Research Article

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Conserved vectors and solutions of the two-dimensional potential KP equation

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Abstract: This article investigates the potential Kadomtsev–Petviashvili (pKP) equation, which describes the evolution of small-amplitude nonlinear long waves with slow transverse coordinate dependence. For the first time, we employ Lie symmetry methods to calculate the Lie point symmetries of the equation, which are then utilized to derive exact solutions through symmetry reductions and with the help of Kudryashov's method. The solutions obtained include exponential, hyperbolic, elliptic, and rational functions. Furthermore, we provide one-parameter group of transformations for the pKP equation. To gain a better understanding of the nature of each solution, we present 3D, 2D, and density plots. These obtained solutions, along with their associated physical characteristics, offer valuable insights into the propagation of small yet finite amplitude waves in shallow water.In addition, the pKP equation conserved vectors are derived by utilizing the multiplier method and the theorems by Noether and Ibragimov.

Keywords: potential Kadomtsev–Petviashvili equation, Lie symmetry methods, Kudryashov's method, conservation law, Noether's theorem, Ibragimov's theorem, multiplier method

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1 Introduction

Nonlinear partial differential equations (NLPDEs) are instrumental in the modeling of a wide range of nonlinear higher-dimensional systems that reflect various natural phenomena. Researchers have continuously studied NLPDEs in recent years, as they are essential in understanding the complicated behavior of such systems. The significance of NLPDEs in our contemporary world has been well established in literature [1–12]. It is, therefore, imperative for scientists and researchers to solve NLPDEs and obtain their exact solutions, as it provides insights into the mechanisms of the phenomena being investigated. Despite the importance of obtaining explicit solutions for NLPDEs, to date, there has been no general method for their determination. However, scientists have developed different special methods such as wavefunction ansatz technique [13], tanh–coth technique [14], extended homoclinic-test approach [15], homotopy perturbation approach [16], rational expansion method [17], Lie point symmetry analysis [18,19], bifurcation approach [20], exponential function technique [21], Kudryashov's technique [22], tanh-function method [23], Painlevé expansion approach [24], Weierstrass elliptic function method [25], tan–cot method [26], extended simplest equation [27], and many more.

Sophus Lie, a prominent mathematician from Norway who lived between 1842 and 1899, is credited with introducing Lie symmetry analysis, which has proven to be a valuable method for deriving closed-form solutions to problems described by differential equations (DEs) in fields like applied mathematics, biology, physics, engineering, and other related areas [28,29]. Sophus Lie was motivated to develop his mathematical work by the achievements of Abel and Galois in the realm of algebraic equations. Specifically, Lie utilized similar mathematical tools to advance the theory of continuous groups, which in turn has proven to be applicable to the study of DEs [30,31].

Conservation laws are of significant importance in the analysis of DEs. One practical application of this is the assessment of the integrability of a partial differential equation (PDE) through an examination of its conservation

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laws [19,29]. Furthermore, conservation laws serve as a tool for assessing the precision of numerical solution techniques, help identify special solutions that have important physical properties and can be used to reduce the order of DEs in a problem, making it easier to solve [28]. Thus, this is the reason researchers find it useful to determine the conservation laws for a given DE. When DEs are derived from variational principles, conservation laws can be determined by invoking Noether's theorem [32], which involves the symmetries admitted by the DE. However, for DEs that do not arise from variational principles, scientists have come up with various techniques to determine conservation laws. For instance, Ibragimov's theorem [33], the general multiplier method [19], and the partial Lagrangian method [34] are some of the techniques used in this regard. The Kadomtsev–Petviashvili (KP) equation was from the study by the two Soviet physicists, Kadomtsev and Petviashvili [35], in (2+1)-dimensional form

$$u_{tx} + 6u_xu_{xx} + \gamma u_{yy} + u_{xxxx} = 0,$$

where $\gamma = \pm 1$. The KP equation has many applications in fields like adaptive optics, plasma physics, phase imaging, and nonlinear mechanics, hence various communities of researchers have employed a variety of effective methods to derive its closed-form solutions, see, *e.g.*, [36–40]. From literature, we see that various forms of KP equation have been considered.

Gupta and Bansal [41] in their work investigated the 2-D variable coefficient potential KP (vcpKP) equation in the form

$$u_{tx} + \omega(t)u_{xxxx} + \mu(t)u_xu_{xx} + \phi(t)u_{yy} = 0,$$

where, $\omega(t)$, $\mu(t)$, and $\phi(t)$ are arbitrary functions of t. The vcpKP was reduced to a lesser-dimensional PDE using Lie group methods, and its solutions were derived through application of the extended (G'/G)-expansion method.

Moreover, Wazwaz [42] examined the 3-D KP equation

$$u_{ty} + u_{tx} + 3(u_xu_y)_x + u_{xxxy} - u_{zz} = 0,$$

where several soliton solutions were obtained using the simplified Hirota's technique. Iqbal and Naeem [43] studied the fourth-order nonlinear generalized KP equation

$$(au_t + \beta(m(u)^a)_x + \gamma(n(u)^b)_{xxx})_x + \sigma u_{yy} = 0,$$

whereby for various choices of $a, b, m(u)$ and $n(u)$, the equation was transformed into several forms of KP-like equations. Using the multiplier method, they obtained conservation laws for unknown functions $m(u)$ and $n(u)$. The obtained conservation laws were then used to construct conservation laws for certain variants of KP-like equations by choosing values of $m(u)$ and $n(u)$. Moreover, implicit and explicit closed-form solutions were obtained for the various KP-like equations through the utilization of the derived conservation laws. Akinyemi and Morazara [44] conducted an in-depth investigation on the extended KP (EKP) equation

$$(E_t - 6EE_x + E_{xxx})_x + \alpha E_{yy} + \gamma E_{tt} + \beta E_{ty} = 0,$$

where α, γ, and β are nonzero constants. They began by validating the integrability of the equation through Painlevé analysis using the WTC-Kruskal algorithm. This analysis confirmed that the EKP equation satisfies the compatibility criteria for integrability. Next, the researchers employed various ansatz functions based on bilinear formalism and symbolic computation to determine analytical solutions for the EKP equation.

Kumar *et al.* [45] conducted a comprehensive study on two novel variable coefficients KP equations in (2+1)-dimensions,

$$(u_t + uu_x + u_{xxx})_x + g(t)u_{xy} = 0,$$
$$(u_t + uu_x + u_{xxx})_x + h(t)u_{xy} = 0,$$

where $g(t)$ and $h(t)$ are functions of variable t. By employing the Lie symmetry technique, they successfully obtained closed-form analytic solutions that exhibit various complex wave structures, including solitons with distinct shapes, both dark and bright soliton shapes, double W-shaped soliton shapes, multi-peakon shapes, curved-shaped multi-wave solitons, and solitary wave solitons.

Ma *et al.* [46] studied the fourth-order NLPDE v

$$\omega\{3(u_xu_t)_x + u_{xxxt}\} + \mu\{3(u_xu_y)_x + u_{xxxy}\} + u_{xxxx} + 6u_xu_{xx} + \gamma_1u_{yt} + \gamma_2u_{xx} + \gamma_3u_{xt} + \gamma_4u_{xy} + \gamma_5u_{yy} + \gamma_6u_{tt} = 0,$$

which possesses diverse lump solutions. For the above equation, when $\omega = \mu = 0$, $\gamma_3 = -\gamma_5 = 1$, and $\gamma_1 = \gamma_2 = \gamma_4 = \gamma_6 = 0$, the potential KP (pKP) equation in (2+1)-dimensional form is obtained, *e.g.*,

$$u_{tx} + 6u_xu_{xx} + u_{xxxx} - u_{yy} = 0. \quad (1.1)$$

The aforementioned pKP equation (1.1) taken from the study by Ma *et al.* [46] is an NLPDE that explains the evolution of nonlinear long waves of small amplitude with slow transverse coordinate dependence [47–49]. In the existing literature, Eq. (1.1) has been mentioned by many authors as a special case of a combined pKp and B-type KP (BKP) equation, see, *e.g.*, [50–53]. For the first time, we derive exact solutions of Eq. (1.1) using symmetry reductions along with the help of Kudryashov's method and construct its conserved vectors.

In this work, we study the pKP equation (1.1). First, we construct exact solutions of Eq. (1.1) by utilizing Lie symmetry analysis along with Kudryashov's method. The corresponding one-parameter group of transformations are also obtained, and utilizing these transformations, new solutions are presented when a solution is given. Furthermore, conserved vectors of Eq. (1.1) are derived using three approaches: Noether, Ibragimov, and multiplier methods.

2 Exact solutions of the pKP equation

We begin by deriving infinitesimal generators of the pKP equation (1.1), which are vector fields that leave the equation invariant. We then utilize them to construct group-invariant solutions.

