Research Article

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Lie symmetry analysis for generalized short pulse equation

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Abstract: Lie symmetry analysis (LSA) is one of the most common, effective, and estimation-free methods to find the symmetries and solutions of the differential equations (DEs) by following an algorithm. This analysis leads to reduce the order of partial differential equations (PDEs). Many physical problems are converted into non-linear DEs and these DEs or system of DEs are then solved with several methods such as similarity methods, Lie Bäcklund transformation, and Lie group of transformations. LSA is suitable for providing the conservation laws corresponding to Lie point symmetries or Lie Bäcklund symmetries. Short pulse equation (SPE) is a non-linear PDE, used in optical fibers, computer graphics, and physical systems and has been generalized in many directions. We will find the symmetries and a class of solutions depending on one-parameter (ε) obtained from Lie symmetry groups. Then we will construct the optimal system for the Lie algebra and invariant solutions (called similarity solutions) from Lie subalgebras of generalized SPE.

Keywords: short pulse equation, lie symmetry, invariant solutions, optimal system, reductions

1 Introduction

Non-linear phenomenon can be translated in terms of non-linear differential equations (DEs). The solutions to these DEs may not be possible to find by using routine methods. There are some numerical as well as analytical methods to solve some particular kinds of non-linear DEs. However, Sophus Lie in 1875 gave a general method to solve these DEs. This method is general, algorithmic, and free of estimations. Lie's method helped to find exact solutions of DEs with the help of Lie group transformations, which is suitable for system of non-linear ordinary differential equations (ODEs) and partial differential equations (PDEs) [1,2]. Inspired by the work of Galois and Abel on the classification of discrete groups, Lie used their work for continuous groups in the nineteenth century. Lie succeeded to reduce order of an ODE by using methodology that the ODE is invariant under one-parameter Lie group of transformations. Lie's fundamental theorems clearly showed that continuous groups of DEs are specified in respect of their symmetry generators. Furthermore, if system of PDEs is invariant under Lie group of transformations, then one can determine solution's classes called invariant (similarity) solutions [1,3,4]. The non-linear PDEs arise in many applied fields such as biology, fluid, engineering, mechanics, plasma physics, and optics. Some physical models can be represented in mathematical form through short pulse equation (SPE). SPE is a non-linear PDE that was derived by Schäfer and Wayne [5] as to model non-linear DE, which describes the propagation of ultra short optical pulses. In optics, pulses are the light flashes. Figure 1 gives an overview of a positively chirped ultra-short pulse of light in the time domain. Thus, physically the short pulses possess the duration of picoseconds, which allows high data transmission in optics [6]. Rauch and Alterman used the approach of short pulses in Fourier domain where pulse is broad [5,7,8]. Their studies motivated other quality researchers to focus on other PDEs rather than cubic non-linear Schrodinger equation (NLSE). This idea inspired Schafer and Wayne to concentrate on ultra short pulses in 2004. They demonstrated that SPE gives strongly preferable approximation to the solution of Maxwell's equation of electric field in fiber as compared with the cubic NLSE [5]. Chung *et al.* proved it numerically that as we shorten the pulse length, the NLSE approximation becomes less precise

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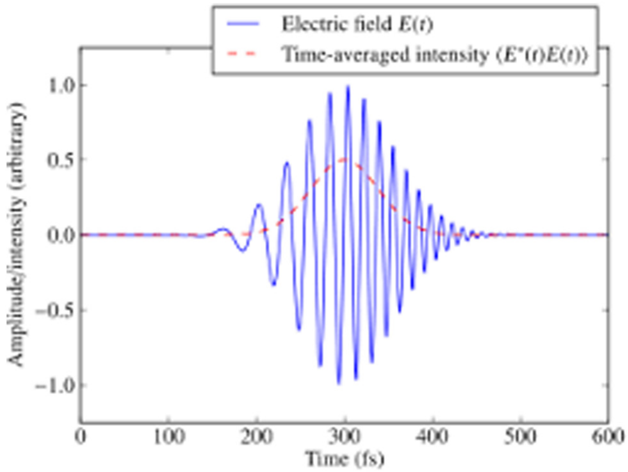

Figure 1: Ultra-short pulse.

as compared with SPE for the solution of Maxwell's equation [9]. Liu and Li evaluated exact travelling wave solution and analytic solution corresponding to SPE [10]. Matsuno derived a class of one- and two-phase non-singular periodic solutions of SP model equation and discussed their properties in ref. [11]. Liu *et al.* determined conditions for wave breaking in SPE [12]. Feng *et al.* worked on integrable discretization of SPE [13]. They have generated determinant formulas of N-soliton solutions of SPE. It is well-known that solitons are the solutions of a particular non-linear PDE. The two-component short pulse, complex short pulse, and coupled complex SPEs were proposed almost 20 years ago in ref. [14]. These equations were derived from the negative-order flows of the WadatiŰKonnoŰIchikawa (WKI) hierarchy. Qiao *et al.* discussed the category of non-linear evolution equations associated with the spectral problem and provided an approach for constructing their algebraic structure and r-matrix. First authors introduced the category of NLEEs, which is composed of various positive-order and negative-order hierarchies of NLEEs both integrable and non-integrable. Authors also proved that the whole category of NLEEs possessed a generalized Lax representation [15]. Feng's paper on complex short pulse and coupled complex SPEs computed the soliton solutions by the typical Hirota methods in ref. [16]. Feng worked on a complex SPE and a coupled complex SPE, which describe propagation of ultra-short pulse in optical fibers [16]. Sakovich concluded that vector form of SPE must be integrable as a result of Painlevé analysis [17]. Liu *et al.* gave a complete Lie group classification for the generalized SPE. They determined the reductions and exact solutions along with power series solutions for SPE [18]. For further details of SPE and its solutions, see refs [19–22].

