Research Article

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Jointly Rayleigh lifetime products in the presence of competing risks model

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Abstract: In this article, we are applying the competing risks model of product from two different lines of production. So, the comparative life test is done under type-II censoring scheme with consideration of only two independent causes of failure. The statistical analysis procedures are developed considering joint sample of production and its life distributed with the Rayleigh lifetime distribution. The point estimation and the corresponding asymptotic confidence interval of the model parameters under maximum likelihood are constructed. Two bootstrap confidence intervals, bootstrap-p, and bootstrap-t, are discussed. Also, Bayesian approach to estimate point and credible interval is constructed. The estimation results are discussed through data set analyses. The validity of theoretical results is assessed and compared through Monte Carlo study. Finally, some of the points are reported as a brief comment.

Keywords: comparative life testing, Rayleigh distribution, competing risks model, Bayes estimation, bootstrap confidence intervals, maximum likelihood estimation

1 Introduction

The units of product produce from different lines of production under the same facility in a competing duration test under joint censoring scheme. Also, the comparative life testing is done on this product to measure the relative merits of two competing durations of life products. Practice, if we have two lines of production then, select two independent sets of product to test under joint censoring scheme. For the time and cost considerations, the experimenter terminates the experiment after recording some failure times. These data and the corresponding statistical inferences were studied previously by Rao *et al.* [1], Basu [2], Johnson and Mehrotra [3], Mehrotra and Johnson [4], Bhattacharyya and Mehrotra [5], and Mehrotra and Bhattacharyya [6]. Also, this problem in modern time was studied by Balakrishnan and Rasouli [7], Rasouli and Balakrishnan [8], and Shafay *et al.* [9]. Recently, it was studied by Bekheet and Abd-Elmougod [10], Soliman *et al.* [11], Hanaa [12], and Algarni *et al.* [13]. Also, the balanced case of the joint censoring scheme was proposed by Mondal and Kundu [14].

Suppose that the first line of production is denoted by the line-Ω_1 (L-Ω_1) and the line-Ω_2 (L-Ω_2) and the joint censoring scheme by JSC. So, two independent samples of products from L-Ω_1 and L-Ω_2 are combined to test under life testing experiment. Hence, the relative merits are measured of two competing durations of life products, which is known by comparative life testing. Under type-II censoring scheme, the JSC is called type-II joint censoring scheme (type-II JCS), which is described as follows.

The total random sample of size N of product consists of m from the L-Ω_1 and n from L-Ω_2. And, let m units have a life $\{X_1, \ldots, X_m\}$ (from L-Ω_1) distributed with cumulative distribution function (CDF) F_1 and probability density function (PDF) f_1. Also, n units with life $\{Y_1, \ldots, Y_n\}$ (from L-Ω_2) are distributed with CDF F_2 and PDF f_2. Suppose that S is the prior integer that represents the number of failure units needed for statistical inference. So, the ordered lifetime sample $\{T_1, T_2, \ldots, T_{\mathbf{S}}\}$ is taken from $\{X_1, \ldots, X_{\mathbf{S}_m}\}$ and $\{Y_1, \ldots, Y_{\mathbf{S}_n}\}$, $\mathbf{S} = \mathbf{S}_m + \mathbf{S}_n$. The random lifetimes with the corresponding type (mean from L-Ω_1 or L-Ω_2) denoted by $(\mathbf{T}, \boldsymbol{\varphi}) = \{(T_1, \varphi_1), (T_2, \varphi_2), \ldots, (T_S, \varphi_{\mathbf{S}})\}$ are called type-II JC sample. The values $\varphi_i = \{1, 2\}$ denoted the line L-Ω_1 or L-Ω_2, respectively. The joint likelihood function of type-II JCS sample $(\mathbf{T}, \boldsymbol{\varphi})$ is given by

$$L((\mathbf{T}, \boldsymbol{\varphi})|\underline{\theta}) = \frac{m!n![S_1(t_S)]^{m-\mathbf{S}_m}[S_2(t_S)]^{n-\mathbf{S}_n}}{(m-\mathbf{S}_m)!(n-\mathbf{S}_n)!} \times \prod_{i=1}^{\mathbf{S}}[f_1(t_i)]^{2-\varphi_i}[f_2(t_i)]^{\varphi_i-1}, \tag{1}$$

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where $S_j(\cdot) = 1 - F_j(\cdot)$, $j = 1, 2$ is the reliability function and $\varphi_i = 1$ or 2, from L-Ω_1 or L-Ω_2, respectively.

In general, in life testing experiment for survival or medical trial the unit may fail due to different causes of failure, and for more details see the studies of AbdullahiBaba *et al.* [15] and Alsulami [16]. The assessment of one cause of failure with respect to other causes is carried out by the competing risks model. Several authors have been interested in this model and its properties, and for the exponential model see the studies of Cox [17] and Crowder [18]. Also, for more details about the competing risks model see the studies of Balakrishnan and Han [19], Modhesh and Abd-Elmougod [20], and Bakoban and Abd-Elmougod [21]. And recently, see the studies of Ganguly and Kundu [22], Abu-Zinadah and Neveen [23], Algarn *et al.* [24], Almalki *et al.* [25], and Soliman *et al.* [26]. In the competing risks model the observed data consist of unit failure time and the corresponding indicator denoting the cause of failure.

For the type-II competing risks sample, suppose the sample of size N is tested under life testing experiment. The pre-fixed number $\mathbf{S}$ needed for statistical inference is proposed. Considering only two independent causes of failure the type-II competing risks sample is defined by $(\mathbf{T}, \rho) = \{(T_1, \rho_1), (T_2, \rho_2), \ldots, (T_\mathbf{S}, \rho_\mathbf{S})\}$, $\rho_2 = \{1, 2\}$. So, the likelihood function of type-II competing risks sample $(\mathbf{T}, \rho)$ is given by

$$\begin{aligned} &f_{1,2,\ldots,\mathbf{S}}((\mathbf{T}, \rho)|\underline{\theta}) \\ &= \frac{N!\,[S_1(t_s)S_2(t_s)]^{(\mathbf{N}-\mathbf{S})}}{(N-\mathbf{S})!} \\ &\times \prod_{i=1}^{\mathbf{S}} [f_1(t_i)S_2(t_i)]^{\omega(\rho_i=1)} \times [f_2(t_i)S_1(t_i)]^{\omega(\rho_i=2)}, \end{aligned} \tag{2}$$

where $0 < t_1 < t_2 < \ldots < t_\mathbf{S} < \infty$ and

$$\omega(\rho_i = j) = \begin{cases} 1, & \rho_i = j \\ 0, & \rho_i \neq j \end{cases} \quad j = 1, 2. \tag{3}$$

Rayleigh distribution (RD) is defined as a special case of the Weibull distribution, which has wide applications in engineering and communication, for instance, see the studies of Dyer and Whisenand [27,28]. The RD has different applications in life testing of electrovacuum devices, see the study of Polovko [29]. The Rayleigh random variable T with parameter β has CDF, which is given by

$$F(t) = 1 - \exp(-\beta t^2), \quad t > 0, \ \beta > 0. \tag{4}$$

RD has a decreasing reliability function with higher rate than in the case of exponential distribution, for more details see Polovko [29], and increasing linear failure rate function. Several authors discussed the statistical inferences of the RD; Harter and Moore [30] derived an explicit form for the maximum likelihood (ML) estimator based on Type II censored data; and Dyer and Whisen [27] and Dyer and Whisen [28] provided the best linear unbiased estimator based on complete sample, censored sample, and selected order statistics. Doubly censored samples were considered, among other authors, by Lalitha and Mishra [31] and Kong and Fei [32]. Bayesian estimation and prediction problems are also important and have been investigated, among others, by Howlader and Hossain [33] and Fernandez [34]. In addition, AL-Hussaini and Ahmad [35] and AL-Hussaini and Ahmad [36] studied Bayesian predictive densities and prediction bounds of generalized order statistics and future records. Also, the estimation under two Rayleigh lifetime population was done by Al-Matrafi and Abd-Elmougod [10].

The type-II JCS in this article, is implemented on two independent samples taken from two Rayleigh lifetime populations to obtain type-II JC sample. We develop classical and Bayesian inference of type-II JC sample when the lifetime of the units has Rayleigh lifetime distribution. Therefore, studying a population which have different lines of production under the same facility, particularly, in competing duration when Rayleigh failure time is happening to two independent causes of failure is the aim of this article. This problem was handheld recently, for exponential by Almarashi *et al.* [37], for Burr XII by Abushal *et al.* [38], and for Weibull by Alghamdi *et al.* [39]. So, we built under these assumptions the model and the corresponding likelihood function. And, the statistical inferences of type-II competing risks samples are presented with classical and Bayesian approach. The point estimators of model parameters are constructed with the corresponding asymptotic confidence/credible intervals (ACIs/CIs). Also, two confidence interval estimation with bootstrap-P (PBCI) and bootstrap-t (PTCI) are discussed. The validity of results of point estimators is measured with the mean squared error (MSE). But interval estimators are measured with coverage probability (CPs) and interval lengths (ILs)

The outline of this article is presented as follows. The model formulation and the corresponding likelihood function are presented in Section 2. Estimation by the ML method for point and the corresponding ACIs are developed in Section 3. Bootstrap confidence intervals of the model parameters with two different methods are constructed in Section 4. Bayesian approach of model parameters are discussed in Section 5. The joint data set is analyzed for illustrative purposes in Section 6. Finally, the numerical simulation study is used to assess the developed methods in Section 7.

