

Research Article

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Fractal-fractional advection–diffusion–reaction equations by Ritz approximation approach

<https://doi.org/10.1515/phys-2022-0221>

received April 15, 2022; accepted November 29, 2022

Abstract: In this work, we propose the Ritz approximation approach with a satisfier function to solve fractal-fractional advection–diffusion–reaction equations. The approach reduces fractal-fractional advection–diffusion–reaction equations to a system of algebraic equations; hence, the system can be solved easily to obtain the numerical solution for fractal-fractional advection–diffusion–reaction equations. With only a few terms of two variables-shifted Legendre polynomials, this method is capable of providing high-accuracy solution for fractal-fractional advection–diffusion–reaction equations. Numerical examples show that this approach is comparable with the existing numerical method. The proposed approach can reduce the number of terms of polynomials needed for numerical simulation to obtain the solution for fractal-fractional advection–diffusion–reaction equations.

Keywords: fractal-fractional derivative, Ritz approximation, satisfier function, fractional advection–diffusion–reaction equations, two variables-shifted Legendre polynomials

1 Introduction

Advection–diffusion–reaction equation is an important class of partial differential equations that have been used to model various physical processes, such as transport dissipative particle dynamics model for simulating mesoscopic

problems [1], in anisotropic media [2], transport of chemical constituents in the Earth's atmosphere [3], bimolecular chemical reactions [4], Brusselator system [5], rubella epidemic [6], tuberculosis transmission modeling [7], COVID-19 mathematical modeling [8] and many more.

This advection–diffusion–reaction equation has been extended to include the fractional derivative, which is called the fractional advection–diffusion–reaction equation. The researchers found that the fractional derivative model was able to describe the transport problems in Earth surface sciences, which include collective behavior of particles in transport [9], heat and mass transfer [10], and the study of the dynamics of cytosolic calcium ion in astrocytes [11]. In this research direction, Caputo fractional derivative is the most common fractional derivative that has been used in this fractional advection–diffusion–reaction equation.

Different from most established work, in this work, we intend to study the fractional advection–diffusion–reaction equation in the fractal-fractional sense as follows:

$$\begin{aligned} {}^{\text{FFM}}D_t^{\alpha,\beta}u(x,t) &= \kappa_1 D_x^{m_1}u(x,t) + \kappa_2 D_x^{m_2}u(x,t) \\ &\quad + \Phi(u(x,t)) + f(x,t), \quad 0 < x, t \leq 1, \\ u(x,0) &= f_0(x), \quad u_t(x,0) = f_1(x), \\ u(0,t) &= g_0(t), \quad u(1,t) = g_1(t), \quad 0 \leq x \leq 1, \end{aligned} \quad (1)$$

where $0 < m_1 \leq 1$, $1 < m_2 \leq 2$, κ_1 , and $\kappa_2 > 0$ are the characteristics speed in advection process and diffusion coefficient, respectively. ${}^{\text{FFM}}D_t^{\alpha,\beta}$ is the fractal-fractional differentiation operator of order (α, β) (where $0 < \alpha, \beta \leq 1$) with respect to t in the sense of Atangana–Riemann–Liouville [12,13], and $\Phi(u(x,t))$ is some reasonable nonlinear function of $u(x,t)$, which may represent reaction process. The fractional derivatives for $D_x^{m_1}$ and $D_x^{m_2}$ are defined in Caputo sense. However, to solve the problem as in ref. [14], one can define the $D_x^{m_1}$ as ${}^{\text{ABC}}D_x^{m_1}$, where ABC denotes Atangana–Baleanu Caputo derivative. This fractional advection–diffusion–reaction equation in the fractal-fractional sense was proposed in ref. [14], while the fractional advection–diffusion–reaction equation's specific applications are discussed in refs [15–17] and the real-world interpretations of this fractal-fractional application are discussed in refs [18–21].