Lie paved the way to solve non-linear system of PDEs by using algorithm of Lie group symmetries. Sun discussed exact and explicit solutions of non-linear Burgers' equation by using the Lie symmetry method [23]. Liu *et al.* established Lie symmetry analysis (LSA) for the non-linear general Burgers' equation. They evaluated its exact solutions and similarity reductions [24]. LSA for the Heisenberg equation is performed by Zhao and Han [25], which allowed them to obtain the corresponding Lie symmetries, optimal system of sub-algebra, invariant solutions, and conservation laws for this equation. Khalique and Adem used LSA to generate exact solutions of generalized (3 + 1)-dimensional Kadomtsev–Petviashivili equation [26]. We actually give some generalizations of the work of Liu and Li [10] on cases of SPE and add more facts such as optimal system of one-dimensional Lie symmetry sub-algebras. The SPE is defined by

$$u_{xt} = \alpha u + \frac{1}{3}\beta(u^3)_{xx}, \tag{1}$$

where $u = u(x, t)$ is the unknown real function and it shows the magnitude of the electric field, x and t are dependent variables, and α and β are non-zero real parameters. In subscripts, x and t denote the differentiation. Some cases have also been done for special SPE for $\alpha = 1$ and $\beta = \frac{1}{2}$ in Eq. (1) which is

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}. \tag{2}$$

For $\alpha = 1$ and $\beta = 3$ in Eq. (1), Liu and Li [10] obtained exact solutions by using LSA for

$$u_{xt} = u + uu_x^2 + u^2u_{xx}. \tag{3}$$

Matsuno worked on modified SPE [27]. The modified short pulse equation is a direct reduction ($u = v$) of a coupled SPE given as

$$u_{xt} = u + \frac{1}{2}u(u^2)_{xx}. \tag{4}$$

Authors studied multi-soliton solutions and the Cauchy problem for a two-component short pulse system, [28]. Zhijun Qiao discussed a finite-dimensional involutive system, and the WKI hierarchy of non-linear evolution equations and their commutator representations in ref. [29]. Recently, the researchers are working on the more general and modified forms of PDEs by taking specific functions to generalize the results corresponding to these PDEs. In the present article, we have considered the generalized form of SPE from Eq. (1), which contains the arbitrary (general) functions $f(u)$ and $g(u)$ by replacing

u in first term and $(u^3)_{xx}$ in second term at right-hand side of Eq. (1), respectively, given as

$$u_{xt} = \alpha f(u) + \frac{1}{3}\beta g(u). \tag{5}$$

Our results will be more general in nature covering a broad class of already available results on SPE. The generalized short pulse equation (GSPE) (5) gives many non-linear wave equations depending upon the functions $f(u)$ and $g(u)$. For the generalized SPE, the following cases have been discussed for the first time:

- For $f(u) = u^n$, $(n \in N)$ and $g(u) = u_{xx}^3$,
- For $f(u) = \ln(u)$ and $g(u) = u_{xx}^3$.

In first case, we will generalize our results for each $n \in \mathbb{N}$. For $n = 1$, we will have the Lie symmetries, infinitesimal generators, and optimal system for (1). By changing functions $f(u)$ and $g(u)$ in Eq. (5), we will have different Lie symmetries corresponding to each case of SPE and this leads to obtain their exact solutions as well.

2 Main results

We focus on LSA on the GSPE given above in two separate cases.

Case 1: We will deal with the situation for $f(u) = u^n$, $(n \in N)$, and $g(u) = u_{xx}^3$

Case 2: We will deal with the situation for $f(u) = \ln(u)$ and $g(u) = u_{xx}^3$.

2.1 LSA for generalized SPE for $f(u) = u^n$, $(n \in N)$, and $g(u) = u_{xx}^3$

The generalized SPE corresponding to Case 1 for $f(u) = u^n$ $(n \in N)$ and $g(u) = u_{xx}^3$, we have

$$u_{xt} = \alpha u^n + \frac{1}{3}\beta u_{xx}^3. \tag{6}$$

First, we consider the Lie group of transformations with independent variables x, t and dependent variable u for the equation

$$\begin{aligned} x^* &= x^*(x, t, u; \varepsilon), \\ t^* &= t^*(x, t, u; \varepsilon), \\ u^* &= u^*(x, t, u; \varepsilon), \end{aligned} \tag{7}$$

where $\varepsilon \in R$ is the group parameter.