2 Model building

Let a joint sample of size N be randomly selected from two lines L-Ω_1 and L-Ω_2 to satisfy m from L-Ω_1 and n from L-Ω_2. Before the test, the integer $\mathbf{S}$ which satisfies the number of failures in statistical inferences procedures is selected. The unit type $\varphi_i = \{1, 2\}$ (mean from L-Ω_1 or L-Ω_2) and the corresponding cause of failure $\rho_i = \{1, 2\}$ (mean first or second cause) are recorded at each observed failure time. Say, the random sample (T_i, φ_i, ρ_i), $i = 1, 2, \ldots, \mathbf{S}$ is observed. So, the competing risks type-II JC sample is presented by

$$\mathbf{T} = \{(T_1, \varphi_1, \rho_1), (T_2, \varphi_2, \rho_2), \ldots, (T_{\mathbf{S}}, \varphi_{\mathbf{S}}, \rho_{\mathbf{S}})\}, \tag{5}$$

where $1 < \mathbf{S} \leq N$. This scheme has the following assumptions:

(1) Number of unit failures from the L-Ω_1 is given by $\kappa_1 = \sum_{i=1}^{\mathbf{S}}(2 - \varphi_i)$.

(2) Number of unit failures from the L-Ω_2 is given by $\kappa_2 = \sum_{i=1}^{\mathbf{S}}(\varphi_i - 1)$.

(3) Number of unit failures from the L-Ω_1 under the cause j is given by $\xi_{1j} = \sum_{i=1}^{\mathbf{S}}(2 - \varphi_i) * \omega(\rho_i = j)$, $j = 1, 2$.

(4) Number of unit failures from the L-Ω_2 under the cause j is given by $\xi_{2j} = \sum_{i=1}^{\mathbf{S}}(\varphi_i - 1) * \omega(\rho_i = j)$, $j = 1, 2$.

Under consideration that the failure time of the unit is distributed with Rayleigh lifetime distribution with PDF given by

$$f_{kj}(t) = 2\beta_{kj} t \exp(-\beta_{kj} t^2), \ t > 0, \ \beta_{kj} > 0, \ j = 1, 2, \ k = 1, 2, \tag{6}$$

where k and j denote the type and cause of the failure, respectively. Also, the observed failure times t_i, $i = 1, 2, \ldots, \mathbf{S}$ are described as $t_i = \min\{t_{ik1}, t_{ik2}\}$ and t_{ikj} is the ith failure time from the line k and cause j. Hence, the minimum value t_i has distribution $G(t)$ given by $G(\cdot) = F_{t_{k1}}(\cdot) + F_{t_{k2}}(\cdot) - F_{t_{k1}}(\cdot) * F_{t_{k2}}(\cdot)$. So, the latent failure time is distributed with RD with scale parameters $\beta_{k1} + \beta_{k2}$. Also, the numbers ξ_{1j} and ξ_{2j} have the binomial distributions described by

$$\xi_{1j} \to \text{Binomial}\left(\mathbf{S} - \boldsymbol{\kappa}_2, \frac{\beta_{k1}}{\beta_{k1} + \beta_{k2}}\right)$$

and

$$\xi_{2j} \to \text{Binomial}\left(\mathbf{S} - \boldsymbol{\kappa}_1, \frac{\beta_{k2}}{\beta_{k1} + \beta_{k2}}\right). \tag{7}$$

From the previous model assumptions the joint likelihood function of $(\mathbf{T}, \varphi, \rho)$ is given by

$$\begin{aligned} L(\mathbf{T}, |\underline{\beta}) = {} & \frac{m!n!}{(m - \mathbf{S}_m)!(n - \mathbf{S}_n)!}[S_{11}(t_{\mathbf{S}})S_{12}(t_{\mathbf{S}})]^{m-\boldsymbol{\kappa}_1} \\ & \times [S_{21}(t_{\mathbf{S}})S_{22}(t_{\mathbf{S}})]^{n-\boldsymbol{\kappa}_2}\prod_{i=1}^{\mathbf{S}}\{(h_{11}(t_i))^{\omega(\rho_i=1)} \\ & \times (h_{12}(t_i))^{\omega(\rho_i=2)}(S_{11}(t_i)S_{12}(t_i))\}^{2-\varphi_i} \\ & \times \{(h_{21}(t_i))^{\omega(\rho_i=1)}(h_{22}(t_i))^{\omega(\rho_i=2)}(S_{21}(t_i)S_{22}(t_i))\}^{\varphi_i-1}, \end{aligned} \tag{8}$$

where $S(\cdot) = 1 - F(\cdot)$, $h(\cdot) = \frac{f(\cdot)}{S(\cdot)}$, $\underline{\beta} = (\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$, and

$$0 < t_1 < t_2 < \ldots < t_{\mathbf{S}} < \infty.$$

3 Maximum likelihood estimation (MLE)

The point estimation and the corresponding ACIs are computed in this section under given competing risks type-II JC sample. Each sample point has the failure time with the corresponding type and failure cause.

3.1 Point estimation

The joint likelihood function (8) under distribution (6) is reduced to

$$\begin{aligned} L(\underline{\beta}|\mathbf{T}) \propto {} & \beta_{11}^{\xi_{11}}\beta_{12}^{\xi_{12}}\beta_{21}^{\xi_{21}}\beta_{22}^{\xi_{22}}\exp\{-(\beta_{11} + \beta_{12}) \\ & \times \sum_{i=1}^{\mathbf{S}}(2 - \varphi_i)t_i^2 - (m - \boldsymbol{\kappa}_1)(\beta_{11} + \beta_{12})t_{\mathbf{S}}^2 \\ & - (n - \boldsymbol{\kappa}_2)(\beta_{21} + \beta_{22})t_{\mathbf{S}}^2 - (\beta_{21} + \beta_{22}) \\ & \times \sum_{i=1}^{\mathbf{S}}(\varphi_i - 1)t_i^2\bigg\}. \end{aligned} \tag{9}$$

The natural log-likelihood function can be presented by

$$\begin{aligned} \ell(\underline{\beta}|\mathbf{T}) \propto {} & \xi_{11}\log\beta_{11} + \xi_{12}\log\beta_{12} + \xi_{21}\log\beta_{21} \\ & + \xi_{22}\log\beta_{22} - (\beta_{11} + \beta_{12})\sum_{i=1}^{\mathbf{S}}(2 - \varphi_i)t_i^2 \\ & - (\beta_{21} + \beta_{22})\sum_{i=1}^{\mathbf{S}}(\varphi_i - 1)t_i^2 - (m - \boldsymbol{\kappa}_1) \\ & \times (\theta_{11} + \theta_{12})t_{\mathbf{S}}^2 - (n - \boldsymbol{\kappa}_2)(\beta_{21} + \beta_{22})t_{\mathbf{S}}^2. \end{aligned} \tag{10}$$

Hence, the likelihood equations are obtained after taking the partial derivatives of (10) with respect to model parameters as follows:

$$\frac{\partial \ell(\underline{\beta}|\mathbf{T})}{\partial \beta_{kj}} = 0, \quad k,\ j = 1, 2. \tag{11}$$

So, the estimators under L-Ω_1 and L-Ω_2, respectively, are given by

$$\hat{\beta}_{1j} = \frac{\xi_{1j}}{\sum_{i=1}^{\mathbf{S}}(2-\varphi_i)t_i^2 + (m-\boldsymbol{\kappa}_1)t_{\mathbf{S}}^2}, \quad j = 1,\ 2, \tag{12}$$

and

$$\hat{\beta}_{2j} = \frac{\xi_{2j}}{\sum_{i=1}^{J}(\varphi_i-1)t_i^2 + (n-\boldsymbol{\kappa}_2)t_{\mathbf{S}}^2}, \quad j = 1, 2. \tag{13}$$

Remark: Eqs (12) and (13) have shown that the estimators depend on the non-zero values of numbers ξ_{1j} and ξ_{2j}. But, if we set $\xi_{1j} = 0$ or S and $\xi_{2j} = 0$ or S, it means the MLEs of $\hat{\beta}_{1j}$ or $\hat{\beta}_{2j}$, $j = 1, 2$ do not exist. Also, the exact probability distributions of the estimators $\hat{\beta}_{1j}$ and $\hat{\beta}_{2j}$ are defined as mixture of discrete and continuous distributions, which are difficult to be derived (see, *e.g.*, Kundu and Joarder [40]).

