Research Article

Peijiang Liu, Haohua Wang*, Ting Cui, and Anwarud Din

Modeling of hepatitis B epidemic model with fractional operator

https://doi.org/10.1515/phys-2022-0219
received June 23, 2022; accepted November 28, 2022

Abstract: In many regions across the world, hepatitis B virus (HBV) infection is still endemic and the transmission rate is much greater than majority of the known epidemic diseases. Numerous mathematical models (utilizing various differential operators) have been put forth over the past 20 years to understand the transmission mechanism of HBV in various nations and geographical areas. In this manuscript, an epidemic model with various novelties for capturing the dynamic of HBV while utilizing Caputo–Fabrizio fractional differential operator with asymptomatic carriers and vaccination effects is introduced. Initially, the model is formulated by using the ordinary derivative, and afterward, the fractional differential operator is applied to transform the model into arbitrary-ordered derivative. A few basic mathematical properties for the proposed integer-ordered model is presented. The existence of solution to the problem and its uniqueness of the fractional order model are established by transforming the problem into integral equations and then applying the standard results of fixed point theory. For boundedness and positivity of model' solution is elaborated utilizing the techniques of fractional calculus. It is too much important to validate the theoretical findings through simulations; therefore, the solution curves of the model under consideration are displayed by using the well-known approach called the Mittag-Leffler. To show the behavior of the order of the operator on the dynamics of the disease, various graphical illustrations are presented at the end of the manuscript. By comparing the findings of the present study with the available literature, it is observed that fractional derivative is better to use than integer-order operator for capturing the realistic scenario of the disease.

Keywords: fractional HBV epidemic model, CF-fractional derivative, asymptomatic carriers, numerical simulation

*** Corresponding author: Haohua Wang,** School of Sciences, Hainan University, Haikou, 570228, Hainan, China, e-mail: huazi8112@hainanu.edu.cn
Peijiang Liu: School of Statistics and Mathematics, Guangdong University of Finance and Economics, Big data and Educational Statistics Application Laboratory, Guangzhou, China; School of Statistics and Mathematics, Guangdong University of Finance and Economics, Guangzhou, China
Ting Cui: Department of Mathematics Sun Yat-sen University, Guangzhou 510275, China; School of Economics, Guangdong University of Finance and Economics, Guangzhou 510320, China
Anwarud Din: Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China, e-mail: anwarud@mail.sysu.edu.cn

1 Introduction

Worldwide there are approximately 350 million of population members suffering from the chronic hepatitis B virus (HBV) [1], and 25–40% of these are ultimately facing chronic liver, hepatocellular carcinoma, and cirrhosis [2]. Due to these and many other reasons, the HBV infection is becoming a major problem related to public health around the world. An infected person in the chronic phase typically has a serum load of 2×10^{11} to 3×10^{12}, according to the study [3]. If we assume that the mean liver' mass is 1.5 kg, then a liver in the human body would normally contain the same amount of cells. We have to employ the conventional rate of incidence instead of the bilinear rate due to the enormous numbers. The time lag associated with the formation of HB virus is only 1–2 days [4], which is significantly smaller than 6–12 months (the life-span of a normal hepatocyte) or even more than that [5].

From a mathematical standpoint, some intriguing findings come from epidemic modeling, specifically in the prevention and control of infectious diseases. Din *et al.* [5] developed and analyzed a stochastic hepatitis B model considering the transmission coefficient's time delay and the cytotoxic T lymphocyte immune response class. Based on the analysis of the model, the authors proved that the extinction of HBV is possible by quite high noise levels. The Caputo–Fabrizio (CF) fractional derivative with non-singular kernel was used in ref. [6] to create the model of the hepatitis B virus, and an iterative strategy is suggested for solving the model. Khan *et al.* [7] used the tools of fractional derivatives and modeled the issue of climate change. For a detailed study of the dynamic behavior of COVID-19 *via* fractional operators, the readers are suggested to see refs. [8–10] and references cited therein. Lu [11] used the techniques of stochastic differential equations and studied the Susceptible–Infectious–Recovered–Susceptible model both with and without distributed time-delay and presented stability conditions for the elimination of the infection out of the community. In the literature, a wide range of models for describing the dynamics of epidemic diseases with numerous features have been formulated and analyzed [9–11]. In human viral illnesses, the elements of stochastic perturbations (like precipitation, temperature, and many others) have a significant effect on the incidence with which a disease spread. By considering these factors in the development of a model will allow us to study models without stochasticity. The environmental variability that influences the dynamics of biological models will be incorporated into the models due to the inclusion of this stochastic fluctuation. The environment's effect on parameters or the system's random noises are the examples of variable influences [11,12]. Additionally, stochastic modeling is more realistic and offers an extra quantity of flexibility compared to deterministic models. Stochastic processes with white noises (or Brownian motions) have a large literature, and ref. [13] provides a detailed examination of a stochastic model. Upon thoroughly studying the literature of stochastic epidemic modeling, one can note that the stochastic models are much close to reality compared to their deterministic counterparts.

Epidemic models with fractional order derivatives are more appropriate for capturing the dynamics of complicated and cross behavior problems due to the availability of the options from a wide range order of the derivatives. Calculus of fractional orders has been used to provide understanding of the spectrum between the integer orders [14–16]. Differential operator with arbitrary order is much better than integer order since it produces a few interesting and nice results about the dynamical behavior of the problems under consideration. As a result, real-world problem and particularly the biological issues can be effectively studied by using the fractional order derivative. This means we have a wide range of options in stochastic models compared to the classical compartmental models. In recent years, scientists have a keen interest in dealing with fractional derivatives rather than traditional differentiation. In this regard, researchers have done sophisticated work on the existence and uniqueness theory of such differential equations, stability, and approximating the solutions of fractional equation. The dynamics of various physical problems have been described by differential or fractional integral equations, such as model of microorganism, a logistic model, and dengue problem, which are the foundation model for modeling the biological situations. Rashid *et al.* [17] used the Antagana–Baleanu fractional operator and the Laplace transformation and derived numerous semi-theoretical solution of the fourth-order time fractional Boussinesq equation in different spaces. Ganji and Jafari [18] used the Atangana–Baleanu operator, operational matrices based on shifted Legendre polynomials and solved a group of non-linear Volterra integro-differential equations. The modeling of Darcy–Forchheimer hybrid nano-liquid flow on a porous spinning disk problem *via* fractional calculus is solved by Li *et al.* [19]. By using the tools of fractional calculus, a few epidemic problems were formulated and solved in refs. [20,21] and references cited therein for a detailed analysis. Additionally, by using various methodologies, the fractional order differential equations were examined for semi-analytical, analytical, and approximate solutions. There are a variety of numerical methods such as Taylor, Euler, predictor–corrector, and Adams–Bashforth, *etc.*, many transformations for achieving semi-analytical solutions, and other well-known approaches [22,23]. Various models with fractional differential operators have recently been built and examined for the recent pandemic COVID-19, see, for example, refs. [24–26]. Interested readers are advised to see refs. [27,28] and references cited therein for a rigorous analysis of epidemic models containing the fractional order derivatives.

In this work, we intend to formulate a novel HBV model with one of the fractional differential operators of CF fractional operator. People with HBV infection can remain symptom-free for 30–180 days and spread the infection to others, which increases the risk of human mortality [29,30] and as a result, literature [9,12,27,28] are of the opinion to include asymptomatic and immunization effects into studies related to HBV infection. Thus, we will include a few biological characteristics of the diseases, namely, the asymptomatic carriers and the effect of vaccination. Each model in the literature offers a unique approach on the dynamics of HBV; however, this study will explain thoroughly and more rigorously the effect of asymptomatic carriers and vaccination on the transmission mechanism of HBV.

Initially, the model is formulated in case of integer-order derivative and then it is extended to model with derivatives of arbitrary order by using the CF differential operator. The objective of this study is to create a public-health plan that relies on vaccinations and which can successfully manage HBV disease through mathematical modeling. Furthermore, we will explain the role of the efficacy of vaccine and the prevention of of the disease from spreading it throughout the country.

The structure of the article is as follows. The model for fractional HBV transmission is developed in Section 2. In Section 3, we discuss the preliminary definitions of the CF derivative. The existence of solution to the model along with its uniqueness and positivity are demonstrated in Section 4. Section 5 provides numerical simulation and also compares the simulation for multiple values of the fractional derivatives. Section 6 contains the conclusion as well as the future study directions.

2 Fractional HBV model

Time-delay and vaccination analyses are perhaps the most cost-effective technique for preventing HBV infection to a significant degree in practical medical therapy. WHO suggested that in high-endemicity areas, the impactful HBV vaccination could be used on an individual basis. Consequently, a number of scientists have formulated and investigated the deterministic HBV model with vaccination as a control strategy (see refs. [5,15,18,19]). While formulating the model, we assumed the non-linear incidence rate beside other characteristics of HBV.

Most recently, Din and Li [31] considered the strategy of the hepatitis B outbreak with vaccination effects and they divided the entire population into five disjoint groups: susceptible/vulnerable $\mathbf{S(t)}$, vaccinated compartment $\mathbf{V\,(t)}$, acutely infected $\mathbf{A(t)}$, chronically infected $\mathbf{C(t)}$, and recovered $\mathbf{R(t)}$. That is, $\mathbf{N(t)} = \mathbf{S(t)} + \mathbf{V(t)} + \mathbf{A(t)} + \mathbf{C(t)} + \mathbf{R(t)}$ at any given time t, which is given by

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{S(t)}}{\mathrm{d}t} &= \kappa\Lambda - \frac{\beta\mathbf{A(t)S(t)}}{\mathbf{N}} + \rho\mathbf{V\,(t)} - (\eta + \mu)\mathbf{S(t)},\\
\frac{\mathrm{d}\mathbf{V\,(t)}}{\mathrm{d}t} &= \Lambda(1-\kappa) - \frac{(1-\tau_1)\beta\mathbf{A(t)V\,(t)}}{\mathbf{N}}\\
&\quad - (\eta + \rho)\mathbf{V\,(t)} + \mu\mathbf{S(t)},\\
\frac{\mathrm{d}\mathbf{A(t)}}{\mathrm{d}t} &= \frac{\beta\mathbf{A(t)S(t)}}{\mathbf{N}} + \frac{(1-\tau_1)\beta\mathbf{A(t)V\,(t)}}{\mathbf{N}}\\
&\quad - \mathbf{A(t)}(\alpha_1 + \alpha_2 + \eta),\\
\frac{\mathrm{d}\mathbf{C(t)}}{\mathrm{d}t} &= \alpha_1\mathbf{A(t)} - \mathbf{C(t)}(\gamma_1 + \gamma_2 + \eta),\\
\frac{\mathrm{d}\mathbf{R(t)}}{\mathrm{d}t} &= \gamma_1\mathbf{C(t)} + \alpha_2\mathbf{A(t)} - \eta\mathbf{R(t)}.
\end{aligned} \tag{1}$$

The parameters used in the model are interpreted in Table 1. Moreover, the vaccine is imperfect and does not provide sufficient prevention. Thus, after sufficiently long time, individuals in the vaccinated compartment become infected after contact with the acute individuals. It could be noted that $0 < \tau_1 < 1$ ($\tau_1 = 1$ means perfect vaccine, however, $\tau_1 = 0$ denotes vaccine with no protection).