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The fractal-fractional derivatives have been found very useful in many science and engineering applications, such as in modeling anomalous diffusion processes [22]. Researchers found that phenomena that are inherent in abnormal exponential or the phenomena with heavy tail decay processes are best described in the fractal-fractional derivatives [23]. Recently, this combination of fractal-fractional derivative was again shown by Atangana [12,13] that this kind of derivative takes into account not only the memory effect but also other characteristics, such as the heterogeneity, elasto-viscosity of the medium, and the fractal geometry of the dynamic system. In this research direction, fractal-fractional derivatives have been used in many phenomena, such as reaction-diffusion model [18], modeling bank data [19], Shinriki's oscillator model [20], and malaria transmission model [21].

On top of that, numerical methods are always needed to solve fractional calculus problems that arise in engineering applications [24–28]. Furthermore, the differential equations that arise from the modeling process, especially in science and engineering applications involving fractal-fractional operators, are often very complex, especially when we intend to obtain their analytical solution. Hence, numerical methods are more applicable and suitable for solving fractal-fractional differential equations. Some numerical methods have been derived to tackle this problem, such as Chebyshev polynomials for solving the model of the nonlinear Ginzburg–Landau equation in a fractal-fractional sense [29], the Crank–Nicolson finite difference scheme is extended to solve the fractal-fractional Boussinesq equation [30], and a numerical method based on the Lagrangian piece-wise interpolation is used to obtain the solution of variable-order fractal-fractional time delay equations [31]. Besides that, the wavelet-based approximation method was used for solving the coupled nonlinear 2D Schrödinger equations in a fractal-fractional sense [32]. Different from the existing methods for solving fractal-fractional differential equations, here we extend the Ritz method to obtain the solution of these fractal-fractional differential equations. More specifically, we use the Ritz approximation approach to solve the fractal-fractional advection–diffusion–reaction equation as in Eq. (1). The approach is very easy to use and able to give high accuracy numerical solutions.

The Ritz approximation approach had been used in solving considerable problems previously. Among those, Rashedi *et al.* used the Ritz–Galerkin approach to handle inverse wave problem [33]. Apart from this, the satisfier function was used in Ritz–Galerkin for the identification of a time-dependent diffusivity [34], the Ritz approxima-

tion has also been applied to solve some fractional partial differential equations [35,36]. Exact and approximation solutions of the heat equation with nonlocal boundary conditions were found in ref. [37] using the Ritz–Galerkin method with Bernoulli polynomials as the basis. Besides that, Genocchi polynomials had been used in the Ritz–Galerkin method for solving the fractional Klein–Gordon equation and fractional diffusion wave equation [38]. Here, for the first time, we propose to use this Ritz approximation approach to solve the fractal-fractional advection–diffusion–reaction equation as in Eq. (1). Apart from this, two variables-shifted Legendre polynomials, which were derived by Khan and Singh [39], will be used. This is different from two-dimensional-shifted Legendre polynomials. In short, the main objective of this article is to solve the fractal-fractional advection–diffusion–reaction equation using the Ritz approximation via two variables-shifted Legendre polynomials.

The rest of the article is organized as follows: Section 2 provides the basic definitions and notations for fractal-fractional derivative and two variables-shifted Legendre polynomials. Section 3 presents the main tool that is used in this article, which is the Ritz approximation and satisfier function for solving fractal-fractional advection–diffusion–reaction equations. Error analysis will also be presented. Section 4 gives some numerical examples to show that the approach is better than some existing methods. Section 5 gives summary and recommendation for future work.

2 Preliminaries

2.1 Fractal-fractional derivative

In this section, we will briefly present some basic definitions related to fractal-fractional derivative.

Definition 1. The Mittag–Leffler function

$$\begin{aligned} E_a(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ak+1)}, \quad \text{where } \operatorname{Re}(a) > 0, \\ E_{a,b}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ak+b)}, \quad \text{where } \operatorname{Re}(a), \operatorname{Re}(b) > 0. \end{aligned} \quad (2)$$

Definition 2. The fractal-fractional derivative of order (α, β) of the continuous function $f(x)$ in the Atangana–Riemann–Liouville sense with Mittag–Leffler kernel [12] is given as follows:

$$\begin{aligned} {}^{\text{FFM}}D_x^{\alpha,\beta}f(x) &= \frac{M(\alpha)}{1-\alpha} \frac{d}{dx^\beta} \int_{x_0}^x f(t) E_\alpha \left(\frac{-\alpha(x-t)^\alpha}{1-\alpha} \right) dt \\ &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \int_{x_0}^x f(t) E_\alpha \left(\frac{-\alpha(x-t)^\alpha}{1-\alpha} \right) dt, \end{aligned} \quad (3)$$

where $\alpha, \beta \in (0, 1)$, $M(\alpha)$ is a normalization function with $M(0) = M(1) = 1$, and $M(\alpha) = 1 - \alpha + \frac{\alpha}{\Gamma(\alpha)}$.