2.1 Lie point symmetries of the pKP equation

The infinitesimal generators admitted by the pKP equation (1.1) are determined by

$$\mathcal{H} = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \psi(t,x,y,u)\partial_y + \eta(t,x,y,u)\partial_u \tag{2.1}$$

if and only if

$$\mathcal{H}^{[4]}(u_{tx} + 6u_x u_{xx} + u_{xxxx} - u_{yy})|_{(1.1)} = 0. \tag{2.2}$$

Here, $\mathcal{H}^{[4]}$ is the fourth prolongation defined by

$$\mathcal{H}^{[4]} = \zeta_x\partial_x + \zeta_{xx}\partial u_{xx} + \zeta_{tx}\partial u_{tx} + \zeta_{yy}\partial u_{yy} + \zeta_{xxxx}\partial u_{xxxx},$$

where ζ_x, ζ_{xx}, ζ_{tx}, ζ_{yy}, and ζ_{xxxx} can be obtained using the prolongation formulas [19] given by

$$\begin{aligned}
\zeta_x &= D_x(\eta) - u_t D_x(\xi) - u_x D_x(\tau) - u_y D_x(\psi),\\
\zeta_y &= D_y(\eta) - u_t D_y(\xi) - u_x D_y(\tau) - u_y D_y(\psi),\\
\zeta_{tx} &= D_x(\zeta_t) - u_{tt} D_x(\xi) - u_{tx} D_x(\tau) - u_{ty} D_x(\psi),\\
\zeta_{yy} &= D_y(\zeta_y) - u_{xy} D_y(\xi) - u_{ty} D_y(\tau) - u_{yy} D_y(\psi),\\
\zeta_{xx} &= D_x(\zeta_x) - u_{tx} D_x(\xi) - u_{xx} D_x(\tau) - u_{xy} D_x(\psi),\\
\zeta_{xxx} &= D_x(\zeta_{xx}) - u_{txx} D_x(\xi) - u_{xxx} D_x(\tau) - u_{xxy} D_z(\psi),\\
\zeta_{xxxx} &= D_x(\zeta_{xxx}) - u_{xxxx} D_x(\xi) - u_{txxx} D_x(\tau) - u_{xxxy} D_x(\psi)
\end{aligned} \tag{2.3}$$

with the total differential operators defined as follows:

$$\begin{aligned}
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial u_t + u_{tx}\partial u_x + u_{ty}\partial u_y + \cdots,\\
D_x &= \partial_x + u_x\partial_u + u_{xx}\partial u_x + u_{tx}\partial u_t + u_{xy}\partial u_y + \cdots,\\
D_y &= \partial_y + u_y\partial_u + u_{yy}\partial u_y + u_{ty}\partial u_t + u_{xy}\partial u_y + \cdots.
\end{aligned} \tag{2.4}$$

By expanding the determining Eq. (2.2) and distributing it among the different derivatives of u, it leads to the linear homogeneous PDEs as follows:

$$\begin{aligned}
&\tau_x = 0, \tau_y = 0, \tau_u = 0, \xi_u = 0, \psi_u = 0, \psi_x = 0, 3\eta_u + \tau_t = 0,\\
&\quad 2\xi_y - \psi_t = 0,\\
&6\eta_x - \xi_t = 0, 3\xi_x - \tau_t = 0, 3\psi_y - 2\tau_t = 0, \psi_u = 0, \psi_v = 0,\\
&\quad \psi_x = 0,\\
&3\tau_t - 5\psi_y = 0, 6\eta_{yy} - \xi_{tt} = 0,
\end{aligned}$$

which upon solving yield the values of the infinitesimals τ, ξ, ψ, and η as follows:

$$\begin{aligned}
\tau &= a(t),\\
\xi &= \frac{1}{3}xa'(t) + \frac{1}{6}y^2a'''(t) + \frac{1}{2}f'(t) + g(t),\\
\psi &= \frac{2}{3}ya'(t) + f(t),\\
\eta &= \frac{1}{36}x^2a''(t) - \frac{1}{3}ua'(t) + \frac{1}{36}xy^2a'''(t) + \frac{1}{6}xg'(t)\\
&\quad + \frac{1}{432}y^4a''''(t) + \frac{1}{72}y^2f'''(t) + \frac{1}{12}g''(t) + yi(t) + j(t),
\end{aligned}$$

and hence the pKP equation (1.1) possesses the following five Lie point symmetries:

$$\mathcal{H}_1 = j(t)\frac{\partial}{\partial u}, \tag{2.5}$$

$$\mathcal{H}_2 = yi(t)\frac{\partial}{\partial u}, \tag{2.6}$$

$$\mathcal{H}_3 = 36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y} + (6xyf''(t) + y^3f'''(t))\frac{\partial}{\partial u}, \tag{2.7}$$

$$\mathcal{H}_4 = 12g(t)\frac{\partial}{\partial x} + (2xg'(t) + y^2g''(t))\frac{\partial}{\partial u}, \tag{2.8}$$

$$\begin{aligned}
\mathcal{H}_5 &= 432a(t)\frac{\partial}{\partial t} + (144xa'(t) + 72y^2a''(t))\frac{\partial}{\partial x}\\
&\quad + 288ya'(t)\frac{\partial}{\partial y} + (12x^2a''(t) - 144ua'(t)\\
&\quad + 12xy^2a'''(t) + y^4a''''(t))\frac{\partial}{\partial u}.
\end{aligned} \tag{2.9}$$

Applying the Lie equations

$$\begin{aligned}
\frac{d\tilde{t}}{da} &= \tau(\tilde{t},\tilde{x},\tilde{y},\tilde{u}),\ \tilde{t}\,|_{a=0} = t,\\
\frac{d\tilde{x}}{da} &= \xi(\tilde{t},\tilde{x},\tilde{y},\tilde{u}),\ \tilde{x}\,|_{a=0} = x,\\
\frac{d\tilde{y}}{da} &= \psi(\tilde{t},\tilde{x},\tilde{y},\tilde{u}),\ \tilde{y}\,|_{a=0} = y,\\
\frac{d\tilde{u}}{da} &= \eta(\tilde{t},\tilde{x},\tilde{y},\tilde{u}),\ \tilde{u}\,|_{a=0} = u,
\end{aligned}$$

we earn the following group of transformations:

$$
\begin{aligned}
&G_{\alpha_1}: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) \longrightarrow (t, x, y, u + aj(t)),\\
&G_{\alpha_2}: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) \longrightarrow (t, x, y, u + ayi(t)),\\
&G_{\alpha_3}: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) \longrightarrow (t, x + 36\alpha_3 yf'(t) + 1{,}296\alpha_3^2 f(t)f'(t), u\\
&\quad + 6\alpha_3 xyf''(t) + 216\alpha_3^2 xf(t)f''(t)\\
&\quad + 108\alpha_3^2 y^2 f'(t)f''(t) + 7{,}776\alpha_3^3 yf(t)f'(t)f''(t)\\
&\quad + 139{,}968\alpha_3^4 f^2(t)f'(t)f''(t) + \alpha_3 y^3 f'''(t)\\
&\quad + 108\alpha_3^2 y^2 f(t)f'''(t) + 5{,}184\alpha_3^3 f(t)f'''(t)\\
&\quad + 93{,}312\alpha_3^4 f^3(t)f'''(t)),\\
&G_{\alpha_4}: (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) \longrightarrow (t, x + 12\alpha_4 g(t), y, u + 2\alpha_4 g'(t)\\
&\quad + 12\alpha_4^2 g(t)g'(t) + \alpha_4 y^2 g''(t)).
\end{aligned}
$$

Using the aforementioned groups, we state the following theorem, which provides new solutions from the known ones:

Theorem 2.1. *If $u = \omega(t, x, y)$ is a solution of the pKP Eq.* (1.1) *then the functions*

$$
\begin{aligned}
u_1 &= \omega_1(t, x, y) + aj(t),\\
u_2 &= \omega_2(t, x, y) + ayi(t),\\
u_3 &= \omega_3(t, x - 36ayf'(t) + 1{,}296a^2 f(t)f'(t), y - 72af(t))\\
&\quad + 6axyf''(t) - 216a^2 xf(t)f'(t) - 108a^2 y^2 f'(t)f''(t)\\
&\quad + 7{,}776a^3 yf(t)f'(t)f''(t) - 139{,}968a^4 f^2(t)f'(t)f''(t)\\
&\quad + ay^3 f'''(t) - 108a^2 y^2 f(t)f'''(t) + 5{,}184a^3 yf^2(t)f'''(t)\\
&\quad - 93{,}312a^4 f^3(t)f'''(t),\\
u_4 &= \omega_4(t, x - 12ag(t), y) + 2axg'(t) - 12a^2 g(t)g'(t)\\
&\quad + ay^2 g''(t),
\end{aligned}
$$

are also solutions of the pKP equation (1.1).

2.2 Constructing group-invariant solutions of Eq. (1.1)

We derive multiple group-invariant solutions of Eq. (1.1) in this section by performing symmetry reductions *via* the characteristic equations.

