} \psi_j(x)\right)\right]. \tag{17}$$

Here the following recurrence relation is defined

$$\left.\begin{aligned} x_0 &= f, \\ x_1 &= N(\psi_0), \\ x_2 &= N(\psi_0 + \psi_1) - N(\psi_0), \\ x_{n+1} &= N(\psi_0 + \psi_1 + \ldots \psi_n) - N(\psi_0 + \psi_1 + \ldots \psi_{n-1}), \\ n &= 1, 2, 3 \ldots, \end{aligned}\right\} \tag{18}$$

Then, we have

$$\left.\begin{aligned} (\psi_0 + \psi_1 + \ldots \psi_n) = N(\psi_0 + \psi_1 + \ldots \psi_n), n = 1, 2, 3 \ldots, \\ \psi = \sum_{i=0}^{\infty} \psi_i(x) = f + N\left(\sum_{i=0}^{\infty} \psi_i(x)\right) \end{aligned}\right\} \tag{19}$$

3.2.1 Basic road map of NIM

In this context, we will explore a fundamental approach to fractional nonlinear PDEs with fractional order using the

NIM method. To illustrate, let us consider the subsequent fractional order PDE as follows:

$$D_t^a \psi(x,t) = A(\psi.\ \partial\psi) + B(x,t), m-1 < a \leq m, \quad m \in n, \tag{20}$$

$$\frac{\partial^k}{\partial t^k} \psi(x,0) = h_k(x), k = 0,1,2,3 \ldots m-1, \tag{21}$$

The function A is nonlinear and dependent on both ψ and its partial derivative ($\partial\psi$), while B serves as the input function. According to the NIM, the problem of finding initial values for Eqs. (20) and (21) can be reformulated as an integral equation.

$$\psi(x,t) = \sum_{k=0}^{m-1} h_k(x)\frac{t^k}{k!} + I_t^\rho(A) + I_t^\rho(B) = f + N(x), \tag{22}$$

where

$$f = \sum_{k=0}^{m-1} h_k(x)\frac{t^k}{k!} + I_t^\rho(B), \tag{23}$$

$$N(\psi) = I_t^\rho(A). \tag{24}$$

3.3 Numerical problem

3.3.1 Implementation of OAFM

Example 1. Consider the following linear time-fractional KG problem

$$D_t^\rho \quad \psi - \psi_{x,x} - \psi = 0, \tag{25}$$

with IC,

$$\psi(x,0) = 1 + \sin(x), \tag{26}$$

Where both linear and nonlinear terms in Eq. (25) are defined as

$$\left.\begin{aligned} L(\psi) &= \frac{\partial^\rho \psi}{\partial t^\rho}, \\ N(\psi) &= -\psi_{x,x} - \psi, \\ g(x,t) &= 0. \end{aligned}\right\} \tag{27}$$

The initial approximation is obtained according to Eq. (9)

$$\frac{\partial^\rho \psi_0(x,t)}{\partial t^\rho} = 0. \tag{28}$$

Using the inverse operator, we get the following solution:

$$\psi_0(x,t) = 1 + \sin(x). \tag{29}$$

By using Eq. (29) in Eq. (27), the nonlinear terms become

$$N[\psi_0(x,t)] = -1. \tag{30}$$

The first approximation is given in Eq. (11).

$$\frac{\partial^\rho \psi_1(x,t)}{\partial t^\rho} = \nabla_1[\psi_0(x,t)] \quad N[\psi_0(x,t)] + \nabla_2[\psi_0(x,t), C_j], \tag{31}$$

we choose the auxiliary function ∇_1 and ∇_2,

$$\left.\begin{aligned} \nabla_1 &= C_1(1+\sin(x))^1 + C_2(1+\sin(x))^3 \\ \nabla_2 &= C_3(1+\sin(x))^5 + C_4(1+\sin(x))^7 \end{aligned}\right\} \tag{32}$$

Using Eqs. (30)–(32), we get

$$\psi_1(x,t) = \frac{1}{a\Gamma[a]}(t^a(c1(1+\sin[x]) + c2(1+\sin[x])^3 - c3(1+\sin[x])^5 - c4(1+\sin[x])^7)). \tag{33}$$

Adding Eqs. (29) and (33), we obtain OAFM solution

$$\psi(x,t) = 1 + \sin(x) + \frac{1}{a\Gamma[a]}(t^a(c1(1+\sin[x]) + c2(1+\sin[x])^3 - c3(1+\sin[x])^5 - c4(1+\sin[x])^7)). \tag{34}$$