Table 1: Parameter description

Parameter	Description
Λ	Recruitment rate
β	Infection rate
η	Mortality rate
μ	Rate at which susceptibles are vaccinated
γ_1	Recovery rate of chronic patients
γ_2	Mortality rate of chronically infected people
α_1	Rate of acute individuals leads to the chronic class
α_2	Severely acute infected people's recovery rate
κ	Fraction of individual born without efficient vaccination
ρ	Rate at which vaccinated individuals are getting vulnerability

For the deterministic system (1), the following analytical findings were obtained by using ref. [31].

Model (1) always has the disease-free equilibrium $\mathbf{E_0}$, also known as the disease-free equilibrium, having the following form:

$$\begin{aligned}
\mathbf{E_0} &= (\mathbf{S_0}, \mathbf{V_0}, \mathbf{A_0}, \mathbf{C_0}, \mathbf{R_0})\\
&= \left(\frac{\Lambda(\rho + \eta\kappa)}{(\rho + \mu + \eta)\eta}, \frac{\eta\Lambda(1-\kappa) + \mu\kappa}{(\rho + \mu + \eta)\eta}, 0, 0, 0\right).
\end{aligned} \tag{2}$$

By utilizing the next-generation approach, the reproductive number symbolized by $\mathbf{R_0^D}$ is calculated and given by

$$\mathbf{R_0^D} = \frac{\beta\mathbf{S_0} + (1-\tau_1)\beta\mathbf{V_0}}{\mathbf{N}(\alpha_1 + \alpha_2 + \eta)}. \tag{3}$$

It is very easy to prove that system (1) has the following dynamical properties:

- The disease free equilibrium $\mathbf{E_0}$ is stable for $\mathbf{R_0^D} < 1$ and unstable for $\mathbf{R_0^D} > 1$.
- If $1 < \mathbf{R_0^D}$, an endemic equilibrium (EE) $\mathbf{E_*} = (\mathbf{S_*}, \mathbf{V_*}, \mathbf{A_*}, \mathbf{C_*}, \mathbf{R_*})$ exists and this equilibrium is asymptotically stable both locally and globally, which biologically shows the presence of HBV in the community.

To proceed further, we wish to incorporate ABC fractional differential operator into model (1), which is based on $\mathbf{S(t)}$, $\mathbf{V(t)}$, $\mathbf{A(t)}$, $\mathbf{C(t)}$, and $\mathbf{V\,(t)}$, which takes the form as follows:

We intend to study and analyze the fractional (differential operator) version of model (1) by incorporating a type of CF fractional operator given by:

$$
\begin{aligned}
{}^{\mathrm{CF}}D_{0,t}(\mathbf{S(t)}) &= \kappa^{\sigma}\Lambda^{\sigma} - \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \rho^{\sigma}\mathbf{V\,(t)} \\
&\quad - (\mu^{\sigma} + \eta^{\sigma})\mathbf{S(t)}, \\
{}^{\mathrm{CF}}D_{0,t}(\mathbf{V\,(t)}) &= \Lambda^{\sigma}(1-\kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(t)} \\
&\quad - \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\rho^{\sigma} + \eta^{\sigma})\mathbf{V\,(t)}, \\
{}^{\mathrm{CF}}D_{0,t}(\mathbf{A(t)}) &= \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} \\
&\quad - \mathbf{A(t)}(\alpha_1^{\sigma} + \alpha_2^{\sigma} + \eta^{\sigma}), \\
{}^{\mathrm{CF}}D_{0,t}(\mathbf{C(t)}) &= \alpha_1^{\sigma}\mathbf{A(t)} - (\gamma_1^{\sigma} + \gamma_2^{\sigma} + \eta^{\sigma})\mathbf{C(t)}, \\
{}^{\mathrm{CF}}D_{0,t}(\mathbf{R(t)}) &= \gamma_1^{\sigma}\mathbf{C(t)} + \alpha_2^{\sigma}\mathbf{A(t)} - \eta^{\sigma}\mathbf{R(t)}.
\end{aligned} \tag{4}
$$

3 Basic definition

In this part of the manuscript, we will present the essential definitions requisite knowing of CF fractional-derivatives to be familiar with common representations taken from refs. [6,25,28].

Definition 1. [6] Let $\phi \in H^1(0,T)$, if $n > \sigma > n-1$, $\sigma > 0$, and $n \in N$, then the CF and Caputo derivative operators are defined by

$$
{}^{\mathrm{CF}}D_{0,t}\{\varphi(t)\} = \frac{K(\sigma)}{(1-\sigma)}\int_0^t \varphi'(z)\exp\left(\frac{(z-t)\rho}{1-\rho}\right)\mathrm{d}z \tag{5}
$$

and

$$
{}^{\mathrm{C}}D_{0,t}^{\sigma}\{\varphi(t)\} = \frac{1}{\Gamma(-\sigma+n)}\int_0^t (t-z)^{-1+n-\sigma}\varphi^{n}(z)\mathrm{d}z, \tag{6}
$$

where CF and C stand for Caputo–Fabrizio and Caputo, respectively.

Definition 2. [6,25] For $\sigma \in (0,1)$, the integral

$$
{}^{\mathrm{RL}}J_{0,t}^{\sigma}\{\varphi(t)\} = \frac{1}{\Gamma(\sigma)}\int_0^t (t-z)^{\sigma-1}\varphi(z)\mathrm{d}z \tag{7}
$$

is the Riemann-Liouville and

$$
{}^{\mathrm{CF}}J_{0,t}^{\sigma}\{\varphi(t)\} = \frac{2}{(2-\sigma)K(\sigma)}\left\{(1-\sigma)\varphi(t) + \sigma\int_0^t \varphi(z)\mathrm{d}z\right\} \tag{8}
$$

is the Caputo–Fabreizio–Caputo (CF) integral operator.

4 Existence and uniqueness

We wish to explain the existence of model' solution as well as its uniqueness in this section. The standard results of fixed point theory will be used to prove the solution' existence as well as its uniqueness. We transform the developed system into the associated integral equation system which looks like:

$$
\begin{aligned}
\mathbf{S(t)} &= {}^{\mathrm{CF}}J_{0,t}^{\sigma}\left\{, \kappa^{\sigma}\Lambda^{\sigma} - \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \rho^{\sigma}\mathbf{V\,(t)} - (\mu^{\sigma}+\eta^{\sigma})\mathbf{S(t)}\right\} + \mathbf{S(0)}, \\
\mathbf{V(t)} &= \mathbf{V(0)} + {}^{\mathrm{CFC}}J_{0,t}^{\sigma}\left\{\Lambda^{\sigma}(1-\kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(t)} - \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\rho^{\sigma}+\eta^{\sigma})\mathbf{V\,(t)}\right\}, \\
\mathbf{A(t)} &= {}^{\mathrm{CFC}}J_{0,t}^{\sigma}\left\{\frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\alpha_1^{\sigma}+\alpha_2^{\sigma}+\eta^{\sigma})\mathbf{A(t)}\right\} + \mathbf{A(0)}, \\
\mathbf{C(t)} &= {}^{\mathrm{CFC}}J_{0,t}^{\sigma}\{\alpha_1^{\sigma}\mathbf{A(t)} - (\gamma_1^{\sigma}+\gamma_2^{\sigma}+\eta^{\sigma})\mathbf{C(t)}\}, \\
\mathbf{R(t)} &= \mathbf{R(0)} + {}^{\mathrm{CFC}}J_{0,t}^{\sigma}\{\gamma_1^{\sigma}\mathbf{C(t)} + \alpha_2^{\sigma}\mathbf{A(t)} - \eta^{\sigma}\mathbf{R(t)}\} + \mathbf{C(0)}.
\end{aligned}
$$

By the application of integration in the sense of CF, the above system will take the form

$$
\begin{aligned}
\mathbf{S(t)} &= \mathbf{S(0)} + \frac{2(1-\alpha)}{K(\alpha)(2-\alpha)}\left\{\kappa^{\sigma}\Lambda^{\sigma} - \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \rho^{\sigma}\mathbf{V(t)} - (\mu^{\sigma}+\eta^{\sigma})\mathbf{S(t)}\right\} \\
&\quad + 2\alpha\int_0^t \left\{\kappa^{\sigma}\Lambda^{\sigma} - \frac{\beta^{\sigma}\mathbf{S(x)A(x)}}{\mathbf{N}} + \rho^{\sigma}\mathbf{V\,(x)} - (\mu^{\sigma}+\eta^{\sigma})\mathbf{S(x)}\right\}\mathrm{d}x\left(\frac{1}{(2-\alpha)K(\alpha)}\right), \\
\mathbf{V(t)} &= \frac{2(1-\alpha)}{K(\alpha)}\left\{\Lambda^{\sigma}(1-\kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(t)} - \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\rho^{\sigma}+\eta^{\sigma})\mathbf{V\,(t)}\right\}\frac{1}{(2-\alpha)} + \mathbf{V(0)} \\
&\quad + \frac{1}{K(\alpha)}\int_0^t \left\{\Lambda^{\sigma}(1-\kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(y)} - \frac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(x)A(x)}}{\mathbf{N}} - (\rho^{\sigma}+\eta^{\sigma})\mathbf{V\,(x)}\right\}\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)}\right),
\end{aligned}
$$

$$\mathbf{A(t)} = \frac{(1-\alpha)}{K(\alpha)}\left\{\frac{\beta^\sigma \mathbf{S(t)A(t)}}{\mathbf{N}} + \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\alpha_1^\sigma + \alpha_2^\sigma + \eta^\sigma)\mathbf{A(t)}\right\}\frac{2}{(2-\alpha)} + \mathbf{A(0)}$$
$$+ \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \left\{\frac{\beta^\sigma \mathbf{S(x)A(x)}}{\mathbf{N}} + \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(x)A(x)}}{\mathbf{N}} - (\alpha_1^\sigma + \alpha_2^\sigma + \eta^\sigma)\mathbf{A(x)}\right\}\mathrm{d}x,$$