Now, consider the corresponding one-parameter Lie group of transformations given as:

$$\begin{aligned} x^* &= x + \varepsilon\xi(x, t, u) + O(\varepsilon^2), \\ t^* &= t + \varepsilon\tau(x, t, u) + O(\varepsilon^2), \\ u^* &= u + \varepsilon\Phi(x, t, u) + O(\varepsilon^2), \end{aligned} \tag{8}$$

where $\varepsilon \in R$ is represented as the group parameter.

The infinitesimal generator of (7) is defined in the vector form as

$$\bar{V} = \xi(x, t, u)\frac{\partial}{\partial x} + \tau(x, t, u)\frac{\partial}{\partial t} + \Phi(x, t, u)\frac{\partial}{\partial u}, \tag{9}$$

where ξ, τ, and Φ are infinitesimal functions of group variables. The generalized SPE involves second-order derivatives, therefore, the second prolongation of the infinitesimal generator has the following form:

$$\begin{aligned} \mathrm{Pr}^{(2)}\bar{V} = \bar{V} &+ \Phi^x\frac{\partial}{\partial u_x} + \Phi^t\frac{\partial}{\partial u_t} + \Phi^{xx}\frac{\partial}{\partial u_{xx}} \\ &+ \Phi^{xt}\frac{\partial}{\partial u_{xt}} + \Phi^{tt}\frac{\partial}{\partial u_{tt}}. \end{aligned} \tag{10}$$

The derivatives w.r.t. xt and xx are involved in (6) when prolongation (10) is applied onto (6). The general formula of nth prolongation of $\bar{V}$ with coefficients Φ_α^j ($j = 1, 2$; $\alpha = 1$) is determined in ref. [1]. We only need the values of Φ^{xt} and Φ^{xx} which are given in ref. [1] as

$$\Phi^{xx} = D_xD_x(\Phi - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt}, \tag{11}$$

$$\Phi^{xt} = D_xD_t(\Phi - \xi u_x - \tau u_t) + \xi u_{xtx} + \tau u_{xtt}, \tag{12}$$

where D_x and D_t are the total derivatives.

In order to calculate the symmetry algebra of (6), apply the second prolongation (10) of the infinitesimal generator (9) along with invariance condition onto Eq. (6):

$$\mathrm{Pr}^{[2]}\bar{V}\left(u_{xt} - \alpha u^n - \frac{1}{3}\beta u_{xx}^3\right)\Big|_{u_{xt}=\alpha u^n+\frac{1}{3}\beta u_{xx}^3} = 0, \tag{13}$$

then simplification gives the following equation:

$$[-\alpha n\Phi u^{n-1} - \beta\Phi^{xx}u_{xx}^2 + \Phi^{xt}]|_{u_{xt}=\alpha u^n+\frac{1}{3}\beta u_{xx}^3} = 0. \tag{14}$$

Two terms Φ^{xx} and Φ^{xt} that are involved in the aforementioned equation are derived as follows:

$$\begin{aligned} \Phi^{xx} = \Phi_{xx} &+ (2\Phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t \\ &+ (\Phi_{uu} - 2\xi_{xu})(u_x)^2 - 2\tau_{xu}u_xu_t - \xi_{uu}(u_x)^3 \\ &- \tau_{uu}(u_x)^2u_t + (\Phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xt} - \tau_uu_tu_{xx} \\ &- 2\tau_uu_xu_{xt} - 3\xi_uu_xu_{xx} - 2\tau_uu_xu_{xt} - 3\xi_uu_xu_{xx}, \end{aligned} \tag{15}$$

$$\begin{aligned} \Phi^{xt} = \Phi_{xt} &+ (\Phi_{tu} - \xi_{xt})u_x + (\Phi_{xu} - \tau_{xt})u_t - \xi_tu_{xx} \\ &+ (\Phi_u - \xi_x - \tau_t)u_{xt} - \tau_xu_{tt} - \xi_uu_tu_{xx} \\ &+ (\Phi_{uu} - \xi_{xu} - \tau_{tu})u_xu_t - 2\tau_uu_tu_{xt} - \xi_{tu}(u_x)^2 - \tau_uu_xu_{tt} \\ &- 2\xi_uu_xu_{xt} - \tau_{xu}(u_t)^2 - \xi_{uu}u_t(u_x)^2 - \tau_{uu}u_x(u_t)^2. \end{aligned} \tag{16}$$