Lemma 1. The fractal-fractional derivative of order $0 < \alpha, \beta < 1$ for $f(x) = x^p$ can be defined as follows:

$${}^{\text{FFM}}D_x^{\alpha,\beta}x^p = \frac{M(\alpha)}{\beta(1-\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i p! x^{ai+p+1-\beta}}{(1-\alpha)^i \Gamma(\alpha i + p + 1)}. \quad (4)$$

Proof. By using Definition in Eqs. (2) and (3),

$$\begin{aligned} {}^{\text{FFM}}D_x^{\alpha,\beta}x^p &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \int_0^x t^p \sum_{i=0}^{\infty} \frac{\left(\frac{-\alpha}{1-\alpha}\right)^i (x-t)^{\alpha i}}{\Gamma(\alpha i + 1)} dt \\ &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \int_0^x \sum_{i=0}^{\infty} \frac{(-1)^i t^p \alpha^i (x-t)^{\alpha i}}{(1-\alpha)^i \Gamma(\alpha i + 1)} dt \\ &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i \int_0^x t^p (x-t)^{\alpha i} dt}{(1-\alpha)^i \Gamma(\alpha i + 1)}, \end{aligned} \quad (5)$$

and knowing that $\int_0^x t^p (x-t)^{\alpha i} dt = \frac{\Gamma(\alpha i + 1) p! x^{ai+p+1}}{\Gamma(\alpha i + p + 2)}$, we obtain

$$\begin{aligned} {}^{\text{FFM}}D_x^{\alpha,\beta}x^p &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i \frac{\Gamma(\alpha i + 1) p! x^{ai+p+1}}{\Gamma(\alpha i + p + 2)}}{(1-\alpha)^i \Gamma(\alpha i + 1)} \\ &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \frac{d}{dx} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i p! x^{ai+p+1}}{(1-\alpha)^i \Gamma(\alpha i + p + 2)} \\ &= \frac{M(\alpha)x^{1-\beta}}{\beta(1-\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i p! (\alpha i + p + 1) x^{ai+p}}{(1-\alpha)^i \Gamma(\alpha i + p + 2)} \\ &= \frac{M(\alpha)}{\beta(1-\alpha)} \sum_{i=0}^{\infty} \frac{(-1)^i \alpha^i p! x^{ai+p+1-\beta}}{(1-\alpha)^i \Gamma(\alpha i + p + 1)}. \quad \square \end{aligned} \quad (6)$$

Eq. (6) can also be written in Mittag–Leffler form, as shown in ref. [29].

2.2 Two variables-shifted Legendre polynomials

The Legendre polynomials, $L_n(x)$, can be defined as the coefficients in a formal expansion in powers of t of the generating function as follows:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} L_n(x) t^n. \quad (7)$$

There are few different ways for the extension of this Legendre polynomials $L_n(x, y)$ (or shifted Legendre polynomials, $P_n(x, y)$) in several variables, such as

(1) Two-variable Legendre polynomials [40]

$$\frac{1}{\sqrt{1-2xt+yt^2}} = \sum_{n=0}^{\infty} L_n(x, y) t^n. \quad (8)$$

(2) Two-dimensional-shifted Legendre polynomials are defined as [41,42] follows:

$$P_n(x, y) = P_n(x) P_k(y). \quad (9)$$

(3) Two variables Legendre polynomials [39]

$$\frac{1}{\sqrt{1-2xs+s^2-2yt+t^2}} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} L_{n,k}(x, y) s^n t^k \quad (10)$$