$$\mathbf{C(t)} = \mathbf{C(0)} + \frac{2(1-\alpha)}{K(\alpha)(2-\alpha)}\{\alpha_1^\sigma \mathbf{A(t)} - \mathbf{C(t)}(\gamma_1^\sigma + \gamma_2^\sigma + \eta^\sigma)\} + \frac{1}{(2-\alpha)}\int_0^t \{\alpha_1^\sigma \mathbf{A(x)} - (\gamma_1^\sigma + \gamma_2^\sigma + \eta^\sigma)\mathbf{C(x)}\}\mathrm{d}x\left(\frac{2\alpha}{K(\alpha)}\right),$$

$$\mathbf{R(t)} = \frac{2}{K(\alpha)}\{\gamma_1^\sigma \mathbf{C(t)} + \alpha_2^\sigma \mathbf{A(t)} - \eta^\sigma \mathbf{R(t)}\} + \frac{2\alpha}{K(\alpha)(2-\alpha)}\int_0^t \{\gamma_1^\sigma \mathbf{C(x)} + \alpha_2^\sigma \mathbf{A(x)} - \eta^\sigma \mathbf{R(x)}\}\mathrm{d}x\left(\frac{(1-\alpha)}{(2-\alpha)}\right) + \mathbf{R(0)}.$$

Let us assume that l_1, l_2, l_3, and l_4 symbolize the kernels and, respectively, are defined by

$$\begin{aligned}
l_1(\mathbf{S(t)}, t) &= \left\{\kappa^\sigma \Lambda^\sigma - \frac{\beta^\sigma \mathbf{S(t)A(t)}}{\mathbf{N}} + \rho^\sigma \mathbf{V\,(t)} - (\mu^\sigma + \eta^\sigma)\mathbf{S(t)}\right\},\\
l_2(\mathbf{V(t)}, t) &= \left\{\Lambda^\sigma(1-\kappa^\sigma) + \mu^\sigma \mathbf{S(t)} - \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\rho^\sigma + \eta^\sigma)\mathbf{V\,(t)}\right\},\\
l_3(\mathbf{A(t)}, t) &= \left\{\frac{\beta^\sigma \mathbf{S(t)A(t)}}{\mathbf{N}} + \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\alpha_1^\sigma + \alpha_2^\sigma + \eta^\sigma)\mathbf{A(t)}\right\},\\
l_4(\mathbf{C(t)}, t) &= \{\alpha_1^\sigma \mathbf{A(t)} - (\gamma_1^\sigma + \gamma_2^\sigma + \eta^\sigma)\mathbf{C(t)}\},\\
l_5(\mathbf{R(t)}, t) &= \{\gamma_1^\sigma \mathbf{C(t)} + \alpha_2^\sigma \mathbf{A(t)} - \eta^\sigma \mathbf{R(t)}\}.
\end{aligned} \tag{9}$$

Theorem 1. *The kernels as symbolized by l_1, l_2, l_3, l_4, and l_5 and stated by the aforementioned equations holds the Lipschitz conditions.*

Proof. We suppose that $\mathbf{S}$ and $\mathbf{S_1}$, $\mathbf{V}$ and $\mathbf{V_1}$, $\mathbf{A}$ and $\mathbf{A_1}$, $\mathbf{C}$ and $\mathbf{C_1}$, $\mathbf{R}$, and $\mathbf{R_1}$, are the functions of l_1, l_2, l_3, l_4, and l_5, therefore, we have

$$\begin{aligned}
l_1(\mathbf{S(t)}, t) &= l_1(\mathbf{S_1(t)}, t) + \left\{\kappa^\sigma \Lambda^\sigma - \frac{\beta^\sigma(\mathbf{S(t)} - \mathbf{S_1(t)})\mathbf{A(t)}}{\mathbf{N}} + \rho^\sigma \mathbf{V\,(t)} - (\mu^\sigma + \eta^\sigma)(\mathbf{S(t)} - \mathbf{S_1(t)})\right\},\\
l_2(\mathbf{V\,(t)}, t) &= l_2(\mathbf{V_2(t)}, t) + \left\{\Lambda^\sigma(1-\kappa^\sigma) + \mu^\sigma \mathbf{S(t)} - \frac{(1-\tau_1^\sigma)\beta^\sigma(\mathbf{V\,(t)} - \mathbf{V_2(t)})\mathbf{A(t)}}{\mathbf{N}} - (\rho^\sigma + \eta^\sigma)(\mathbf{V\,(t)} - \mathbf{V_2(t)})\right\},\\
l_3(\mathbf{A(t)}, t) &= l_3(\mathbf{A31(t)}, t) + \left\{\frac{\beta^\sigma \mathbf{S(t)}}{\mathbf{N}} + \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(t)}}{\mathbf{N}} - (\alpha_1^\sigma + \alpha_2^\sigma + \eta^\sigma)\right\}(\mathbf{A(t)} - \mathbf{A_3(t)}),\\
l_4(\mathbf{C(t)}, t) &= l_4(\mathbf{C_1(t)}, t) + \{\alpha_1^\sigma \mathbf{A(t)} - (\gamma_1^\sigma + \gamma_2^\sigma + \eta^\sigma)\mathbf{C(t)} - \mathbf{C_4(t)}\},\\
l_5(\mathbf{R(t)}, t) &= l_5(\mathbf{R_1(t)}, t) + \{\gamma_1^\sigma \mathbf{C(t)} + \alpha_2^\sigma \mathbf{A(t)} - \eta^\sigma \mathbf{R(t)} - \mathbf{R_5(t)}\}.
\end{aligned}$$

By applying the result of Cauchy's inequality, we have

$$\begin{aligned}
&\|l_1(\mathbf{S(t)}, t) - l_1(\mathbf{S_1(t)}, t)\|\\
&\quad\le \left\|\kappa^\sigma \Lambda^\sigma - \frac{\beta^\sigma(\mathbf{S(t)} - \mathbf{S_1(t)})\mathbf{A(t)}}{\mathbf{N}} + \rho^\sigma \mathbf{V\,(t)} - (\mu^\sigma + \eta^\sigma)(\mathbf{S(t)} - \mathbf{S_1(t)})\right\|,\\
&\|l_2(\mathbf{V\,(t)}, t) - l_2(\mathbf{V_1(t)}, t)\|\\
&\quad\le \left\|\Lambda^\sigma(1-\kappa^\sigma) + \mu^\sigma \mathbf{S(t)} - \frac{(1-\tau_1^\sigma)\beta^\sigma(\mathbf{V\,(t)} - \mathbf{V_1(t)})\mathbf{A(t)}}{\mathbf{N}} - (\rho^\sigma + \eta^\sigma)(\mathbf{V\,(t)} - \mathbf{V_1(t)})\right\|,\\
&\|l_3(\mathbf{A(t)}, t) - l_3(\mathbf{A_1(t)}, t)\|\\
&\quad= \left\|\left\{\frac{\beta^\sigma \mathbf{S(t)}}{\mathbf{N}} + \frac{(1-\tau_1^\sigma)\beta^\sigma \mathbf{V\,(t)}}{\mathbf{N}} - (\alpha_1^\sigma + \alpha_2^\sigma + \eta^\sigma)\right\}(\mathbf{A(t)} - \mathbf{A_1(t)})\right\|,
\end{aligned}$$

$$\|l_4(\mathbf{C(t)},t) - l_4(\mathbf{C_1(t)},t)\| \le \|\alpha_1^\sigma \mathbf{A(t)} - (\gamma_1^\sigma + \gamma_2^\sigma + \eta^\sigma)\mathbf{C(t)} - \mathbf{C_1(t)}\|,$$
$$\|l_5(\mathbf{R(t)},t) - l_5(\mathbf{R_1(t)},t)\| \le \|\gamma_1^\sigma \mathbf{C(t)} + \alpha_2^\sigma \mathbf{A(t)} - \eta^\sigma \mathbf{R(t)} - \mathbf{R_1(t)}\|.$$

From the above, the following expression could be readily obtained

$$\begin{aligned}
\mathbf{S(t)} &= \frac{2(1-\sigma)l_1(\mathbf{S}_{n-1}(t),t)}{K(\sigma)(2-\sigma)} + 2\sigma\frac{1}{K(\sigma)(-\sigma+2)}\int_0^t l_1(\mathbf{S}_{n-1}(x),x)\mathrm{d}x,\\
\mathbf{V(t)} &= \frac{2(1-\alpha)l_2(\mathbf{A}_{n-1}(t),t)}{K(\sigma)(2-\sigma)} + 2\sigma\frac{1}{K(\sigma)(-\sigma+2)}\int_0^t l_2(\mathbf{V}_{n-1}(x),x)\mathrm{d}x,\\
\mathbf{A(t)} &= \frac{2(1-\alpha)l_3(\mathbf{B}_{n-1}(t),t)}{K(\sigma)(2-\sigma)} + 2\sigma\frac{1}{K(\sigma)(-\sigma+2)}\int_0^t l_3(\mathbf{A}_{n-1}(x),x)\mathrm{d}x,\\
\mathbf{C(t)} &= \frac{2(1-\alpha)l_4(\mathbf{C}_{n-1}(t),t)}{K(\sigma)(2-\sigma)} + 2\sigma\frac{1}{K(\sigma)(-\sigma+2)}\int_0^t l_4(\mathbf{C}_{n-1}(x),x)\mathrm{d}x,\\
\mathbf{R(t)} &= \frac{2(1-\alpha)l_5(\mathbf{R}_{n-1}(t),t)}{K(\sigma)(2-\sigma)} + 2\sigma\frac{1}{K(\sigma)(-\sigma+2)}\int_0^t l_5(\mathbf{R}_{n-1}(x),x)\mathrm{d}x.
\end{aligned} \tag{10}$$