Substituting their values in (14), we have an equation

$$\begin{aligned}&\Phi_{xt}+\alpha(\Phi_u-\xi_x-\tau_t)u^n-\alpha n\Phi u^{n-1}\\&\quad+(\Phi_{tu}-\xi_{xt}-2\alpha\xi_u u^n)u_x+(\Phi_{xu}-\tau_{xt}-2\alpha\tau_u u^n)u_t\\&\quad-\xi_t u_{xx}-\tau_x u_{tt}+(\Phi_{uu}-xi_{xu}-\tau_{tu})u_x u_t-\xi_u u_t u_{xx}\\&\quad-\tau_{xu}(u_t)^2-\xi_{tu}(u_x)^2-\tau_u u_x u_{tt}-\xi_{uu}u_t(u_x)^2\\&\quad-\tau_{uu}u_x(u_t)^2+\beta(-\Phi_{xx}+2\beta\tau_x u^n)(u_{xx})^2\\&\quad+\beta\tau_{xx}u_t(u_{xx})^2-\beta(2\Phi_{xu}-\xi_{xx}-2\alpha\tau_u u^n)u_x(u_{xx})^2\\&\quad-\beta(\Phi_{uu}-2\xi_{xu})(u_x)^2(u_{xx})^2+2\beta\tau_{xu}u_x u_t(u_{xx})^2\\&\quad+\beta\xi_{uu}(u_x)^3(u_{xx})^2+\beta\tau_{uu}u_t(u_x)^2(u_{xx})^2\\&\quad+\beta\left(-\frac{2}{3}\Phi_u+\frac{5}{3}\xi_x-\frac{1}{3}\tau_t\right)(u_{xx})^3+\frac{2}{3}\beta^2\tau_x(u_{xx})^5\\&\quad+\frac{2}{3}\beta\tau_u u_t(u_{xx})^3+\frac{2}{3}\beta^2\tau_u u_x(u_{xx})^5+\frac{7}{3}\beta\xi_u u_x(u_{xx})^3=0.\end{aligned}\tag{17}$$

The coefficients of monomials are then compared and this comparison will give us the following overdetermined system of equations:

$$\begin{aligned}&\tau_{xx}=0,\quad \tau_{xu}=0,\quad \tau_{uu}=0,\quad \xi_{uu}=0,\quad \tau_u=0,\\&\xi_u=0,\quad \xi_{tu}=0,\quad \tau_x=0,\quad \Phi_{uu}=0,\quad \Phi_{xu}=0,\\&\Phi_{xx}=0,\quad \Phi_{tu}=0,\quad \xi_t=0,\quad \xi_{xx}=0,\\&-\alpha\Phi n u^{n-1}+\Phi_{xt}+\alpha(\Phi_u-\xi_x-\tau_t)u^n=0,\\&-\frac{2}{3}\Phi_u+\frac{5}{3}\xi_x-\frac{1}{3}\tau_t=0.\end{aligned}\tag{18}$$

These are the determining equations in terms of infinitesimals and derivatives of infinitesimals with respect to independent and dependent variables. One can obtain the most general Lie groups for (6) by finding the values of infinitesimal functions from the above determined equations. There is no hard and fast method to solve the above system of equations. Therefore, to solve the above system, we are considering such a value of Φ, which satisfies the above system of equations involving Φ in (18)

$$\Phi=A(t)x+Bu+C(t),\tag{19}$$

where $A(t)$, B, and $C(t)$ are arbitrary constants. Using this above value of Φ in the last two equations of system (18), we will achieve the values of ξ,τ, and Φ that are given as

$$\begin{aligned}\xi(x)&=\frac{n-3}{5n-3}c_1x+c_3,\\\tau(t)&=c_1t+c_2,\\\Phi(x,t,u)&=\frac{6c_1}{-5n+3}u.\end{aligned}\tag{20}$$

c_1, c_2, and c_3 are arbitrary constants. Consequently, the infinitesimal generators of SPE (6) for the one-parameter Lie groups of transformations by (6) are obtained by substituting the values of infinitesimal functions (20) into (9) and simplifying this equation. Thus, these infinitesimal generators are given as follows:

$$\begin{aligned}O_1&=\frac{n-3}{5n-3}x\partial_x+t\partial_t+\frac{6u}{-5n+3}\partial_u,\\O_2&=\partial_t,\\O_3&=\partial_x.\end{aligned}\tag{21}$$

Theorem 2.1.1. *The set of generators is closed under the one-parameter Lie groups G_i^ε that are generated by infinitesimal generators O_i $(i=1,2,3)$ where the entries will give the transformed points* $\exp(\varepsilon O_i)(x,t,u)=(x^*,t^*,u^*)$.

$$\begin{aligned}&G_1^\varepsilon:(x,t,u)\to(e^{\varepsilon\frac{n-3}{5n-3}}x,e^\varepsilon t,e^{\varepsilon\frac{6}{-5n+3}}u),\\&G_2^\varepsilon:(x,t,u)\to(x,t+\varepsilon,u),\\&G_3^\varepsilon:(x,t,u)\to(x+\varepsilon,t,u).\end{aligned}\tag{22}$$

$\varepsilon\in R$ is a group parameter and $n\in N$.

Theorem 2.1.2. *For each symmetry group G_i^ε, if $\sqcap=f(x,t)$ satisfies* (6), *then $\sqcap^{(i)}$ for $i=1,2,3$ are solutions of* (6).

Some special solutions and their graphs are demonstrated in Figures 2–5.

$$\begin{aligned}\sqcap^{(1)}&=e^{\varepsilon\left(\frac{6}{-5n+3}\right)}f\left(e^{-\varepsilon\left(\frac{n-3}{5n-3}\right)}x,e^{-\varepsilon}t\right),\\\sqcap^{(2)}&=f(x,t-\varepsilon),\\\sqcap^{(3)}&=f(x-\varepsilon,t),\end{aligned}\tag{23}$$

where $\sqcap^i=G_i^\varepsilon\cdot f(x,t)$ and $\varepsilon\ll 1$ is any positive number and $n\in N$.