The significant advantage for the two variables Legendre polynomials derived by Khan and Singh [39] in 2010 is that the polynomials can be obtained via generating function in two variables form as shown in Eq. (10). Here, we apply the definition of two variables Legendre polynomials derived by Khan and Singh [39] to define the two variables-shifted Legendre polynomials, $P_{n,k}(x, y)$, as follows:

$$\begin{aligned} \frac{1}{\sqrt{1-2(2x-1)s+s^2-2(2y-1)t+t^2}} \\ = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} P_{n,k}(x, y) s^n t^k. \end{aligned} \quad (11)$$

As discussed in Section 3 of ref. [39], we have the following analytical expression for two variables-shifted Legendre polynomials.

$$\begin{aligned} P_{n,k}(x, y) \\ = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(1/2)_{n+k-r-j} (4x-2)^{n-2r} (4y-2)^{k-2j} (-1)^{r+j}}{r! j! (n-2r)! (k-2j)!}, \end{aligned} \quad (12)$$

where $(1/2)_{n+k-r-j}$ denotes the falling factorial. When $k=0$, we have $P_{n,0}(x, y) = P_n(x)$, where $P_n(x)$ is the well known shifted Legendre polynomials. These two variables-shifted Legendre polynomials also can be expressed in terms of hypergeometric function as follows:

$$\begin{aligned} P_{n,k}(x, y) \\ = \frac{2^{n+k} (1/2)_{n+k} (2x-1)^n (2y-1)^k}{n! k!} \\ \times {}_F_{1,0,0}^{0,1,1} \left[\begin{matrix} -; & -\frac{n}{2}, \frac{1-n}{2}; & -\frac{k}{2}, \frac{1-k}{2}; & 1 \\ 1-2n-2k; & -; & -; & (2x-1)^2 (2y-1)^2 \end{matrix} \right]. \end{aligned} \quad (13)$$

For function $f(x, y) = x^a y^b$ with a and b as positive integers, it is easy to see that it can be obtained via

$$f(x, y) = x^a y^b = \frac{1}{(a+1)(b+1)} \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} c_{n,k} P_{n,k}(x, y), \quad (14)$$

where

$$c_{n,k} = \frac{\int_0^1 \int_0^1 f(x, y) P_{n,k}(x, y) dx dy}{\int_0^1 \int_0^1 P_{n,k}(x, y) P_{n,k}(x, y) dx dy}. \quad (15)$$

3 Ritz method

3.1 Ritz approximation approach and satisfier function for fractal-fractional advection–diffusion–reaction equation

In this Ritz approximation approach, we use two variables-shifted Legendre polynomials $P_{p,q}(x, t)$, as explained in Section 2 and the approximate solution $u(x, t)$ for fractal-fractional advection–diffusion–reaction Eq. (1) is denoted as $\tilde{u}(x, t)$. Hence, we have

$$\tilde{u}(x, t) = \sum_{p=0}^M \sum_{q=0}^M K_{pq} \omega_{pq}(x, t) + \zeta(x, t), \quad (16)$$

$$(x, t) \in [0, L] \times [0, T],$$

where $\omega_{pq}(x, t) = x(x-L)t^2 P_{p,q}(x, t)$ and the satisfier function is represented by $\zeta(x, t)$. $P_{p,q}(x, t)$ denotes two variables-shifted Legendre polynomials, and K_{pq} is the coefficient that needs to be calculated. The main purpose of using the satisfier function is that the satisfier function satisfies all the initial and boundary conditions.

Based on the initial and boundary condition as in Eq. (1), the satisfier equation, $\zeta(x, t)$ can be found using the following procedure:

Step 1: First, we find

$$\varphi(x, t) = \left(1 - \frac{x}{L}\right) g_0(t) + \frac{x}{L} g_1(t). \quad (17)$$

Step 2: Next, we determine

$$\begin{aligned} F_0(x) &= f_0(x) - \varphi(x, 0), \\ F_1(x) &= f_1(x) - \varphi_t(x, 0). \end{aligned} \quad (18)$$