Also, the exact solution for this problem reads

$$\psi(x,t) = 1 + \sin(x) \quad t. \tag{35}$$

Example 2. Consider the following time-fractional nonlinear KG problem

$$D_t^\rho \quad \psi - \psi_{x,x} + \psi^2 = 0, \tag{36}$$

with IC,

$$\psi(x,0) = 1 + \sin(x), \tag{37}$$

whereas both linear and nonlinear terms read

$$\left.\begin{aligned} L(\psi) &= \frac{\partial^\rho \psi}{\partial t^\rho}, \\ N(\psi) &= \psi^2, \\ g(x,t) &= 0. \end{aligned}\right\} \tag{38}$$

The initial approximation is obtained from Eq. (9) as follows:

$$\frac{\partial^\rho \psi_0(x,t)}{\partial t^\rho} = 0. \tag{39}$$

Using the inverse operator, we get the following solution:

$$\psi_0(x,t) = 1 + \sin(x). \tag{40}$$

From Eqs. (40) and (38), the nonlinear terms become

$$N[\psi_0(x,t)] = (1+\sin(x))^2. \tag{41}$$

The first approximation is given in Eq. (11).

$$\frac{\partial^\rho \psi_1(x,t)}{\partial t^\rho} = \nabla_1[\psi_0(x,t)]N[\psi_0(x,t)] + \nabla_2[\psi_0(x,t), C_j], \tag{42}$$

we choose the auxiliary function ∇_1 and ∇_2,

$$\left.\begin{aligned} \nabla_1 &= C_1 + C_2(1+\sin(x))^4, \\ \nabla_2 &= C_3(1+\sin(x))^5 + C_4(1+\sin(x))^7 \end{aligned}\right\} \tag{43}$$

Using Eqs. (41)–(43), we get

$$\begin{aligned} \psi_1(x,t) = {} & \frac{1}{a\Gamma[a]}(t^a(-c3(1+\sin[x])^5 \\ & - c4(1+\sin[x])^7 - (1+\sin[x])^2(c1 \\ & + c2(1+\sin[x])^4))). \end{aligned} \tag{44}$$

Adding Eqs. (40) and (44), we obtained OAFM solution.

$$\begin{aligned} \psi(x,t) = 1 + \sin(x) + {} & \frac{1}{a\Gamma[a]}(t^a(-c3(1+\sin[x])^5 \\ & - c4(1+\sin[x])^7 - (1+\sin[x])^2(c1 \\ & + c2(1+\sin[x])^4))). \end{aligned} \tag{45}$$

Example 3. Consider the following time-fractional nonlinear KG problem

$$D_t^\rho \psi - \psi_{x,x} + \psi - \psi^3 = 0, \tag{46}$$

with IC,

$$\psi(x,0) = -\text{sech}(x), \tag{47}$$

whereas the linear and nonlinear terms read

$$\left.\begin{aligned} L(\psi) &= \frac{\partial^\rho \psi}{\partial t^\rho}, \\ N(\psi) &= -\psi^3, \\ g(x,t) &= 0. \end{aligned}\right\} \tag{48}$$

The initial approximation is obtained from Eq. (9).

$$\frac{\partial^\rho \psi_0(x,t)}{\partial t^\rho} = 0. \tag{49}$$

Using the inverse operator, we get the following solution:

$$\psi_0(x,t) = -\text{sech}(x). \tag{50}$$

Inserting Eq. (50) in Eq. (48), the nonlinear terms become

$$N[\psi_0(x,t)] = (\text{sech}\,(x))^3. \tag{51}$$

The first approximation is given in Eq. (11).

$$\begin{aligned} \frac{\partial^\rho \psi_1(x,t)}{\partial t^\rho} = {} & \nabla_1[\psi_0(x,t)]N[\psi_0(x,t)] \\ & + \nabla_2[\psi_0(x,t), C_j], \end{aligned} \tag{52}$$

we choose the auxiliary function ∇_1 and ∇_2,

$$\left.\begin{aligned} \nabla_1 &= C_1(-\text{sech}\,(x))^1 + C_2(-\text{sech}\,(x))^3, \\ \nabla_2 &= C_3(-\text{sech}\,(x))^5 + C_4(-\text{sech}\,(x))^7. \end{aligned}\right\} \tag{53}$$