2.1.1 Optimal system of one-dimensional subalgebra for generalized SPE (6)

Optimal system provides us the classes of equivalent Lie subalgebras and each similar class of Lie subalgebra in a set forms an optimal system. Optimal system leads to reduce the number of independent variables in DE, which makes it easier to obtain the solution of this DE. As we know that symmetry group of system of PDEs maps one solution of the system to another one of that system of PDEs. In this subsection, we will find a one-dimensional optimal system of Lie subalgebras admitted by (6) with the help of Lie brackets and values in adjoint table.

Lie brackets for the commutator table are defined as

$$[O_i,O_j]=O_iO_j-O_jO_i,\quad (i,j=1,2,3).\tag{24}$$

The commutator table for O_i is easy to be seen

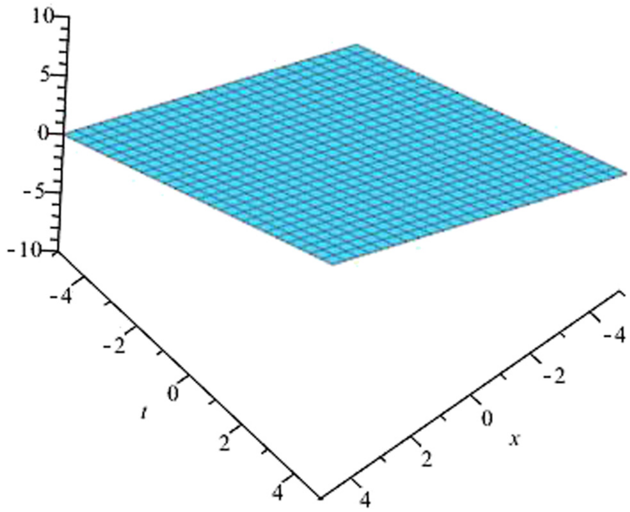

Figure 2: For $\sqcap^{(1)} = e^{-\frac{6}{7}\varepsilon}\left[e^{\frac{1}{7}\varepsilon}x + e^{-\varepsilon}\cos(t)\right]$, $n = 2$, and $\varepsilon = 0.00002$.

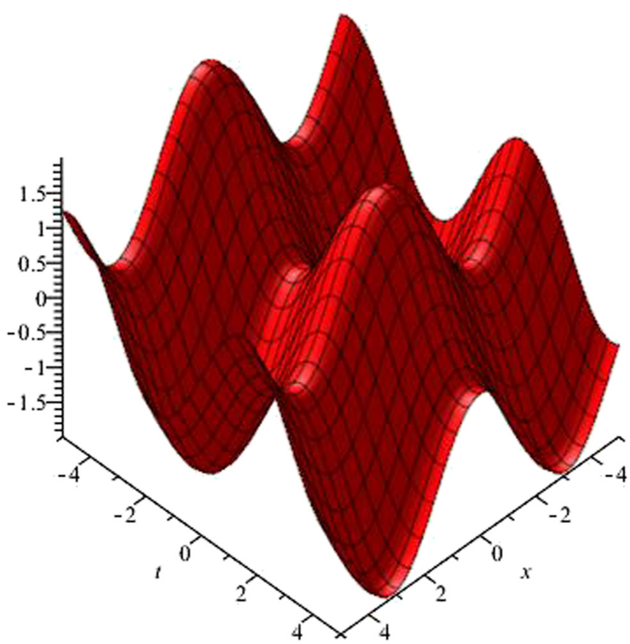

Figure 3: For $\sqcap^{(1)} = e^{-\frac{6}{237}\varepsilon}\left(e^{-\frac{45}{237}\varepsilon}\cos(x) + e^{-t}\sin(t)\right)$, $n = 48$, and $\varepsilon = 0.000001$.

$[..,..]$	O_1	O_2	O_3
O_1	0	$-O_2$	$-\dfrac{n-3}{5n-3}O_3$
O_2	O_2	0	0
O_3	$\dfrac{n-3}{5n-3}O_3$	0	0

for $n \in N$. Here, we will generate the Adjoint table for infinitesimal operators O_i for $i = 1, 2, 3$ by using the formula given as

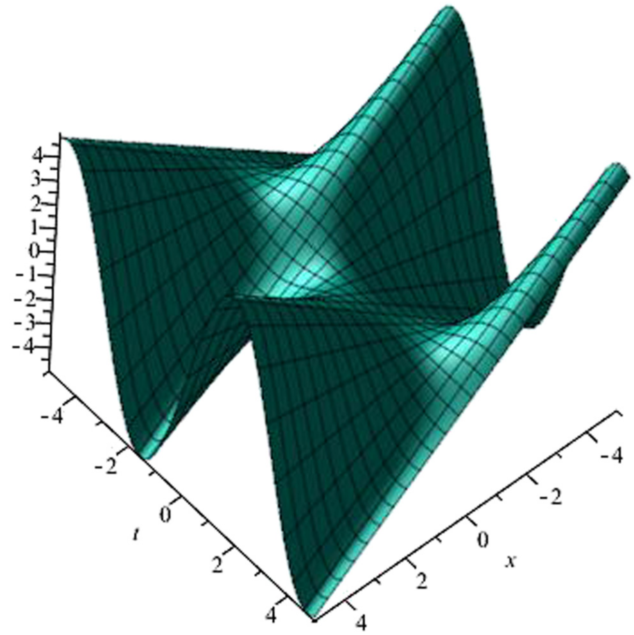

Figure 4: For $\sqcap^{(2)} = x\sin(t - \varepsilon)$ and $\varepsilon = 0.00002$.