Step 3: Calculate

$$R(x, t) = F_0(x) + t F_1(x). \quad (19)$$

Step 4: Finally, we determine

$$\zeta(x, t) = R(x, t) + \varphi(x, t). \quad (20)$$

Furthermore, the coefficients K_{pq} in Eq. (16) can also be calculated by using the following inner product as follows:

$$\langle F(\tilde{u}), P_{p,q}(x, t) \rangle = 0, \quad (21)$$

where

$$\begin{aligned} F(\tilde{u}) &= {}^{\text{FFM}}D_t^{\alpha, \beta} \tilde{u}(x, t) - \kappa_1 D_x^{m_1} \tilde{u}(x, t) \\ &\quad - \kappa_2 D_x^{m_2} \tilde{u}(x, t) - \Phi(\tilde{u}(x, t)) - f(x, t) \end{aligned} \quad (22)$$

and

$$\langle F(\tilde{u}), P_{p,q}(x, t) \rangle = \int_0^L \int_0^T F(\tilde{u}) P_{p,q}(x, t) dt dx, \quad (23)$$

where $P_{p,q}(x, t)$ are the two variables-shifted Legendre polynomials. A linear system of equations can be formed by using Eq. (21). Solving this linear system, we can obtain the entries of K_{pq} where $p = 0, \dots, M$, and $q = 0, \dots, M$. Hence, by putting the value obtained for K_{pq} into Eq. (16), we can obtain the approximate solution for fractal-fractional advection–diffusion–reaction equation in (1).

3.2 Error analysis

Lemma 2. *Let the solution of fractal-fractional advection–diffusion–reaction equation is $u(x, t)$, where $u(x, t) \in C^{m+1}[0, 1] \times [0, 1]$, suppose $Y = \text{span}\{P_{0,m}(x), P_{1,m}(x), \dots, P_{m,m}(x)\} \subset L^2[0, 1]$ and $Y' = \text{span}\{P_{0,m}(t), P_{1,m}(t), \dots, P_{m,m}(t)\} \subset L^2[0, 1]$. We have $u_m(x, t) = Y \times Y'$ is the best approximation of $u(x, t)$ by means of two variables-shifted Legendre polynomials, the error bound is given as follows:*

$$\|u(x, t) - u_m(x, t)\|_2 \leq \frac{M}{(m+1)!} \frac{\sqrt{2} 2^{m+1}}{\sqrt{(2m+3)(m+2)}}. \quad (24)$$

Proof. By using the Taylor series, we have

$$u(x, t) = \sum_{j=0}^{\infty} \frac{1}{j!} \left[(x-a) \frac{\partial}{\partial x} + (t-b) \frac{\partial}{\partial t} \right]^j u(x, t) \quad (25)$$

For simplicity, let $a = b = 0$, and in practical, we estimate $u(x, t)$ up to m order as follows:

$$u_m(x, t) = \sum_{j=0}^m \frac{1}{j!} \left[x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \right]^j u(x, t). \quad (26)$$

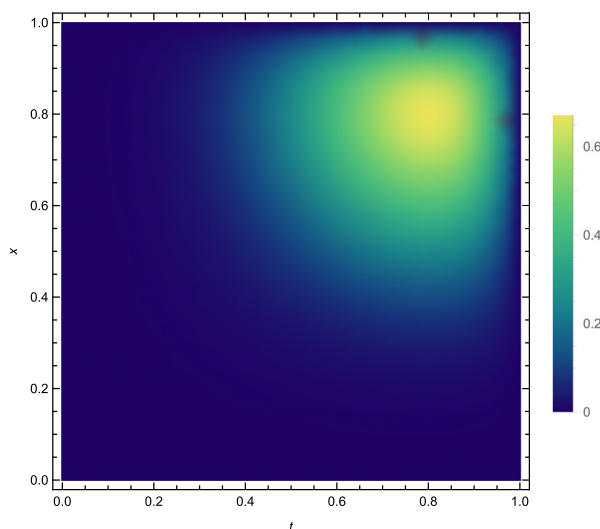
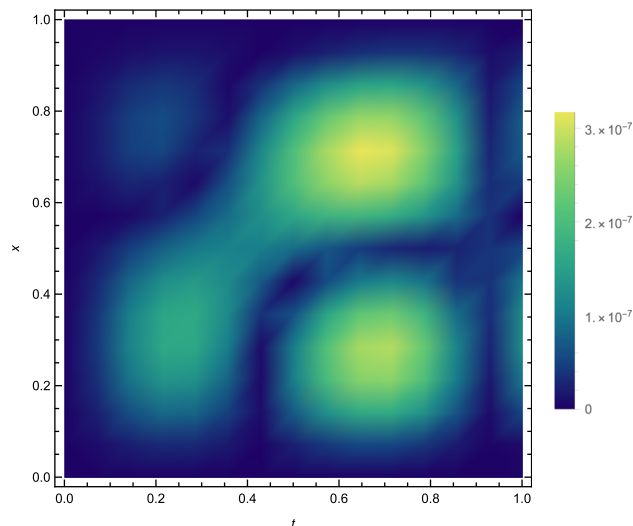
Since $u_m(x, t)$ is the best approximation $u(x, t)$ out of $Y \times Y'$, we have