Case 1. We consider $\mathcal{H}_3 = 36yf'(t)\partial/\partial x + 72f(t)\partial/\partial y + (6xyf''(t) + y^3 f'''(t))\partial/\partial u$. Solving the characteristic equations associated with $\mathcal{H}_3$ leads to the following invariants:

$$
\begin{aligned}
J_1 &= t, \quad J_2 = \frac{y^2}{2} - \frac{2xf(t)}{f'(t)},\\
J_3 &= u - \frac{3x^2 f''(t)}{36f'(t)} + \frac{2x^2 f(t)f'''(t)}{36f'^2(t)} - \frac{xy^2 f'''(t)}{36f'(t)}.
\end{aligned}
$$

The above invariants imply that

$$
\begin{aligned}
u &= \Phi(t, \varphi) + \frac{3x^2 f''(t)}{36f'(t)} - \frac{2x^2 f(t)f'''(t)}{36f'^2(t)}\\
&\quad + \frac{xy^2 f'''(t)}{36f'(t)}, \quad \varphi = \frac{y^2}{2} - \frac{2xf(t)}{f'(t)},
\end{aligned}
$$

where Φ is an arbitrary function of t and φ. Substituting the value of u into the pKP equation (1.1), we obtain the NLPDE

$$
\begin{aligned}
&864f^4(t)\Phi_{\varphi\varphi\varphi\varphi} - 2\varphi f(t)f'(t)f'''(t) - 108f(t)f'^3(t)\Phi_{t\varphi}\\
&\quad -2{,}592f^3(t)f'(t)\Phi_{\varphi\varphi}\Phi_\varphi + 72f^2(t)f'(t)f'''(t)\Phi_\varphi\\
&\quad + 72\varphi f^2(t)f'''(t)\Phi_\varphi + 3\varphi f'^3(t)f''''(t)\\
&\quad - 162f'^4(t)\Phi_\varphi - 108\varphi f'^4(t)\Phi_{\varphi\varphi} = 0.
\end{aligned} \tag{2.10}
$$

As a result, the similarity solution of Eq. (1.1) is

$$
u = \Phi(t, \varphi) + \frac{3x^2 f''(t)}{36f'(t)} - \frac{2x^2 f(t)f'''(t)}{36f'^2(t)} + \frac{xy^2 f'''(t)}{36f'(t)},
$$

where Φ represents any solution of the NLPDE (2.10).

Particular case $f(t) = t$. We consider the particular case $f(t) = t$, which transforms the NLPDE (2.10) into

$$
\begin{aligned}
&16t^4\Phi_{\varphi\varphi\varphi\varphi} - 2t\Phi_{t\varphi} - 48t^3\Phi_{\varphi\varphi}\Phi_\varphi - 3\Phi_\varphi - 2\varphi\Phi_{\varphi\varphi} = 0,\\
&\varphi = \frac{y^2}{2} - 2tx.
\end{aligned} \tag{2.11}
$$

The NLPDE (2.11) has the following five symmetries:

$$
\begin{aligned}
\Gamma_1 &= t\frac{\partial}{\partial\varphi}, \quad \Gamma_2 = h(t)\frac{\partial}{\partial\Phi},\\
\Gamma_3 &= 3t\frac{\partial}{\partial t} + 4\varphi\frac{\partial}{\partial\varphi} - \phi\frac{\partial}{\partial\phi},\\
\Gamma_4 &= 48t^{3/2}\frac{\partial}{\partial\varphi} + \varphi t^{-3/2}\frac{\partial}{\partial\Phi},\\
\Gamma_5 &= 192t^{3/2}\frac{\partial}{\partial t} + 288\sqrt{t}\frac{\partial}{\partial\varphi} + (\varphi t^{-5/2} - 96\Phi\sqrt{t})\frac{\partial}{\partial\Phi}.
\end{aligned}
$$

We perform reductions using the last Lie point symmetry Γ_5. This gives us two invariants $I_1 = \varphi t^{-3/2}$ and $I_2 = \Phi\sqrt{t} - \varphi^2/96t^{5/2}$, and consequently, we obtain the group invariant solution:

$$
\Phi = \frac{\varphi^2}{96t^3} + \frac{1}{\sqrt{t}}G\left(\frac{\varphi}{t^{3/2}}\right).
$$

Substituting the value of Φ into Eq. (2.11), we obtain the fourth-order nonlinear ordinary differential equation (NLODE) as follows:

$$
G''''(\xi) - 3G'(\xi)G''(\xi) = 0, \quad \xi = \frac{\varphi}{t^{3/2}}.
$$

When the above equation is integrated twice with respect to ξ, we obtain the NLODE as follows:

$$
\frac{1}{2}G''^2(\xi) - \frac{1}{2}G'^3(\xi) + k_1 G'(\xi) + k_2 = 0,
$$

where k_1 and k_2 are constants. Solving the above NLODE with the help of Maple and reverting to the original variables t, x, y, and u, we obtain our solution of Eq. (1.1) as follows:

$$u(t,x,y) = \frac{(y^2 - 4tx)\left\{(27k_2 + 3\sqrt{81k_2^2 - 24k_1^3})^{\frac{2}{3}} + k_1\right\}}{6t^2(27k_2 + 3\sqrt{81k_2^2 - 24k_1^3})^{\frac{1}{3}}} + \frac{(y^2 - 4tx)^2}{384t^3} + k_3, \tag{2.12}$$

where k_3 is a constant. A dynamical picture of the solution (2.12) is shown in Figure 1.

Case 2. We now consider the symmetry $\mathcal{H}_4 = 12g(t)\partial/\partial x + (2xg'(t) + y^2g''(t))\partial/\partial u$. Resolving the Lagrange system that corresponds to it yields three invariants:

$$J_1 = t,\ J_2 = y,\ J_3 = u - \frac{x^2g'(t) + xy^2g''(t)}{12g(t)},$$

and hence the group-invariant solution is given as follows:

$$u = \Phi(t,y) + \frac{x^2g'(t) + xy^2g''(t)}{12g(t)},$$

where Φ is any function of t and y. Inserting this value of u into Eq. (1.1) gives

$$12g(t)\Phi_{yy} - y^2g'''(t) = 0,$$

whose solution is

$$\Phi(t,y) = \frac{y^4g'''(t)}{144g(t)} + yk(t) + l(t),$$

where k and l are any functions of t. Hence, the group-invariant solution under $\mathcal{H}_4$ is

$$u(t,x,y) = \frac{y^4g'''(t)}{144g(t)} + \frac{x^2g'(t)}{12g(t)} + \frac{xy^2g''(t)}{12g(t)} + yk(t) + l(t). \tag{2.13}$$

A dynamical picture of the solution (2.13) is shown in Figure 2.

Case 3. We consider

$$\mathcal{H}_5 = 432a(t)\frac{\partial}{\partial t} + (144xa'(t) + 72y^2a''(t))\frac{\partial}{\partial x} + 288ya'(t)\frac{\partial}{\partial y} + (12x^2a''(t) - 144ua'(t) + 12xy^2a'''(t) + y^4a''''(t))\frac{\partial}{\partial u}.$$

Resolving the associated characteristic equations to $\mathcal{H}_5$, we obtain the following three invariants:

$$J_1 = \frac{y}{a(t)^{2/3}},\quad J_2 = xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}},$$

$$J_3 = ua(t)^{1/3} - \frac{x^2a'(t)}{36a(t)} + \frac{xy^2a'^2(t)}{54a^2(t)} - \frac{xy^2a''(t)}{36a(t)} - \frac{5y^4a'^3(t)}{1{,}944a^3(t)} - \frac{y^4a'''(t)}{432a(t)} + \frac{y^4a'(t)a''(t)}{2{,}116a^2(t)}.$$

The similarity solution is

$$u = \frac{1}{a(t)^{1/3}}\Phi(\sigma,\varphi) + \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} + \frac{5y^4a'^3(t)}{1{,}944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2{,}116a^2(t)},$$

with

$$\sigma = \frac{y}{a(t)^{2/3}},\ \varphi = xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}},$$

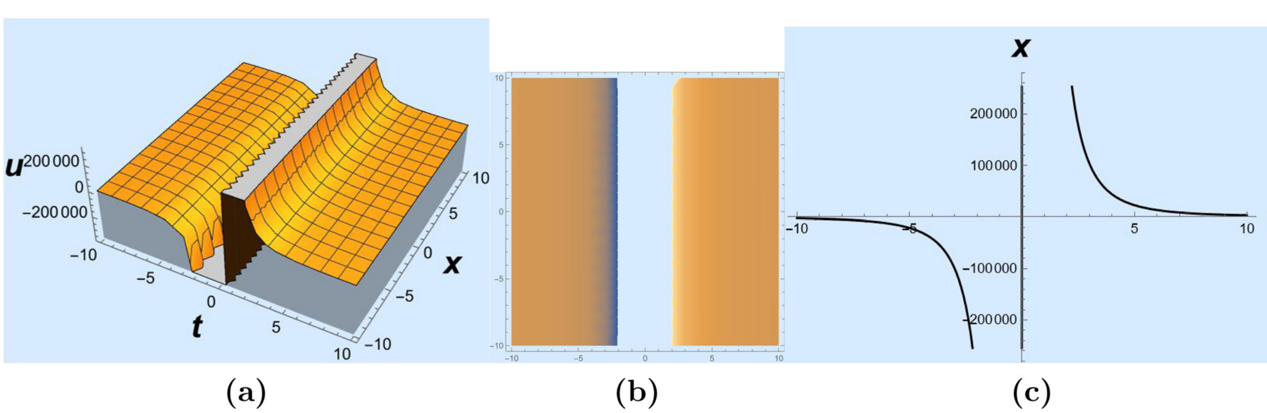


Figure 1: (a) 3D graph of singular solution (2.12) for $k_1 = 0.4$, $k_2 = 0.6$, $k_3 = 0.8$, and $y = 180$, within the interval $-10 \le t, x \le 10$. (b) 2D density plot of solution (2.12). (c) 2D graph of solution (2.12) for $x = 0$ within the interval $-10 \le t \le 10$.