3.2 Interval estimation

The minus expectation of second derivatives of the log-likelihood function given by (10) presents the Fisher information matrix of the model parameters. We denote the Fisher information matrix by $\mathbf{\Pi}(\underline{\beta})$, $\underline{\beta} = \{\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}\}$, which is given by

$$\mathbf{\Pi}(\underline{\beta}) = -E\left(\frac{\partial^2 \ell(\underline{\beta}|\mathbf{T})}{\partial \beta_{kj}^2}\right). \tag{14}$$

The second derivative of log-likelihood function of the model parameters is presented by

$$\frac{\partial^2 \ell(\underline{\beta}|\mathbf{T})}{\partial \beta_{1j}^2} = \frac{-\xi_{1j}}{\beta_{2j}^2}, \quad j = 1, 2, \tag{15}$$

$$\frac{\partial^2 \ell(\underline{\beta}|\mathbf{T})}{\partial \beta_{1j}^2} = \frac{-\xi_{2j}}{\beta_{2j}^2}, \quad j = 1, 2, \tag{16}$$

and

$$\frac{\partial^2 \ell(\underline{\beta}|\mathbf{T})}{\partial \beta_{ij}\partial \beta_{kl}} = 0, \quad \text{for each } ij \neq kl. \tag{17}$$

Then, the Fisher information matrix when $\underline{\beta} = (\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22})$ is given by

$$\mathbf{\Pi}_0(\underline{\beta}) = \begin{pmatrix} \frac{-\xi_{11}}{\hat{\beta}_{11}^2} & 0 & 0 & 0 \\ 0 & \frac{-\xi_{12}}{\hat{\beta}_{12}^2} & 0 & 0 \\ 0 & 0 & \frac{-\xi_{21}}{\hat{\beta}_{21}^2} & 0 \\ 0 & 0 & 0 & \frac{-\xi_{22}}{\hat{\beta}_{22}^2} \end{pmatrix}. \tag{18}$$

Hence, the variance–covariance matrix of the MLE of model parameters can be obtained from the inverse Fisher information matrix when the diagonal elements have non-zero values. Under the property that ML estimators have a normal distribution with mean $(\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22})$ and variance obtained from $\mathbf{\Pi}_0^{-1}(\underline{\beta})$ from (18). Therefore, the approximate $(1-\gamma)\%$ CIs of the parameters $\beta_{11}, \beta_{12}, \beta_{21}$, and β_{22} under normal property of MLEs $\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}$, and $\hat{\beta}_{22}$ are given by

$$\hat{\beta}_{1j} \mp z_{\frac{\gamma}{2}}\frac{\hat{\beta}_{1j}}{\sqrt{\xi_{1j}}} \quad \text{and} \quad \hat{\beta}_{2j} \mp z_{\frac{\gamma}{2}}\frac{\hat{\beta}_{2j}}{\sqrt{\xi_{2j}}}, \quad j = 1, 2, \tag{19}$$

where $z_{\frac{\gamma}{2}}$ is the percentile value computed from standard normal distribution.

4 Bootstrap confidence intervals

Bootstrap technique is a commonly resembling method used in parameter estimation as well as estimation of the bias and variance of an estimator or calibrate hypothesis tests. Parametric and nonparametric bootstrap techniques are defined in the studies by Davison and Hinkley [41] and Efron and Tibshirani [42]. In this section, we adopted the problem of statistical inference in bootstrap techniques in the form of parameter estimation. To formulate confidence intervals under parametric bootstrap technique, we are adopted the percentile bootstrap-p and bootstrap-t methods. For more details, see the studies of Efron [43] and Hall [44]. The confidence interval estimation can be formulated with the following algorithms:

1. For given original competing risks type-II JC sample $(\mathbf{t}, \varphi, \rho) = \{(t_1, \varphi_1, \rho_1), (t_2, \varphi_2, \rho_2), \ldots, (t_{\mathbf{S}}, \varphi_{\mathbf{S}}, \rho_{\mathbf{S}})\}$, the estimators of parameter vector $\underline{\hat{\beta}} = \{\hat{\beta}_{11}, \hat{\beta}_{12}, \hat{\beta}_{21}, \hat{\beta}_{22}\}$ are obtained.

2. Considering that the censoring scheme N, n, m, and $\mathbf{S}$ and estimate $\underline{\hat{\beta}}$ generate a sample of size n from RD with scale parameters $\hat{\beta}_{11} + \hat{\beta}_{12}$ and a sample of size m from RD with scale parameters $\hat{\beta}_{21} + \hat{\beta}_{22}$. The $\mathbf{S}$-bootstrap competing risks type-II JC sample is obtained from the generated joint sample as a small $\mathbf{S}$ failure denoted by $(\mathbf{T}^*, \boldsymbol{\varphi}^*) = \{(T_1^*, \varphi_1^*), (T_2^*, \varphi_2^*), \ldots, (T_{\mathbf{S}}^*, \varphi_{\mathbf{S}}^*)\}$, $1 < \mathbf{S} \leq N$.
3. For the bootstrap random sample $(\mathbf{T}^*, \varphi^*) = \{(T_1^*, \varphi_1^*), (T_2^*, \varphi_2^*), \ldots, (T_{\mathbf{S}}^*, \varphi_{\mathbf{S}}^*)\}$, $1 < \mathbf{S} \leq N$, the two numbers $\boldsymbol{\kappa}_1^*$ and $\boldsymbol{\kappa}_2^*$ (number of failures taken from L-Ω_1 and L-Ω_2, respectively) are reported.
4. The four numbers ξ_{1j}^* and ξ_{2j}^*, $j = 1, 2$ are randomly generated from binomial distribution with size $\mathbf{S} - \kappa_{3-k}^*$ and probability $\frac{\hat{\beta}_{kj}}{\hat{\beta}_{kj} + \hat{\beta}_{kj}}$, $k, j = 1, 2$.
5. The last steps are used to obtain the bootstrap estimate sample $\underline{\hat{\beta}}^* = \{\hat{\beta}_{11}^*, \hat{\beta}_{12}^*, \hat{\beta}_{21}^*, \hat{\beta}_{22}^*\}$.
6. Repeat steps 2–5 $\mathbf{M}$ times.
7. The values $(\hat{\beta}_{11}^{[i]*}, \hat{\beta}_{12}^{[i]*}, \hat{\beta}_{21}^{[i]*}, \hat{\beta}_{22}^{[i]*})$, $i = 1, 2, \ldots, \mathbf{M}$, are arranged in ascending order to obtain $\underline{\tilde{\beta}}^* = \{\hat{\beta}_{11}^{(i)*}, \hat{\beta}_{12}^{(i)*}, \hat{\beta}_{21}^{(i)*}, \hat{\beta}_{22}^{(i)*}\}$.

4.1 Percentile bootstrap confidence interval (PBCI)

Define the empirical distribution $\Phi(x) = P(\tilde{\beta}_l^* \leqslant x)$, $l = 1, 2, 3, 4$ as a CDF of $\tilde{\beta}_l^*$, where $\tilde{\beta}_1^*$ mean $\hat{\beta}_{11}^*$ and others. So, the point bootstrap **estimators** are defined by

$$\bar{\tilde{\beta}}_l^* = \frac{1}{\mathbf{M}} \sum_{i=1}^{\mathbf{M}} \tilde{\beta}_l^{(i)*}. \tag{20}$$

Also, the $100(1-\gamma)\%$ PBCIs are given by

$$\left(\tilde{\beta}^*_{l\,\mathrm{boot}\left(\frac{\gamma}{2}\right)},\quad \tilde{\beta}^*_{l\,\mathrm{boot}\left(1-\frac{\gamma}{2}\right)}\right), \tag{21}$$

where $\tilde{\beta}^*_{l\,\mathrm{boot}} = \Phi^{-1}(x)$.

4.2 Bootstrap-t confidence interval (PTCI)

Also, from the sample $\underline{\tilde{\beta}}^* = \{\hat{\beta}_{11}^{(i)*}, \hat{\beta}_{12}^{(i)*}, \hat{\beta}_{21}^{(i)*}, \hat{\beta}_{22}^{(i)*}\}$, we built the order statistics values $\Psi_l^{*(1)} < \Psi_l^{*(2)} < \ldots < \Psi_l^{*(\mathbf{M})}$, where

$$\Psi_l^{*[i]} = \frac{\tilde{\beta}_l^{*(i)} - \hat{\beta}_l}{\sqrt{\mathrm{var}(\tilde{\beta}_l^{*(i)})}}, \quad j = 1, 2, \ldots, N,\ l = 1, 2, 3, 4. \tag{22}$$

There the $100(1-\gamma)\%$ PTCIs are given by

$$(\tilde{\beta}^*_{l\,\mathrm{boot\text{-}t}(\gamma)},\quad \tilde{\beta}^*_{l\,\mathrm{boot\text{-}t}(1-\gamma)}), \tag{23}$$

where the value $\tilde{\beta}^*_{l\,\mathrm{boot\text{-}t}}$ is given by

$$\tilde{\beta}^*_{l\,\mathrm{boot\text{-}t}} = \hat{\beta}_l + \sqrt{\mathrm{Var}(\hat{\beta}_l)}\,\Phi^{-1}(x), \tag{24}$$

and $\Phi(x) = P(\tilde{\beta}_l^* \leqslant x)$ is the CDF of $\tilde{\beta}_l^*$.

