Research Article

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Bayesian and non-Bayesian estimation of dynamic cumulative residual Tsallis entropy for moment exponential distribution under progressive censored type II

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Abstract: The dynamic cumulative residual (DCR) entropy is a helpful randomness metric that may be used in survival analysis. A challenging issue in statistics and machine learning is the estimation of entropy measures. This article uses progressive censored type II (PCT-II) samples to estimate the DCR Tsallis entropy (DCRTE) for the moment exponential distribution. The non-Bayesian and Bayesian approaches are the recommended estimating strategies. We obtain the DCRTE Bayesian estimator using the gamma and uniform priors *via* symmetric and asymmetric (linear exponential and general entropy) loss functions (LoFs). The Metropolis–Hastings algorithm is employed to generate Markov chain Monte Carlo samples from the posterior distribution. The accuracy of different estimates for various sample sizes, is implemented *via* Monte Carlo simulations. Generally, we note based on the simulation study that, in the majority of cases, the DCRTE Bayesian estimates under general entropy followed by linear exponential LoFs are preferable to the others. The accuracy measure of DCRTE Bayesian estimates using a gamma prior has smaller values than the others using a uniform prior. As sample sizes grow, the Bayesian estimates of the DCRTE are closer to the true value. Finally, analysis of the leukemia data confirmed the proposed estimators.

Keywords: Tsallis entropy, moment exponential distribution, squared error loss function, Markov chain Monte Carlo, maximum likelihood, general entropy loss function

1 Introduction

The concept of Tsallis (T) entropy, which can be employed to calculate the uncertainty in a random observation, was first proposed in a study by Tsallis [1]. It has been widely employed in the research of quantum communication protocols, quantum systems, and quantum correlations [2,3]. The entropy (T) of order q is provided *via*

$$T(q) = (q-1)^{-1}\left(1 - \int_0^\infty g^q(z)\mathrm{d}z\right), \quad (1)$$

where $q > 0, q \neq 1$, and $g(z)$ is the probability density function (PDF). A number of authors have recently examined the estimation of the entropy measures utilizing various statistical distributions and sampling techniques (see, for instance, [4–14]). Recently, several researchers have become interested in alternative uncertainty measures of probability distributions, particularly in reliability and survival analysis research. This is why [15] suggested cumulative residual (CR) entropy. The following is the definition for the dynamic form of CR entropy, known as dynamic cumulative residual Tsallis entropy (DCRTE), as proposed by Sunoj and Linu [16]

$$\gamma(q,t) = (q-1)^{-1}\left(1 - \int_t^\infty \frac{\bar{G}^q(z)}{\bar{G}^q(t)}\mathrm{d}z\right), \quad (2)$$

where $q > 0, q \neq 1$, and $\bar{G}(t) = 1 - G(t)$ is the survival function (SF) and for $t = 0$, the DCRTE leads to CR Tsallis entropy (CRTE).

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Only a few studies from the literature were considered in the case of inferential approaches to dynamic CR (DCR) entropy for probability distributions. Kamari [17] demonstrated the properties of the DCR entropy derived from order statistics. The CR has been established in a study by Kundu *et al.* [18] as extensions of the cumulative entropies for truncated random variables. The DCR entropy of the Pareto model was studied by the Bayesian technique using a range of sampling approaches in a few studies [19–23]. Almarashi *et al.* [24] used Bayesian inference to estimate the DCR entropy of the Lindley distribution.

A flexible strategy for analyzing lifetime datasets is typically appealing to researchers. The moment exponential (MExp) distribution (or length-biased exponential distribution) was established in ref. [25] by weighting the exponential distribution in accordance with Fisher's (1934). Dara and Ahmad in their study [25] offered many characteristics and applications of the MExp distribution. The PDF of the MExp distribution is

$$g(z;\lambda) = \frac{z}{\lambda^2}e^{-\frac{z}{\lambda}}; \quad z, \lambda > 0. \tag{3}$$

The cumulative distribution function (CDF) and the SF of the MExp model are as follows:

$$G(z;\lambda) = 1 - e^{-\frac{z}{\lambda}}\left(1 + \frac{z}{\lambda}\right); \quad z, \lambda > 0, \tag{4}$$

and

$$\bar{G}(z;\lambda) = e^{-\frac{z}{\lambda}}\left(1 + \frac{z}{\lambda}\right); \quad z, \lambda > 0. \tag{5}$$

Due to the MExp model's versatility, it attracted a lot of attention, and as a result, other authors investigated and further generalized it for more complicated datasets. For instance, the generalized exponentiated MExp distribution [26], the Marshall–Olkin length-biased exponential distribution [27], the Topp–Leone MExp distribution [28], the Kumaraswamy MExp distribution [29], the Marshall–Olkin Kumaraswamy MExp distribution [30], the Burr XII-MExp distribution [31], and the alpha power MExp distribution [32] and references therein.

In reliability experiments, investigators seek to know how long it takes for units to malfunction. However, investigators are unable to observe the life of all units due to time and money constraints, as well as a number of other issues. This leads to the availability of data that have been censored. Censoring types I (CT-I) and II (CT-II) are the two most reasonably available methods. But in some situations, like medical/engineering survival analysis, units might be taken out at intermediate stages for a number of reasons that are beyond the experimenter's control. Since it allows for the removal of surviving items prior to the test's conclusion, a progressive censoring (PC) system is a suitable censoring method in this context. The PC type-I (PCT-I) takes place when the percentage of survivors falls to predetermined levels, whereas the PC type-II (PCT-II) takes place when the percentage of survivors falls to specific values.

A PCT-II sample is prepared in the manner described below: A life-testing experiment that has n units and the PC strategy $r_i, i = 1, 2, \ldots, m$ is implemented. Units are picked at random from the remaining $n - 1$ surviving units at the moment of the first failure $z_{(1)}$. After the second failure $z_{(2)}$, units from the remaining $n - 2 - r_1$ are also randomly removed. The test goes on until the mth failure happens, at which point all remaining $n - m - r_1 - r_2 - \ldots - r_{m-1}$ units are discarded. The number of failures m in addition to the PC design $r_1, r_2, \ldots, r_m$ is predetermined and fixed. Assume that $z_{(1)}, z_{(2)}, \ldots, z_{(m)}$ indicate a PCT-II sample with $(r_1, r_2, \ldots, r_m)$ being the PC. A useful description and good summary of PC can be found in in a study by Balakrishnan and Aggrawala [33]. It should be noted that, when $r_1 = r_2 = \ldots = r_{m-1} = 0$ and $r_m = n - m$, this censoring reduces to CT-II. In addition, for $m = n$ and $r_i = 0$, $i = 1, 2, \ldots, n$, it reduces to a complete sample. Figure 1 represents this PC strategy.

This article looks into the DCRTE estimators for the MExp model in the presence of PCT-II data. We construct

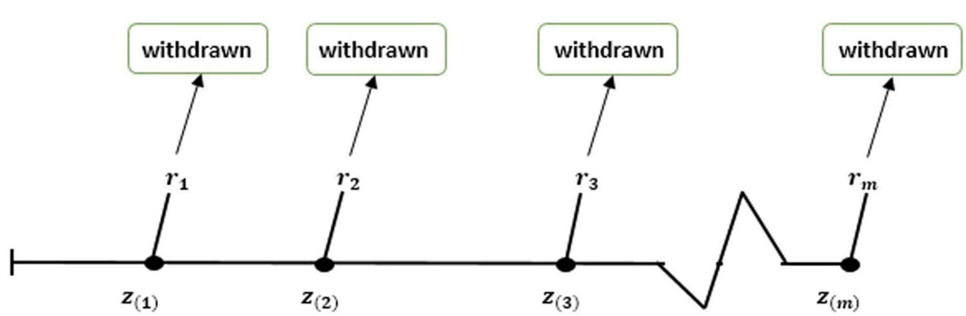

Figure 1: PCT-II scheme.

analytical formulas for the proposed DCRTE metric. The Bayesian estimator (BE) of the DCRTE measure is derived, along with its maximum likelihood estimator (MLE). The BE is regarded under squared error (SE), linear exponential (LIN), and general entropy (GE) loss functions (LoFs). Using various sample sizes and censoring strategies, Monte Carlo simulations are utilized to assess the performance of the DCRTE measure. Using actual data, the inferential techniques described in this study were also shown.

The article is broken down into six sections. Section 2 presents the formula for the DCRTE for the MExp distribution. PCT-II is used in Sections 3 and 4 to give DCRTE estimators by using the proposed approaches to estimation. In Section 5, data analysis and the numerical study of the produced estimators are covered. The study's summary and conclusions are included in Section 6.