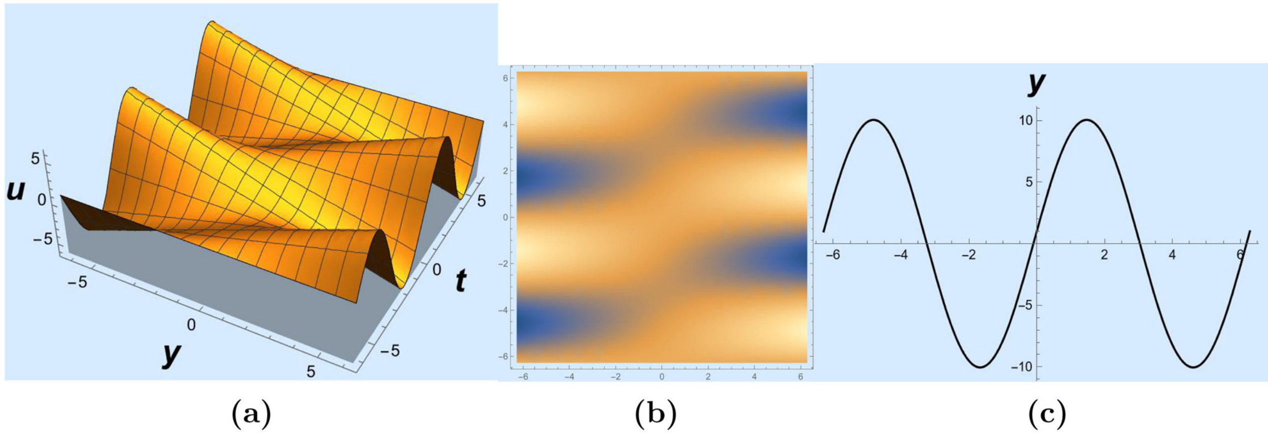


Figure 2: (a) 3D graph of periodic solution (2.13) for $g(t) = 1$, $x = 1$, $k(t) = \sin(t)$, and $l(t) = \cos(t)$ within the interval $-2\pi \le t, y \le 2\pi$. (b) 2D density plot of solution (2.13). (c) 2D graph of solution (2.13) for $y = 10$ within the interval $-2\pi \le t \le 2\pi$.

where Φ is any function of σ and φ. Substituting this value of u into Eq. (1.1), we obtain

$$\Phi_{\sigma\sigma} - 6\Phi_{\varphi}\Phi_{\varphi\varphi} - \Phi_{\varphi\varphi\varphi\varphi} = 0. \tag{2.14}$$

The NLPDE (2.14) has five infinitesimal generators as follows:

$$\mathcal{P}_1 = \frac{\partial}{\partial\sigma}, \mathcal{P}_2 = \frac{\partial}{\partial\varphi},$$
$$\mathcal{P}_3 = \frac{\partial}{\partial\Phi}, \mathcal{P}_4 = \sigma\frac{\partial}{\partial\Phi},$$
$$\mathcal{P}_5 = \varphi\frac{\partial}{\partial\varphi} + 2\sigma\frac{\partial}{\partial\sigma} - \Phi\frac{\partial}{\partial\Phi}.$$

Utilizing the translational symmetries as $\mathcal{P} = \mathcal{P}_1 + c\mathcal{P}_2$, yields the invariants $J_1 = \Phi$ and $J_2 = \varphi - c\sigma$, and consequently, we have $\Phi = \chi(\xi)$ and $\xi = \varphi - c\sigma$. Substituting the value of Φ into Eq. (2.14), we obtain the NLODE as follows:

$$\chi'''' + 6\chi'\chi'' - c^2\chi'' = 0. \tag{2.15}$$

We can now use Kudryashov's method as outlined in [54] to find the exact solution of Eq. (2.15). We start by assuming that the solution of Eq. (2.15) takes the form

$$\chi(\xi) = \sum_{i=0}^{n} B_i \mathcal{Y}(\xi)^i, \tag{2.16}$$

where $\mathcal{Y}$ satisfies

$$\mathcal{Y}'(\xi) = \mathcal{Y}^2(\xi) - \mathcal{Y}(\xi), \tag{2.17}$$

whose solution is given by

$$\mathcal{Y}(\xi) = \frac{1}{1 + \exp(\xi)}. \tag{2.18}$$

Using the balancing procedure, Eq. (2.15) gives $n = 1$, and hence

$$\chi(\xi) = B_o + B_1\mathcal{Y}(\xi). \tag{2.19}$$

Applying the above value of $\chi(\xi)$ into Eq. (2.15) and using Eq. (2.17), we obtain an equation that, when split on on powers of $\mathcal{Y}$, yields

$$\mathcal{Y}^5(\xi) : B_1^2 + 2B_1 = 0,$$
$$\mathcal{Y}^4(\xi) : B_1^2 + 2B_1 = 0,$$
$$\mathcal{Y}^3(\xi) : c^2B_1 - 12B_1^2 - 25B_1 = 0,$$
$$\mathcal{Y}^2(\xi) : c^2B_1 - 2B_1^2 - 5B_1 = 0,$$
$$\mathcal{Y}(\xi) : c^2B_1 - B_1 = 0.$$

Solving the above equations, we obtain $B_1 = -2$ and $c^2 = 1$, and hence Eq. (2.19) becomes

$$\chi(\xi) = B_0 - \frac{2}{1 + \exp(\xi)}. \tag{2.20}$$

Reverting to the original variables, the exact solutions of Eq. (1.1) are

$$\begin{aligned} u(t,x,y) = {} & \frac{1}{a(t)^{1/3}} \\ & \times \left(K - 2\left[1 + \exp\left\{xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}} + \frac{cy}{a(t)^{2/3}}\right\}\right]^{-1}\right) \\ & + \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)} \\ & + \frac{5y^4a'^3(t)}{1{,}944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2{,}116a^2(t)}, \end{aligned} \tag{2.21}$$

where $K = B_0$ is an arbitrary constant and $c = \pm 1$. The solution profile of (2.21) is presented in Figure 3.

Direct integration of (2.15).

Integrating NLODE equation (2.15) twice with respect to ξ yields

$$\frac{1}{2}\chi''^2 + \chi'^3 - \frac{1}{2}c^2\chi'^2 + k_1\chi' + k_2 = 0, \tag{2.22}$$

where k_1 and k_2 are constants of integration. By letting $v = \chi'$, Eq. (2.22) becomes

$$v'^2 + 2v^3 - c^2v^2 + 2k_1v + 2k_2 = 0. \tag{2.23}$$

If the algebraic equation

$$v^3 - \frac{c^2}{2}v^2 + k_2v + k_2 = 0$$

has the real roots α_1, α_2, and α_3 such that $\alpha_1 > \alpha_2 > \alpha_3$, then the NLODE equation (2.23) becomes

$$v'^2 = 2(v - \alpha_1)(v - \alpha_2)(v - \alpha_3),$$

whose solution can be written as follows [55,56]:

$$\begin{aligned} v(\xi) &= \alpha_2 - (\alpha_1 - \alpha_2)\mathrm{cn}^2\left\{\sqrt{\frac{\alpha_1 - \alpha_3}{2}}\,\xi \,\middle|\, M^2\right\},\\ M^2 &= \frac{\alpha_1 - \alpha_2}{\alpha_1 - \alpha_3}, \end{aligned} \tag{2.24}$$

where cn denotes the cosine elliptic function. Since $v = \chi'$, the above expression (2.24) is integrated with respect to ξ to obtain the solution to the pKP equation after returning to the original variables as follows:

$$\begin{aligned} u(t,x,y) = {}&\frac{1}{a(t)^{1/3}}\Big(\mathcal{P}_1[\,\text{EllipticE }\{\mathrm{sn}(\mathcal{P}_2\xi|M^2), M^2\}]\\ &+ \left\{\alpha_2 - (\alpha_1 - \alpha_2)\frac{1 - M^4}{M^4}\right\}\xi + k_3\Big)\\ &+ \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)}\\ &+ \frac{5y^4a'^3(t)}{1{,}944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2{,}116a^2(t)}, \end{aligned} \tag{2.25}$$

where $\xi = xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}} - \frac{cy}{a(t)^{2/3}}$, $\mathcal{P}_1 = \sqrt{\frac{2(\alpha_1 - \alpha_2)^2}{(\alpha_1 - \alpha_3)M^8}}$, $\mathcal{P}_2 = \sqrt{\frac{\alpha_1 - \alpha_3}{2}}$, k_3 is a constant, and

$$\text{EllipticE }[q,r] = \int_0^r \sqrt{\frac{1 - r^2n^2}{1 - n^2}}\,\mathrm{d}n$$

is the incomplete elliptic integral [57]. Figure 4 depicts the wave profile of the periodic solution (2.25).