By assuming the difference of successive terms, taking the norm and considering the majorization of the resultant expression, we obtain

$$\begin{aligned}
\|\mathbf{U}_n(t)\| = \|\mathbf{S}_n(t) - \mathbf{S}_{1,n-1}(t)\| &\le (1-\sigma)\frac{2}{(-\sigma+2)K(\sigma)}\|l_1(\mathbf{S}_{n-1}(t),t) - l_1(\mathbf{S}_{1,n-2}(t),t)\|\\
&\quad + \frac{2\sigma}{K(\sigma)(2-\sigma)}\left\|\int_0^t [l_1(\mathbf{S}_{n-1}(x),x) - l_1(\mathbf{S}_{1,n-2}(x),x)]\mathrm{d}x\right\|,\\
\|\mathbf{W}_n(t)\| = \|\mathbf{V}_n(t) - \mathbf{A}_{1,n-1}(t)\| &\le \|l_2(\mathbf{V}_{n-1}(t),t) - l_2(\mathbf{V}_{1,n-2}(t),t)\|(1-\sigma)\frac{2}{(-\sigma+2)K(\sigma)}\\
&\quad + \sigma\frac{2}{(-\sigma+2)K(\sigma)}\left\|\int_0^t [l_2(\mathbf{V}_{n-1}(x),x) - l_2(\mathbf{V}_{1,n-2}(x),x)]\mathrm{d}x\right\|,\\
\|\mathbf{X}_n(t)\| = \|\mathbf{A}_n(t) - \mathbf{B}_{1,n-1}(t)\| &\le \frac{2(1-\sigma)}{K(\sigma)(3-\sigma)}\|l_3(\mathbf{A}_{n-1}(t),t) - l_1(\mathbf{A}_{1,n-2}(t),t)\|\\
&\quad + \frac{2\sigma}{K(\sigma)(2-\sigma)}\left\|\int_0^t [l_3(\mathbf{A}_{n-1}(x),x) - l_3(\mathbf{A}_{1,n-2}(x),x)]\mathrm{d}x\right\|,\\
\|\mathbf{Y}_n(t)\| = \|\mathbf{C}_n(t) - \mathbf{C}_{1,n-1}(t)\| &\le \|l_4(\mathbf{C}_{n-1}(t),t) - l_4(\mathbf{C}_{1,n-2}(t),t)\|(1-\sigma)\frac{2}{(-\sigma+2)K(\sigma)}\\
&\quad + \sigma\frac{2}{(-\sigma+2)K(\sigma)}\left\|\int_0^t [l_4(\mathbf{C}_{n-1}(x),x) - l_4(\mathbf{C}_{1,n-2}(x),x)]\mathrm{d}x\right\|,\\
\|\mathbf{Z}_n(t)\| = \|\mathbf{R}_n(t) - \mathbf{R}_{1,n-1}(t)\| &\le \|l_5(\mathbf{R}_{n-1}(t),t) - l_5(\mathbf{R}_{1,n-2}(t),t)\|(1-\sigma)\frac{2}{(-\sigma+2)K(\sigma)}\\
&\quad + \sigma\frac{2}{(-\sigma+2)K(\sigma)}\left\|\int_0^t [l_5(\mathbf{R}_{n-1}(x),x) - l_5(\mathbf{R}_{1,n-2}(x),x)]\mathrm{d}x\right\|.
\end{aligned} \tag{11}$$

In the above system

$$\sum_{i=0}^{\infty}\mathbf{X}_i(t) = \mathbf{A}_n(t),\ \ \sum_{i=0}^{\infty}\mathbf{W}_i(t) = \mathbf{V}_n(t),\ \ \sum_{i=0}^{\infty}\mathbf{Y}_i(t) = \mathbf{C}_n(t),\ \ \sum_{i=0}^{\infty}\mathbf{Z}_i(t) = \mathbf{R}_n(t),\ \ \sum_{i=0}^{\infty}\mathbf{U}_i(t) = \mathbf{S}_n(t). \tag{12}$$

As the kernels $\ell_1, \ldots, \ell_5$ satisfy the Lipschitz conditions, so

$$\|\mathbf{U}_n(t)\| = \|\mathbf{S}_n(t) - \mathbf{S}_{1,n-1}(t)\| \le \frac{2(1-\alpha)}{(2-\alpha)K(\alpha)}\tau_1\|\mathbf{S}_{n-1}(t) - \mathbf{S}_{1,n-2}(t)\| + \frac{2\alpha}{(2-\alpha)K(\alpha)}\tau_2\int_0^t \|\mathbf{S}_{n-1}(x) - \mathbf{S}_{1,n-2}(x)\|\mathrm{d}x,$$
$$\|\mathbf{W}_n(t)\| = \|\mathbf{V}_n(t) - \mathbf{V}_{1,n-1}(t)\| \le \tau_3\|\mathbf{V}_{n-1}(t) - \mathbf{V}_{1,n-2}(t)\|\frac{2(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\tau_4\int_0^t \|\mathbf{V}_{n-1}(x) - \mathbf{V}_{1,n-2}(x)\|\mathrm{d}x,$$
$$\|\mathbf{X}_n(t)\| = \|\mathbf{A}_n(t) - \mathbf{A}_{1,n-1}(t)\| \le \tau_5\|\mathbf{A}_{n-1}(t) - \mathbf{A}_{1,n-2}(t)\|\frac{2(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\tau_6\int_0^t \|\mathbf{A}_{n-1}(x) - \mathbf{A}_{1,n-2}(x)\|\mathrm{d}x, \tag{13}$$
$$\|\mathbf{Y}_n(t)\| = \|\mathbf{C}_n(t) - \mathbf{C}_{1,n-1}(t)\| \le \tau_7\|\mathbf{C}_{n-1}(t) - \mathbf{C}_{1,n-2}(t)\|\frac{(1-\alpha)(2)}{(2-\alpha)K(\alpha)} + \tau_8\int_0^t \|\mathbf{C}_{n-1}(x) - \mathbf{C}_{1,n-2}(x)\|\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)K(\alpha)}\right),$$
$$\|\mathbf{Z}_n(t)\| = \|\mathbf{R}_n(t) - \mathbf{R}_{1,n-1}(t)\| \le \frac{2}{K(\alpha)}\tau_7\|\mathbf{R}_{n-1}(t) - \mathbf{R}_{1,n-2}(t)\|\frac{(1-\alpha)}{(2-\alpha)} + \tau_8\int_0^t \|\mathbf{R}_{n-1}(x) - \mathbf{R}_{1,n-2}(x)\|\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)K(\alpha)}\right).$$
□

Theorem 2. *Model* (4) *possess a solution.*

Proof. By using expressions in (12) with recursive schemes lead to the assertions given as

$$\|\mathbf{U}_n(t)\| \le \|\mathbf{S}(0)\| + \left\{\left(\frac{2\tau_1}{K(\alpha)(-\alpha+2)}(1-\alpha)\right)^n\right\} + \left\{\left(2\frac{t\tau_2\alpha}{K(\alpha)(-\alpha+2)}\right)^n\right\},$$
$$\|\mathbf{W}_n(t)\| \le \|\mathbf{V}(0)\| + \left\{\left(2\frac{\tau_3}{K(\alpha)(-\alpha+2)}(1-\alpha)\right)^n\right\} + \left\{\left(\frac{2\tau_4 t\alpha}{(2-\alpha)M(\alpha)}\right)^n\right\},$$
$$\|\mathbf{X}_n(t)\| \le \|\mathbf{A}(0)\| + \left\{\left(2\frac{\tau_5}{K(\alpha)(-\alpha+2)}(1-\alpha)\right)^n\right\} + \left\{\left(2\frac{t\alpha\tau_6}{K(\alpha)(-\alpha+2)}\right)^n\right\}, \tag{14}$$
$$\|\mathbf{Y}_n(t)\| \le \|\mathbf{C}(0)\| + \left\{\left(\frac{2(1-\alpha)\tau_7}{(2-\alpha)K(\alpha)}\right)^n\right\} + \left\{\left(\frac{2\tau_8 t\alpha}{K(\alpha)(2-\alpha)}\right)^n\right\},$$
$$\|\mathbf{Z}_n(t)\| \le \|\mathbf{R}(0)\| + \left\{\left(\frac{2(1-\alpha)\tau_9}{(2-\alpha)K(\alpha)}\right)^n\right\} + \left\{\left(\frac{2\alpha\tau_{10}t}{(2-\alpha)K(\alpha)}\right)^n\right\}.$$

Consider that $\Upsilon_{1,i}(t)$, $i = 1,\ldots,5$ represent the remaining terms, then for proving that inequalities (14) satisfy model (4), the following expressions are utilized

$$\begin{aligned}&\mathbf{S}(t) = \Upsilon_{1,n}(t) + \mathbf{S}_n(t), \quad \mathbf{V}(t) = \mathbf{V}_n(t) + \Upsilon_{2,n}(t), \quad \mathbf{A}(t) + \Upsilon_{3,n}(t) = \mathbf{A}_n(t),\\ &\mathbf{C}(t) = +\Upsilon_{4,n}(t) + \mathbf{C}_n(t), \quad \mathbf{R}(t) + \Upsilon_{4,n}(t) = \mathbf{R}_n(t).\end{aligned} \tag{15}$$

Thus,

$$\mathbf{S}(t) - \mathbf{S}_{n-1}(t) = \frac{2(1-\alpha)l_1(\mathbf{S}(t) - \Upsilon_{1,n}(t))}{K(\alpha)(2-\alpha)} + \frac{2\alpha}{K(\alpha)(2-\alpha)}\int_0^t l_1(\mathbf{S}(x) - \Upsilon_{1,n}(x))\mathrm{d}x,$$
$$\mathbf{V}(t) - \mathbf{V}_{n-1}(t) = \frac{2l_2(\mathbf{V}(t) - \Upsilon_{2,n}(t))(1-\alpha)}{K(\alpha)(2-\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_2(\mathbf{V}(x) - \Upsilon_{2,n}(x))\mathrm{d}x,$$
$$\mathbf{A}(t) - \mathbf{A}_{n-1}(t) = \frac{2l_3(\mathbf{A}(t) - \Upsilon_{3,n}(t))(1-\alpha)}{K(\alpha)(2-\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t ll_3(\mathbf{A}(x) - \Upsilon_{3,n}(x))\mathrm{d}x, \tag{16}$$
$$\mathbf{C}(t) - \mathbf{C}_{n-1}(t) = \frac{2(1-\alpha)l_4(R(t) - \Upsilon_{4,n}(t))}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_4(\mathbf{C}(x) - \Upsilon_{4,n}(x))\mathrm{d}x,$$
$$\mathbf{R}(t) - \mathbf{R}_{n-1}(t) = \frac{2(1-\alpha)l_5(\mathbf{R}(t) - \Upsilon_{5,n}(t))}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_5(\mathbf{R}(x) - \Upsilon_{5,n}(x))\mathrm{d}x.$$