Using Eqs. (51)–(53), we get

$$\begin{aligned} \psi_1(x,t) = {} & \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^4(c1 + c3\,\text{sech}[x] \\ & + c2\,\text{sech}[x]^2 + c4\,\text{sech}[x]^3)). \end{aligned} \tag{54}$$

Adding Eqs. (40) and (44), we obtain the OAFM solution.

$$\begin{aligned} \psi(x,t) = -\text{sech}\,(x) + {} & \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^4(c1 \\ & + c3\,\text{sech}[x] + c2\,\text{sech}[x]^2 + c4\,\text{sech}[x]^3)). \end{aligned} \tag{55}$$

The values of C_1, C_2, C_3 and C_4 is define in Table 1.

3.3.2 Implementation of NIM

Applying Riemann–Liouville integral to example (1) (*i.e.*, Eqs. (25) and (26)), we get

$$\psi(x,t) = 1 + \sin(x) + J_t^\rho[-\psi_{x,x} + \psi]. \tag{56}$$

From NIM algorithm, the zeroth-order problem $\psi(x,t)$ reads

$$\psi_0(x,t) = 1 + \sin(x). \tag{57}$$

The first-order component of the solution reads

$$\psi_1(x,t) = \frac{1}{a\Gamma[a]}(t^a). \tag{58}$$

The second-order component of the solution reads

$$\psi_2(x,t) = -\frac{1}{\Gamma[1+2a]}(t^{2a}). \tag{59}$$

Therefore, three terms approximate solution $\psi(x,t)$ reads

$$\psi(x,t) = \left(1 + \sin(x) + \frac{1}{a\Gamma[a]}(t^a) + -\frac{1}{\Gamma[1+2a]}(t^{2a})\right). \tag{60}$$

Applying Riemann–Liouville integral to example (2) (*i.e.*, Eqs. (36) and (37)), we get

$$\psi(x,t) = 1 + \sin(x) + J_t^\rho[\psi_{x,x} - \psi^2]. \tag{61}$$

From NIM algorithm, the zeroth-order problem $\psi(x,t)$ reads

$$\psi_0(x,t) = 1 + \sin(x). \tag{62}$$

The first-order component of solution reads

$$\psi_1(x,t) = -\frac{t^a}{a\Gamma[a]}(1 + 3\sin[x] + \sin[x]^2). \tag{63}$$

The second-order component of solution reads

$$\psi_2(x,t) = \frac{1}{a\Gamma[a]}\Bigg(t^a(1+\sin[x])^2 + \frac{t^a}{6\Gamma[a]}\Bigg(-\frac{6(\cos[\frac{x}{2}] + \sin[\frac{x}{2}])^4}{a} - \frac{t^{2a}\Gamma[2a](-3+\cos[2x]-6\sin[x])^2}{a^2\Gamma[a]\Gamma[3a]} - \frac{3t^a\Gamma[a](-12+12\cos[2x]-25\sin[x]+\sin[3x])}{\Gamma[1+2a]}\Bigg). \tag{64}$$

Therefore, three terms approximate solution $\psi(x,t)$ reads

$$\psi(x,t) = 1 + \sin(x) - \frac{t^a}{a\Gamma[a]}(1 + 3\sin[x] + \sin[x]^2) + \frac{1}{a\Gamma[a]} + \Bigg(t^a(1+\sin[x])^2 + \frac{1}{6\Gamma[a]}\Bigg(t^a\Bigg(-\frac{6(\cos[\frac{x}{2}] + \sin[\frac{x}{2}])^4}{a} - \frac{t^{2a}\Gamma[2a](-3+\cos[2x]-6\sin[x])^2}{a^2\Gamma[a]\Gamma[3a]} - \frac{3t^a\Gamma[a](-12+12\cos[2x]-25\sin[x]+\sin[3x])}{\Gamma[1+2a]}\Bigg)\Bigg)\Bigg). \tag{65}$$

Applying Riemann–Liouville integral to example (3) (*i.e.*, Eqs. (46) and (47)), we get

$$\psi(x,t) = -\text{sech}(x) + J_t^{\rho}[\psi_{x,x} - \psi + \psi^3]. \tag{66}$$

From NIM algorithm, the zeroth-order problem $\psi(x,t)$ reads

$$\psi_0(x,t) = -\text{sech}\,(x) \tag{67}$$

The first-order component of solution reads

$$\psi_1(x,t) = \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^3). \tag{68}$$