Special case $k_1 = k_2 = 0$.

We consider the special case of Eq. (2.22) where $k_1 = k_2 = 0$, which upon solving yields the solution

$$\chi(\xi) = c\tanh\left\{\frac{1}{2}c(\xi + A_1)\right\} + A_2,$$

where A_1 and A_2 are constants. Returning to the original variables yields

$$\begin{aligned} u(t,x,y) = {}&\frac{1}{a(t)^{1/3}}\left(c\tanh\left\{\frac{1}{2}c\left(xa(t)^{-1/3} - \frac{y^2a'(t)}{6a(t)^{4/3}} - \frac{cy}{a(t)^{2/3}} + A_1\right)\right\} + A_2\right)\\ &+ \frac{x^2a'(t)}{36a(t)} - \frac{xy^2a'^2(t)}{54a^2(t)} + \frac{xy^2a''(t)}{36a(t)}\\ &+ \frac{5y^4a'^3(t)}{1{,}944a^3(t)} + \frac{y^4a'''(t)}{432a(t)} - \frac{y^4a'(t)a''(t)}{2{,}116a^2(t)}. \end{aligned} \tag{2.26}$$

The solution profile of Eq. (2.26) is presented in Figure 5.

2.3 Traveling wave solution

Traveling wave solutions of Eq. (1.1) are obtained by considering the special values of the functions $f(t)$, $g(t)$, and $a(t)$ in the symmetries $\mathcal{H}_3$, $\mathcal{H}_4$, and $\mathcal{H}_5$, respectively. By taking $f(t) = 1/72$, $g(t) = 1/12$, and $a(t) = 1/432$ in Eqs. (2.7), (2.8), and (2.9), we obtain

$$\mathcal{H}_3 = \frac{\partial}{\partial y}, \mathcal{H}_4 = \frac{\partial}{\partial x}, \mathcal{H}_5 = \frac{\partial}{\partial t}.$$

We now take the linear combination

$$\mathcal{H}_5 + a\mathcal{H}_4 + b\mathcal{H}_3 = \frac{\partial}{\partial t} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y},$$

whose associated Lagrange system gives the similarity variables and solution

$$p = x - at, \quad q = y - bt, u = \Theta(p,q). \tag{2.27}$$

Utilizing these invariants, Eq. (1.1) transforms into the following NLPDE in two independent variables:

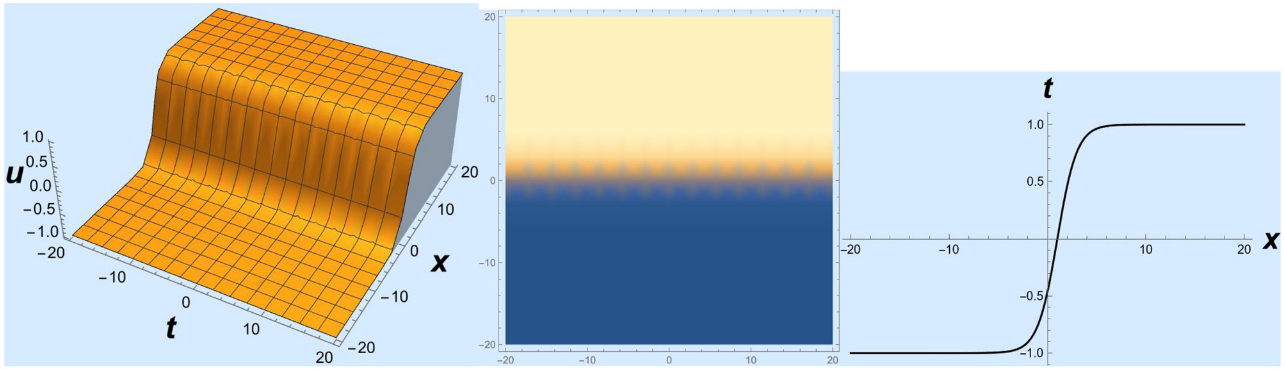
$$\Theta_{pppp} + 6\Theta_p\Theta_{pp} - a\Theta_{pp} - b\Theta_{pq} - \Theta_{qq} = 0. \tag{2.28}$$

The above equation has five point symmetries, namely

$$\begin{aligned} &\mathcal{S}_1 = \frac{\partial}{\partial p},\ \mathcal{S}_2 = \frac{\partial}{\partial q},\ \mathcal{S}_3 = \frac{\partial}{\partial \Theta},\ \mathcal{S}_4 = p\frac{\partial}{\partial \Theta},\\ &\mathcal{S}_5 = (6bq + 12p)\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} - (pb^2 - 4pa + 12\Theta)\frac{\partial}{\partial \Theta}. \end{aligned}$$

The symmetry $\mathcal{S} = \mathcal{S}_1 + c\mathcal{S}_2$ gives two invariants $I_1 = q - cp$ and $I_2 = \Theta$, and consequently, the invariant solution is $\Theta = F(q - cp)$. Substituting the value of Θ into Eq. (2.28), we obtain the fourth-order NLODE as follows:

$$(bc - ac^2 - 1)F''(z) - 6c^3F'(z)F''(z) + c^4F''''(z) = 0, \tag{2.29}$$



(a) (b) (c)

Figure 3: (a) 3D graph of a kink-shaped soliton solution (2.21) for $a(t) = 1$, $k = 1$, $c = 1$, and $y = 1$ within the interval $-20 \le t, x \le 20$. (b) 2D density plot of solution (2.21). (c) 2D graph of solution (2.21) for $t = 0$ within the interval $-20 \le x \le 20$.

which we rewrite as

$$AF''(z) - BF'(z)F''(z) + CF''''(z) = 0, \qquad (2.30)$$

where $A = bc - ac^2 - 1$, $B = 6c^3$, $C = c^4$, and $z = (ac - b)t - cx + y$. We observe that Eq. (2.30) takes the same form as the NLODE equation (2.15). Therefore, following similar steps as presented in the section “Direct integration of (2.15)”, the pKP equation (1.1) solution is given by

$$\begin{aligned} u(t,x,y) = {} & \sqrt{\frac{12C(r_1 - r_2)^2}{B(r_1 - r_3)S^8}} \\ & \times \left\{ \text{EllipticE}\left[\text{sn}\left(\frac{B(r_1 - r_3)}{12C} z, S^2 \right), S^2 \right] \right\} \\ & + \left\{ r_2 - (r_1 - r_2)\frac{1 - S^4}{S^4} \right\} z + k_3, \end{aligned} \qquad (2.31)$$

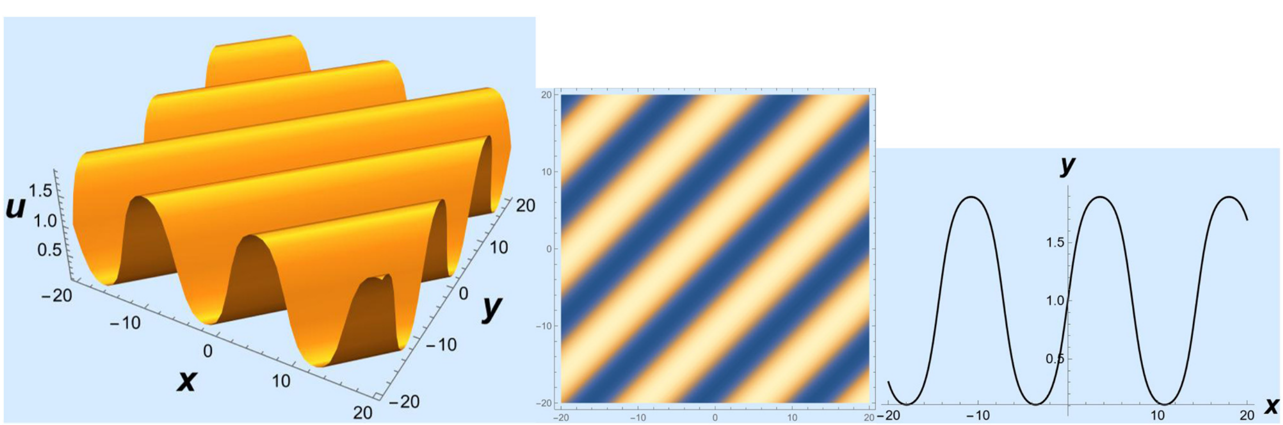
where k_3 is a constant and EllipticE$[q, v]$ represents the incomplete elliptic integral as follows [57]:

$$\text{EllipticE}[q, v] = \int_0^v \sqrt{\frac{1 - v^2 n^2}{1 - n^2}}\, dn.$$

The wave profile of solution (2.31) is illustrated in Figure 6.

3 Graphical and physical explanation of the obtained solutions

In this section, we provide more details on the obtained group-invariant solutions to the pKP equation (1.1) by



(a) (b) (c)

Figure 4: (a) 3D graph of periodic soliton solution (2.25) for $a(t) = 1$, $\alpha_1 = 10$, $\alpha_2 = 5$, $\alpha_3 = 2$, $k_3 = 0$, and $c = 1$ within the interval $-20 \le x, y \le 20$. (b) 2D density plot of solution (2.25). (c) 2D graph of solution (2.25) for $y = 0$ within the interval $-20 \le x \le 20$.