5 Bayesian estimation

Considering the available prior information about the model parameters, Bayesian approach can be used not only in parameter estimation but also in the problem of estimating a common functions of the parameters such as readability, hazard rate, and lifetime performance index functions; see ref. [45]. Therefore, we consider the competing risks type-II JC sample obtained from life testing experiment under random joint sample selected from two lines L-Ω_1 and L-Ω_2 of production . The point and corresponding CI estimates are constructed. The common prior densities of the model parameters are considered to be independent gamma distributions. The independent gamma distributions are chosen for each of the model parameters as follows:

$$h_i^*(\beta_i) = \frac{b_i^{a_i}}{\Gamma(a_i)} \beta_i^{a_i - 1} \exp(-b_{i\beta i}), \quad \beta > 0,\ a_i,\ b_i > 0, \tag{25}$$

where $\underline{\beta} = \{\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}\}$ and $(a_i,\ b_i)$, $i = 1, 2, \ldots, 4$ are prior parameters that present prior information of the model parameters. The joint prior distribution is given by $h^*(\underline{\beta}) = \prod_{i=1}^4 h_i^*(\beta_i)$. In gereneral, the posterior distribution of model parameter is given by

$$h(\underline{\beta}|\mathbf{T}) = \frac{L(\underline{\beta}|(\mathbf{T}, \varphi, \rho))h^*(\underline{\beta})}{\int \cdots \int_{\underline{\beta}} L(\underline{\beta}|(\mathbf{T}, \varphi, \rho))h^*(\underline{\beta})\mathrm{d}\underline{\beta}}. \tag{26}$$

Hence, from (9) and (26) the corresponding posterior distribution is given by

$$\begin{aligned} h(\underline{\beta}|\mathbf{T}) \propto{}& \beta_{11}^{\xi_{11}+a_1-1} \beta_{12}^{\xi_{12}+a_2-1} \beta_{21}^{\xi_{21}+a_3-1} \beta_{22}^{\xi_{22}+a_4-1} \\ & \times \exp\left\{ -(\beta_{11} + \beta_{12}) \sum_{i=1}^{\mathbf{S}} (2 - \varphi_i) t_i^2 - (m - \boldsymbol{\kappa}_1) \right. \\ & \times (\beta_{11} + \beta_{12}) t_{\mathbf{S}}^2 - (n - \boldsymbol{\kappa}_2)(\beta_{21} + \beta_{22}) t_{\mathbf{S}}^2 \\ & - (\beta_{21} + \beta_{22}) \sum_{i=1}^{\mathbf{S}} (\varphi_i - 1) t_i^2 - b_1 \beta_{11} - b_2 \beta_{12} \\ & \left. - b_3 \beta_{21} - b_4 \beta_{22} \right\}. \end{aligned} \tag{27}$$

The function given by (27) shows that the joint posterior distribution has independent gamma distributions with shape parameters given by

$$Q_1 = \xi_{11} + a_1, \quad Q_2 = \xi_{12} + a_2, \quad Q_3 = \xi_{21} + a_3, \quad Q_4 = \xi_{22} + a_4 \tag{28}$$

and scale parameters given by

$$\begin{cases} W_1 = b_1 + \sum_{i=1}^{\mathbf{s}} (2-\varphi_i)t_i^2 + (m-\boldsymbol{\kappa}_1)t_{\mathbf{s}}^2, \\ W_2 = b_2 + \sum_{i=1}^{\mathbf{s}} (2-\varphi_i)t_i^2 + (m-\boldsymbol{\kappa}_1)t_{\mathbf{s}}^2, \\ W_3 = b_3 + \sum_{i=1}^{\mathbf{s}} (\varphi_i-1)t_i^2 + (n-\boldsymbol{\kappa}_2)t_{\mathbf{s}}^2, \\ W_4 = b_4 + \sum_{i=1}^{\mathbf{s}} (\varphi_i-1)t_i^2 + (n-\boldsymbol{\kappa}_2)t_{\mathbf{s}}^2. \end{cases} \tag{29}$$

5.1 Point estimation

The results of the point estimators of model parameters are given by

1. Under squared error (SE) loss function, the point estimators (PEs) of model parameters are given by

$$\hat{\beta}_{lB1} = \frac{Q_l}{W_l}, \quad l = 1, 2, 3, 4. \tag{30}$$

2. Under LINEX loss function, the PEs of the model parameters are given by

$$\hat{\beta}_{lB2} = \frac{-Q_l}{c}\log\left[\frac{W_l}{W_l + c}\right], \tag{31}$$

where c is the parameter of LINEX loss function.

5.2 Interval estimation

In this subsection, we built interval estimation of model parameters under consideration that the posterior distribution is joint-independent gamma distribution. Therefore, the probability credible interval (PCI) of model parameters is given by

$$\int_L^U h_l(\underline{\beta}|(\mathbf{T}, \varphi, \rho))\mathrm{d}\beta_l = 1 - \gamma, \quad l = 1, 2, 3, 4, \tag{32}$$

where h_l is the marginal posterior distribution (marginal gamma density). So, Bayes credible intervals are constructed based on the following lemma.

Lemma: *If X has gamma distribution as parameters m and B, then the pivotal quantity $\Pi = 2Bx$ is distributed with χ^2 distribution with $2m$ degrees of freedom* (*see Kundu and Joarder* [40]).

Therefore, the parameter β_l has gamma distribution with shape parameter Q_l and scale parameter W_l. So, the pivotal quantity $\Pi = 2W_l\beta_l$ is distributed with χ^2 distribution with $2Q_l$ degrees of freedom. And hence, the $100(1-\gamma)\%$ PCI is reduced to

$$\left(\frac{\chi^2_{2Q_l,\gamma}}{2W_l}, \quad \frac{\chi^2_{2Q_l,1-\gamma}}{2W_l}\right), \quad l = 1, 2, 3, 4. \tag{33}$$

6 Data analysis

In this section, we analyzed a real data set presented by Hoel [46], which has presented the failure times and the corresponding cause of failure for two groups of strain male mice under laboratory experiment, which is received a radiation dose of $300r$ at an age of 5–6 weeks. Let L-Ω_1 be considered as the first group which lived in a conventional laboratory environment and L-Ω_2 be the second group lived in a germ-free environment. The complete data of two lines are presented in ref. [46]. Let the data be classified into two causes of failure: thymic lymphoma with reticulum cell sarcoma is the first cause of death (failure) and the second cause is presented by other causes of death (failure), and more details are presented in ref. [47]. These data are divided by 100 and after that, the validity of modeling these data by Rayleigh lifetime distribution is tested by plotting the fitted survival function with empirical survival function. The data are tested to follow Rayleigh lifetime distribution with scale parameters $\beta_{k1} + \beta_{k2}$, $k = 1, 2$ (k mean from L-Ω_1 or L-Ω_2). Figures 1 and 2 show the fitted

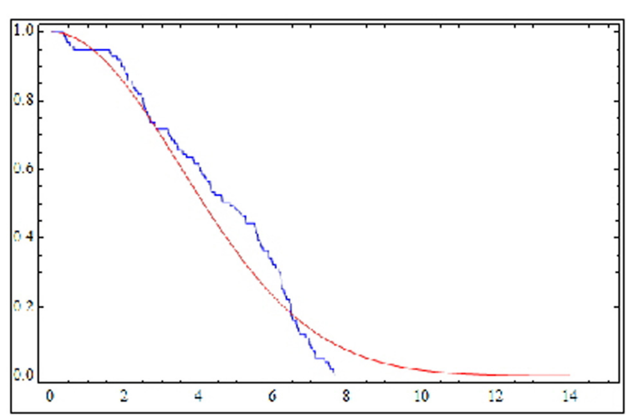

Figure 1: The empirical and fitted survival.

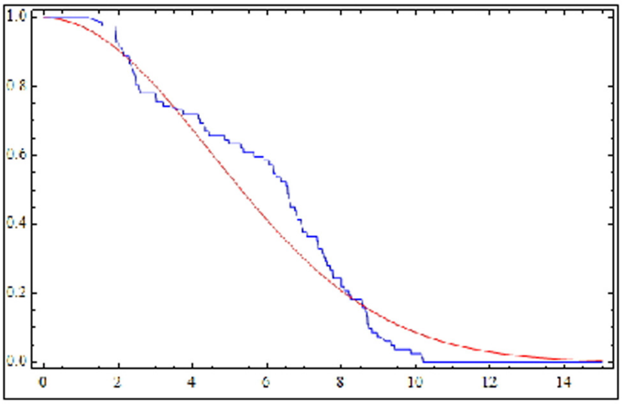

Figure 2: The empirical and fitted survival.