2 Formula of the DCRTE for MExp distribution

The DCRTE formula for MExp distribution is shown in this section. The MExp distribution's DCRTE is determined by inserting Eq. (5) into Eq. (2):

$$\begin{aligned}\gamma(q,t) &= \frac{1}{(q-1)}\left(1-\frac{1}{\bar{G}^q(t)}\int_t^{\infty} e^{-\frac{qz}{\lambda}}\left(1+\frac{z}{\lambda}\right)^q \mathrm{d}z\right)\\ &= \frac{1}{(q-1)}\left(1-\frac{1}{\bar{G}^q(t)}I\right),\end{aligned} \tag{6}$$

where, $I=\int_t^{\infty} e^{-\frac{qz}{\lambda}}\left(1+\frac{z}{\lambda}\right)^q \mathrm{d}z$. To obtain I, we utilize the transformation $y = 1+\frac{z}{\lambda}$, which gives us $z = \lambda(y-1)$

$$I = \int_{1+\frac{t}{\lambda}}^{\infty} e^{-\frac{q\lambda(y-1)}{\lambda}}\lambda y^q\,\mathrm{d}y = \lambda e^q\int_{1+\frac{t}{\lambda}}^{\infty} y^q e^{-qy}\mathrm{d}y. \tag{7}$$

Let $x = qy$, then, (7) can be expressed as follows:

$$\begin{aligned}I &= \lambda e^q\int_{q\left(1+\frac{t}{\lambda}\right)}^{\infty}\left(\frac{x}{q}\right)^q e^{-x}\frac{1}{q}\mathrm{d}x\\ &= \frac{\lambda e^q}{[q]^{q+1}}\Gamma\left(q+1, q\left(1+\frac{t}{\lambda}\right)\right),\end{aligned} \tag{8}$$

where $\Gamma(.,x)$ stands for incomplete gamma function. The DCRTE for MExp distribution is expressed, as follows, after putting Eq. (8) into Eq. (6):

$$\gamma(q,t) = \frac{1}{q-1}\left(1-\frac{\lambda e^{A(\lambda,t,q)}\Gamma(q+1, A(\lambda,t,q))}{q(A(\lambda,t,q))^q}\right), \tag{9}$$

where $A(\lambda,t,q) = q\left(1+\frac{t}{\lambda}\right)$.

The plots of the DCRTE for the MExp distribution at various values of q and λ are shown in Figure 2. This figure shows that when the value of q is less than 1, the DCRTE for the MExp distribution takes increasing and then decreasing shapes. On the other hand, when the value of q is greater than 1, the DCRTE for the MExp distribution has rising forms. In addition, we see that when q is less than 1 and as the value of λ increases, the DCRTE for the MExp distribution exhibits increasing behavior.

3 Maximum likelihood estimation

Suppose $z_{(1)} \le z_{(2)} \le \ldots \le z_{(m)}$ represent a PCT-II sample of size m from a sample of size n with PDF (3), CDF (4), and censoring scheme $r_1, r_2, \ldots, r_m$. The likelihood function, under the PCT-II sample, is as follows:

$$\begin{aligned}L(\lambda) &= c\prod_{i=1}^{m} f(z_{(i)})[1-F(z_{(i)})]^{r_i}\\ &= c\prod_{i=1}^{m}\frac{z_{(i)}}{\lambda^2}e^{-\frac{(r_i+1)z_{(i)}}{\lambda}}\left(1+\frac{z_{(i)}}{\lambda}\right)^{r_i},\end{aligned} \tag{10}$$

where $c = n(n-r_1-1)(n-r_1-r_2-2)\ldots n-m+1-\sum_{i=1}^{m-1}r_i$. As a result, the constant is the number of different ways in which the m PCT-II order statistics might arise if the observed failure times are $z_{(1)}, z_{(2)}, \ldots, z_{(m)}$. The log-likelihood function of Eq. (10) is supplied by

$$\begin{aligned}\ln L(\lambda) = {} & \ln c - 2m\ln\lambda + \sum_{i=1}^{m}\ln z_{(i)}\\ & - \sum_{i=1}^{m}\frac{(r_i+1)z_{(i)}}{\lambda} + \sum_{i=1}^{m} r_i\ln\left[1+\frac{z_{(i)}}{\lambda}\right].\end{aligned} \tag{11}$$

Form Eq. (11), we derive the likelihood equation for λ as follows:

$$\frac{\mathrm{d}\ln L(\lambda)}{\mathrm{d}\lambda} = -\frac{2m}{\lambda} + \sum_{i=1}^{m}\frac{(r_i+1)z_{(i)}}{\lambda^2} - \frac{1}{\lambda}\sum_{i=1}^{m}\frac{r_i z_{(i)}}{\lambda + z_{(i)}}. \tag{12}$$

By solving the nonlinear Eq. (12) after setting it to 0, the MLE of λ can be found using the numerical technique. We can acquire the MLE of the DCRTE measure specified in Eq. (9) after the MLE of λ, say $\hat{\lambda}$, is computed. As a result, the invariance property is used to generate the MLE of DCRTE, designated by $\hat{\gamma}(q,t)$, by putting in Eq. (9) as shown below:

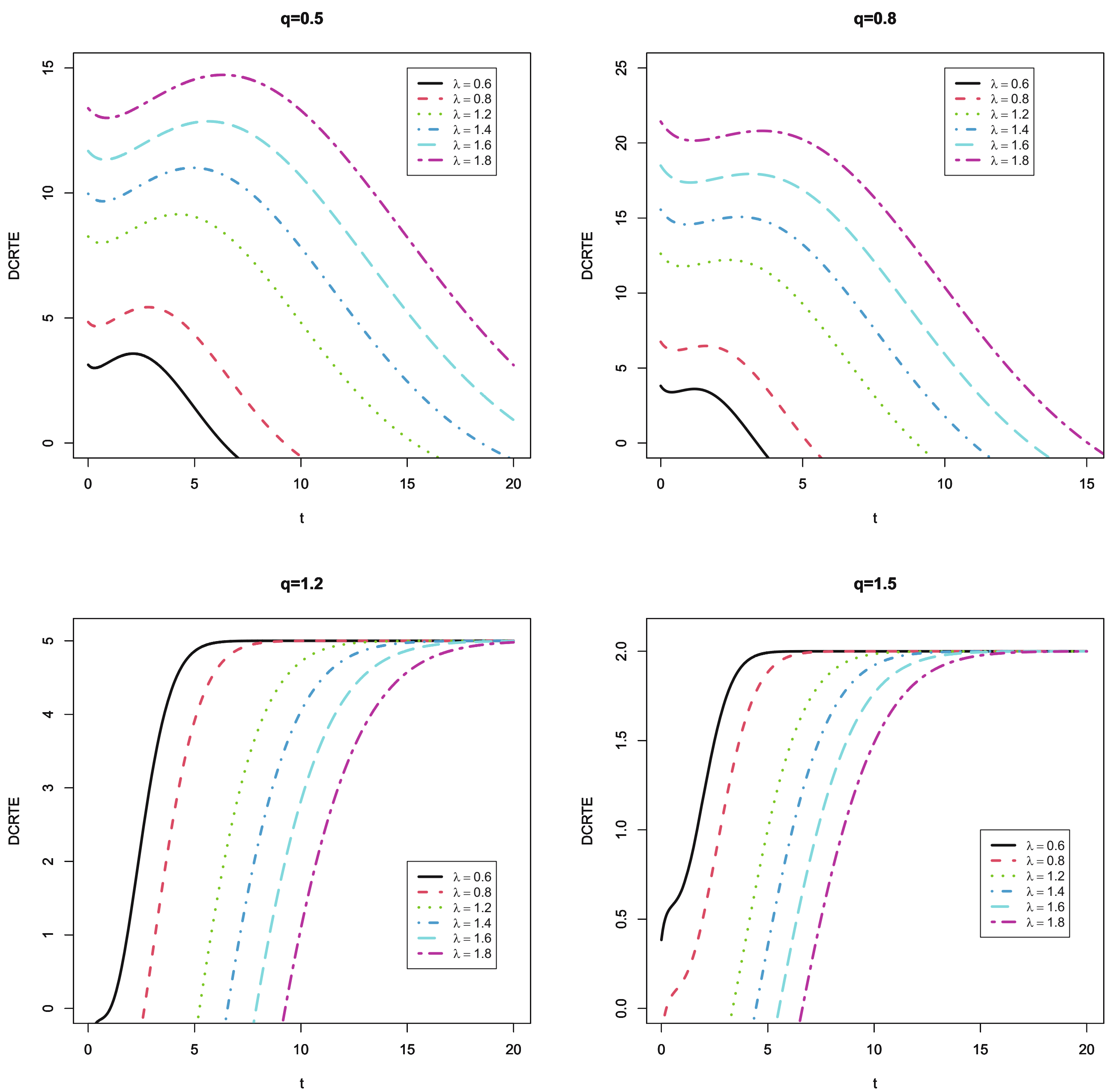

Figure 2: Plots of DCRTE for the MExp distribution.

$$\hat{\gamma}(q,t) = \frac{1}{q-1}\left(1 - \frac{\hat{\lambda}e^{A(\hat{\lambda},q,t)}\Gamma(q+1, A(\hat{\lambda},q,t))}{q(A(\hat{\lambda},q,t))^{q}}\right), \quad (13)$$

where $A(\hat{\lambda},q,t) = q\left(1+\frac{t}{\hat{\lambda}}\right)$.

The theoretical findings presented above can be further specialized in two situations. First, the MLE $\hat{\lambda}$ and $\hat{\gamma}(q,t)$ are yielded when $r_1 = r_2 = \ldots = r_{m-1} = 0$ and $r_m = n - m$ *via* CT-II. Second, we obtain the recommended MLE of λ and $\gamma(q,t)$ for $r_1 = r_2 = \ldots = r_m = 0$.