The second-order component of solution reads

$$\psi_2(x,t) = \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^3) + \frac{t^a\,\text{sech}[x]^3}{a^3\Gamma[a]}\Bigg(-a^2 - \frac{t^{2a}\Gamma[1+2a]\,\text{sech}[x]^4}{\Gamma[a]\Gamma[3a]} + \frac{t^{3a}\Gamma[1+3a]\,\text{sech}[x]^6}{\Gamma[a]^2\Gamma[1+4a]} + \frac{4^{-a}a^2\sqrt{\pi}\,t^a(-1+9\text{Tanh}[x]^2)}{\Gamma[\frac{1}{2}+a]}\Bigg). \tag{69}$$

Therefore, three terms approximate solution $\psi(x,t)$ reads

$$\psi(x,t) = -\text{sech}(x) + \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^3) + \frac{1}{a\Gamma[a]}(t^a\,\text{sech}[x]^3) + \frac{t^a\,\text{sech}[x]^3}{a^3\Gamma[a]}\Bigg(-a^2 - \frac{t^{2a}\Gamma[1+2a]\,\text{sech}[x]^4}{\Gamma[a]\Gamma[3a]} + \frac{t^{3a}\Gamma[1+3a]\,\text{sech}[x]^6}{\Gamma[a]^2\Gamma[1+4a]} + \frac{4^{-a}a^2\sqrt{\pi}\,t^a(-1+9\text{Tanh}[x]^2)}{\Gamma\left[\frac{1}{2}+a\right]}\Bigg). \tag{70}$$

4 Numerical results and discussion

In this work, we derived some approximations to the FOKGE using both OAFM and NIM approaches. The obtained results are compared with each other and with the exact solutions. It is found that there is a good agreement with all the obtained approximations. Tables 1–3