Table 3: MLEs, bootstrap estimate, and PEs of the model parameters

	ML	Bootstrap	SE	LINX	
				C = 2.0	C = −2.0
β_{11}	0.01532	0.01621	0.01533	0.01532	0.01533
β_{12}	0.01415	0.01524	0.01415	0.01414	0.01415
β_{21}	0.01604	0.01714	0.01604	0.01603	0.01605
β_{22}	0.00321	0.00421	0.00282	0.00282	0.00283

survival function with empirical survival function. The Kolmogorov-Smirnov distances of fit data between the empirical distribution functions and the fitted

Table 1: Time-to failure of male mice which received a radiation dose at age 5–6 weeks

Group 1	Cause 1	59	189	191	198	200	207	220	235	245	250	256	261	265
		266	280	317	318	343	356	383	399	403	414	428	432	495
		525	536	549	552	554	557	558	571	586	594	596	605	612
		621	628	631	636	643	647	648	649	661	663	666	670	695
		697	700	705	712	713	738	748	753					
	Cause 2	40	42	51	62	163	179	206	222	228	249	252	282	324
		333	341	366	385	407	420	431	441	461	462	482	517	517
		524	564	567	586	619	620	621	622	647	651	686	761	763
Group 2	Cause 1	158	192	193	194	195	202	212	215	229	230	237	240	244
		247	259	300	301	321	337	415	430	434	444	485	496	529
		537	590	606	624	638	655	679	691	693	696	707	747	752
		760	778	800	821	986								
	Cause 2	136	246	255	376	421	565	616	617	652	655	658		
		660	662	675	681	734	736	737	757	769	777	800		
		807	825	855	857	864	868	870	873	882	895	910		
		934	942	1,015	1,019									

Table 2: Competing risks type-II JPC sample for radiation dose data with S = 80

t_i	0.4	0.42	0.51	0.62	1.36	1.58	1.59	1.63	1.79	1.89	1.91	1.92	1.93	1.94
φ_i	1	1	1	1	2	2	1	1	1	1	1	2	2	2
ρ_i	2	2	2	2	2	1	1	2	2	1	1	1	1	1
t_i	1.95	1.98	2.00	2.02	2.06	2.07	2.12	2.15	2.2	2.22	2.28	2.29	2.3	2.35
φ_i	2	1	1	2	1	1	2	2	1	1	1	2	2	1
ρ_i	1	1	1	1	2	1	1	1	1	2	2	1	1	1
t_i	2.37	2.4	2.44	2.45	2.46	2.47	2.49	2.5	2.52	2.55	2.56	2.59	2.61	2.65
φ_i	2	2	2	1	2	2	1	1	1	2	1	2	1	1
ρ_i	1	1	1	1	2	1	2	1	2	2	1	1	1	1
t_i	2.66	2.8	2.82	3.0	3.01	3.17	3.18	3.21	3.24	3.33	3.37	3.41	3.43	3.56
φ_i	1	1	1	2	2	1	1	2	1	1	2	1	1	1
ρ_i	1	1	2	1	1	1	1	1	2	2	1	2	1	1
t_i	3.66	3.76	3.83	3.85	3.99	4.03	4.07	4.14	4.15	4.2	4.21	4.28	4.3	4.31
φ_i	1	2	1	1	1	1	1	1	2	1	2	1	2	1
ρ_i	2	2	1	2	1	1	2	1	1	2	2	1	1	2
t_i	4.32	4.34	4.41	4.44	4.61	4.62	4.82	4.85	4.95,	4.96				
φ_i	1	2	1	2	1	1	1	2	1	2				
ρ_i	1	1	2	1	2	2	2	1	1	1				

Table 4: 95% CIs ML, boot, and BCs of the model parameters

	CIs	CIs-Boo-p	CIs-Boo-t	BCs
β_{11}	(0.0094, 0.0212)	(0.0102, 0.0325)	(0.0085, 0.0215)	(0.0107, 0.0206)
β_{12}	(0.0085, 0.0198)	(0.0102, 0.0215)	(0.0075, 0.0191)	(0.0098, 0.0192)
β_{21}	(0.0097, 0.0223)	(0.0120, 0.0291)	(0.0082, 0.0235)	(0.0112, 0.0217)
β_{22}	(0.0004, 0.0060)	(0.0011, 0.0095)	(0.0011, 0.0067)	(0.0011, 0.0052)

distribution functions satisfy (K-S) equal to 0.1513 under L-Ω_1 and (K-S) equal to 0.1805 under L-Ω_2. From Figures 1 and 2, the values of (K-S) and the tested P-value, we accept that the null hypothesis from transformed data set is drawn from Rayleigh lifetime distribution with scale parameter $\beta_{k1} + \beta_{k2}$.

Table 5: MEs and MSEs when $\varphi = \{1.0, 0.5, 2.0, 1.5\}$

(*m*, *n*, S)		$(\cdot)_{\mathbf{MLE}}$		$(\cdot)_{\mathbf{Boot}}$		$(\cdot)_{\mathbf{B(p0)}}$		$(\cdot)_{\mathbf{B(p1)}}$	
(20, 20, 15)	β_{11}	1.312	0.234	1.412	0.324	1.291	0.218	1.200	0.189
	β_{12}	0.724	0.124	0.812	0.164	0.703	0.121	0.652	0.098
	β_{21}	2.521	0.453	2.621	0.512	2.501	0.445	2.412	0.332
	β_{22}	1.874	0.421	1.942	0.487	1.855	0.395	1.752	0.241
(20, 20, 25)	β_{11}	1.295	0.202	1.397	0.301	1.275	0.190	1.189	0.157
	β_{12}	0.708	0.100	0.800	0.139	0.686	0.095	0.638	0.062
	β_{21}	2.503	0.423	2.602	0.492	2.482	0.417	2.400	0.299
	β_{22}	1.858	0.392	1.925	0.452	1.843	0.361	1.737	0.208
(30, 30, 25)	β_{11}	1.288	0.195	1.391	0.303	1.271	0.187	1.184	0.152
	β_{12}	0.712	0.101	0.795	0.132	0.680	0.091	0.632	0.057
	β_{21}	2.497	0.427	2.597	0.495	2.481	0.410	2.402	0.291
	β_{22}	1.853	0.387	1.921	0.448	1.839	0.363	1.733	0.203
(30, 30, 40)	β_{11}	1.252	0.151	1.361	0.258	1.225	0.139	1.141	0.114
	β_{12}	0.671	0.072	0.770	0.100	0.647	0.071	0.601	0.049
	β_{21}	2.467	0.375	2.564	0.450	2.439	0.371	2.350	0.252
	β_{22}	1.815	0.345	1.875	0.403	1.801	0.325	1.682	0.169
(50, 50, 50)	β_{11}	1.233	0.122	1.343	0.229	1.204	0.109	1.120	0.085
	β_{12}	0.652	0.041	0.749	0.078	0.632	0.045	0.579	0.023
	β_{21}	2.448	0.346	2.548	0.401	2.418	0.343	2.329	0.213
	β_{22}	1.801	0.317	1.861	0.371	1.781	0.300	1.678	0.135
(50, 50, 70)	β_{11}	1.195	0.101	1.301	0.210	1.185	0.091	1.089	0.066
	β_{12}	0.613	0.023	0.711	0.061	0.613	0.031	0.558	0.011
	β_{21}	2.404	0.327	2.510	0.386	2.400	0.325	2.307	0.193
	β_{22}	1.762	0.300	1.823	0.349	1.759	0.281	1.661	0.118
(50, 50, 100)	β_{11}	1.145	0.070	1.254	0.177	1.137	0.073	1.065	0.042
	β_{12}	0.565	0.010	0.675	0.043	0.554	0.009	0.529	0.004
	β_{21}	2.351	0.301	2.466	0.349	2.361	0.294	2.240	0.165
	β_{22}	1.711	0.270	1.771	0.318	1.704	0.242	1.618	0.104
(75, 75, 100)	β_{11}	1.138	0.068	1.249	0.179	1.132	0.070	1.061	0.041
	β_{12}	0.557	0.012	0.671	0.041	0.549	0.011	0.532	0.005
	β_{21}	2.354	0.304	2.460	0.351	2.357	0.291	2.243	0.161
	β_{22}	1.713	0.268	1.768	0.314	1.692	0.240	1.615	0.099
(75, 75, 130)	β_{11}	1.122	0.053	1.231	0.162	1.118	0.051	1.035	0.031
	β_{12}	0.541	0.008	0.632	0.029	0.535	0.008	0.517	0.004
	β_{21}	2.338	0.280	2.438	0.325	2.341	0.276	2.218	0.142
	β_{22}	1.691	0.251	1.749	0.291	1.781	0.235	1.600	0.091