4 Bayesian estimation

The BE of the DCRTE for MExp distribution under the PCT-II scheme will be covered in this section. Different LoFs, such as the SE, LIN, and GE, can be taken into consideration for Bayesian estimation. For an informative prior (IP) distribution of the parameter λ of the MExp distribution, we can suggest using independent gamma prior for λ having PDF

$$\pi(\lambda) \propto \lambda^{a}e^{-b\lambda} \quad \lambda > 0,\ a > 0,\ b > 0,$$

where the parameters a and b are chosen to represent the prior knowledge about the unknown parameter. The corresponding posterior density given the observed data, $\mathbf{z} = (z_{(1)}, z_{(2)}, \ldots, z_{(m)})$, is given as follows:

$$\pi(\lambda|\mathbf{z}) = \frac{\pi(\lambda)L(\lambda)}{\int_0^\infty \pi(\lambda)L(\lambda)d\lambda} \propto \pi(\lambda)L(\lambda).$$

As a result, the posterior density function may be expressed as follows:

$$\pi(\lambda|\mathbf{z}) \propto \lambda^{-2m+a} e^{-b\lambda} \prod_{i=1}^{m} z_{(i)} e^{-\frac{(r_i+1)z_{(i)}}{\lambda}} \left(1 + \frac{z_{(i)}}{\lambda}\right)^{r_i}. \tag{14}$$

The BE of $\gamma(q,t)$, denoted by $\hat{\gamma}_{\mathrm{BE_{SE}}}(q,t)$, under the SE LoF, is given as follows:

$$\hat{\gamma}_{\mathrm{BE_{SE}}}(q,t) = E[\gamma(q,t)|\mathbf{z}] = \int_0^\infty \gamma(q,t)\pi(\lambda|z)\mathrm{d}\lambda. \tag{15}$$

Based on LIN LoF, the Bayes estimator of $\gamma(q,t)$, say $\hat{\gamma}_{\mathrm{BE_{LIN}}}(q,t)$, is as follows:

$$\begin{aligned}\hat{\gamma}_{\mathrm{BE_{LIN}}}(q,t) &= -\frac{1}{\nu}\log E[\mathrm{e}^{\{-\nu\gamma(q,t)\}}|\mathbf{z}] \\ &= -\frac{1}{\nu}\log\left[\int_0^\infty \mathrm{e}^{\{-\nu\gamma(q,t)\}}\pi(\lambda|z)\mathrm{d}\lambda\right].\end{aligned} \tag{16}$$

Furthermore, we consider the GE LoF where the Bayes estimator of $\gamma(q,t)$, say $\hat{\gamma}_{\mathrm{BE}_{GE}}(q,t)$, is as follows:

$$\begin{aligned}\hat{\gamma}_{\mathrm{BE}_{GE}}(q,t) &= [E((\gamma(q,t))^{-\tau}|\mathbf{z})]^{-1/\tau} \\ &= \left[\int_0^\infty (\gamma(q,t))^{-\tau}\pi(\lambda|z)\mathrm{d}\lambda\right]^{-1/\tau}.\end{aligned} \tag{17}$$

It is seen that the estimates given by Eqs. (15)–(17) cannot be simplified into closed form expressions. Therefore, we next apply the Markov chain Monte Carlo (MCMC) approach and generate a posterior sample using Metropolis–Hasting (MH) algorithm to obtain the desired DCRTE under BEs. Note that, by selecting the hyper parameters $a = b = 0$ and using the same MCMC approach, the previous BE is obtained in the case of noninformative prior (Non-IP).

4.1 MH algorithm

In order to execute the MH algorithm for the DCRTE of MExp distribution, a suggested distribution and the initial values of the unknown parameter λ have to be specified. For the proposal distribution, a normal distribution will be taken into account, *i.e.*, $q(\lambda'|\lambda) \equiv N(\lambda, S_\lambda)$, where S_λ represents the variance–covariance matrix (V-CM). For the initial values, the MLE for λ is considered, *i.e.*, $\lambda^{(0)} = \hat{\lambda}_{\mathrm{MLE}}$. The choice of S_λ is thought to be the asymptotic V-CM, say $I^{-1}(\hat{\lambda}_{\mathrm{MLE}})$, where $I(.)$ is the Fisher information matrix. In this manner, the MH method uses the stages listed below to draw a sample from the posterior density provided by Eq. (14):

Step 1. Set initial value of λ as $\lambda^{(0)} = \hat{\lambda}_{\mathrm{MLE}}$.

Step 2. For $i = 1, 2, \ldots, M$ repeat the following steps:

- Set $\lambda = \lambda^{(i-1)}$.
- Create a new candidate parameter value using $N(\lambda, S_\lambda)$.
- Compute the formula $\delta = \frac{\pi(\lambda' \mid z)}{\pi(\lambda|z)}$, where $\pi(\cdot)$ is the posterior density (Eq. (14)).
- Create a sample u from the uniform $U(0,1)$ distribution.
- Accept or reject the new candidate λ'

$$\begin{cases}\text{If} \quad u \le \delta & \text{put} \quad \lambda^{(i)} = \lambda' \\ \text{elsewhere} & \text{put} \quad \lambda^{(i)} = \lambda.\end{cases}$$

Eventually, a portion of the initial samples can indeed be removed (burn-in), and the remaining samples can be used to calculate Bayes estimates using random samples of size M drawn from the posterior density. The BE of $\gamma(q,t)$ is assessed by employing MCMC with SE, LIN, and GE LoFs as follows:

$$\hat{\gamma}_{\mathrm{BE_{SE}}}(q,t) = \frac{1}{M - l_{\mathrm{B}}}\sum_{i=l_{\mathrm{B}}}^{M} \gamma(q,t)^{(i)}, \tag{18}$$

$$\hat{\gamma}_{\mathrm{BE_{LIN}}}(q,t) = -\frac{1}{\nu}\log\left[\frac{1}{M - l_{\mathrm{B}}}\sum_{i=l_{\mathrm{B}}}^{M} \mathrm{e}^{\{-\nu\gamma(q,t)^{(i)}\}}\right], \tag{19}$$

$$\hat{\gamma}_{\mathrm{BE}_{GE}}(q,t) = \left[\frac{1}{M - l_{\mathrm{B}}}\sum_{i=l_{\mathrm{B}}}^{M} (\gamma(q,t)^{(i)})^{-\tau}\right]^{-1/\tau}, \tag{20}$$

where l_{B} represents the number of burn-in samples.

5 Data analysis and simulation study

This section's goal is to examine the behavior of the suggested DCRTE estimators for the MExp distribution under the PCT-II scheme that was addressed in earlier parts. For demonstration purposes, we investigate an actual dataset. Furthermore, we conducted a simulation study to investigate the behavior of the proposed approaches and evaluate the estimates' performance under various censoring schemes. We used R, a statistical programming language, for calculation. Furthermore, one can utilize *bbmle* package to compute MLEs in R-language.

5.1 Data analysis

A real dataset is analyzed for illustrative purposes as well as to assess the statistical performances of the MLE and BEs for DCRTE in the case of the MExp distribution under different PCT-II schemes.

According to the International Bone Marrow Transplant Registry, 101 patients with advanced acute myelogenous leukemia are linked to the following datasets (see [34]). Fifty-one of these patients have received an autologous bone marrow transplant. Fifty-one patients had an allogeneic bone marrow transplant. The 51 autologous transplant patients' leukemia-free survival periods (in months) are shown in Table 1.

We begin by determining if the MExp distribution is appropriate for evaluating this dataset. To assess the quality of fit, we provide the MLE of λ and the Kolmogorov–Smirnov (K–S) test statistic value. The estimated K–S distance for the MExp distribution between the empirical and fitted distributions is 0.13301, and its p-value is 0.3276 where $\hat{\lambda} = 8.36519$, indicating that this distribution may be deemed an appropriate model for the current dataset. For graphically fitting the given dataset, the empirical CDF (ECDF), the empirical PDF (EPDF), the empirical SF (ECDF), and probability–probability (PP)-plots are represented in Figure 3 for the MExp distribution. This figure also indicates that the MExp distribution provides a good fit for this dataset.

For convergence of BEs, we divide real data by 100, from the original data, hence one can generate, *e.g.*, four schemes, namely, Scheme 1 (Sch.1), Scheme 2 (Sch.2), Scheme 3 (Sch.3), and Scheme 4 (Sch.4) from PCT-II samples with the number of stages $m = 26$ and eliminated items r_i are presumed to be the following:

Sch.1: $r_1 = r_2 = \ldots = r_{25} = 0, r_{26} = 25$
Sch.2: $r_1 = r_2 = \ldots = r_{25} = 1, r_{26} = 0$
Sch.3: $r_1 = 26, r_2 = \ldots = r_{25} = 0$
Sch.4: $r_1 = r_2 = \ldots = r_{26} = 0, n = m = 51$.

Note that: Sch.1 represents the usual CT-II and Sch.4 represents a complete sampling case. Two distinct values of the constant t are used, 0.5 and 1.5 given $q = 2$.

In Table 2, the MLEs of λ have been investigated and then plugged into DCRTE measures at the proposed schemes for PCT-II samples as in the suggested real dataset. In addition, BEs can be computed employing the MH algorithm under Non-IP (uniform prior), *i.e.*, the hyper-parameter values are taken as $a = b = 0$. Different LoFs, including SE, LIN-1 ($\nu = 0.5$), LIN-2 ($\nu = -0.5$), GE-1 ($\tau = -0.5$), and GE-2 ($\tau = 0.5$) are assumed. It is indicated that, while generating samples from the posterior distribution utilizing the MH algorithm, initial values of λ are considered as $\lambda^{(0)} = \hat{\lambda}_{\text{MLE}}$. Eventually, among the total 10,000 samples generated by the posterior density, 2,000 burn-in samples were removed, and a DCRTE estimate was derived.