Table 1: Absolute errors obtained by proposed method with $M = 2, 3$ for Example 1 for $\theta = 0.5$

(x, t)	Exact	Abs. error, $M = 2$	Abs. error, $M = 3$
(0.1, 0.1)	0.00000081	9.85884×10^{-5}	2.29895×10^{-8}
(0.2, 0.2)	0.00016384	7.45007×10^{-4}	1.22948×10^{-7}
(0.3, 0.3)	0.00321489	4.95369×10^{-3}	1.56609×10^{-7}
(0.4, 0.4)	0.02359296	3.82750×10^{-3}	8.62400×10^{-8}
(0.5, 0.5)	0.09765625	1.70405×10^{-2}	8.87000×10^{-8}
(0.6, 0.6)	0.26873856	4.86057×10^{-2}	2.29700×10^{-7}
(0.7, 0.7)	0.51883209	5.01841×10^{-2}	3.15500×10^{-7}
(0.8, 0.8)	0.67108864	5.28156×10^{-3}	1.92000×10^{-7}
(0.9, 0.9)	0.43046721	7.89781×10^{-3}	2.74000×10^{-8}

$$\begin{aligned}
& \|u(x, t) - u_m(x, t)\|_2 \\
& \leq \left\| \frac{1}{(m+1)!} \left[x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \right]^{m+1} u(\xi, \eta) \right\|_2 \\
& = \left(\int_0^1 \int_0^1 \left(\frac{1}{(m+1)!} \left[x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t} \right]^{m+1} u(\xi, \eta) \right)^2 dx dt \right)^{1/2} \\
& = \left(\int_0^1 \int_0^1 \left[\sum_{r=0}^{m+1} \frac{\binom{m}{r}}{(m+1)!} \frac{\partial^{m+1} u(\xi, \eta) t^r x^{m+1-r}}{\partial t^r \partial x^{m+1-r}} \right]^2 dx dt \right)^{1/2} \quad (27) \\
& \leq \left(\int_0^1 \int_0^1 \left(\frac{M}{(m+1)!} (x+t)^{m+1} \right)^2 dx dt \right)^{1/2} \\
& = \frac{M}{(m+1)!} \left(\int_0^1 \int_0^1 (x+t)^{2m+2} dx dt \right)^{1/2} \\
& \leq \frac{M}{(m+1)!} \frac{\sqrt{2} 2^{m+1}}{\sqrt{(2m+3)(m+2)}},
\end{aligned}$$

where $0 \leq \xi \leq x$ and $0 \leq \eta \leq t$, we obtain the above error bound. \square

**Figure 1:** Diagram of the approximate solution for Example 1 using $M = 3$ and $\theta = 0.5$.**Figure 2:** Diagram of the absolute error for Example 1 using $M = 3$ and $\theta = 0.5$.

4 Numerical examples

In this section, we solve two benchmark examples taken from published work. Our calculation shows that the proposed method is comparable with the published work. Here, we conduct numerical experiments using Ritz approximation via satisfier function as explained in Section 3 for solving the fractal-fractional advection–diffusion–reaction equation. Here, we use Maple to perform all the computations.