Table 1: Convergence control parameter values obtained by the collocation method

C_1	C_2	C_3	C_4
1.3893225828	−0.538801623	−0.141206521	0.0166159469

Table 2: Convergence-control parameter values obtained by collocation method

C_1	C_2	C_3	C_4
0.9808794933	−0.152779348	0.1245001438	0.0442022463

Table 3: Convergence control parameter values obtained by collocation method

C_1	C_2	C_3	C_4
7.3254640962	10.241536004	−13.85392749	−2.758428144

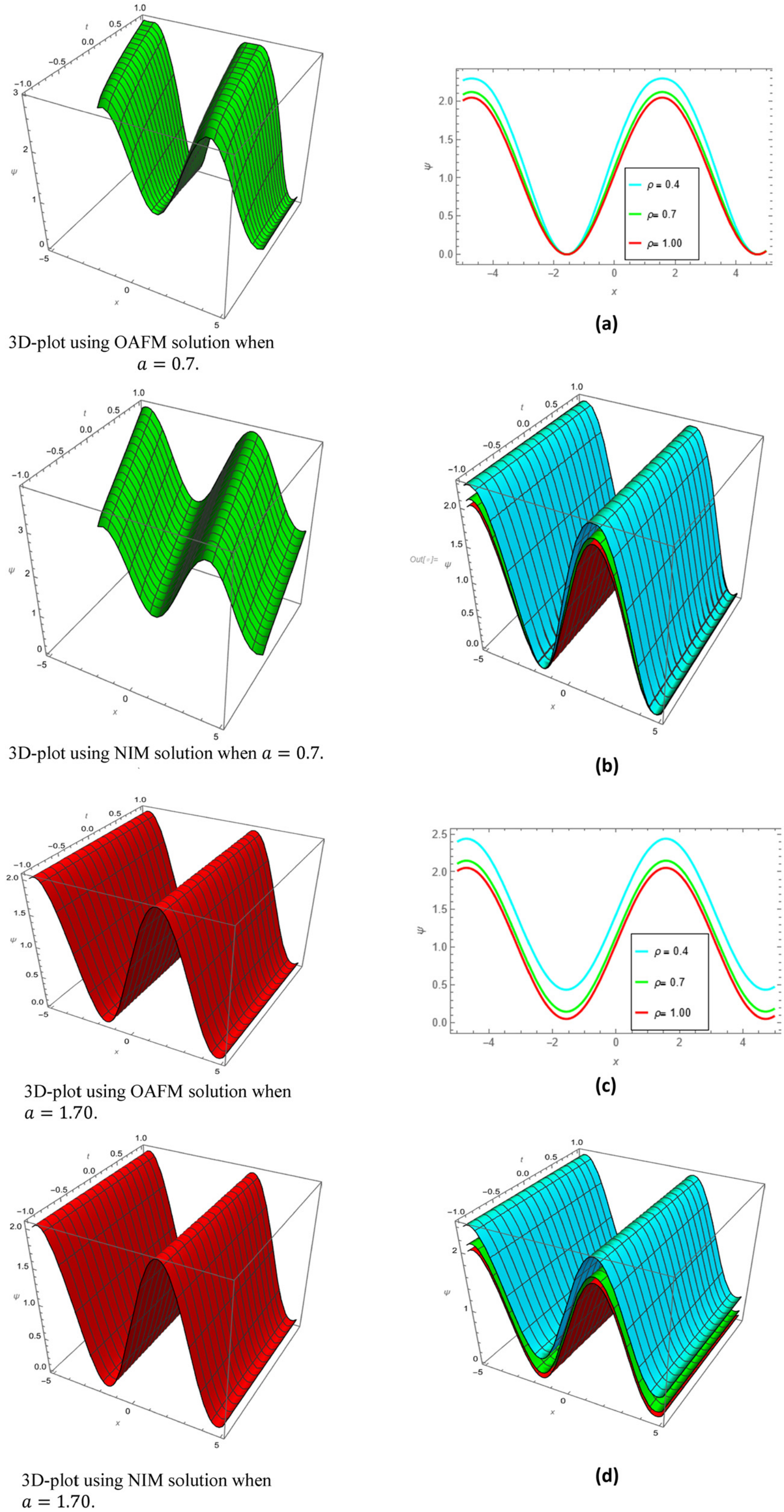

Figure 1: (a) 2D periodic approximation (34), (b) 3D periodic approximation (34) using OAFM technique, (c) 2D periodic approximation (62), and (d) 3D periodic approximation (62) using NIM technique.