From Table 2, one can infer that the behavior of Sch.3 is better for estimating $\gamma(q, t)$. In addition, as t increases, the estimates of $\gamma(q, t)$ increase. The convergence (scatter plot, histogram, and cumulative mean) of MCMC estimation for λ and $\gamma(q, t)$ is shown in Figure 4 in the case of complete sampling. From Figure 4, the normality of MCMC estimates can be observed.

5.2 Simulation study

In this part, we use a Monte Carlo simulation analysis to assess the effectiveness of estimate techniques, specifically ML and Bayesian using MCMC, for MExp distribution under the PCT-II scheme. We produce 1,000 MLEs from the MExp distribution with the following principles:

1) The parameter of the MExp distribution λ is assumed to be 0.4, 0.8, and 1.
2) The value of q is equal to 2, and the constant value of t is equal 0.5, 1.5.
3) The true value of $\gamma(q, t)$, say γ_1 for simplified form, at $t = 0.5, q = 2$, and $\lambda = 0.4$ is $\gamma_1 = 0.69135$. The true value of $\gamma(q, t)$, say γ_2 for simplified form, at $t = 1.5, q = 2$, and $\lambda = 0.4$, is $\gamma_2 = 0.75346$.
4) The true value of $\gamma(q, t)$, say γ_3 for simplified form, at $t = 0.5, q = 2$, and $\lambda = 0.8$ is $\gamma_3 = 0.27811$. The true value of $\gamma(q, t)$, say γ_4 for simplified form, at $t = 1.5, q = 2$, and $\lambda = 0.8$ is $\gamma_4 = 0.43667$.

Table 1: The leukemia-free survival times (in months) dataset

0.658	0.822	1.414	2.500	3.322	3.816	4.737	4.836	4.934	5.033	5.757
5.855	5.987	6.151	6.217	6.447	8.651	8.717	9.441	10.329	11.480	12.007
12.007	12.237	12.401	13.059	14.474	15.000	15.461	15.757	16.480	16.711	17.204
17.237	17.303	17.664	18.092	18.092	18.750	20.625	23.158	27.730	31.184	32.434
35.921	42.237	44.638	46.480	47.467	48.322	56.086				

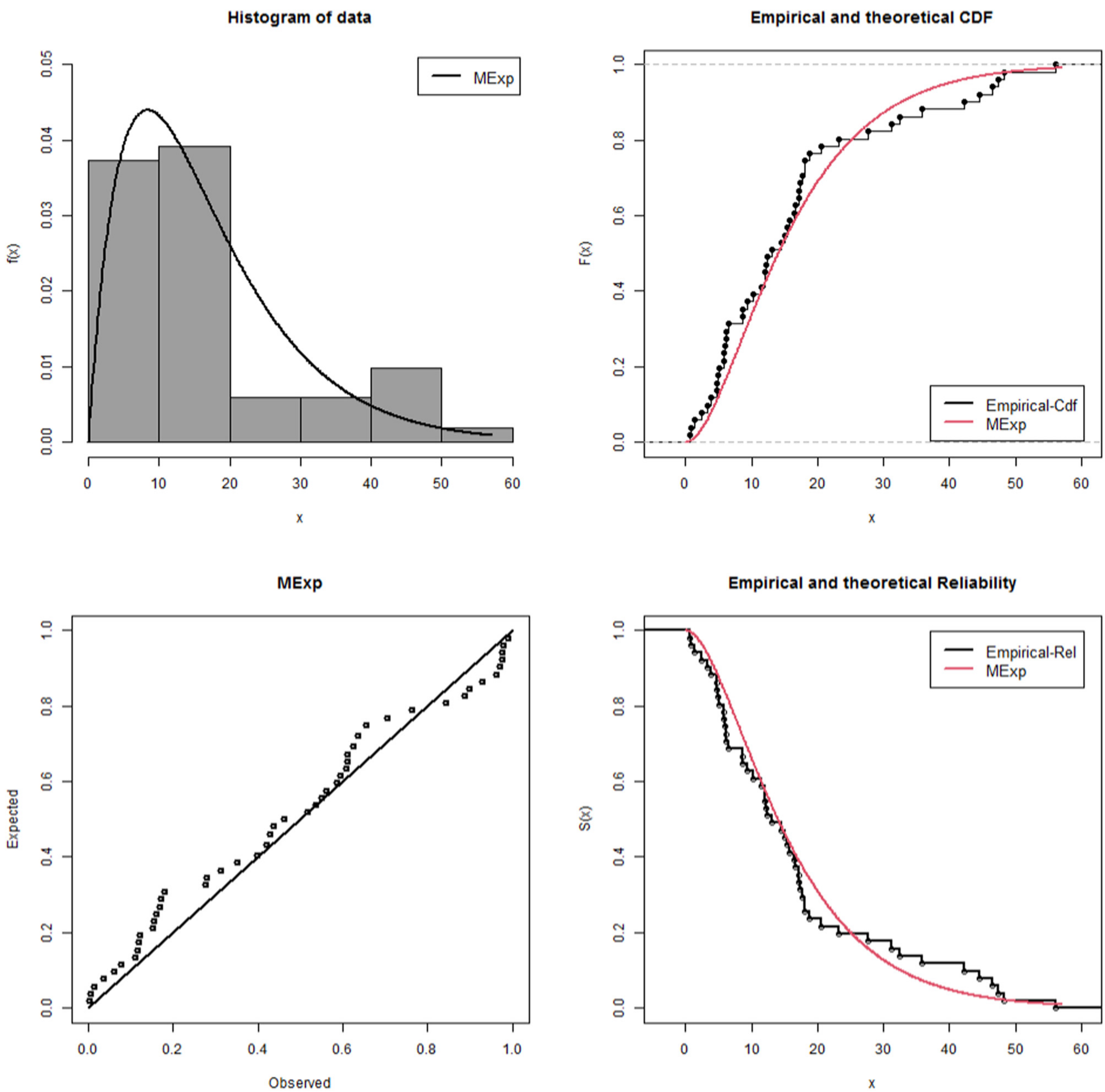

Figure 3: The EPDF, ECDF, ESF, and PP plots for MExp model.

Table 2: The estimate values of ($\gamma(q,t)$) for MExp distribution under different PCT-II schemes for leukemia-free survival times data

t	Sch.	ML	BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2
0.5	Sch.1	0.84347	0.84124	0.84142	0.84106	0.84103	0.84059
	Sch.2	0.93009	0.92865	0.92868	0.92861	0.92861	0.92854
	Sch.3	0.95671	0.95574	0.95576	0.95573	0.95573	0.95570
	Sch.4	0.95176	0.95116	0.95117	0.95115	0.95115	0.95113
1.5	Sch.1	0.84347	0.84739	0.84750	0.84729	0.84727	0.86702
	Sch.2	0.93772	0.93660	0.93662	0.93658	0.93658	0.93653
	Sch.3	0.96017	0.95939	0.95940	0.95938	0.95937	0.95935
	Sch.4	0.95176	0.95543	0.95543	0.95542	0.95542	0.95541