Table 6: ILs and PCs when $\boldsymbol{\varphi} = \{1.0, 0.5, 2.0, 1.5\}$

$(m, n, \mathbf{S})$		$(\cdot)_{\text{MLE}}$		$(\cdot)_{\text{Boot-}P}$		$(\cdot)_{\text{Boot-}t}$		$(\cdot)_{\text{B}(p0)}$		$(\cdot)_{\text{B(p1)}}$	
(20, 20, 15)	β_{11}	3.124	0.88	3.171	0.89	3.111	0.90	3.103	0.89	3.071	0.91
	β_{12}	1.624	0.88	1.674	0.90	1.620	0.89	1.605	0.90	1.570	0.90
	β_{21}	4.852	0.89	4.899	0.87	4.843	0.89	4.832	0.89	4.801	0.92
	β_{22}	3.512	0.87	3.561	0.89	3.501	0.90	3.507	0.90	3.449	0.91
(20, 20, 25)	β_{11}	3.085	0.89	3.124	0.90	3.063	0.91	3.063	0.93	3.024	0.92
	β_{12}	1.570	0.90	1.627	0.91	1.572	0.92	1.540	0.92	1.518	0.95
	β_{21}	4.803	0.89	4.851	0.89	4.795	0.90	4.800	0.90	4.748	0.91
	β_{22}	3.477	0.90	3.514	0.89	3.451	0.91	3.451	0.96	3.403	0.93
(30, 30, 25)	β_{11}	3.082	0.89	3.133	0.89	3.054	0.93	3.051	0.92	3.017	0.91
	β_{12}	1.564	0.91	1.624	0.91	1.566	0.90	1.529	0.92	1.512	0.92
	β_{21}	4.807	0.90	4.847	0.90	4.790	0.90	4.803	0.92	4.741	0.94
	β_{22}	3.471	0.89	3.508	0.90	3.443	0.92	3.447	0.91	3.411	0.91
(30, 30, 40)	β_{11}	3.037	0.93	3.100	0.91	3.003	0.93	3.012	0.97	2.980	0.92
	β_{12}	1.525	0.91	1.582	0.91	1.524	0.96	1.492	0.94	1.476	0.93
	β_{21}	4.762	0.91	4.801	0.93	4.751	0.90	4.769	0.94	4.700	0.94
	β_{22}	3.434	0.92	3.478	0.91	3.403	0.91	3.408	0.95	3.382	0.95
(50, 50, 50)	β_{11}	3.019	0.90	3.087	0.92	2.987	0.94	2.985	0.92	2.957	0.94
	β_{12}	1.508	0.93	1.561	0.91	1.505	0.90	1.471	0.95	1.451	0.93
	β_{21}	4.741	0.91	4.792	0.90	4.729	0.92	4.745	0.94	4.685	0.95
	β_{22}	3.415	0.94	3.456	0.94	3.384	0.95	3.382	0.91	3.361	0.91
(50, 50, 70)	β_{11}	2.970	0.95	3.048	0.94	2.951	0.92	2.954	0.93	2.921	0.95
	β_{12}	1.476	0.95	1.527	0.92	1.456	0.92	1.439	0.94	1.417	0.92
	β_{21}	4.701	0.91	4.755	0.93	4.691	0.93	4.708	0.92	4.645	0.92
	β_{22}	3.385	0.92	3.421	0.93	3.351	0.94	3.335	0.93	3.324	0.93
(50, 50, 100)	β_{11}	2.917	0.90	3.001	0.93	2.902	0.95	2.901	0.95	2.871	0.95
	β_{12}	1.428	0.95	1.481	0.91	1.403	0.92	1.391	0.94	1.365	0.91
	β_{21}	4.649	0.92	4.701	0.93	4.628	0.94	4.361	0.92	4.600	0.94
	β_{22}	3.334	0.93	3.372	0.95	3.300	0.92	3.291	0.94	3.281	0.94
(75, 75, 100)	β_{11}	2.921	0.91	3.007	0.92	2.899	0.94	2.904	0.90	2.869	0.92
	β_{12}	1.432	0.94	1.484	0.92	1.401	0.92	1.395	0.94	1.369	0.93
	β_{21}	4.644	0.92	4.703	0.92	4.622	0.92	4.364	0.92	4.612	0.94
	β_{22}	3.329	0.92	3.369	0.94	3.303	0.91	3.287	0.95	3.277	0.92
(75, 75, 130)	β_{11}	2.891	0.92	2.971	0.93	2.865	0.93	2.882	0.92	2.838	0.95
	β_{12}	1.399	0.94	1.451	0.94	1.372	0.92	1.367	0.94	1.335	0.92
	β_{21}	4.615	0.94	4.671	0.92	4.594	0.93	4.341	0.93	4.582	0.94
	β_{22}	3.302	0.91	3.341	0.92	3.276	0.94	3.252	0.94	3.252	0.93

Therefore, the observed joint type-II competing risks data taken from two lines of production L-Ω_1 and L-Ω_2 with censoring scheme $N = 181, m = 99, n = 82$, and $\mathbf{S} = 80$ are reported in Table 1. The data show that $(\boldsymbol{\kappa}_1, \boldsymbol{\kappa}_2) = (50, 30)$ and $(\xi_{11}, \xi_{11}, \xi_{11}, \xi_{11}) = \{26, 24, 25, 5\}$ (Table 2). The point estimators under ML, bootstrap, and Bayes estimators with non-informative prior information (mean $a_i = b_i = 0.001$, $i = 1, 2, 3, 4$) are reported in Table 3. The 95% approximate ML, two bootstrap and credibly intervals, respectively, are reported in Table 4.

7 Simulation study

The performances of developed results presented in this article are assessed and compared through Monte Carlo simulations study. Therefore, we test the estimators by changing sample size $m + n$, effected sample size $\mathbf{S}$, and different choices of the initial parameter values. For the point estimate, we reported the mean estimate (ME) and MSE. But, for interval estimate, we reported the mean interval length (MIL) and coverage percentage (CP). So,

Table 7: MEs and MSEs when $\varphi = \{2.0, 3.0, 1.0, 0.8\}$

(m, n, S)		$(\cdot)_{MLE}$		$(\cdot)_{Boot}$		$(\cdot)_{B(p0)}$		$(\cdot)_{B(p1)}$	
(20, 20, 15)	β_{11}	2.451	0.425	2.492	0.479	2.442	0.419	2.403	0.364
	β_{12}	3.524	0.621	3.569	0.669	3.515	0.618	3.475	0.561
	β_{21}	1.354	0.235	1.388	0.281	1.343	0.229	1.302	0.185
	β_{22}	1.213	0.214	1.248	0.271	1.202	0.215	1.165	0.152
(20, 20, 25)	β_{11}	2.424	0.394	2.461	0.451	2.415	0.389	2.365	0.334
	β_{12}	3.500	0.591	3.541	0.641	3.485	0.589	3.418	0.535
	β_{21}	1.327	0.208	1.361	0.254	1.304	0.201	1.251	0.152
	β_{22}	1.185	0.184	1.219	0.242	1.161	0.187	1.109	0.125
(30, 30, 25)	β_{11}	2.429	0.398	2.467	0.454	2.410	0.384	2.364	0.330
	β_{12}	3.502	0.590	3.542	0.639	3.489	0.582	3.412	0.531
	β_{21}	1.324	0.214	1.360	0.251	1.308	0.212	1.255	0.147
	β_{22}	1.181	0.180	1.213	0.246	1.167	0.183	1.111	0.131
(30, 30, 40)	β_{11}	2.401	0.371	2.441	0.427	2.382	0.349	2.351	0.309
	β_{12}	3.475	0.569	3.517	0.615	3.461	0.551	3.288	0.504
	β_{21}	1.303	0.187	1.332	0.225	1.281	0.187	1.229	0.121
	β_{22}	1.154	0.154	1.185	0.221	1.142	0.151	1.082	0.108
(50, 50, 50)	β_{11}	2.385	0.355	2.427	0.415	2.368	0.332	2.334	0.291
	β_{12}	3.462	0.551	3.503	0.604	3.449	0.536	3.275	0.289
	β_{21}	1.287	0.168	1.315	0.208	1.270	0.170	1.209	0.104
	β_{22}	1.139	0.134	1.169	0.209	1.126	0.137	1.066	0.092
(50, 50, 70)	β_{11}	2.361	0.329	2.403	0.392	2.344	0.308	2.311	0.274
	β_{12}	3.439	0.524	3.481	0.581	3.425	0.509	3.249	0.265
	β_{21}	1.261	0.146	1.290	0.186	1.247	0.149	1.200	0.083
	β_{22}	1.119	0.109	1.145	0.184	1.103	0.112	1.045	0.071
(50, 50, 100)	β_{11}	2.339	0.329	2.282	0.369	2.323	0.291	2.289	0.257
	β_{12}	3.415	0.508	3.459	0.560	3.407	0.289	3.231	0.246
	β_{21}	1.237	0.127	1.268	0.169	1.224	0.124	1.179	0.065
	β_{22}	1.104	0.091	1.127	0.165	1.082	0.100	1.027	0.057
(75, 75, 100)	β_{11}	2.331	0.322	2.284	0.363	2.320	0.293	2.285	0.253
	β_{12}	3.412	0.502	3.452	0.561	3.401	0.284	3.228	0.242
	β_{21}	1.231	0.123	1.265	0.162	1.220	0.120	1.171	0.060
	β_{22}	1.101	0.092	1.121	0.164	1.079	0.101	1.024	0.054
(75, 75, 130)	β_{11}	2.311	0.308	2.255	0.351	2.300	0.274	2.261	0.241
	β_{12}	3.385	0.491	3.431	0.539	3.284	0.268	3.204	0.229
	β_{21}	1.209	0.109	1.241	0.145	1.200	0.105	1.142	0.041
	β_{22}	1.079	0.072	1.101	0.147	1.057	0.079	1.005	0.039