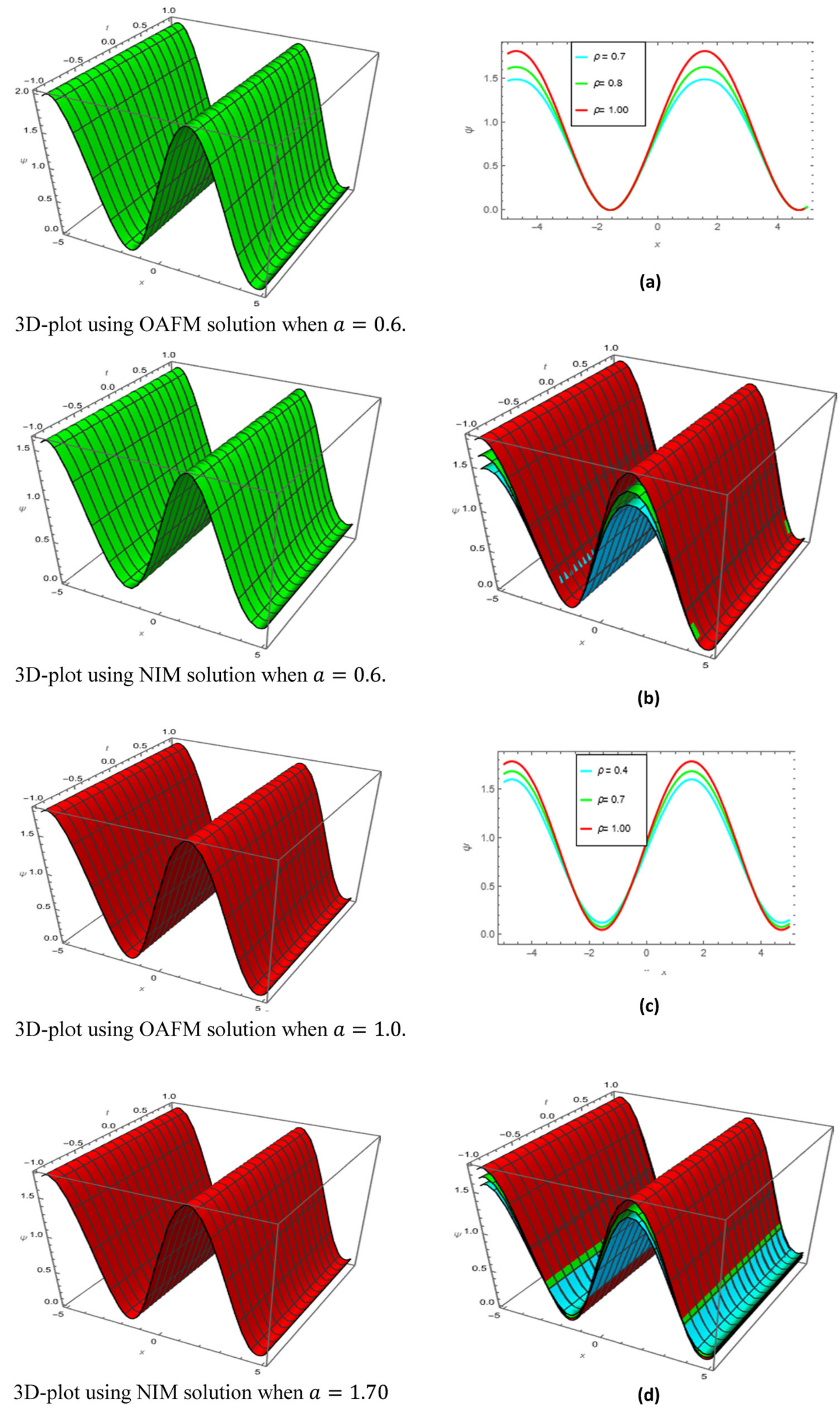

Figure 2: (a) 2D periodic approximation (45), (b) 3D periodic approximation (45) using OAFM technique, (c) 2D periodic approximation (69), and (d) 3D periodic approximation (69) using NIM technique.