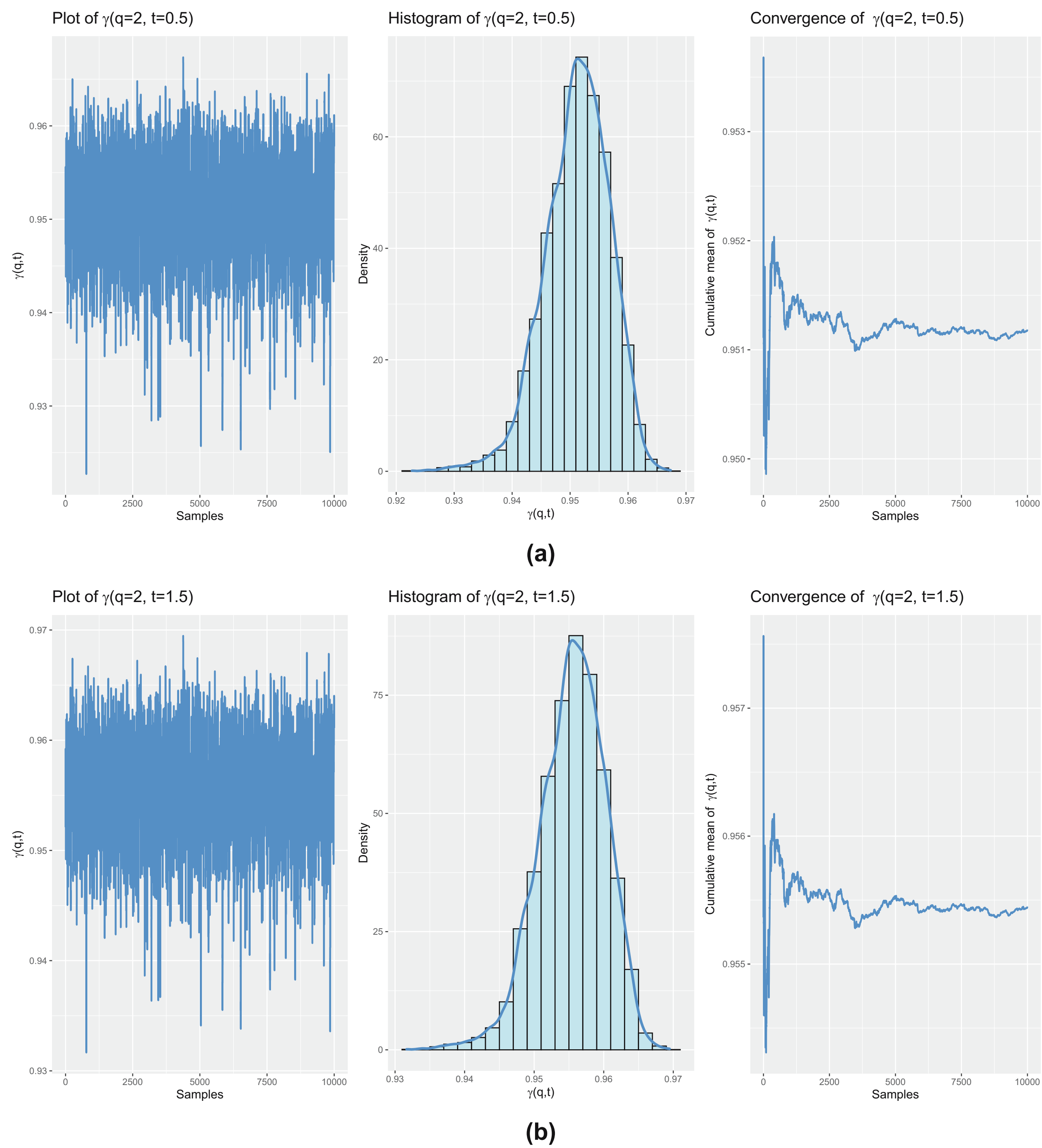

Figure 4: Convergence of MCMC estimates for $\gamma(q,t)$ using MH algorithm. (a) Graphs of MCMC estimates for $\gamma(q=2, t=0.5)$, (b) Graphs of MCMC estimates for $\gamma(q=2, t=1.5)$.

5) The true value of $\gamma(q,t)$, say γ_5 for simplified form, at $t = 0.5$, $q = 2$, and $\lambda = 1$ is $\gamma_5 = 0.05556$. The true value of $\gamma(q,t)$, say γ_6 for simplified forms, at $t = 1.5$, $q = 2$, and $\lambda = 1$ is $\gamma_6 = 0.26000$.
6) The sample sizes are $n = 30$ and $n = 60$.
7) The number of stages of the PCT-II scheme are $m = 10, 20$, and 40.
8) Removed items r_i are assumed at n and m values as shown in Table 3 where $r_m = n - (\sum_{i=1}^{m-1} r_i + m)$ and r is the number of failure items.

Table 3: Different patterns for removing items from life test at different number of stages

n	m	Censoring schemes			
		Sch.1	Sch.2	Sch.3	Sch.4
30	10	$(20, 0^{*9})$	$(10, 0^{*8}, 10)$	$(0^{*4}, 10, 10, 0^{*4})$	$(0^{*9}, 20)$
	20	$(10, 0^{*19})$	$(5, 0^{*18}, 5)$	$(0^{*9}, 5, 5, 0^{*9})$	$(0^{*19}, 10)$
60	20	$(40, 0^{*19})$	$(20, 0^{*18}, 20)$	$(0^{*9}, 20, 20, 0^{*9})$	$(0^{*19}, 40)$
	40	$(20, 0^{*39})$	$(10, 0^{*38}, 10)$	$(0^{*19}, 10, 10, 0^{*19})$	$(0^{*39}, 20)$

Here, $(1^{(3)}, 0)$, for example, means that the censoring scheme employed is $(1, 1, 1, 0)$.

MLEs are determined employing PCT-II based on the generated data and the preceding assumptions. When calculating MLEs, the initial estimate values are assumed to be the same as the actual parameter values. These values, known as MLEs, are then employed to compute the DCRTE $(\gamma(q, t))$ given q and t.

For the Bayesian method, BEs using the MH algorithm using Non-IP and IP under different LoFs are computed. Thus:

- For Non-IP, the hyper-parameter values are $a = b = 0$, thus $\pi(\lambda) = \frac{1}{\lambda}$.
- For IP, the hyper-parameter values are taken as $a = 0.5$ and $b = 1.5$.
- Different LoFs, including SE, LIN-1 ($\nu = 0.5$), LIN-2 ($\nu = -0.5$), GE-1 ($\tau = -0.5$), and GE-2 ($\tau = 0.5$) are assumed.

These values are then employed to determine the estimated values. When using the MH technique, the MLEs take into consideration the related V-CM S_λ of $\hat{\lambda}_{\text{MLE}}$ as initial estimate values. Finally, 2,000 burn-in samples are removed from the total 10,000 samples created by the posterior density, and the estimates of DCRTE $(\gamma(q, t))$ are derived.

The mean squared error (MSEr) for all DCRTE estimates is reported in Tables 1–6 covering all inputs of Monte Carlo simulation. From tabulated result values, it can be noted that:

1) For MLEs, higher values of n and m lead to a decrease in MSEr and convergence to the initial values of $\gamma(q, t)$ as expected.
2) The MSEr of Bayes estimates under IP gradually decreases as n and m increase.
3) As the true value of $\gamma(q, t)$ increases, the values of $(\hat{\gamma}(q, t))$ approach to true values implying that there is more information.
4) As can be seen in Tables 4 and 6, in the majority of cases, the precision measure of DCRTE under GE-1 and LIN-1 for IP and Non-IP are more effective than those under GE-2 and LIN-2 for all schemes (Table 7).
5) At true value $\gamma_5 = 0.05556$, the Bayes estimates of $\gamma(q, t)$ under LIN-1 and GE-2 are preferable to the corresponding estimates under LIN-2 and GE-1 in both IP and Non-IP (see Tables 8 and 9) for all schemes.

Table 4: MSEr of DCRTE estimates based on MExp distribution at $\lambda = 0.4$ under different PCT-II schemes at $n = 30$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_1 = 0.69135$												
10	Sch.1	0.00562	0.00691	0.00669	0.00735	0.00718	0.00814	0.00759	0.00716	0.00785	0.00903	0.01026
	Sch.2	0.00499	0.00594	0.00579	0.00643	0.00628	0.00704	0.00661	0.00610	0.00681	0.00738	0.00853
	Sch.3	0.00493	0.00583	0.00568	0.00625	0.00615	0.00688	0.00644	0.00600	0.00666	0.00719	0.00847
	Sch.4	0.00431	0.00490	0.00479	0.00546	0.00511	0.00589	0.00562	0.00503	0.00580	0.00590	0.00706
20	Sch.1	0.00281	0.00302	0.00298	0.00310	0.00311	0.00325	0.00314	0.00306	0.00319	0.00333	0.00350
	Sch.2	0.00273	0.00295	0.00291	0.00294	0.00303	0.00307	0.00298	0.00299	0.00303	0.00321	0.00332
	Sch.3	0.00267	0.00289	0.00285	0.00289	0.00296	0.00302	0.00293	0.00293	0.00297	0.00316	0.00323
	Sch.4	0.00261	0.00286	0.00282	0.00298	0.00295	0.00313	0.00303	0.00291	0.00308	0.00314	0.00335
$t = 1.5$, $\gamma_2 = 0.75346$												
10	Sch.1	0.00318	0.00388	0.00380	0.00433	0.00404	0.00467	0.00445	0.00397	0.00457	0.00441	0.00520
	Sch.2	0.00267	0.00321	0.00315	0.00344	0.00334	0.00369	0.00352	0.00328	0.00362	0.00366	0.00410
	Sch.3	0.00282	0.00347	0.00339	0.00389	0.00363	0.00417	0.00399	0.00355	0.00408	0.00403	0.00466
	Sch.4	0.00262	0.00319	0.00312	0.00331	0.00331	0.00353	0.00338	0.00325	0.00346	0.00360	0.00396
20	Sch.1	0.00171	0.00189	0.00187	0.00194	0.00192	0.00200	0.00196	0.00191	0.00198	0.00199	0.00208
	Sch.2	0.00160	0.00167	0.00166	0.00177	0.00169	0.00182	0.00179	0.00168	0.00181	0.00174	0.00188
	Sch.3	0.00161	0.00177	0.00175	0.00170	0.00179	0.00174	0.00172	0.00178	0.00173	0.00186	0.00180
	Sch.4	0.00157	0.00166	0.00165	0.00170	0.00168	0.00174	0.00171	0.00167	0.00173	0.00173	0.00180