we used two sets of parameter values $\varphi = \{\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}\} = \{\mathbf{1.0}, \mathbf{0.5}, \mathbf{2.0}, \mathbf{1.5}\}$ and $\{\mathbf{2.0}, \mathbf{3.0}, \mathbf{1.0}, \mathbf{0.8}\}$. Different censoring schemes are used in the simulation tables. About the prior information in Bayesian approach, we adopted non-informative priors (mean $a_i = b_i = 0.0001$) and informative priors (mean a_i, b_i take non-zero values). For the first choice of the parameter values the prior information is taken to be $\{(2, 2), (2, 3), (4, 2), (3, 2)\}$. Also, for the second choice of the parameter values the prior information is taken to be $\{(4, 2), (3, 1), (2, 2), (2, 3)\}$. Our studying dependent on simulate samples with iteration number equal to 1,000. Monte Carlo simulation study has algorithm described as follows:

1. For given prior information, the initial parameter values can be selected randomly by generating value from prior distribution. Also, some authors selected the mean of some generated values. In this article, we selected the initial parameter value to satisfy almost the mean of prior distribution. Therefore, for given parameter vector $\varphi = \{\beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}\}$, the informative prior information satisfy $\mathrm{E}(\varphi_i) \simeq \frac{a_i}{b_i}$.

Table 8: ILs and PCs when $\boldsymbol{\varphi} = \{2.0, 3.0, 1.0, 0.8\}$

$(m, n, \mathbf{S})$		$(\cdot)_{\mathrm{MLE}}$		$(\cdot)_{\text{Boot-}P}$		$(\cdot)_{\text{Boot-}t}$		$(\cdot)_{\mathrm{B(p0)}}$		$(\cdot)_{\mathrm{B(p1)}}$	
(20, 20, 15)	β_{11}	4.725	0.88	4.754	0.89	4.711	0.89	4.714	0.90	4.650	0.90
	β_{12}	5.541	0.89	5.569	0.89	5.528	0.90	5.528	0.91	5.470	0.91
	β_{21}	2.842	0.87	2.867	0.89	2.825	0.90	2.829	0.90	2.761	0.89
	β_{22}	2.102	0.89	2.132	0.90	2.085	0.89	2.101	0.89	2.025	0.91
(20, 20, 25)	β_{11}	4.689	0.90	4.721	0.91	4.684	0.90	4.687	0.92	4.618	0.92
	β_{12}	5.513	0.90	5.541	0.89	5.501	0.93	5.501	0.93	5.439	0.92
	β_{21}	2.815	0.89	2.842	0.90	2.794	0.92	2.804	0.92	2.728	0.91
	β_{22}	2.071	0.91	2.105	0.92	2.054	0.91	2.076	0.91	2.001	0.93
(30, 30, 25)	β_{11}	4.684	0.89	4.724	0.92	4.688	0.91	4.681	0.91	4.614	0.94
	β_{12}	5.517	0.91	5.538	0.90	5.512	0.91	5.497	0.92	5.431	0.92
	β_{21}	2.811	0.90	2.835	0.91	2.789	0.90	2.801	0.92	2.723	0.93
	β_{22}	2.074	0.92	2.101	0.90	2.051	0.92	2.078	0.90	2.005	0.93
(30, 30, 40)	β_{11}	4.649	0.91	4.691	0.91	4.651	0.92	4.649	0.91	4.581	0.92
	β_{12}	5.484	0.91	5.504	0.92	5.472	0.91	5.462	0.91	5.401	0.95
	β_{21}	2.781	0.91	2.801	0.92	2.757	0.91	2.768	0.93	2.691	0.93
	β_{22}	2.041	0.93	2.069	0.91	2.018	0.93	2.042	0.91	1.972	0.91
(50, 50, 50)	β_{11}	4.638	0.92	4.678	0.94	4.638	0.93	4.632	0.93	4.565	0.93
	β_{12}	5.471	0.90	5.492	0.93	5.466	0.91	5.448	0.92	5.386	0.94
	β_{21}	2.769	0.91	2.789	0.92	2.741	0.92	2.751	0.93	2.679	0.92
	β_{22}	2.028	0.94	2.054	0.92	2.004	0.94	2.027	0.93	1.958	0.93
(50, 50, 70)	β_{11}	4.611	0.91	4.651	0.92	4.609	0.91	4.603	0.93	4.539	0.95
	β_{12}	5.445	0.92	5.466	0.92	5.441	0.95	5.421	0.94	5.347	0.93
	β_{21}	2.741	0.91	2.761	0.92	2.715	0.92	2.723	0.93	2.645	0.92
	β_{22}	2.001	0.92	2.024	0.93	1.956	0.92	2.002	0.94	1.931	0.94
(50, 50, 100)	β_{11}	4.582	0.95	4.622	0.91	4.582	0.92	4.572	0.93	4.511	0.92
	β_{12}	5.417	0.92	5.437	0.93	5.415	0.95	5.591	0.92	5.318	0.95
	β_{21}	2.715	0.91	2.729	0.92	2.691	0.93	2.691	0.93	2.617	0.95
	β_{22}	1.976	0.94	2.001	0.94	1.927	0.93	1.959	0.92	1.903	0.92
(75, 75, 100)	β_{11}	4.548	0.93	4.591	0.92	4.553	0.93	4.540	0.93	4.479	0.96
	β_{12}	5.381	0.92	5.403	0.93	5.388	0.94	5.555	0.91	5.292	0.93
	β_{21}	2.687	0.92	2.700	0.92	2.657	0.91	2.654	0.93	2.589	0.92
	β_{22}	1.941	0.91	1.659	0.90	1.889	0.92	1.918	0.94	1.872	0.94
(75, 75, 130)	β_{11}	4.521	0.93	4.558	0.93	4.515	0.91	4.517	0.93	4.447	0.93
	β_{12}	5.342	0.94	5.371	0.92	5.357	0.94	5.519	0.91	5.256	0.93
	β_{21}	2.651	0.92	2.669	0.94	2.619	0.92	2.621	0.92	2.549	0.94
	β_{22}	1.914	0.93	1.618	0.91	1.862	0.94	1.900	0.95	1.839	0.93

2. Generate two samples of size m and n from RD with scale parameters $\beta_{k1} + \beta_{k2}$, $k = 1, 2$, respectively. Hence, from the joint sample of size $m + n$ select a small value of size $\mathbf{S}$.
3. From Step 2 compute the two integer values $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$.
4. The binomial distributions are used to generate discrete random variables ξ_{kj}, $k, j = 1, 2$, sample size $\boldsymbol{\kappa}_1$ and $\boldsymbol{\kappa}_2$ under probability of success $\frac{\beta_{k1}}{\beta_{k1}+\beta_{k2}}$ and $\frac{\beta_{k2}}{\beta_{k1}+\beta_{k2}}$, respectively.
5. Steps 2–4 are repeated N times, then N competing risks type-II JC samples with the corresponding N times of $\boldsymbol{\kappa}_1$, $\boldsymbol{\kappa}_2$, and ξ_{kj} are available.
6. The MLE, bootstrap, and Bayes point and interval estimates are computed for each sample.
7. Compute the values of each ME, MSEs, ILs, and PCs, which are reported in Tables 5–8.

8 Conclusions and remarks

Estimating problem under joint type-II competing risks sample is considered in this article. The unknown parameters under consideration of product life distributed with Rayleigh lifetime population are estimated under competing risks model with two different failure modes. Estimation problem is discussed with, three different estimation methods, ML, bootstrap, and Bayesian methods . Two different loss functions are considered in Bayesian context. The ACIs, two bootstrap, and Bayes credible intervals are also discussed. Bayesian estimation under SE loss function or LINEX loss functions for the parameters of two competing RDs was discussed for given competing risks type-II JC sample. Bayes results were compared with the ML and bootstrap estimates through a Monte Carlo simulation study. The computational results show that the Bayesian estimation under two proposed loss functions is more precise than the MLE. The results in both ML and Bayes with non-informative prior are closed for the information are obtained only from likelihood function. It is evident from the numerical results that while the MLEs are close to the ones based on the Bayes method under non-informative prior P^0, the Bayes estimators under informative prior and bootstrap-t perform well when compared to the MLEs and bootstrap-p. Statistic defined by (22) show that bootstrap-t confidence intervals combined with the mean do not necessarily deal with skewness as well as other methods, which makes it better than other methods. It is observed that when the effective sample m increases, the MSEs of the estimates and the ALs of the confidence intervals decreased. Moreover, the MSEs of the point estimates as well as the ILs and CPs of CIs, approximate boot-p and boot-t as well as CIs improve when **S** is large. This observation is expected due to the fact that when **S** increases, more information on the data is obtained.