By using the Lipschitz conditions along with norm could lead to

$$
\begin{aligned}
&\left\| \mathbf{S}(t) - \frac{2l_1(\mathbf{S}(t), t)(1-\alpha)}{K(\alpha)}\frac{1}{(2-\alpha)} - \mathbf{S}(0) - \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_1(\mathbf{S}(x), x)\mathrm{d}x \right\| \\
&\quad \leq \|\Upsilon_{1,n}(t)\|\left\{1 + \left(\frac{2(1-\alpha)\tau_1}{(2-\alpha)K(\alpha)} + \frac{2\alpha\tau_2 t}{(2-\alpha)K(\alpha)}\right)\right\}, \\
&\left\| \mathbf{V}(t) - \frac{2l_2(\mathbf{V}(t), t)(1-\alpha)}{K(\alpha)(2-\alpha)} - \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_2(\mathbf{V}(x), x)\mathrm{d}x \right\| - \mathbf{V}(0) \\
&\quad \leq \|\Upsilon_{2,n}(t)\|\left\{1 + \left(\frac{2(1-\alpha)\tau_3}{(2-\alpha)K(\alpha)} + \frac{2\alpha\tau_4 t}{(2-\alpha)K(\alpha)}\right)\right\}, \\
&\left\| \mathbf{A}(t) - \frac{2l_3(\mathbf{A}(t), t)(1-\alpha)}{(2-\alpha)K(\alpha)} - \mathbf{A}(0) - \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t l_3(\mathbf{A}(x), x)\mathrm{d}x \right\| \\
&\quad \leq \|\Upsilon_{3,n}(t)\|\left\{1 + \left(\frac{2\tau_5(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha\tau_6 t}{(2-\alpha)K(\alpha)}\right)\right\}, \\
&\left\| \mathbf{C}(t) - \frac{2l_4(\mathbf{C}(t), t)(1-\alpha)}{(2-\alpha)K(\alpha)} - \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \ell_4(\mathbf{C}(x), x)\mathrm{d}x \right\| - \mathbf{C}(0) \\
&\quad \leq \|\Upsilon_{4,n}(t)\|\left\{1 + \left(\frac{2(1-\alpha)\tau_7}{(2-\alpha)K(\alpha)} + \frac{2\alpha\tau_8 t}{(2-\alpha)K(\alpha)}\right)\right\}, \\
&\left\| \mathbf{R}(t) - \frac{2l_5(\mathbf{R}(t), t)(1-\alpha)}{K(\alpha)(2-\alpha)} - \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \ell_5(\mathbf{R}(x), x)\mathrm{d}x \right\| - \mathbf{R}(0) \\
&\quad \leq \|\Upsilon_{5,n}(t)\|\left\{1 + \left(\frac{2(1-\alpha)\tau_8}{(2-\alpha)K(\alpha)} + \frac{2\alpha\tau_{10} t}{(2-\alpha)K(\alpha)}\right)\right\}.
\end{aligned}
\tag{17}
$$

We assume $\Upsilon = K(\alpha)(2-\alpha)$ for the sake of simplicity and by applying lim without bounds, we have

$$
\begin{aligned}
\mathbf{S}(t) &= \frac{2l_1(\mathbf{S}(t), t)(1-\alpha)}{\Upsilon} + \int_0^t l_1(\mathbf{S}(x), x)\mathrm{d}x\left(\frac{2\alpha}{\Upsilon}\right) + \mathbf{S}(0), \\
\mathbf{V}(t) &= \frac{2l_2(\mathbf{V}(t), t)(1-\alpha)}{\Upsilon} + \int_0^t l_2(\mathbf{V}(x), x)\mathrm{d}x\left(\frac{2\alpha}{\Upsilon}\right) + \mathbf{V}(0), \\
\mathbf{A}(t) &= \frac{2l_3(\mathbf{A}(t), t)(1-\alpha)}{\Upsilon} + \frac{2\alpha}{\Upsilon}\int_0^t l_3(\mathbf{A}(x), x)\mathrm{d}x + \mathbf{A}(0), \\
\mathbf{C}(t) &= \frac{2(1-\alpha)l_4(\mathbf{C}(t), t)}{\Upsilon} + \frac{2\alpha}{\Upsilon}\int_0^t l_4(\mathbf{C}(x), x)\mathrm{d}x + \mathbf{C}(0), \\
\mathbf{R}(t) &= \frac{2l_5(\mathbf{R}(t), t)(1-\alpha)}{\Upsilon} + \frac{2\alpha}{\Upsilon}\int_0^t l_5(\mathbf{R}(x), x)\mathrm{d}x + \mathbf{R}(0),
\end{aligned}
\tag{18}
$$

which is enough for the conclusion regarding the existence of solution to the model under consideration. □

Theorem 3. *The proposed model* (4) *possess a unique solution.*

Proof. On the basis of contradiction basis, let us consider another solution as symbolized by $(S^{+}(t), V^{+}(t), A^{+}(t), C^{+}(t), R^{+}(t))$, then

$$
\begin{aligned}
\mathbf{S}(t) &= \mathbf{S}^{+}(t) + \frac{\{l_1(\mathbf{S}(t), t) - l_1(\mathbf{S}^{+}(t), t)\}2(1-\alpha)}{\Upsilon} + \int_0^t \{l_1(\mathbf{S}(y), y) - l_1(\mathbf{S}^{+}(y), y)\}\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)K(\alpha)}\right),\\
\mathbf{V}(t) &= \mathbf{V}^{+}(t) + \frac{\{l_2(\mathbf{V}(t), t) - l_2(\mathbf{V}^{+}(t), t)\}2(1-\alpha)}{\Upsilon} + \int_0^t \{l_2(\mathbf{V}(y), y) - l_2(\mathbf{V}^{+}(x), x)\}\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)K(\alpha)}\right),\\
\mathbf{A}(t) &= \mathbf{A}^{+}(t) + \frac{\{l_3(\mathbf{A}(t), t) - l_3(\mathbf{C}^{+}(t), t)\}2(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \{l_3(\mathbf{A}(x), x) - l_3(\mathbf{A}^{+}(x), x)\}\mathrm{d}x,\\
\mathbf{C}(t) &= \mathbf{C}^{+}(t) + \frac{\{l_4(\mathbf{C}(t), t) - l_4(\mathbf{C}^{+}(t), t)\}2(1-\alpha)}{K(\alpha)(2-\alpha)} + \int_0^t \{l_4(\mathbf{C}(x), x) - l_4(\mathbf{C}^{+}(x), x)\}\mathrm{d}x\left(\frac{2\alpha}{(2-\alpha)K(\alpha)}\right),\\
\mathbf{R}(t) - \mathbf{R}^{+}(t) &= \frac{2(1-\alpha)\{-l_5(\mathbf{R}^{+}(t), t) + l_5(\mathbf{R}(t), t)\}}{\Upsilon} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \{l_5(\mathbf{R}(x), x) - l_5(\mathbf{R}^{+}(x), x)\}\mathrm{d}x.
\end{aligned}
\tag{19}
$$

The majorizing of the above system leads to the system given by

$$
\begin{aligned}
\|\mathbf{S}(t) - \mathbf{S}^{+}(t)\| &= \frac{\|l_1(\mathbf{S}(t), t) - l_1(\mathbf{S}^{+}(t), t)\|2(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{K(\alpha)(2-\alpha)}\int_0^t \|l_1(\mathbf{S}(x), x) - l_1(\mathbf{S}^{+}(x), x)\|\mathrm{d}x,\\
\|\mathbf{V}(t) - \mathbf{V}^{+}(t)\| &= \frac{\|l_2(\mathbf{V}(t), t) - l_2(\mathbf{V}^{+}(t), t)\|2(1-\alpha)}{\Upsilon} + \frac{2\alpha}{K(\alpha)(2-\alpha)}\int_0^t \|l_2(\mathbf{V}(x), x) - l_2(\mathbf{V}^{+}(x), x)\|\mathrm{d}x,\\
\|\mathbf{A}(t) - \mathbf{A}^{+}(t)\| &= \frac{\|l_3(\mathbf{A}(t), t) - l_3(\mathbf{A}^{+}(t), t)\|2(1-\alpha)}{(2-\alpha)K(\alpha)} + \frac{2\alpha}{(2-\alpha)K(\alpha)}\int_0^t \|l_3(\mathbf{A}(x), x) - l_3(\mathbf{A}^{+}(x), x)\|\mathrm{d}x,\\
\|\mathbf{C}(t) - \mathbf{C}^{+}(t)\| &= \frac{\|l_4(\mathbf{C}(t), t) - l_4(\mathbf{C}^{+}(t), t)\|2(1-\alpha)}{\Upsilon} + \frac{2\alpha}{K(\alpha)(2-\alpha)}\int_0^t \|l_4(\mathbf{C}(x), x) - l_4(\mathbf{C}^{+}(x), x)\|\mathrm{d}x,\\
\|\mathbf{R}(t) - \mathbf{R}^{+}(t)\| &= \frac{\|l_5(\mathbf{R}(t), t) - l_4(\mathbf{R}^{+}(t), t)\|2(1-\alpha)}{\Upsilon} + \frac{2\alpha}{\Upsilon}\int_0^t \|l_5(\mathbf{R}(x), x) - l_5(R^{+}(x), x)\|\mathrm{d}x.
\end{aligned}
\tag{20}
$$

We now use the results stated by Theorems 1 and 2 and obtain

$$
\begin{aligned}
&\|\mathbf{S}(t) - \mathbf{S}^{+}(t)\| \le \frac{2\tau_1\psi_1(1-\alpha)}{\Upsilon} + \left(\frac{2\alpha\tau_2 t\phi_2}{\Upsilon}\right)^n, \|\mathbf{V}(t) - \mathbf{V}^{+}(t)\| \le \frac{2\tau_3(1-\alpha)\psi_3}{\Upsilon} + \left(\frac{2\tau_4\alpha\phi_4 t}{\Upsilon}\right)^n,\\
&\|\mathbf{A}(t) - \mathbf{A}^{+}(t)\| \le \frac{2\tau_5\psi_5(1-\alpha)}{\Upsilon} + \left(\frac{2\alpha\tau_6\phi_6 t}{\Upsilon}\right)^n, \|\mathbf{C}(t) - \mathbf{C}^{+}(t)\| \le \frac{2\tau_7\psi_7(1-\alpha)}{\Upsilon} + \left(\frac{2\alpha\tau_8\phi_8 t}{\Upsilon}\right)^n,\\
&\|\mathbf{R}(t) - \mathbf{R}^{+}(t)\| \le \frac{2\tau_9\psi_9(1-\alpha)}{\Upsilon} + \left(\frac{2\alpha\tau_{10}\phi_{10} t}{\Upsilon}\right)^n.
\end{aligned}
\tag{21}
$$

From the aforementioned assertions, we can note that the result holds for all n, and hence

$$
\begin{aligned}
&\mathbf{S}(t) = \mathbf{S}^{+}(t), \quad \mathbf{V}(t) = \mathbf{V}^{+}(t),\\
&\mathbf{A}(t) = \mathbf{A}^{+}(t), \quad \mathbf{C}(t) = \mathbf{C}^{+}(t), \quad \mathbf{R}(t) = \mathbf{R}^{+}(t).
\end{aligned}
\tag{22}
$$

This shows that if a solution to the problem exists, it must be unique. □

For the feasibility and posedness of the system, we are stating and prove the following lemma.