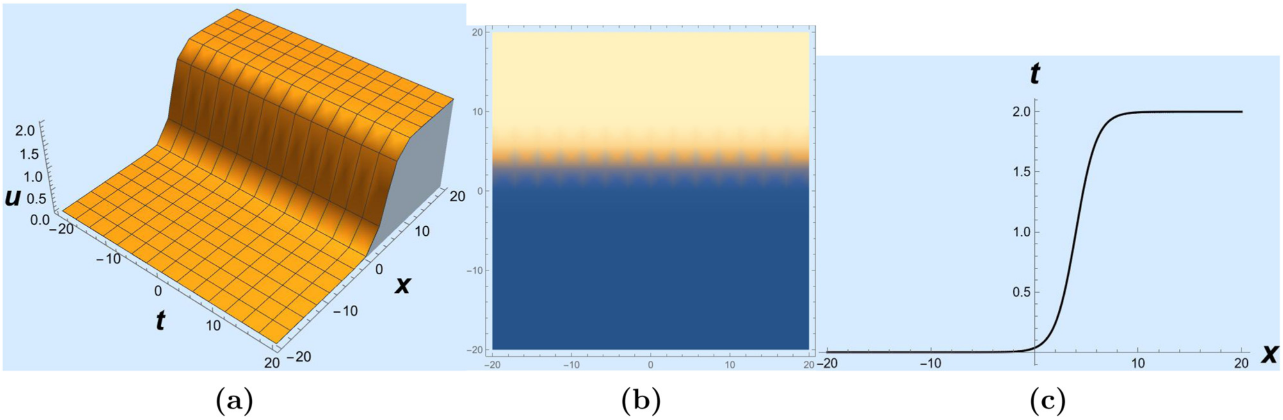


Figure 5: (a) 3D graph of a kink-shaped soliton solution (2.26) for $a(t) = 1, A_1 = 1, A_2 = 1, c = 1$, and $y = 5$ within the interval $-20 \le t, x \le 20$. (b) 2D density plot of solution (2.26). (c) 2D graph of solution (2.26) for $t = 0$ within the interval $-20 \le x \le 20$.

discussing their geometrical representation. 3D, 2D, and corresponding density plots in Figures 1–6 are constructed by utilizing the mathematical software tool Mathematica. This involves taking acceptable values of the parameters under certain limits in order to visualize the mechanism of the equation under study. Graphs of solution (2.12) are shown in Figure 1, which represent singular solitons. Figure 2 shows graphs of the periodic solution (2.13). The solutions given in Eqs. (2.25) and (2.31) are periodic solitons shown in Figures 4 and 6. The kink-shaped soliton solutions (2.21) and (2.26) are presented in Figures 3 and 5.

4 Conservation laws for the pKP equation

We derive the conserved vectors of the pKP equation (1.1) by using three approaches: the theorem by Noether [32], Ibragimov's theorem [33], and the multiplier method [19] as given in their respective references.

4.1 Conservation laws for pKP *via* Noether's theorem

Here, we apply Noether's theorem [32] to construct conservation laws for the pKP equation (1.1). It is easy to verify that Eq. (1.1) has the Lagrangian

$$\mathcal{L} = \frac{1}{2}u_{xx}^2 - \frac{1}{2}u_t u_x + \frac{1}{2}u_y^2 - u_x^3. \tag{4.1}$$

We now use the Lagrangian (4.1) in the determining equation

$$\mathcal{G}^{[2]}\mathcal{L} + \mathcal{L}\{D_t(\tau) + D_x(\xi) + D_y(\psi)\} - D_t(B^t) - D_x(B^x) - D_y(B^y) = 0, \tag{4.2}$$

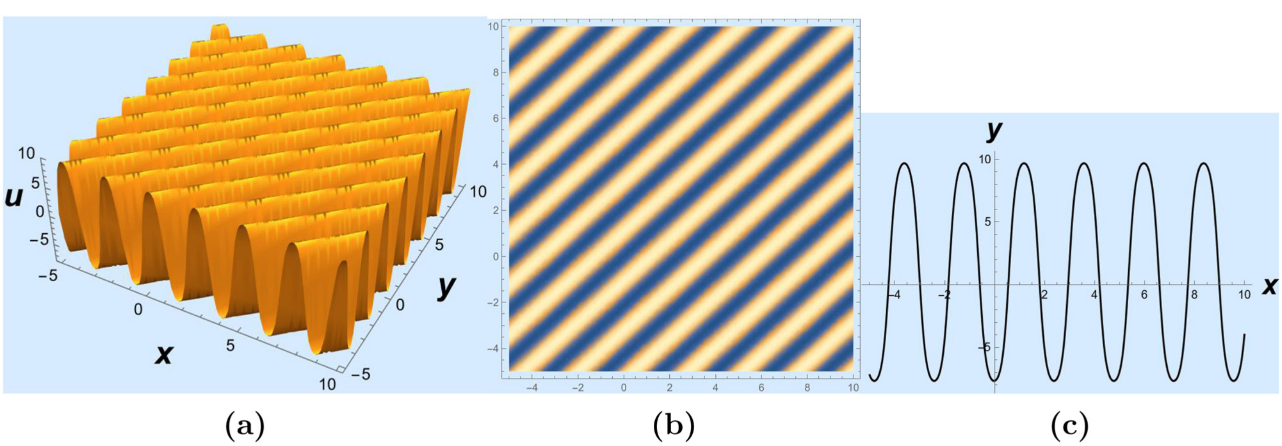


Figure 6: (a) 3D graph of periodic soliton solution (2.31) for $r_1 = 10, r_2 = 5, r_3 = 3, k_3 = 1, c = 0.9$, and $t = 1$ within the interval $-5 \le x, y \le 10$. (b) 2D density plot of solution (2.31). (c) 2D graph of solution (2.31) for $y = 0$ within the interval $-5 \le x \le 10$.

where $B^t = B^t(t,x,y,u)$, $B^x = B^x(t,x,y,u)$, and $B^y = B^y(t,x,y,u)$ are gauge functions and $\mathcal{G}^{[2]}$ is the second-order prolongation given by

$$\mathcal{G}^{[2]} = \mathcal{G} + \zeta_t \partial u_t + \zeta_x \partial u_x + \zeta_y \partial u_y + \zeta_{xx} \partial u_{xx}.$$

Expanding Eq. (4.2) and splitting on derivatives of u, we obtain the system of PDEs as follows:

$$\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \xi_u = 0, \quad \psi_x = 0, \quad \psi_u = 0,$$
$$\eta_{xx} = 0,$$
$$\psi_y + 2\eta_u = 0, \quad \xi_{xx} - \eta_{xu} = 0, \quad \psi_t - 2\xi_y = 0, \quad \xi_t - 6\eta_x = 0,$$
$$\tau_t - \xi_x + \psi_y = 0, \quad \tau_t + \xi_x - \psi_y + 2\eta_u = 0,$$
$$2\xi_x - \tau_t - \psi_y - 3\eta_u = 0,$$
$$\eta_y - B_u^y = 0, \quad \eta_t + 2B_u^x = 0, \quad \eta_x + 2B_u^t = 0,$$
$$B_t^t + B_x^x + B_y^y = 0.$$

Solving the above system of PDEs, we gain gauge functions and Noether symmetries listed below:

$$\mathcal{G}_1 = \frac{\partial}{\partial t}, B_1^t = 0, B_1^x = 0, B_1^y = 0,$$
$$\mathcal{G}_2 = \frac{1}{2} y i'(t) \frac{\partial}{\partial x} + i(t) \frac{\partial}{\partial y} + \left(\frac{1}{12} xy i''(t) + \frac{1}{72} y^3 i'''(t)\right) \frac{\partial}{\partial u},$$
$$B_2^t = -\frac{1}{24} yu i''(t), B_2^x = -\frac{1}{24} xyu i'''(t) - \frac{1}{144} u y^3 i''''(t),$$
$$B_2^y = \frac{1}{12} xu i''(t) + \frac{1}{24} u y^2 i'''(t),$$
$$\mathcal{G}_3 = j(t) \frac{\partial}{\partial x} + \left(\frac{1}{6} x j'(t) + \frac{1}{12} y^2 j''(t)\right) \frac{\partial}{\partial u},$$
$$B_3^t = -\frac{1}{12} u j'(t), B_3^x = -\frac{1}{12} xu j''(t) - \frac{1}{24} u y^2 j'''(t),$$
$$B_3^y = \frac{1}{6} uy j''(t),$$
$$\mathcal{G}_4 = y n(t) \frac{\partial}{\partial u}, B_4^t = 0, B_4^x = -\frac{1}{2} uy n'(t), \; B_4^y = u n(t),$$
$$\mathcal{G}_5 = m(t) \frac{\partial}{\partial u}, B_5^t = 0, B_5^x = -\frac{1}{2} u m'(t), B_5^y = 0.$$