Table 5: MSEr for DCRTE estimates based on MExp distribution at $\lambda = 0.4$ under different PCT-II schemes at $n = 60$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_1 = 0.69135$												
20	Sch.1	0.00260	0.00291	0.00287	0.00307	0.00301	0.00324	0.00313	0.00296	0.00319	0.00327	0.00351
	Sch.2	0.00249	0.00264	0.00262	0.00265	0.00269	0.00273	0.00268	0.00267	0.00270	0.00281	0.00287
	Sch.3	0.00247	0.00271	0.00268	0.00269	0.00278	0.00278	0.00272	0.00275	0.00275	0.00296	0.00293
	Sch.4	0.00238	0.00257	0.00254	0.00257	0.00263	0.00265	0.00260	0.00260	0.00263	0.00276	0.00279
40	Sch.1	0.00163	0.00168	0.00168	0.00166	0.00170	0.00168	0.00166	0.00169	0.00167	0.00173	0.00171
	Sch.2	0.00133	0.00134	0.00134	0.00133	0.00135	0.00134	0.00133	0.00134	0.00134	0.00136	0.00136
	Sch.3	0.00136	0.00135	0.00135	0.00138	0.00136	0.00140	0.00139	0.00136	0.00139	0.00138	0.00142
	Sch.4	0.00141	0.00137	0.00137	0.00141	0.00138	0.00142	0.00141	0.00138	0.00142	0.00140	0.00145
$t = 1.5$, $\gamma_2 = 0.75346$												
20	Sch.1	0.00173	0.00191	0.00189	0.00199	0.00195	0.00206	0.00202	0.00193	0.00204	0.00203	0.00215
	Sch.2	0.00136	0.00148	0.00147	0.00152	0.00151	0.00156	0.00153	0.00150	0.00155	0.00156	0.00162
	Sch.3	0.00129	0.00137	0.00136	0.00141	0.00140	0.00145	0.00142	0.00139	0.00144	0.00144	0.00150
	Sch.4	0.00136	0.00148	0.00147	0.00147	0.00150	0.00150	0.00148	0.00150	0.00150	0.00155	0.00155
40	Sch.1	0.00089	0.00091	0.00090	0.00090	0.00091	0.00091	0.00091	0.00091	0.00091	0.00092	0.00092
	Sch.2	0.00080	0.00081	0.00080	0.00080	0.00081	0.00081	0.00080	0.00081	0.00081	0.00082	0.00081
	Sch.3	0.00074	0.00076	0.00075	0.00079	0.00076	0.00079	0.00079	0.00076	0.00079	0.00077	0.00081
	Sch.4	0.00078	0.00078	0.00078	0.00079	0.00078	0.00080	0.00079	0.00078	0.00080	0.00079	0.00081

6) In both IP and Non-IP, Bayes estimates of $\gamma(q, t)$ under LIN-1 and GE-2 are commonly chosen over similar estimates under those LIN-2 and GE-1 at true value $\gamma_6 = 0.26000$ for all schemes (see Tables 8 and 9).
7) Bayes estimate under Non-IP has a decreasing pattern as compared to the others under IP. However, Bayesian estimates under Non-IP provide larger estimate values.

Table 6: MSEr for DCRTE estimates based on MExp distribution at $\lambda = 0.8$ under different PCT-II schemes at $n = 30$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_3 = 0.27811$												
10	Sch.1	0.02942	0.03725	0.03627	0.04669	0.01988	0.01905	0.04843	0.03834	0.05032	0.02386	0.02461
	Sch.2	0.02481	0.02967	0.02897	0.03516	0.01682	0.01682	0.03653	0.03048	0.03808	0.02079	0.02262
	Sch.3	0.02613	0.03250	0.03167	0.03965	0.01723	0.01718	0.04108	0.03346	0.04265	0.02117	0.02269
	Sch.4	0.02436	0.03032	0.02950	0.03558	0.01746	0.01659	0.03685	0.03127	0.03825	0.02184	0.02162
20	Sch.1	0.01589	0.01940	0.01911	0.02021	0.01331	0.01308	0.02065	0.01973	0.02114	0.01725	0.01796
	Sch.2	0.01488	0.01709	0.01688	0.01828	0.01195	0.01179	0.01865	0.01734	0.01906	0.01553	0.01623
	Sch.3	0.01503	0.01763	0.01739	0.01742	0.01285	0.01193	0.01774	0.01792	0.01809	0.01670	0.01641
	Sch.4	0.01464	0.01638	0.01620	0.01756	0.01233	0.01198	0.01792	0.01659	0.01832	0.01630	0.01692
$t = 1.5$, $\gamma_4 = 0.43667$												
10	Sch.1	0.01941	0.02356	0.02305	0.02801	0.01948	0.02036	0.02889	0.02411	0.02985	0.02683	0.02852
	Sch.2	0.01540	0.01999	0.01956	0.02278	0.01687	0.01775	0.02350	0.02046	0.02429	0.02368	0.02626
	Sch.3	0.01444	0.01844	0.01793	0.02171	0.01554	0.01709	0.02250	0.01900	0.02334	0.02239	0.02560
	Sch.4	0.01522	0.01807	0.01766	0.02261	0.01517	0.01814	0.02326	0.01853	0.02396	0.02134	0.02630
20	Sch.1	0.01014	0.01156	0.01140	0.01223	0.01146	0.01191	0.01247	0.01175	0.01273	0.01535	0.01620
	Sch.2	0.00890	0.01013	0.00998	0.01138	0.01007	0.01126	0.01163	0.01031	0.01190	0.01369	0.01570
	Sch.3	0.00896	0.01053	0.01039	0.01097	0.01038	0.01081	0.01120	0.01068	0.01147	0.01342	0.01528
	Sch.4	0.00861	0.00975	0.00962	0.01053	0.00945	0.01040	0.01075	0.00990	0.01099	0.01239	0.01430

Table 7: MSEr for DCRTE estimates based on MExp distribution at $\lambda = 0.8$ under different PCT-II schemes at $n = 60$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_3 = 0.27811$												
20	Sch.1	0.01483	0.01623	0.01604	0.01756	0.01200	0.01185	0.01792	0.01647	0.01834	0.01595	0.01663
	Sch.2	0.01267	0.01496	0.01473	0.01518	0.01080	0.01065	0.01549	0.01522	0.01584	0.01492	0.01550
	Sch.3	0.01401	0.01657	0.01628	0.01784	0.01172	0.01170	0.01828	0.01690	0.01877	0.01605	0.01695
	Sch.4	0.01183	0.01336	0.01313	0.01444	0.00978	0.00984	0.01480	0.01363	0.01521	0.01412	0.01515
40	Sch.1	0.00767	0.00793	0.00792	0.00833	0.00710	0.00728	0.00839	0.00795	0.00846	0.00944	0.01008
	Sch.2	0.00771	0.00780	0.00779	0.00849	0.00707	0.00747	0.00855	0.00782	0.00861	0.00927	0.01029
	Sch.3	0.00806	0.00862	0.00857	0.00882	0.00780	0.00787	0.00889	0.00867	0.00898	0.01060	0.01105
	Sch.4	0.00719	0.00752	0.00749	0.00775	0.00691	0.00697	0.00779	0.00755	0.00785	0.00923	0.00968
$t = 1.5$, $\gamma_4 = 0.43667$												
20	Sch.1	0.00909	0.01108	0.01090	0.01147	0.01074	0.01140	0.01172	0.01128	0.01201	0.01419	0.01638
	Sch.2	0.00765	0.00862	0.00851	0.00951	0.00880	0.00952	0.00970	0.00874	0.00991	0.01153	0.01300
	Sch.3	0.00835	0.00917	0.00904	0.01026	0.00905	0.01030	0.01046	0.00931	0.01068	0.01168	0.01397
	Sch.4	0.00684	0.00746	0.00739	0.00890	0.00749	0.00898	0.00908	0.00755	0.00927	0.00945	0.01217
40	Sch.1	0.00468	0.00486	0.00485	0.00512	0.00497	0.00535	0.00516	0.00487	0.00520	0.00562	0.00628
	Sch.2	0.00471	0.00478	0.00477	0.00523	0.00490	0.00541	0.00526	0.00479	0.00530	0.00545	0.00628
	Sch.3	0.00493	0.00528	0.00526	0.00543	0.00543	0.00568	0.00547	0.00532	0.00552	0.00617	0.00656
	Sch.4	0.00439	0.00461	0.00459	0.00477	0.00472	0.00498	0.00480	0.00463	0.00483	0.00527	0.00573

Table 8: MSEr for DCRTE estimates based on MExp distribution at $\lambda = 1$ under different PCT-II schemes at $n = 30$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_5 = 0.05556$												
10	Sch.1	0.04913	0.06328	0.06216	0.07966	0.03555	0.03164	0.08217	0.06460	0.08494	0.02255	0.01960
	Sch.2	0.04143	0.05078	0.05001	0.05995	0.02982	0.02630	0.06189	0.05171	0.06410	0.01862	0.01586
	Sch.3	0.04361	0.05532	0.05434	0.06697	0.03016	0.02681	0.06910	0.05650	0.07147	0.01863	0.01630
	Sch.4	0.04070	0.05226	0.05123	0.06142	0.03050	0.02649	0.06330	0.05350	0.06541	0.01975	0.01616
20	Sch.1	0.02655	0.03405	0.03363	0.03589	0.02076	0.01851	0.03664	0.03455	0.03750	0.01277	0.01073
	Sch.2	0.02562	0.03206	0.03153	0.03572	0.01875	0.01645	0.03669	0.03269	0.03778	0.01134	0.00964
	Sch.3	0.02368	0.02824	0.02774	0.03289	0.01627	0.01410	0.03386	0.02883	0.03493	0.00942	0.00765
	Sch.4	0.02111	0.02530	0.02491	0.02821	0.01591	0.01460	0.02889	0.02578	0.02967	0.00914	0.00799
$t = 1.5$, $\gamma_6 = 0.26000$												
10	Sch.1	0.03169	0.04137	0.04060	0.05345	0.02259	0.02097	0.05506	0.04223	0.05678	0.02526	0.02507
	Sch.2	0.02667	0.03313	0.03258	0.03986	0.01885	0.01846	0.04109	0.03375	0.04244	0.02162	0.02285
	Sch.3	0.02819	0.03628	0.03561	0.04489	0.01938	0.01877	0.04625	0.03704	0.04773	0.02222	0.02295
	Sch.4	0.02615	0.03409	0.03339	0.04102	0.01968	0.01793	0.04222	0.03489	0.04354	0.02283	0.02199
20	Sch.1	0.01704	0.02208	0.02178	0.02353	0.01459	0.01389	0.02401	0.02241	0.02454	0.01774	0.01812
	Sch.2	0.01647	0.02085	0.02051	0.02358	0.01353	0.01303	0.02418	0.02125	0.02483	0.01727	0.01776
	Sch.3	0.01530	0.01838	0.01805	0.02178	0.01211	0.01181	0.02237	0.01874	0.02301	0.01616	0.01695
	Sch.4	0.01357	0.01640	0.01614	0.01847	0.01125	0.01170	0.01890	0.01670	0.01937	0.01474	0.01667