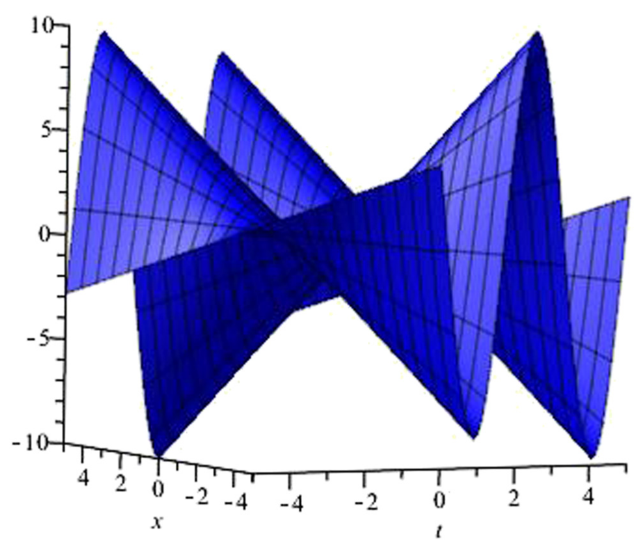

Figure 5: For $\sqcap^{(3)} = 2t\cos(x - \varepsilon)$ and $\varepsilon = 0.00002$.

$$\mathrm{Adj}_{e^{\varepsilon O_i}}(O_j) = O_j - \varepsilon[O_i, O_j] + \frac{\varepsilon^2}{2!}[O_i, [O_i, O_j]] - \ldots. \quad (25)$$

Thus, the adjoint table is as follows:

Adj	O_1	O_2	O_3
O_1	O_1	$e^{\varepsilon}O_2$	$e^{\varepsilon\frac{n-3}{5n-3}}O_3$
O_2	$O_1 - \varepsilon O_2$	O_2	O_3
O_3	$O_1 - \varepsilon\left(\dfrac{n-3}{5n-3}\right)O_3$	O_2	O_3.

Taking a non-zero generator of the form

$$O = \alpha_1 O_1 + \alpha_2 O_2 + \alpha_3 O_3, \tag{26}$$

we suppose that the coefficients $\alpha_i \neq 0$ for $i = 1, 2, 3$.

Case 1: If we take $\alpha_1 = 1$, we have the symmetry generator as follows:

$$O = O_1 + \alpha_2 O_2 + \alpha_3 O_3. \tag{27}$$

Applying the $\mathrm{Adj}_{e^{\alpha_2 O_2}}$ on O and representing it with O'. Therefore,

$$O' = \mathrm{Adj}_{e^{\alpha_2 O_2}}(O), \tag{28}$$

$$O' = O_1 + \alpha_3' O_3. \tag{29}$$

Now we apply $\mathrm{Adj}_{e^{\frac{5n-3}{n-3}\alpha_3' O_3}}$ on O'. Therefore, we have the symmetry generator O'' as

$$O'' = \mathrm{Adj}_e^{\frac{5n-3}{n-3}\alpha_3' O_3} O', \tag{30}$$

$$\Rightarrow O'' = O_1. \tag{31}$$

According to this algorithm, any subalgebra obtained from the above O is equivalent to O_1.

Case 2: If we take $\alpha_1 = 0$ and $\alpha_2 \neq 0$, in this case, we have the symmetry generator of the form:

$$O = O_2 + \alpha_3 O_3. \tag{32}$$

Applying the $\mathrm{Adj}_{e^{\varepsilon O_1}}$ onto O:

$$O''' = \mathrm{Adj}_{e^{\varepsilon O_1}}(O), \tag{33}$$

$$O''' = O_2 + \alpha_3 e^{-\frac{4n}{5n-3}\varepsilon} O_3. \tag{34}$$

Subcase 2.1: If $\alpha_3 < 0$, then

$$O''' \simeq O_2 - O_3. \tag{35}$$

Subcase 2.2: If $\alpha_3 > 0$, then

$$O''' \simeq O_2 + O_3. \tag{36}$$

Case 3: If we take $\alpha_1 = 0 = \alpha_2$ and let $\alpha_3 \neq 0$. Then, the generator becomes $O \simeq O_3$.

Case 4: Let us consider $\alpha_1 = 0 = \alpha_3$ and $\alpha_2 \neq 0$. In this case, the generator is $X \simeq O_2$.

In this way, we will have the comprising form of optimal system for the symmetries of generalized SPE (6):

$$O = \begin{cases} O_1, \\ O_2, \\ O_3, \\ O_2 \pm O_3. \end{cases} \tag{37}$$

2.1.2 Reductions and invariant solutions

Now, we reduce non-linear PDE (6) into a non-linear ODE against each symmetry by constructing new invariant solutions from the characteristic equation.

Reduction through $O_3 = \partial_x$.