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References

[1] Rao UVR, Savage IR, Sobel M. Contributions to the theory of rank order statistics: the two-sample censored case. Ann Math Stat. 1960;31(2):415–511.
[2] Basu AP. On a generalized savage statistic with applications to life testing. Ann Math Stat. 1968;39(5):1591–613.
[3] Johnson RA, Mehrotra KG. Locally most powerful rank tests for the two-sample problem with censored data. Ann Math Stat. 1972;43(3):823–8.
[4] Mehrotra KG, Johnson RA. Asymptotic sufficiency and asymptotically most powerful tests for the two sample censored situation. Ann Stat. 1976;4(3):589–97.
[5] Bhattacharyya GK, Mehrotra KG. On testing equality of two exponential distributions under combined type-II censoring. J Am Stat Assoc. 1981;76(376):886–8.
[6] Mehrotra KG, Bhattacharyya GK. Confidence intervals with jointly type-II censored samples from two exponential distributions. J Am Stat Assoc. 1982;77(378):441–5.
[7] Balakrishnan N, Rasouli A. Exact likelihood inference for two exponential populations under joint type-II censoring. Comput Stat Data Anal. 2008;52(5):2725–313.
[8] Rasouli A, Balakrishnan N. Exact likelihood inference for two exponential populations under joint progressive type-II censoring. Commun Stat Theory Methods. 2010;39(12):2172–219.
[9] Shafaya AR, Balakrishnanbc N, Abdel-Atyd Y. Bayesian inference based on a jointly type-II censored sample from two exponential populations. J Statist Comput Simul. 2014;84(11):2427–313.
[10] Bekheet NA, Abd-Elmougod GA. Statistical inferences with jointly type-II censored samples from two Rayleigh distributions. Global J Pure Appl Math. 2017;13(12):8361–411.
[11] Soliman AA, Abd-Elmougod GA, Al-Sobhi MM. Estimation in step-stress partially accelerated life tests for the Chen distribution using progressive type-II censoring. Appl Math Inform Sci. 2017;11(1):325–7.
[12] Abu-Zinadah HH. Statistical inferences with jointly type-II censored samples from two Pareto distributions. Open Phys. 2017;15:557–8.
[13] Algarni A, Almarashi AM, Abd-Elmougod GA, Abo-Eleneen ZA. Two compound Rayleigh lifetime distributions in analyses the jointly type-II censoring samples. J Math Chem. 2019;58:950–1016.
[14] Mondal S, Kundu D. A new two sample type-II progressive censoring scheme. Commun Stat Theory Methods. 2019;48(10):2602–16.
[15] AbdullahiBaba IA, Ahmad H, Alsulami MD Abualnaja KM, Altanji M. A mathematical model to study resistance and non-resistance strains of influenza. Results Phys. 2021;26:1–6.
[16] Alsulami MD. Stochastic modeling of infectious disease. Int J Innovat Sci Math. 2020;8:1–28.
[17] Cox DR. The analysis of exponentially distributed lifetimes with two types of failures. J R Stat Soc. 1959;21(2):411–510.
[18] Crowder MJ. Classical competing risk. London: Chapman and Hall; 2021.
[19] Balakrishnan N, Han D. Exact inference for a simple step-stress model with competing risks for failure from exponential distribution under type-II censoring. J Stat Plann Inference. 2008;138(12):4172–214.
[20] Modhesh AA, Abd-Elmougod GA. Analysis of progressive first-failure-censoring in the Burr XII model for competing risk data. Am J Theoret Appl Stat. 2015;4(6):610–8.
[21] Bakoban RA, Abd-Elmougod GA. MCMC in analysis of progressively first failure censored competing risk data for Gompertz model. J Comput Theoret Nanosci. 2016;13(10):6662–8.

[22] Ganguly A, Kundu D. Analysis of simple step-stress model in presence of competing risks. J Statist Comput Simul. 2016;86(10):1989–2017.
[23] Abu-Zinadah HH, Neveen SA. Competing risk model with partially step-stress accelerate life tests in analyses lifetime Chen data under type-II censoringscheme. Open Phys. 2019;17:192–7.
[24] Algarn A, Almarashi AM, Abd-Elmougod GA, Abo-Eleneen ZA. Partially constant stress accelerate life tests model in analyses lifetime competing risk with a bathtub shape lifetime distribution in presence of type-I censoring. Transylvanian Rev. 2019;1:202798080.
[25] Almalki SJ, Abushal TA, Alsulami MD, Abd-Elmougod GA. Analysis of type-II censored competing risk' data under reduced new modified Weibull distribution. Complexity. 2021;2021:1–13.
[26] Soliman AA, Farghal AA, Abd-Elmougod GA. Statistical inference under Copula approach of accelerated dependent generalized inverted exponential failure time with progressive hybrid censoring scheme. Appl Math Inform Sci. 2021;15(6):687–712.
[27] Dyer DD, Whisenand CW. Best linear unbiased estimator of the parameter of the Rayleigh distribution. part I: small sample theory for censored order statistics. IEEE Trans Reliab. 1973;22(4):27–37.
[28] Dyer DD, Whisenand CW. Best linear unbiased estimator of the parameter of the Rayleigh distribution. part II: optimum theory for selected order statistics. IEEE Trans Reliab. 1973;22(4):229–32.
[29] Polovko AM. Fundamentals of reliability theory. New York: Academic Press; 1968.
[30] Leon HL, Harter AH. Moore maximum-likelihood estimation of the parameters of gamma and Weibull populations from complete and from censored samples. Technometrics. 1965;7(4):639–44.
[31] Lalitha S, Mishra A. Modified maximum likelihood estimation for Rayleigh distribution. Commun Stat Theory Methods. 1996;25(2):389–412.
[32] Kong F, Fei H. Limits theorems for the maximum likelihood estimate under general multiply type II censoring. Annals Inst Stat Math. 1996;48(4):731–24.
[33] Howlader HA, Hossain A. On Bayesian estimation and prediction from Rayleigh based on type II censored data. Commun Stat Theory Methods. 1995;24(9):2249–310.
[34] Fernandez AJ. Bayesian inference from type II doubly censored Rayleigh data. Statt Probab Lett. 2000;48(4):393–6.
[35] AL-Hussaini EK, Ahmad AA. On Bayes predictive distributions of generalized order statistics. Metrika. 2003;57:165–211.
[36] AL-Hussaini EK, Ahmad AA. On Bayes interval prediction of future records. Test. 2003;12:79–99.
[37] Almarashi AM, Algarni A, Daghistani AM, Abd-Elmougod GA, Abdel-Khalek S, Raqab MZ. Inferences for joint hybrid progressive censored exponential lifetimes under competing risk model. Math Probl Eng. 2021;2021:1–12.
[38] Abushal TA, Soliman AA, Abd-Elmougod GA. Statistical inferences of Burr XII lifetime models under joint type-1 competing risk samples. J Math. 2021;2021:1–16.
[39] Alghamdi AS, Abd-Elmougod GA, Kundu D, Marin M. Statistical inference of jointly type-II lifetime samples under Weibull competing risk models. Symmetry. 2022;14(4):1–17.
[40] Kundu D, Joarder A. Analysis of type II progressively hybrid censored data. Comput Statist Data Anal. 2006;50(10):2509–19.
[41] Davison AC, Hinkley DV. Bootstrap methods and their applications. 2nd edition. Cambridge, United Kingdom: Cambridge University Press; 1997.
[42] Efron B, Tibshirani RJ. An introduction to the bootstrap. New York Chapman and Hall: CHAPMAN and HALL/CRC; 1953.
[43] Efron B. The jackknife, the bootstrap and other resampling plans. In: CBMS-NSF Regional Conference Series in Applied Mathematics. Vol. 38. Philadelphia, PA: SIAM; 1982.
[44] Hall P. Theoretical comparison of bootstrap confidence intervals. Ann Stat. 1988;16(3):927–26.
[45] Majdah MB, Shawky AI, Abd-Elmougod GA. Hybrid censoring samples in assessment the lifetime performance index of Chen distributed products. Open Phys. 2019;17:607–9.
[46] Hoel DG. A representation of mortality data by competing risk. Biometrics. 1972;28(2):475–513.
[47] Koley A, Kundu D. On generalized progressive hybrid censoring in presence of competing risk. Metrika. 2017;80:401–25.