Utilizing the formulae [58]

$$T^t = \tau \mathcal{L} + W \frac{\partial \mathcal{L}}{\partial u_t} - B^t,$$
$$T^x = \xi \mathcal{L} + W \left\{\frac{\partial \mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial \mathcal{L}}{\partial u_{xx}}\right)\right\} + W_x \frac{\partial}{\partial u_{xx}} - B^x,$$
$$T^y = \psi \mathcal{L} + W \frac{\partial Ł}{\partial u_y} - B^y,$$

where $W = \eta - u_t \tau - u_x \xi - u_y \psi$, the corresponding conserved vectors are given, respectively, as follows:

$$T_1^t = \frac{1}{2} u_y^2 + \frac{1}{2} u_{xx}^2 - u_x^3,$$
$$T_1^x = 3u_x^2 u_t + \frac{1}{2} u_t^2 + u_{xxx} u_t - u_{xx} u_{tx},$$
$$T_1^y = -u_t u_y;$$

$$T_2^t = -\frac{1}{144} y^3 i'''(t) u_x - \frac{1}{24} xy i''(t) u_x + \frac{1}{24} y i''(t) u + \frac{1}{4} y i'(t) u_x^2 + \frac{1}{2} i(t) u_y u_x,$$
$$T_2^x = \frac{1}{144} y^3 i''''(t) u - \frac{1}{24} y^3 i'''(t) u_x^2 - \frac{1}{72} y^3 i'''(t) u_{xxx} - \frac{1}{144} y^3 i'''(t) u_t + \frac{1}{24} xy i'''(t) u - \frac{1}{4} xy i''(t) u_x^2$$
$$+ \frac{1}{12} y i''(t) u_{xx} - \frac{1}{12} xy i''(t) u_{xxx} - \frac{1}{24} xy i''(t) u_t + y i'(t) u_x^3 + \frac{1}{4} y i'(t) u_y^2 - \frac{1}{4} y i'(t) u_{xx}^2 + \frac{1}{2} y i'(t) u_x u_{xxx} + 3i(t) u_y u_x^2$$
$$- i(t) u_{xy} u_{xx} + i(t) u_y u_{xxx} + \frac{1}{2} i(t) u_y u_t,$$
$$T_2^y = \frac{1}{72} y^3 i'''(t) u_y - \frac{1}{24} y^2 i'''(t) u + \frac{1}{12} xy i''(t) u_y - \frac{1}{12} x i''(t) u - \frac{1}{2} y i'(t) u_y u_x - i(t) u_x^3$$
$$- \frac{1}{2} i(t) u_y^2 + \frac{1}{2} i(t) u_{xx}^2 - \frac{1}{2} i(t) u_x u_t;$$

$$T_3^t = -\frac{1}{24} y^2 j''(t) u_x - \frac{1}{12} x j'(t) u_x + \frac{1}{12} j'(t) u + \frac{1}{2} j(t) u_x^2,$$
$$T_3^x = \frac{1}{24} y^2 j'''(t) u - \frac{1}{4} y^2 j''(t) u_x^2 - \frac{1}{12} y^2 j''(t) u_{xxx} - \frac{1}{24} y^2 j''(t) u_t + \frac{1}{12} x j''(t) u$$
$$- \frac{1}{2} x j'(t) u_x^2 + \frac{1}{6} j'(t) u_{xx} - \frac{1}{6} x j'(t) u_{xxx} - \frac{1}{12} x j'(t) u_t + 2j(t) u_x^3 + j(t) u_{xxx} u_x + \frac{1}{2} j(t) u_y^2 - \frac{1}{2} j(t) u_{xx}^2,$$
$$T_3^y = \frac{1}{12} y^2 j''(t) u_y - \frac{1}{6} y j''(t) u + \frac{1}{6} x j'(t) u_y - j(t) u_y u_x;$$

$$T_4^t = -\frac{1}{2} y n(t) u_x,$$
$$T_4^x = \frac{1}{2} y n'(t) u - 3y n(t) u_x^2 - y n(t) u_{xxx} - \frac{1}{2} y n(t) u_t,$$
$$T_4^y = y n(t) u_y - n(t) u;$$

$$T_5^t = -\frac{1}{2} m(t) u_x,$$
$$T_5^x = \frac{1}{2} m'(t) u - 3m(t) u_x^2 - m(t) u_{xxx} - \frac{1}{2} m(t) u_t,$$
$$T_5^y = m(t) u_y.$$

4.2 Conservation laws for pKP equation by applying Ibragimov's theorem

We invoke the conservation theorem of Ibragimov, which has been outlined in [33], to find conserved vectors for the pKP equation (1.1). The Euler–Lagrange operator for the pKP equation (1.1) is given as follows:

$$\begin{aligned}\delta u &= \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_tD_x\frac{\partial}{\partial u_{tx}}\\ &\quad + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_x^4\frac{\partial}{\partial u_{xxxx}} + \cdots ,\end{aligned} \tag{4.3}$$

where D_t, D_x, and D_y are given in Eq. (2.4). The adjoint equation for Eq. (1.1) is given by the formula

$$F^* = \delta u[v(u_{tx} + 6u_xu_{xx} + u_{xxxx} - u_{yy})] = 0, \tag{4.4}$$

where $v = v(t, x, y)$. Eq. (4.4) yields

$$F^* = v_{tx} + 6v_xu_{xx} + 6u_xv_{xx} + v_{xxxx} - v_{yy} = 0. \tag{4.5}$$

It is observed that the pKP equation is not self-adjoint. The formal Lagrangian for Eqs (1.1) and (4.5) is

$$\mathcal{L} = v(u_{tx} + 6u_xu_{xx} + u_{xxxx} - u_{yy}). \tag{4.6}$$

The conserved vectors for the pKP equation (1.1) are formulated as follows [58,59]:

$$\begin{aligned}T^t &= \tau\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_t} - D_x\left(\frac{\partial L}{\partial u_{xt}}\right)\right\},\\ T^x &= \xi\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_x} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xx}}\right) + D_{xx}\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right.\\ &\quad \left. - D_{xxx}\left(\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)\right\} + W_x\left\{\frac{\partial\mathcal{L}}{\partial u_{xx}} - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right)\right.\\ &\quad \left. + D_{xx}\left(\frac{\partial L}{\partial u_{xxxx}}\right)\right\} + W_t\frac{\partial\mathcal{L}}{\partial u_{xt}} + W_{xx}\left\{\frac{\partial\mathcal{L}}{\partial u_{xxx}}\right.\\ &\quad \left. - D_x\left(\frac{\partial\mathcal{L}}{\partial u_{xxxx}}\right)\right\} + W_{xxx}\frac{\partial\mathcal{L}}{\partial u_{xxxx}},\\ T^y &= \psi\mathcal{L} + W\left\{\frac{\partial\mathcal{L}}{\partial u_y} - D_y\left(\frac{\partial\mathcal{L}}{\partial u_{yy}}\right)\right\} + W_y\frac{\partial\mathcal{L}}{\partial u_{yy}},\end{aligned} \tag{4.7}$$

where $W = \eta - \tau u_t - \xi u_x - \psi u_y$.

Case 1. For the symmetry $\mathcal{H}_1 = j(t)\partial/\partial u$, the conserved vector (T_1^t, T_1^x, T_1^y) using Eq. (4.7) is

$$\begin{aligned}T_1^t &= -j(t)v_x,\\ T_1^x &= j'(t)v - 6j(t)v_xu_x - j(t)v_{xxx},\\ T_1^y &= j(t)v_y.\end{aligned}$$

Case 2. For $\mathcal{H}_2 = yi(t)\partial/\partial u$, the conserved vector is

$$\begin{aligned}T_2^t &= -yi(t)v_x,\\ T_2^x &= yi'(t)v - 6yi(t)v_xu_x - yi(t)v_{xxx},\\ T_2^y &= yi(t)v_y - i(t)v.\end{aligned}$$

Case 3. For the symmetry

$$\begin{aligned}\mathcal{H}_3 &= 36yf'(t)\frac{\partial}{\partial x} + 72f(t)\frac{\partial}{\partial y}\\ &\quad + (6xyf''(t) + y^3f'''(t))\frac{\partial}{\partial u},\end{aligned}$$

we obtain

$$\begin{aligned}T_3^t &= 36yf'(t)u_xv_x - y^3f'''(t)v_x\\ &\quad - 6xyf''(t)v_x + 72f(t)u_yv_x,\\ T_3^x &= y^3f''''(t)v - 6y^3f'''(t)u_xv_x\\ &\quad - y^3f'''(t)v_{xxx} + 6xyf'''(t)v - 36xyf''(t)u_xv_x\\ &\quad + 6yf''(t)v_{xx} - 6xyf''(t)v_{xxx} + 216yf'(t)u_x^2v_x\\ &\quad - 36yf'(t)u_{xx}v_{xx} + 36yf'(t)u_{xxx}v_x\\ &\quad + 36yf'(t)u_xv_{xxx} - 36yf'(t)u_{yy}v - 72f'(t)u_yv\\ &\quad + 432f(t)u_yu_xv_x - 72f(t)u_{ty}v_{xx}\\ &\quad + 72f(t)u_{xxy}v_x + 72f(t)u_yv_{xxx} - 432f(t)u_xu_{ty}v\\ &\quad - 72f(t)u_{xxxy}v - 72f(t)u_{ty}v,\\ T_3^y &= 72f(t)vu_{tx} + 36f'(t)vu_x - 72f(t)v_yu_y\\ &\quad - 6xf''(t)v - 3y^2f'''(t)v + 6xyf''(t)v_y\\ &\quad + y^3f'''(t)v_y - 36yf'(t)v_yu_x + 36yf'(t)vu_{xy}\\ &\quad + 432f(t)vu_xu_{xx} + 72f(t)vu_{xxxx}.\end{aligned}$$