The corresponding characteristic equation is

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0}, \tag{38}$$

$$\Rightarrow dt = 0dx, \quad du = 0dx. \tag{39}$$

Eq. (40) gives invariants $t = r$ and $u = s$. So, the invariant solution will be

$$\Rightarrow s = f(r), \tag{40}$$

where r and s are any constants. The aforementioned equation will reduce Eq. (7) in

$$\alpha f^n(r) = 0. \tag{41}$$

This implies that $s = f(r)$ is a solution of trivial ODE and it gives $u = 0$.

Reduction through $O_2 = \partial_t$

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0}, \tag{42}$$

which implies that

$$dx = 0dt, \quad du = 0dt, \tag{43}$$

$$\Rightarrow x = r_1, \quad u = s_1, \tag{44}$$

where r_1 and s_1 are any constants. This will lead to similarity solution of the form:

$$s_1 = f(r_1). \tag{45}$$

Thence, substituting this equation into (6)

$$\beta(f''(r_1))^3 + 3\alpha f^n(r_1) = 0. \tag{46}$$

Hence, the solution set will be

$$\begin{gathered} \int^{f(r_1)} \pm \frac{(n+3)\beta}{\sqrt{(n+3)\beta(6a3^{1/3}(-\alpha a^n\beta^2)^{1/3} - c_1)}} da - r_1 - c_2 = 0, \\ \int^{f(r_1)} \pm \frac{I\beta(n+3)}{\sqrt{-I\beta(n+3)(33^{5/6}(-\alpha a^n\beta^2)^{1/3}a + 3I(-\alpha a^n\beta^2)^{1/3}a3^{1/3} - Ic_1)}} \\ da - r_1 - c_2 = 0, \\ \int^{f(r_1)} \pm \frac{I\beta(n+3)}{\sqrt{I\beta(n+3)(33^{5/6}(-\alpha a^n\beta^2)^{1/3}a + -3I(-\alpha a^n\beta^2)^{1/3}a3^{1/3} - Ic_1)}} \\ da - r_1 - c_2 = 0. \end{gathered} \tag{47}$$

Reduction through $O_1 = \frac{n-3}{5n-3}x\partial_x + t\partial_t + \frac{6u}{-5n+3}\partial_u$

The corresponding characteristic equation of O_1 will be written as:

$$\frac{5n-3}{n-3}\frac{dx}{x} = \frac{dt}{t} = \frac{(-5n+3)}{6}\frac{du}{u}. \tag{48}$$

Therefore, we have the invariants

$$r_2 = \frac{x^{\frac{5n-3}{n-3}}}{t}, \tag{49}$$

$$s_2 = ux^{\frac{6}{n-3}}. \tag{50}$$

Here, in the aforementioned equations r_2 and s_2 are arbitrary constants.

The invariant solution in the form of $s_2 = f(r_2)$ is

$$u = x^{\frac{-6}{n-3}}f(r_2), \quad n \in N. \tag{51}$$

We need the derivatives u_x, u_{xx}, and u_{xt} of the above value of u for substitution into (6). Therefore, we have

$$\begin{aligned} u_x &= \frac{x^{-\left(\frac{n+3}{n-3}\right)}}{n-3}[(5n-3)r_2f'(r_2) - 6f(r_2)], \\ u_{xx} &= \frac{x^{-\frac{2n}{n-3}}}{(n-3)^2}[(5n-3)^2r_2^2f''(r_2) + 4(n-3) \\ &\quad \times (5n-3)r_2f'(r_2) + 6(n+3)f(r_2)], \\ u_{xt} &= \frac{x^{-\frac{n+3}{n-3}}}{t(n-3)}[(9-5n)r_2f'(r_2) - (5n-3)r_2^2f''(r_2)], \end{aligned} \tag{52}$$

inserting these values in (6) and simplifying the calculations lead to

$$\begin{aligned} &3(n-3)^5r_2[(9-5n)r_2f'(r_2) - (5n-3)r_2^2f''(r_2)] \\ &\quad - 3\alpha(n-3)^6f^n(r_2) - \beta[6(n+3)f(r_2) + 4(n-3) \\ &\quad \times (5n-3)r_2f'(r_2) + (5n-3)^2r_2^2f''(r_2)]^3 = 0. \end{aligned} \tag{53}$$

Thus, it reduced our PDE into an ODE, which is nonlinear.

Reduction through $O_2 + O_3 = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}$

The corresponding Lagrange equation is

$$\frac{dx}{1} = \frac{dt}{1} = \frac{du}{0}, \tag{54}$$

and the invariants are

$$x = t + r_3, \quad u = s_3. \tag{55}$$

r_3, s_3 are arbitrary constants and the corresponding solution to this in the form of function f will be written as:

$$s_3 = f(r_3), \tag{56}$$

substituting this value into (6), we will have

$$3f''(r_3) + \beta(f''^3(r_3))^3 + 3\alpha f^n(r_3) = 0. \tag{57}$$

Reduction through $O_2 - O_3 = \frac{\partial}{\partial t} - \frac{\partial}{\partial x}$