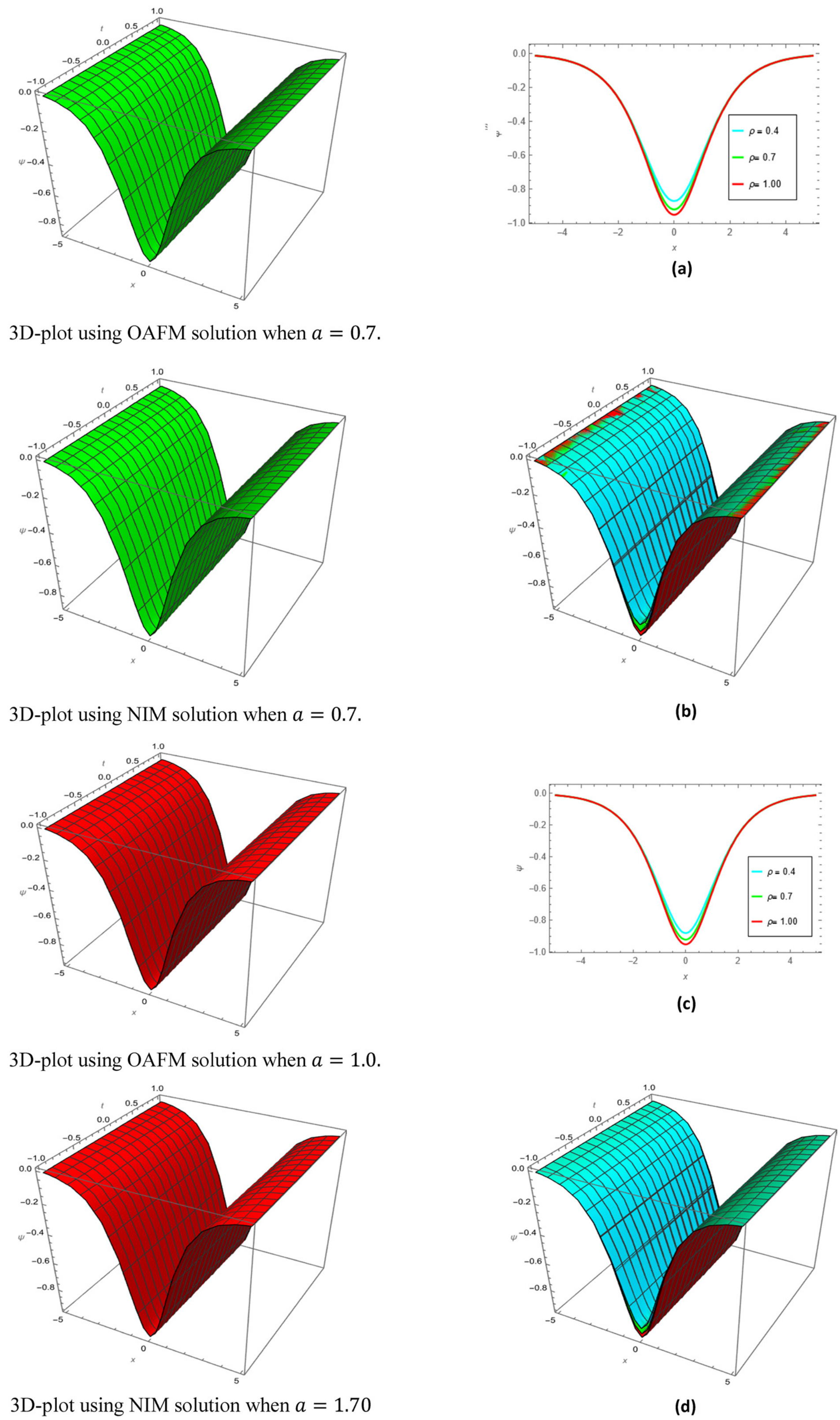

Figure 3: (a) 2D localized approximation (negative soliton) (55), (b) 3D localized approximation (negative soliton) (55) using OAFM technique, (c) 2D localized approximation (negative soliton) (76), and (d) 3D localized approximation (negative soliton) (76) using NIM technique.

Table 4: Comparison between the approximations of example (1) or (4) using OAFM and NIM

x	OAFM (α = 1)	NIM (α = 1)
0	1.00878	1.00904
0.1	1.10889	1.10887
0.2	1.20792	1.20771
0.3	1.3049	1.30456
0.4	1.39887	1.39846
0.5	1.48891	1.48847
0.6	1.57412	1.57368
0.7	1.65364	1.65326
0.8	1.72666	1.7264
0.9	1.79247	1.79237
1	1.00878	1.00904

Table 5: Comparison between the approximations of example (2) or (5) using OAFM and NIM

x	OAFM (α = 1)	NIM (α = 1)
0	0.991029	0.991
0.1	1.08901	1.0881
0.2	1.18588	1.18406
0.3	1.28068	1.27793
0.4	1.37249	1.36879
0.5	1.46041	1.45574
0.6	1.54357	1.53795
0.7	1.62113	1.6146
0.8	1.69233	1.68495
0.9	1.75645	1.74834
1	0.991029	0.991

Table 6: Comparison between the approximations of example (3) or (6) using OAFM and NIM

x	OAFM (α = 1)	NIM (α = 1)
0	−0.99141	−0.99104
0.1	−0.98653	−0.98619
0.2	−0.97212	−0.97187
0.3	−0.94887	−0.94876
0.4	−0.91782	−0.91788
0.5	−0.88031	−0.88052
0.6	−0.83776	−0.83811
0.7	−0.79165	−0.79211
0.8	−0.74337	−0.74389
0.9	−0.69415	−0.69469
1	−0.99141	−0.99104

represent numerical values for auxiliary constant using collocation method. Both Figures 1–3 and Tables 4–6 indicate the comparison between the obtained approximations using both OAFM and NIM techniques for examples 1–6. Both approximation (34) using OAFM and the approximation (62) using NIM for example (1) or (4) are illustrated in Figures 1(a and b) and 1(c–d), respectively, against the fractional-order ρ. It is clear that the oscillation amplitude decreases with the increase in ρ.

Figures 2(a and b) and 2(c and d), respectively, demonstrate both approximation (45) using OAFM and the approximation (69) using NIM for example (2) or (5), against the fractional-order ρ.

Both approximation (55) using OAFM and the approximation (76) using NIM for example (3) or (6) are, respectively, introduced in Figures 3(a and b) and 3(c and d) and 6 at different values of the fractional-order ρ. It is seen in this example that the magnitude of oscillation amplitude increases with the enhancement of ρ.

5 Conclusion

In this work, the OAFM and NIM have been carried out for analyzing and solving the FOKGE. The comparative study demonstrated that both mentioned methods produce results in excellent agreement. The obtained results provided accurate and reliable solutions to the FOKGE. NIM and OAFM have proven to be effective and efficient methods for solving this equation. Based on the obtained results, we expect that these results will serve many researchers who are interested in studying nonlinear phenomena in plasma physics and optical fibers.

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