8) The MSEr of all estimates based on Sch.3 has the smallest values, at true value $\gamma_2 = 0.75346$, when compared to others at $m = 20, 40$, and $n = 60$ (see Table 5). In addition, at $\gamma_1 = 0.69135$, as seen in Table 5, the MSEr of all estimates based on Sch.3 also obtains the smallest values in comparison to other estimates at $m = 40$, and $n = 60$.
9) As shown in Table 6, the MSEr of all estimates based on Sch.3 typically has the fewest values, when compared to other estimates, in the majority of cases, for $m = 10$, and $n = 30$ with a real value of $\gamma_4 = 0.43667$.
10) The MSEr of all estimates based on Sch.2 generally yields the smallest values when compared to others,

Table 9: MSEr for DCRTE estimates based on MExp distribution at $\lambda = 1$ under different PCT-II schemes at $n = 60$

m	Sch.	MLE	BE-MCMC: IP					BE-MCMC: Non-IP				
			SE	LIN-1	LIN-2	GE-1	GE-2	SE	LIN-1	LIN-2	GE-1	GE-2
$t = 0.5$, $\gamma_5 = 0.05556$												
20	Sch.1	0.02692	0.03384	0.03326	0.03810	0.01965	0.01730	0.03916	0.03452	0.04035	0.01194	0.01018
	Sch.2	0.02112	0.02494	0.02453	0.02827	0.01453	0.01252	0.02906	0.02543	0.02994	0.00834	0.00671
	Sch.3	0.02000	0.02383	0.02347	0.02635	0.01501	0.01380	0.02697	0.02427	0.02768	0.00861	0.00761
	Sch.4	0.02109	0.02608	0.02569	0.02613	0.01530	0.01385	0.02669	0.02655	0.02733	0.00908	0.00778
40	Sch.1	0.01380	0.01543	0.01542	0.01546	0.01061	0.00935	0.01561	0.01547	0.01579	0.00632	0.00525
	Sch.2	0.01258	0.01422	0.01419	0.01373	0.00937	0.00833	0.01382	0.01428	0.01395	0.00520	0.00452
	Sch.3	0.01405	0.01519	0.01516	0.01551	0.01016	0.00948	0.01562	0.01525	0.01577	0.00603	0.00547
	Sch.4	0.01218	0.01348	0.01346	0.01368	0.00906	0.00833	0.01377	0.01352	0.01388	0.00525	0.00465
$t = 1.5$, $\gamma_6 = 0.26000$												
20	Sch.1	0.01731	0.02202	0.02164	0.02518	0.01404	0.01351	0.02584	0.02245	0.02655	0.01771	0.01825
	Sch.2	0.01364	0.01621	0.01594	0.01866	0.01113	0.01095	0.01914	0.01651	0.01966	0.01513	0.01605
	Sch.3	0.01285	0.01545	0.01521	0.01724	0.01077	0.01126	0.01762	0.01572	0.01805	0.01426	0.01622
	Sch.4	0.01355	0.01694	0.01669	0.01709	0.01154	0.01110	0.01744	0.01724	0.01783	0.01497	0.01567
40	Sch.1	0.00885	0.00991	0.00989	0.01000	0.00857	0.00819	0.01010	0.00995	0.01022	0.01109	0.01144
	Sch.2	0.00808	0.00917	0.00914	0.00887	0.00753	0.00737	0.00894	0.00922	0.00902	0.00973	0.01029
	Sch.3	0.00902	0.00977	0.00974	0.01003	0.00833	0.00829	0.01010	0.00982	0.01020	0.01074	0.01120
	Sch.4	0.00781	0.00867	0.00865	0.00883	0.00755	0.00758	0.00889	0.00870	0.00897	0.00986	0.01039

in the majority of cases, at $m = 20$ and $n = 30$ with a real value of $\gamma_3 = 0.27811$, as seen in Table 6.

11) The MSEr of all estimates based on Sch.2 typically produces the smallest values when compared to other estimates at $m = 10$ and $n = 30$, in most of the situations, as seen in Table 8, when $\gamma_5 = 0.05556$ and $\gamma_6 = 0.26000$. Also, we observe that estimates for Sch.4 provide the least MSEr compared to others at $m = 20$ and $n = 30$, when $\gamma_5 = 0.05556$ and $\gamma_6 = 0.26000$.
12) The MSEr of all estimates based on Sch.4 generally yields the smallest values when compared to other estimates at $m = 10, 20$ and $n = 30$, as seen in Table 4.
13) The smallest MSEr results are often produced for all estimates in Sch.2 when compared to other estimates, in a majority of situations at $m = 10$, as provided in Table 9.
14) Figure 5 demonstrates that the precision measure of the Bayesian estimates in the Non-IP case has the biggest values when compared to the other estimates in the IP case. Also, as the value of m increases, the MSEr for all estimates decreases.
15) As t increases, a slight decrease in most estimates (MLE, SE, LIN-1, and LIN-2) and a slight increase in GE estimates were observed.

Figure 5 represents the MSEr for DCRTE at different sample sizes (n) and different number of stages (m) when $\lambda = 0.8$ under scheme Sch.1 as a PCT-II.

6 Concluding remarks

The PCT-II method is used in this study to provide Bayesian and non-Bayesian estimations of the DCRTE measure for a MExp distribution. In the case of IP and Non-IP functions for three LoFs, the BEs of the DCRTE measure are produced. The MH algorithm-based MCMC approach is used to compute the BEs. The accuracy of the DCRTE estimations for the MExp distribution is examined, and applications to the leukemia data and simulation issues are given. We deduce from the simulation results that, as sample sizes grow, the Bayesian estimate of DCRTE gets closer to the genuine value. DCRTE Bayesian estimates under GE LoF followed by LIN LoF are generally preferred to the other competing estimates. The DCRTE Bayesian estimates using IP have lower values than those using Non-IP in terms of their accuracy. Sch.3 is frequently the one that yields the lowest MSEr findings, followed by Sch.2, when compared to