For this generator, the invariants are

$$r_4 = x + t, \quad u = s_4, \tag{58}$$

where r_4 and s_4 are any constants. The invariant solution corresponding to generator $O_2 - O_3$ is found to be

$$s_4 = f(r_4), \tag{59}$$

substituting this solution into Eq. (6) gives an ODE of the following form:

$$\beta(f''(r_4))^3 - 3f''(r_4) + 3\alpha f^n(r_4) = 0. \tag{60}$$

2.2 LSA for $f(u) = ln(u)$ and $g(u) = u_{xx}^3$

We deal some part of the second case and leave the remaining as an open case to discuss. The non-linear SPE for the functions $f(u) = \ln(u)$ and $g(u) = u_{xx}^3$ is

$$u_{xt} = \alpha\ln(u) + \frac{1}{3}\beta u_{xx}^3. \tag{61}$$

For this, we have discussed the one-parameter Lie group of transformations, infinitesimal generator, and second prolongation with coefficients in (8), (9), (10), (11), and (12), respectively. Applying the second-order prolongation onto (61):

$$\mathrm{Pr}^{[2]}\bar{V}(u_{xt} - \alpha\ln(u) - \frac{1}{3}\beta u_{xx}^3)|_{u_{xt}=\alpha\ln(u)+\frac{1}{3}\beta u_{xx}^3} = 0, \tag{62}$$

simplification allows the equation of the form:

$$-\alpha\Phi + u\Phi^{xt} - \beta\Phi^{xx}uu_{xx}^2\ |_{u_{xt}=\alpha\ln(u)+\frac{1}{3}\beta u_{xx}^3} = 0, \tag{63}$$

solving this equation in the aforementioned manner of Section 2.1 for values of Φ^{xt} and Φ^{xx}. Also, comparing the values of coefficients will generate a system of equations which is:

$$\begin{aligned} &\tau_x = 0,\ \xi_u = 0,\ \xi_t = 0,\ \xi_{tu} = 0,\ \xi_{uu} = 0, \\ &\tau_{xu} = 0,\ \tau_{uu} = 0,\ \xi_{xx} = 0,\ \tau_{xx} = 0,\ \Phi_{tu} = 0, \\ &\Phi_{xu} = 0,\ \Phi_{xx} = 0,\ \Phi_{uu} = 0, \\ &-\alpha\Phi + u\Phi_{xt} + \alpha(\Phi_u - \xi_x - \tau_t)u\ln(u) = 0, \\ &\beta(-2\Phi_u + 5\xi_x - \tau_t)u = 0. \end{aligned} \tag{64}$$

For the same value of Φ in (19), infinitesimals are

$$\begin{aligned} \xi(x) &= c_4, \\ \tau(t) &= c_5, \\ \Phi(x,t,u) &= 0. \end{aligned} \tag{65}$$

The infinitesimal generators for the one-parameter Lie group of transformations are as follows:

$$\acute{X}_1 = \partial_t, \tag{66}$$

$$\acute{X}_2 = \partial_x. \tag{67}$$

The commutator table for $\acute{X}_i$ $(i = 1, 2)$ is

[..,..]	$\acute{X}_1$	$\acute{X}_2$
$\acute{X}_1$	0	0
$\acute{X}_2$	0	0.

The adjoint table has been constructed by using the above definition in (25).

Adj	$\acute{X}_1$	$\acute{X}_2$
$\acute{X}_1$	$\acute{X}_1$	$\acute{X}_2$
$\acute{X}_2$	$\acute{X}_1$	$\acute{X}_2$.

2.2.1 Proposition

The symmetry generators $\acute{X}_1 = \partial_t$ and $\acute{X}_2 = \partial_x$ form a two-dimensional abelian Lie symmetry algebra.

3 Conclusion

In this article, LSA is performed for the generalized non-linear SPE by taking the general functions $f(u) = u^n$ for $n \in N$, $f(u) = \ln(u)$ in place of u and $g(u) = u_{xx}^3$ in place of $(u^3)_{xx}$, respectively. Lie point symmetries are evaluated for these cases. Exact solutions are found from one-parameter Lie groups G_i^{ε}. In first case, optimal system of one-dimensional Lie sub-algebra is constructed for GSPE, which leads to their invariant solutions. The graphs are obtained for the functions $\sqcap^{(i)}$ $(i = 1, 2, 3)$, which are solutions of our GSPE. By using the symmetries, the reductions are made which basically converts non-linear SPE into non-linear ODE. LSA is a powerful tool to deal with PDEs especially non-linear and fractional PDEs that represent many physical models. LSA enables us to find the symmetries and solutions for the travelling waves. Many of the non-linear PDEs have been solved by this algorithm of LSA. Physicists and mathematicians are working on different forms of non-linear PDEs from nineteenth century to the present such as coupled PDEs, complex PDEs, modified forms of non-linear PDEs, vector forms of PDEs, solitary wave solutions, multi-soliton solutions, N-soliton solutions, their relations with other PDEs, *etc*. Their work is helping to overcome many difficulties in mathematics and applied sciences. Others can do work on generalized forms of different non-linear PDEs to evaluate their symmetries and exact solutions. Also, they can find their solitary solutions and construct conservation laws corresponding to the symmetries of non-linear PDEs and enable us to attain interesting results related to these PDEs.

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