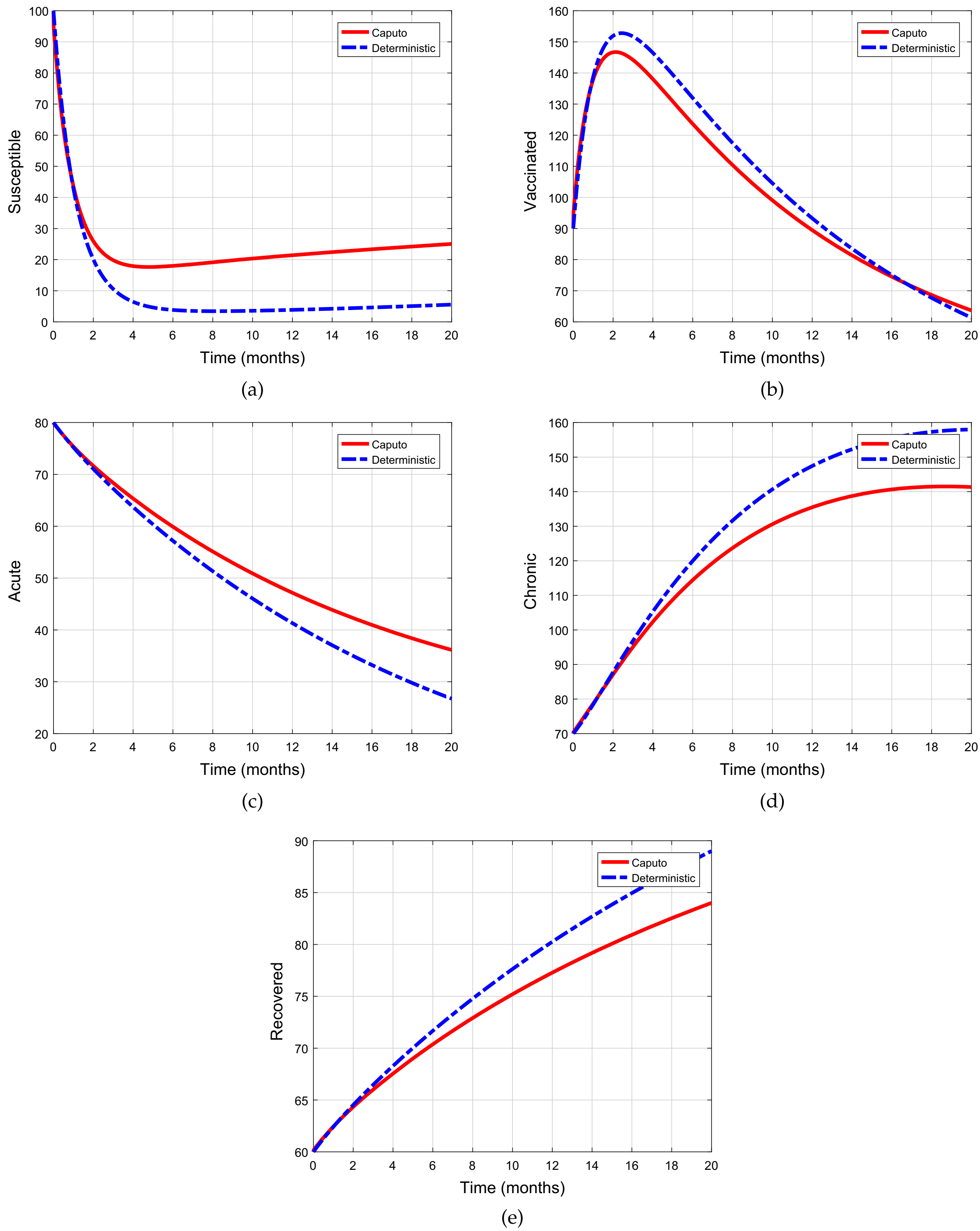

Figure 1: The visualization of the dynamics of system (4), when $\sigma = 0.95$. (a) Susceptible individuals, (b) vaccinated individuals, (c) acute individuals, (d) chronically individuals, and (e) recovered individuals.

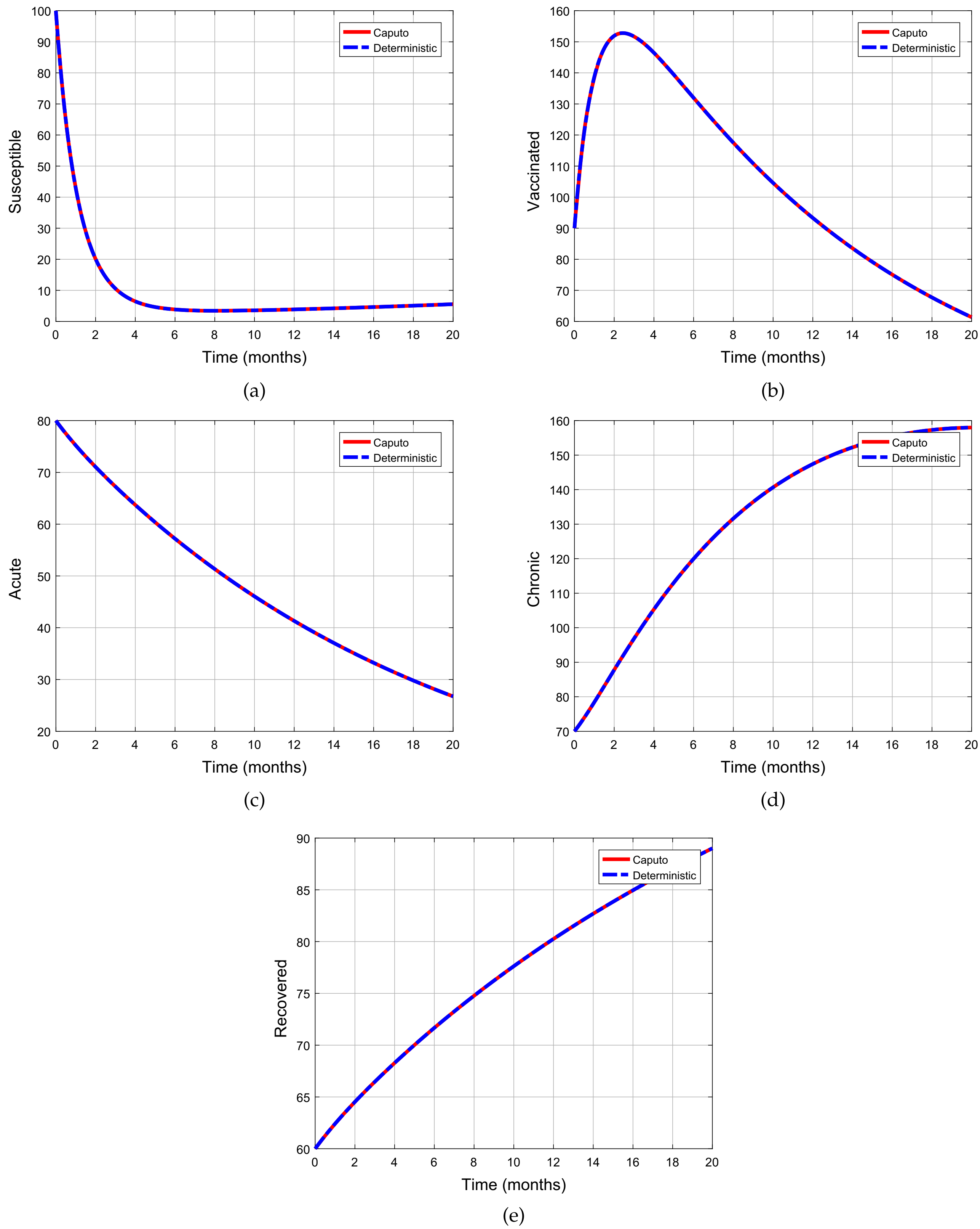

Figure 2: The visualization of the dynamics of system (4), when $\sigma = 1.0$. (a) Susceptible individuals, (b) vaccinated individuals, (c) acute individuals, (d) chronically individuals, and (e) recovered individuals.