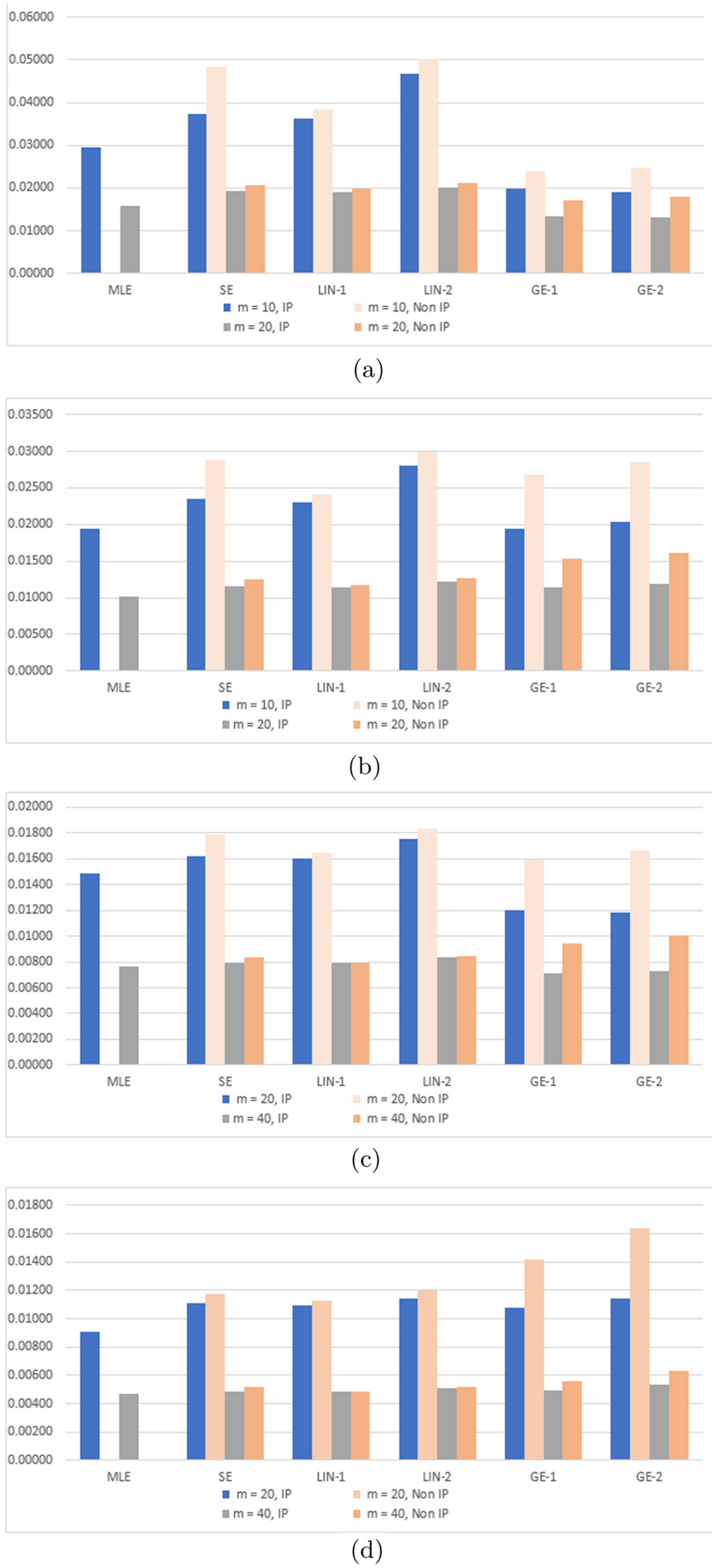

Figure 5: The MSEr for different estimates of DCRTE when $\lambda = 0.8$ under Sch.1. (a) MSEr for $n = 30$ and $t = 0.5$, (b) MSEr for $n = 30$ and $t = 1.5$, (c). MSEr for $n = 60$ and $t = 0.5$, (d) MSEr for $n = 60$ and $t = 1.5$.

other estimates, in most cases. In future research, one may use the E-Bayesian technique to estimate additional metrics of uncertainty, such as DCR Shannon entropy. Bayesian and non-Bayesian estimation of the DCRTE in the presence of outliers can be considered [35]. In addition, other methods such as Tierney–Kadane approximations can be used along with the MCMC approach.

Abbreviations

BE	Bayesian estimator
CDF	cumulative distribution function
CR	cumulative residual
CT-I	censoring type I
CT-II	censoring type II
DCR	dynamic cumulative residual
DCRTE	DCR Tsallis entropy
ECDF	empirical cumulative distribution function
EPDF	empirical probability density function
ESF	empirical survival function
IP	informative prior
GE	general entropy
K–S	Kolmogorov–Smirnov
LIN	linear exponential
LoFs	loss functions
MCMC	Markov chain Monte Carlo
MLE	maximum likelihood estimator
MH	Metropolis–Hasting
MSEr	mean squared error
MExp	moment exponential
Non-IP	non-informative prior
PC	progressive censoring
PCT-I	PC type I
PCT-II	PC type II
PDF	probability density function
PP	probability–probability
Sch.	scheme
SE	squared error
SF	survival function
T	tsallis
V-CM	variance–covariance matrix

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References

[1] Tsallis C. The role of constraints within generalized nonextensive statistics. Phys A Stat Mech Appl. 1998;261:547–61.
[2] Renner R, Gisin N, Kraus, B. Information-theoretic security proof for quantum-key-distribution protocols. Phys Rev A. 2005;72:1–18.
[3] Lévay P, Nagy S, Pipek J. Elementary formula for entanglement entropies of fermionic systems. Phys Rev A. 2005;72:1–8.
[4] Baratpour S, Ahmadi J, Arghami NR. Entropy properties of record statistics. Stat Pap. 2007;48:197–213.
[5] Abo-Eleneen ZA. The entropy of progressively censored samples. Entropy. 2011;13(2):437–49.
[6] Seo JI, Lee HJ, Kan SB. Estimation for generalized half logistic distribution based on records. J Korea Inf Sci Soc. 2012;23(6):1249–57.
[7] Cho Y, Sun H, Lee K. Estimating the entropy of a Weibull distribution under generalized progressive hybrid censoring. Entropy. 2015;17:101–22.
[8] Lee K. Estimation of entropy of the inverse Weibull distribution under generalized progressive hybrid censored data. J Korea Inf Sci Soc. 2017;28(3):659–68.
[9] Al-Babtain AA, Elbatal I, Chesneau Ch, Elgarhy M. Estimation of different types of entropies for the Kumaraswamy distribution. PLoS One. 2021;16(3):1–21.
[10] Hassan AS, Zaky AN. Entropy Bayesian estimation for Lomax distribution based on record. Thail Stat. 2021;19(1):96–115.
[11] Hassan AS, Zaky AN. Estimation of entropy for inverse Weibull distribution under multiple censored data. J Taibah Univ Sci. 2019;13(1):331–7.
[12] Shrahili M, El-Saeed AR, Hassan AS, Elbatal I, Elgarhy M. Estimation of entropy for log-logistic distribution under progressive type II censoring. J Nanomater. 2022;2022:Article ID 2739606, 10 pages.
[13] Helmy BA, Hassan AS, El-Kholy AK. Analysis of uncertainty measure using unified hybrid censored data with applications. J Taibah Univ Sci. 2021;15(1):1130–43.
[14] Hassan AS, Elsherpieny EA, Mohamed RE. Estimation of information measures for power-function distribution in presence of outliers and their applications. J Inf Commun Technol. 2022;21(1):1–25.
[15] Rao M, Chen Y, Vemuri BC, Wang F. Cumulative residual entropy: a new measure of information. IEEE Trans Inf Theory. 2004;50(6):1220–8.
[16] Sunoj, SM, Linu MN. Dynamic cumulative residual Rényias entropy. Statistics. 2012;46(1):41–56.
[17] Kamari O. On dynamic cumulative residual entropy of order statistics. J Stat Appl Prob. 2016;5(3):515–9.

[18] Kundu C, Crescenzo AD, Longobardi M. On cumulative residual (past) inaccuracy for truncated random variables. Metrika. 2016;79(3):335–56.
[19] Renjini KR, Abdul-Sathar, EI, Rajesh G. Bayes estimation of dynamic cumulative residual entropy for Pareto distribution under type-II right censored data. Appl Math Model. 2016;40(19–20):8424–34.
[20] Renjini KR, Abdul-Sathar, EI, Rajesh G. A study of the effect of loss functions on the Bayes estimates of dynamic cumulative residual entropy for Pareto distribution under upper record values. J Stat Comput Sim. 2016;86(2):324–39.
[21] Renjini KR, Abdul-Sathar, EI, Rajesh G. Bayesian estimation of dynamic cumulative residual entropy for classical Pareto distribution. AM J MATH-S. 2018;37:1–13.
[22] Ahmadini AAH, Hassan AS, Zaki AN, Alshqaq SS. Bayesian inference of dynamic cumulative residual entropy from Parto II distribution with application to Covid-19. AIM Math. 2020;6(3):2196–216.
[23] Al-Babtain AA, Hassan AS, Zaky AN, Elbatal I, Elgarhy M. Dynamic cumulative residual Rényi entropy for Lomax distribution: Bayesian and non-Bayesian methods, AIM Math. 2021;6(3):3889–914.
[24] Almarashi AM, Algarni A, Hassan AS, Zaky AN, Elgarhy M. Bayesian analysis of dynamic cumulative residual entropy for Lindley distribution. Entropy. 2021;23:1256.
[25] Dara T, Ahmad M. Recent advances in moment distributions and their hazard rate. Ph.D. Thesis, National College of Business Administration and Economics, Lahore, Pakistan; 2012.
[26] Iqbal Z, Hasnain SA, Salman M, Ahmad M, Hamedani GG. Generalized exponentiated moment exponential distribution. Pak J Stat. 2014;30(4):537–54.
[27] Haq MA, Usman RM, Hashmi S, Al-Omeri AI. The Marshall–Olkin length-biased exponential distribution and its applications. J King Saud Univ Sci. 2017;4763(October):1–11.
[28] Abbas S, Jahngeer A, Shahbaz SH, Afify AZ, Shahbaz MQ. Topp-Leone moment exponential properties and applications. Foundation Sri Lanka. 2020;48(3):265–74.
[29] Hashmi B, Hashmi S, ul Haq MA, Muhammad Usman R. A generalized exponential distribution with increasing, decreasing, and constant shape hazard curves. Electron J Appl Stat Anal. 2019;12:223–44.
[30] Ahmadini AAH;Hassan AS, Mohamed RE, Alshqaq SS, Nagy HF. A new four-parameter moment exponential model with applications to lifetime data. Intell Autom Soft Comput. 2021;29(1):131–46.
[31] Bhatti FA, Hamedani GG, Korkmaz MC, Sheng W, Ali A. On the Burr XII-moment exponential distribution. PLoS One 2021;16(2):e0246935.
[32] Shrahili M, Hassan, AS, Almetwally EM, Ghorbal AB, Elbatal I. Alpha power moment exponential model with application to biomedical science. Sci Program. 2022;2022:Article ID 6897405. doi: 10.1155/2022/6897405.
[33] Balakrishnan N, Aggrawala R. Progressive censoring. Theory methods and applications. Birkhauser, Boston, 2000.
[34] Ghitany ME, Al-Awadhi S. Maximum likelihood estimation of Burr XII distribution parameters under random censoring. Appl Stat. 2002;29(7):955–65.
[35] Hassan AS, El-Sherpieny EA, Mohamed RE. Classical and Bayesian estimation of entropy for Pareto distribution in presence of outliers with application. Sankhya A Indian J Stat. 2023;85(1):707–40. doi: 10.1007/s13171-021-00274-z.