Example 1. Consider a fractal-fractional advection–diffusion–reaction equation as in Example 3 [14]:

$$\begin{aligned}
& {}^{\text{FFM}}D_t^{0.65, 0.35} u(x, t) \\
& = {}^{\text{ABC}}D_x^\theta u(x, t) + {}^C D_x^{1.85} u(x, t) + \sin(u(x, t)) \\
& \quad + \sin(2u(x, t)) + f(x, t), \quad 0 < x, t \leq 1, \quad (28) \\
& u(x, 0) = 0, \quad u_t(x, 0) = 0, \\
& u(0, t) = 0, \quad u(1, t) = 0.
\end{aligned}$$

Here, we refer the reader to Example 3 [14] for the long expression of $f(x, t)$. The exact solution is given by $u(x, t) = 100x^4 t^4 (1-x)(1-t)$.

Table 2: Comparison of the maximum absolute errors obtained by proposed method with $M = 2$ for Example 2 with ref. [14]

(α, β)	m	n	MAE [14]	M	MAE (proposed method)
(0.75, 0.25)	7	7	5.9238×10^{-6}	2	8.99969×10^{-6}
(0.75, 0.5)	7	7	7.0270×10^{-6}	2	9.16790×10^{-6}
(0.45, 0.35)	7	7	2.5772×10^{-6}	2	9.15874×10^{-6}
(0.65, 0.35)	7	7	4.3460×10^{-6}	2	9.12105×10^{-6}

When $\theta = 0.5$, we obtained the numerical result as in Table 1. In this example, we are using $M = 2$, which means the two variables-shifted Legendre polynomials is only up to quadratic power. The approximation solution is increasing in terms of accuracy as M is increased to $M = 3$. The calculation is done by using Maple and applying Lemma 2.4 and Corollary 2.5 in ref. [32] and Corollary 2.5 in ref. [43]. Figures 1 and 2 show the graph for the approximate solution and absolute error using $M = 3$ for the Ritz approximation and $\theta = 0.5$ for Example 1.

Example 2. Consider a fractal-fractional advection–diffusion–reaction equation as in Example 1 [14]:

$$\begin{aligned} & {}^{\text{FFM}}D_t^{\alpha,\beta}u(x,t) \\ &= {}^{\text{ABC}}D_x^{0.25}u(x,t) + 2^CD_x^{1.95}u(x,t) \\ &+ \exp(u(x,t)) + f(x,t), \quad 0 < x, t \leq 1, \quad (29) \\ &u(x,0) = 0, \quad u_t(x,0) = \sin(x), \\ &u(0,t) = 0, \quad u(1,t) = \sin(1)\sin(t). \end{aligned}$$

Here, again, we refer the reader to Example 1 [14] for the long expression of $f(x,t)$. The exact solution is given by $u(x,t) = \sin(x)\sin(t)$. By using the procedure as explained in Section 3.1, the satisfier function, $\zeta(x,t)$ is $x \sin(1)\sin(t) - tx \sin(1) + t \sin(x)$.

In order to obtain the numerical solution, we need Lemma (2.4) as in ref. [32]. From the numerical result as shown in Table 2, with only few terms of two variables-shifted Legendre polynomials via Ritz approximation, our proposed method is comparable with the method via Bernstein polynomials and its operational matrix as in ref. [14], using more terms of polynomials. Similar to Example 1, the calculation is done by using Maple and applying Lemma 2.4 and Corollary 2.5 in ref. [32] and Corollary 2.5 in [43].

5 Conclusion

In this article, we successfully used the Ritz approximation approach to solve fractal-fractional advection–diffusion–reaction equations via two variables-shifted Legendre polynomials. With Ritz approximation, greatly reduces the number of terms of polynomials needed for numerical simulation to obtain the solution for fractal-fractional advection–diffusion–reaction equations. With only few terms of two variables-shifted Legendre polynomials, we were able to obtain the numerical solution with high accuracy. The proposed procedure can be easily extended to solve fractal-fractional advection–diffusion–reaction equations in variable order. Furthermore, we hope to extend the method to tackle

the inverse problems related to fractional partial differential equations such as those in ref. [44–46], or more complicated scenarios and problems, such as those in ref. [47].

Funding information: The authors would like to thank Universiti Teknologi MARA, Johor Branch for supporting this research under Geran Penyelidikan Bestari Vot No. 29000.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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