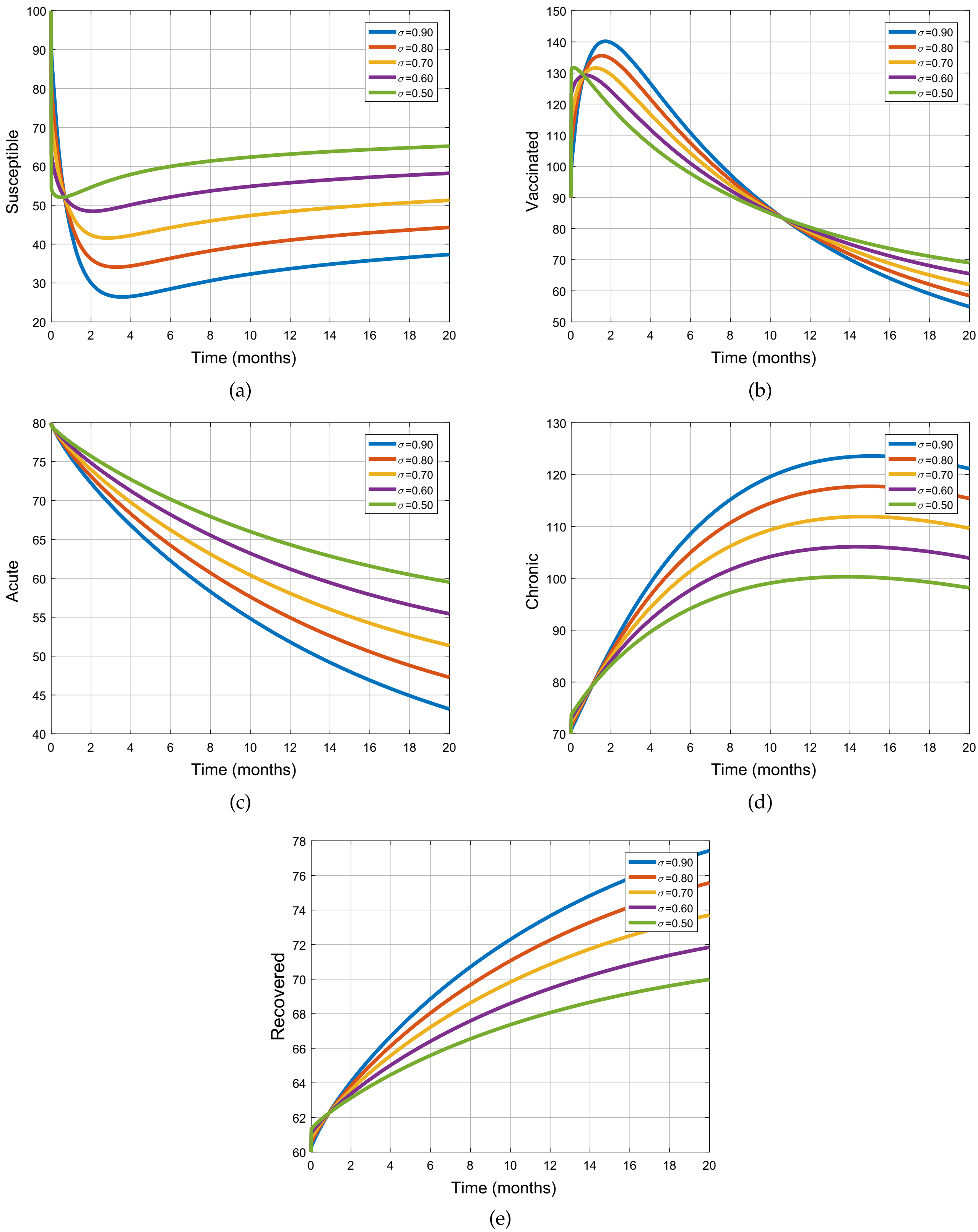

Figure 3: The visualization of the dynamics of system (4), when $\sigma = 0.90, 0.80, 0.70, 0.60, 0.50$. (a) Susceptible individuals, (b) vaccinated individuals, (c) acute individuals, (d) chronically individuals, and (e) recovered individuals.

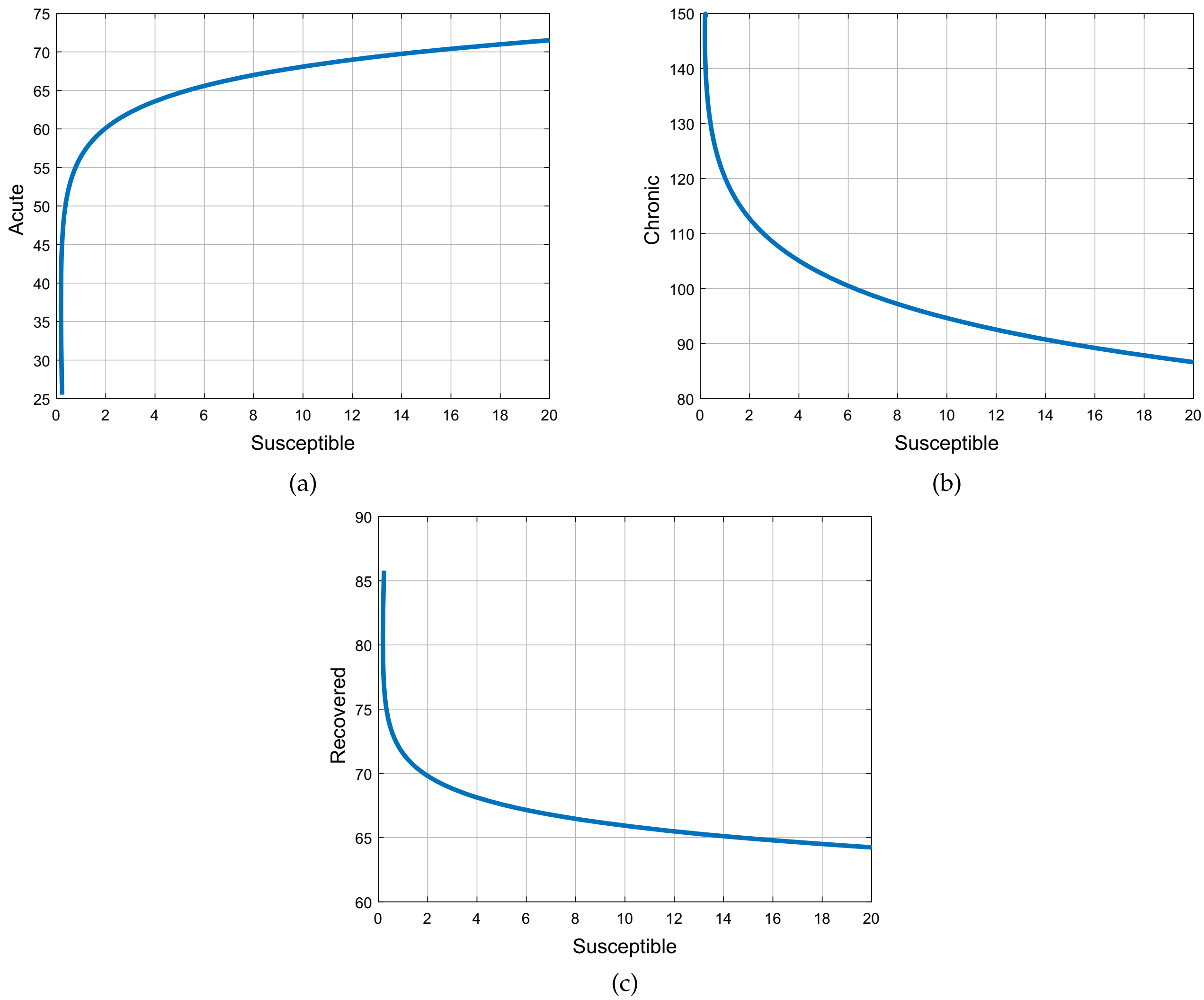

Figure 4: Phase portrait of system (4). (a) The graph represents the phase portrait of the acute and susceptible population. (b) The graph represents the phase portrait of the acute and susceptible population. (c) The graph represent the phase portrait of the recovered and susceptible population.

Lemma 1. *Assume that* $(\mathbf{S(t)}, \mathbf{V\,(t)}, \mathbf{A(t)}, \mathbf{C(t)}, \mathbf{R(t)})$ *be a solution of the model and let the system be accompanied with non-negative initial population, then* $(\mathbf{S(t)}, \mathbf{V\,(t)}, \mathbf{A(t)}, \mathbf{C(t)}, \mathbf{R(t)})$ *is non-negative for every* $t \geq 0$.

Proof. Let us assume that σ is the fractional parameter for system (4), then

$$\begin{aligned}
{}^{G}D_{0,t}^{\sigma}(\mathbf{S(t)}) &= \kappa^{\sigma}\Lambda^{\sigma} - \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \rho^{\sigma}\mathbf{V\,(t)} - (\mu^{\sigma} + \eta^{\sigma})\mathbf{S(t)},\\
{}^{G}D_{0,t}^{\sigma}(\mathbf{V\,(t)}) &= \Lambda^{\sigma}(1 - \kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(t)} - \frac{(1 - \tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\rho^{\sigma} + \eta^{\sigma})\mathbf{V\,(t)},\\
{}^{G}D_{0,t}^{\sigma}(\mathbf{A(t)}) &= \frac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} + \frac{(1 - \tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} - (\alpha_1^{\sigma} + \alpha_2^{\sigma} + \eta^{\sigma})\mathbf{A(t)},\\
{}^{G}D_{0,t}^{\sigma}(\mathbf{C(t)}) &= \alpha_1^{\sigma}\mathbf{A(t)} - (\gamma_1^{\sigma} + \gamma_2^{\sigma} + \eta^{\sigma})\mathbf{C(t)},\\
{}^{G}D_{0,t}^{\sigma}(\mathbf{R(t)}) &= \gamma_1^{\sigma}\mathbf{C(t)} + \alpha_2^{\sigma}\mathbf{A(t)} - \eta^{\sigma}\mathbf{R(t)},
\end{aligned} \tag{23}$$

Table 2: Parameter values used in simulations

Parameters	Value	Source
μ	0.030	All values of the parameters are assumed.
Λ	0.800	
ρ	0.005	
γ	0.030	
β	0.050	
η	0.070	
κ	0.009	
λ	0.070	
δ	0.070	
Π	0.040	
$S(0)$	100.0	
$V(0)$	90.00	
$I^a(0)$	80.00	
$I^c(0)$	70.00	
$R(0)$	60.00	

where G represents the fractional operator having order ω, so we have

$$\begin{cases} {}^{G}D_{0,t}^{\sigma}(\mathbf{S(t)})|_{\kappa(\mathbf{S(t)})} = \kappa^{\sigma}\Lambda^{\sigma} > 0, \\ {}^{G}D_{0,t}^{\sigma}(\mathbf{V\,(t)})|_{\kappa(\mathbf{V\,(t)})} = \Lambda^{\sigma}(1-\kappa^{\sigma}) + \mu^{\sigma}\mathbf{S(t)} \geq 0 \\ {}^{G}D_{0,t}^{\sigma}(\mathbf{A(t)})|_{\kappa(\mathbf{A(t)})} = \dfrac{\beta^{\sigma}\mathbf{S(t)A(t)}}{\mathbf{N}} \\ \qquad + \dfrac{(1-\tau_1^{\sigma})\beta^{\sigma}\mathbf{V\,(t)A(t)}}{\mathbf{N}} \geq 0, \\ {}^{G}D_{0,t}^{\sigma}(\mathbf{C(t)})|_{\kappa(\mathbf{C(t)})} = \alpha_1^{\sigma}\mathbf{A(t)} \geq 0, \\ {}^{G}D_{0,t}^{\sigma}(\mathbf{R(t)})|_{\kappa(\mathbf{R(t)})} = \gamma_1^{\sigma}\mathbf{C(t)} + \alpha_2^{\sigma}\mathbf{A(t)} \geq 0, \end{cases} \tag{24}$$

where $\kappa(\xi) = \{\xi = 0$ and $\mathbf{S(t)}, \mathbf{V\,(t)}, \mathbf{A(t)}, \mathbf{C(t)}, \mathbf{R(t)}$ is in $C(R_+ \times R_+)\}$ and $\xi \in \{\mathbf{S(t)}, \mathbf{V\,(t)}, \mathbf{A(t)}, \mathbf{C(t)}, \mathbf{R(t)}\}$. Following the methodology proposed in [25], we reach to the conclusion that the solution to the model is non-negative for any positive initial condition and $t \geq 0$.