Case 4. For the symmetry

$$\mathcal{H}_4 = 12g(t)\frac{\partial}{\partial x} + (2xg'(t) + y^2g''(t))\frac{\partial}{\partial u},$$

we have

$$\begin{aligned}T_4^t &= 12g(t)v_xu_x + g'(t)v_x - y^2g''(t)v_x - 2xg'(t)v_x,\\ T_4^x &= y^2g'''(t)v - 6y^2g''(t)u_xv_x\\ &\quad - y^2g''(t)v_{xxx} + 2xg''(t)v - 12xg'(t)u_xv_x + 2g'(t)v_{xx}\\ &\quad - 2xg'(t)v_{xxx} + 72g(t)u_x^2v_x - 12g(t)u_{xx}v_{xx}\\ &\quad + 12g(t)u_{xxx}v_x + 12g(t)u_xv_{xxx} - 12g(t)u_{yy}v,\\ T_4^y &= 12g(t)vu_{xy} - 2yg''(t)v + 2xg'(t)v_y - 12g(t)v_yu_x.\end{aligned}$$

Case 5. Finally, for the symmetry

$$\begin{aligned}\mathcal{H}_5 &= 432a(t)\frac{\partial}{\partial t} + (144xa'(t) + 72y^2a''(t))\frac{\partial}{\partial x} + 288ya'(t)\frac{\partial}{\partial y}\\ &\quad + (12x^2a''(t) - 144ua'(t) + 12xy^2a'''(t) + y^4a''''(t))\frac{\partial}{\partial u},\end{aligned}$$

we obtain

$$T_5^t = 72y^2a''(t)u_xv_x - y^4a''''(t)v_x - 12xy^2a'''(t)v_x$$
$$- 12x^2a''(t)v_x + 288ya'(t)u_yv_x + 144xa'(t)u_xv_x$$
$$+ 144a'(t)uv_x + 432a(t)u_tv_x - 432a(t)u_{yy}v$$
$$+ 2592a(t)u_xu_{xx}v + 432a(t)u_{xxxx}v + 432a(t)u_{tx}v,$$

$$T_5^x = y^4va^{(5)}(t) - 6y^4a''''(t)u_xv_x - y^4a''''(t)v_{xxx}$$
$$+ 12xy^2va''''(t) - 72y^2va''(t)u_{yy}$$
$$+ 432y^2a''(t)u_x^2v_x - 72xy^2a'''(t)u_xv_x$$
$$+ 12y^2a'''(t)v_{xx} - 72y^2a''(t)u_{xx}v_{xx}$$
$$+ 72y^2a''(t)v_xu_{xxx} - 12xy^2a'''(t)v_{xxx}$$
$$+ 72y^2a''(t)u_xv_{xxx} - 288yva''(t)u_y$$
$$+ 1728ya'(t)u_yu_xv_x - 1728yva'(t)u_xu_{xy}$$
$$- 288ya'(t)u_{xy}v_{xx} + 288ya'(t)v_xu_{xxy}$$
$$+ 288ya'(t)u_yv_{xxx} - 288yva'(t)u_{xxxy} - 288yva'(t)u_{ty}$$
$$- 1728va'(t)u_x^2 - 144uva''(t) + 12x^2va'''(t)$$
$$- 144xva'(t)u_{yy} + 864xa'(t)u_x^2v_x - 24a''(t)v_x$$
$$+ 864ua'(t)u_xv_x - 72x^2a''(t)u_xv_x + 432a'(t)v_xu_{xx}$$
$$+ 24xa''(t)v_{xx} - 288a'(t)u_xv_{xx} - 144xa'(t)u_{xx}v_{xx}$$
$$- 576va'(t)u_{xxx} + 144xa'(t)v_xu_{xxx} + 144ua'(t)v_{xxx}$$
$$- 12x^2a''(t)v_{xxx} + 144xa'(t)u_xv_{xxx}$$
$$- 576va'(t)u_t + 2592a(t)u_xv_xu_t + 432a(t)v_{xxx}u_t$$
$$- 2592a(t)vu_xu_{tx} - 432a(t)v_{xx}u_{tx}$$
$$+ 432a(t)v_xu_{txx} - 432a(t)vu_{txxx} - 432a(t)vu_{tt},$$

$$T_5^y = 12x^2a''(t)v_y + 432a'(t)vu_y - 24xya'''(t)v$$
$$- 4y^3a''''(t)v - 144a'(t)v_yu + 12xy^2a''(t)v_y$$
$$+ y^4a''''(t)v_y - 288ya'(t)v_yu_y + 144ya''(t)vu_x$$
$$- 144xa'(t)v_yu_x - 72y^2a''(t)v_yu_x + 144xa'(t)vu_{xy}$$
$$+ 72y^2a''(t)vu_{xy} + 1728ya'(t)vu_xu_{xx} - 432a(t)v_yu_t$$
$$+ 432a(t)vu_{ty} + 288ya'(t)vu_{ty} + 288ya'(t)vu_{xxxx}.$$

4.3 Conservation laws for pKP equation using the multiplier method

We utilize the multiplier method to obtain the conserved vectors of the pKP equation (1.1) by seeking zeroth-order multipliers $Q = Q(t, x, y, u)$. The determining equation for gaining multipliers of the pKP is

$$\delta_u\{Q(u_{tx} + 6u_xu_{xx} + u_{xxxx} - u_{yy})\} = 0, \tag{4.8}$$

with δ_u being the Euler operator given as follows:

$$\delta_u = \frac{\partial}{\partial u} - D_x\frac{\partial}{\partial u_x} + D_tD_x\frac{\partial}{\partial u_{tx}} + D_x^2\frac{\partial}{\partial u_{xx}} + D_y^2\frac{\partial}{\partial u_{yy}} + D_x^4\frac{\partial}{\partial u_{xxxx}},$$

where D_t, D_x, and D_y are total differential operators. Expanding Eq. (4.8) and splitting on derivatives of u yields the following system of PDEs:

$$Q_{yy} = 0, \quad Q_x = 0, \quad Q_u = 0.$$

Solving the above PDE system, we obtain

$$Q = f(t) + yg(t),$$

where, f and g are arbitrary functions of t. The conservation laws of Eq. (1.1) are then derived by invoking the divergence identity

$$D_tT^t + D_xT^x + D_yT^y = Q(u_{tx} + 6u_xu_{xx} + u_{xxxx} - u_{yy}). \tag{4.9}$$

Here, T^x and T^y represent the spatial fluxes and T^t represent the conserved density [19]. Utilizing Eq. (4.9), we gain conserved vectors under each multiplier.

Case 1. For $Q_1 = f(t)$, we have

$$T_1^t = f(t)u_x,$$
$$T_1^x = f(t)u_{xxx} + 3f(t)u_x^2 - f'(t)u,$$
$$T_1^y = -f(t)u_y.$$

Case 2. For $Q_2 = yg(t)$, we obtain

$$T_2^t = g(t)yu_x,$$
$$T_2^x = 3g(t)yu_x^2 + yg'(t)u_{xxx} - g(t)yu,$$
$$T_2^y = ug(t) - yg(t)u_y.$$

Remark 4.1. We observe that the conserved quantities obtained through Noether's theorem, Ibragimov's approach, and the multiplier method contain arbitrary functions, and the presence of these functions in the conserved vectors indicates the existence of infinitely many conserved quantities in the pKp equation. Moreover, it is well known that these conserved quantities have diverse applications in physical systems. They play a crucial role in establishing the existence and uniqueness of solutions, investigating integrability and linearization mappings, analyzing the stability and global behavior of solutions, and so on. In addition, we note that some of these conserved quantities represent momentum and energy, which makes them very useful in studying physical systems.

5 Conclusion

In this article, we studied the pKP equation (1.1). Using Lie symmetry methods, its symmetries were computed and used to obtain exact solutions through symmetry reductions and with the aid of Kudryashov's method. Moreover, its one-parameter group of transformation is given. The constructed solutions were in terms of rational, exponential, elliptical, and hyperbolic functions, which were presented in 3D, 2D and density plots to help analyze the diverse nature of each obtained solution. We noted that from Figures 1–6, the achieved solutions of the pKP equation comprised singular, periodic, periodic soliton, and kink-shaped solitons. Finally, we derived its conservation laws using Noether's theorem, Ibragimov's theorem, and the multiplier method. In addition to the numerous advantages of the obtained solutions presented in this study across various scientific fields, the investigated conservation laws hold significant importance. In classical physics, these laws encompass the conservation of energy, linear momentum, and angular momentum. Conserved quantities play a vital role in our understanding of the physical world, representing fundamental laws of nature. As a result, they have a broad range of applications in physics and various other fields of study. Therefore, the outcomes of this research can be employed for experimental and applied purposes, facilitating further investigations in diverse areas of scientific research.

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