5 Numerical simulations

This section is devoted to the justification and feasibility of the proposed work through simulations. We will present simulations of the fractional HBV epidemic model (4) by using the standard numerical procedures. We assume the level of time 20 units, while the epidemic parameters value are given in Table 2. The various graphs of the compartments are presented for distinct fractional parameter value σ to describe that the fractional order derivative provide more useful outcomes in describing the dynamics of infectious diseases. The dynamical behavior of various compartmental individuals of the epidemic problem (4) is presented in Figures 1–4.

Figure 1(a)–(e) represents the comparison of models (1) and (4), which indicates that the the curves obtained from the ordinary derivatives stabilize itself very quickly. In contrast, the solution curves obtained from the fractional model show a complex dynamics and the stability of the curves could be seen in the long run. Figure 2(a)–(e) represents the dynamics of compartmental population for the value of $\sigma = 1.01$ to show the effect of fractional order on the dynamics of the disease transmission. This means that the dynamics of both the models with ordinary and fractional derivatives are the same. In Figure 3(a), the temporal dynamics of the susceptible individuals are described, which show that an increase or decrease in the value of α causing the decreasing behavior, which shows that there is an inverse relation between the susceptible and derivative' order. Figure 3(b) represents the dynamics of vaccinated population, which is also inversely proportional to the order of the derivative. Similarly, the dynamics of acute, chronic, and recovered individuals are represented in Figure 3(c)–(e), respectively, which shows that increasing the value of fractional parameter causes the decrease or increase in acute, chronic while increase or decrease in the recovered population. Hence, it could be observed from the numerical study that the model has a strong relation and notably depends on the fractional parameter, which is clearly visible in the simulations being presented. Figure 4 represents the phase portrait of model (4).

6 Conclusion

The HBV is a potentially fatal liver infection that increases the risk of liver cancer and cirrhosis significantly, as well as the likelihood of chronic infection. It is a significant issue for global health and to decrease its spread, WHO advises using antiviral prophylaxis from mother to child besides infant vaccination. Keeping in view the impact of vaccination on the dynamics and control of HBV, we formulated a non-integer-order model (based on the CF derivative) for the HBV transmission with asymptomatic carriers and vaccination classes. Initially, a base model is developed by using the ordinary derivatives and the equilibria of the same model were studied for the stability analysis. The essential mathematical properties for the proposed fractional model were investigated by using the concept of fractional calculus. The existence and uniqueness of solution to the model are investigated by transforming the problem into integral equations and then applying the standard results of fixed point theory. It is was shown that the model' solution is positive and bounded for any non-negative initial data and $t \geq 0$. The model was simulated and the effect of the order of fractional derivative was illustrated. The predictions of the model are also investigated through simulations and the efficacy of the vaccination strategy was explained. Simulations suggest that increasing the value of fractional parameter causes the decrease/increase in acute, chronic while increase/decrease in the recovered population. Furthermore, numerical investigation reveals that the dynamics of the model have a strong dependency on the fractional parameter. The present work focuses on the concept of asymptomatic carriers and immunization with CF derivatives, thus, this work is assumed more useful than previous studies which are based on integer-order derivatives.

The findings of the study indicate that vaccination is the most common technique for reducing the spread of HBV in the community, and thus timely and properly implementing the vaccines are very important. Thus, it is recommended that the government and public-health authorities should focus on this in order to eliminate HBV disease out of the community. In addition, the government must take steps in educating people of the rural areas particularly, provision of vaccines, and provision of proper treatment at the doors, hospitals, and in other health-care centers.

Funding information: This work was supported by the Hainan Province Science and Technology Special Fund (No. ZDYF2021SHFZ231) and the National Natural Science Foundation of China (No. 12261028, No. 11901114).

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

References

[1] WHO, 2008. Fact sheet N204 Hepatitis B, available at http://www.who.int/mediacen-tre/factsheets/fs204/en/index.html.
[2] Nowak M, May RM. Virus dynamics: mathematical principles of immunology and virology: mathematical principles of immunology and virology. UK: Oxford University Press; 2000.
[3] Wodarz D, May RM, Nowak MA. The role of antigen-independent persistence of memory cytotoxic T lymphocytes. Int Immunol. 2000;12(4):467–77.
[4] Nowak MA, Bonhoeffer S, Hill AM, Boehme R, Thomas HC, McDade H. Viral dynamics in hepatitis B virus infection. Proc National Acad Sci. 1996;93(9):4398–402.
[5] Din A, Li Y, Yusuf A. Delayed hepatitis B epidemic model with stochastic analysis. Chaos Solitons Fractals. 2021;146:110839.
[6] Ullah S, Altaf Khan M, Farooq M. A new fractional model for the dynamics of the hepatitis B virus using the Caputo-Fabrizio derivative. European Phys J Plus. 2018;133(6):1–14.
[7] Khan FM, UllahKhan Z, Yusuf A, Munir T. The mathematical study of climate change model under nonlocal fractional derivative. Partial Differential Equations Appl Math. 2022;5:100204.
[8] Din A, Li Y, Khan T, Zaman G. Mathematical analysis of spread and control of the novel corona virus (COVID-19) in China. Chaos Solitons Fractals. 2020:110286.
[9] Khan MA, Atangana A. Mathematical modeling and analysis of COVID-19: A study of new variant Omicron. Phys A Statist Mech Appl. 2022;599:127452.
[10] Liu P, ur Rahman M. Fractal fractional based transmission dynamics of COVID-19 epidemic model. Comput Meth Biomech Biomed Eng. 2022;25:1–18.
[11] Lu Q. Stability of SIRS system with random perturbations. Phys A Statist Mech Appl. 2009;388(18):3677–86.
[12] Din A, Li Y. Stationary distribution extinction and optimal control for the stochastic hepatitis B epidemic model with partial immunity. Phys Scr. 2021;96(7):074005.
[13] Zhang X-B, Wang X-D, Huo H-F. Extinction and stationary distribution of a stochastic SIRS epidemic model with standard incidence rate and partial immunity. Phys A Stat Mech Appl. 2019;531:121548.
[14] Ganji RM, Jafari H, Baleanu D. A new approach for solving multi variable orders differential equations with Mittag-Leffler kernel. Chaos Solitons Fractals. 2020;130:109405.
[15] Atangana A. Modelling the spread of COVID-19 with new fractal-fractional operators: can the lockdown save mankind before vaccination?. Chaos Solitons Fractals. 2020;136:109860.
[16] Tuan NH, Aghdam YA, Jafari H, Mesgarani H. A novel numerical manner for two-Řdimensional space fractional diffusion equation arising in transport phenomena. Numer Meth Partial Differential Equations. 2021;37(2):1397–406.
[17] Rashid S, Kubra KT, Jafari H, UllahLehre S. A semi-analytical approach for fractional order Boussinesq equation in a gradient unconfined aquifers. Math Meth Appl Sci. 2022;45(2):1033–62.
[18] Ganji RM, Jafari H. A new approach for solving nonlinear Volterra integro-differential equations with Mittag-Leffler kernel. Proc Inst Math Mech. 2020;46(1):144–58.
[19] Li Y-X, Muhammad T, Bilal M, Khan MA, Ahmadian A, Pansera BA. Fractional simulation for Darcy–Forchheimer hybrid nanoliquid flow with partial slip over a spinning disk. Alex Eng J. 2021;60(5):4787–96.
[20] Shen H, Chu Y-M, Khan MA, Muhammad S, Al-Hartomy OA, Higazy M. Mathematical modeling and optimal control of the COVID-19 dynamics. Results Phy. 2021;31Article ID 105026, 9 pp.
[21] Tuan NH, Ganji RM, Jafari H. A numerical study of fractional rheological models and fractional Newell-Whitehead-Segel equation with non-local and non-singular kernel. Chinese J Phys. 2020;68:308–20.

[22] Ganji RM, Jafari H, Nemati S. A new approach for solving integro-differential equations of variable order. J Comput Appl Math. 2020;379:112946.
[23] Atangana A, Koca I. Chaos in a simple nonlinear system with Atangana–Baleanu derivatives with fractional order. Chaos Solitons Fractals. 2016;89:447–54.
[24] Tuan NH, Aghdam YE, Jafari H, Mesgarani H. A novel numerical manner for two-Řdimensional space fractional diffusion equation arising in transport phenomena. Numer Meth Partial Differential Equations. 2021;37(2):1397–406.
[25] Ain QT, Anjum N, Din A, Zeb A, Djilali S, Khan ZA. On the analysis of Caputo fractional order dynamics of Middle East Lungs Coronavirus (MERS-CoV) model. Alexandria Eng J. 2022;61(7):5123–131.
[26] Li X.-P., Al Bayatti H., Din A, Zeb A. A vigorous study of fractional order COVID-19 model *via* ABC derivatives. Results Phys. 2021;29.
[27] Li Y, Khan FM, Khan ZU, Liu P. On Analysis of fractional order mathematical model of Hepatitis B using Atangana–Baleanu Caputo (ABC) derivative. Fractals. 2021:2240017.
[28] Din A, Yusuf A, Ali AI. Caputo type fractional operator applied to Hepatitis B system. Fractals. 2021;2240023.
[29] Shepard CW, Simard EP, Finelli L, Fiore AE, Bell BP. Hepatitis B virus infection: epidemiology and vaccination. Epidemiologic Reviews. 2006;28(1):112–25.
[30] Shapiro CN. Epidemiology of hepatitis B. Pediatric Infect Disease J. 1993;12(5):433–37.
[31] Din A, Li Y. Mathematical analysis of a new nonlinear stochastic hepatitis B epidemic model with vaccination effect and a case study. European Phys J Plus. 2022;137(5):1–24.