Research Article

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Exploring the new optical solitons to the time-fractional integrable generalized (2+1)-dimensional nonlinear Schrödinger system *via* three different methods

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Abstract: This current research is about some new optical solitons to the time-fractional integrable generalized (2+1)-dimensional nonlinear Schrödinger (NLS) system with novel truncated M-fractional derivative. The obtained results may be used in the description of the model in fruitful way. The novel derivative operator is applied to study the aforementioned model. The achieved results are in the form of dark, bright, and combo optical solitons. The achieved solutions are also verified by using the MATHEMATICA software. The obtained solutions are explained with different plots. Modified integration methods, function, extended (G'/G)-expansion, and extended sinh-Gordon equation expansion method are applied to achieve the results. These exact solitons suggest that these methods are effective, straight forward, and reliable compared to other methods.

Keywords: integrable generalized NLS system, Exp_a function method, extended (G'/G)-expansion method, extended sinh-Gordon equation expansion method, optical solitons

1 Introduction

Fractional calculus [1–12] has become very popular due to its many applications in different areas of sciences. Many models have been made in the area of physical sciences and engineering that are representing the different phenomenon. For example, mostly naturally occurring phenomena are modeled in the form of nonlinear Schrödinger equations [13–15]. To determine the exact solutions of the models, a lot of schemes have been developed. Instantly, the modified extended tanh expansion scheme [16] has been applied to discuss the Biswas–Arshed model. Some different wave solutions of the perturbed Gerdjikov-Ivanov equation are gained with the help of the semi-inverse variational method [17]. Various solitons of the new coupled evolution equation were explained [18]. Distinct solitons are investigated by applying the sine-Gordon equation method [19]. Two types of soliton solutions have been obtained by using $\mathrm{Exp}(-\phi(\eta))$-expansion and generalized Kudryashov methods in ref. [20]. New kinds of general solutions have been achieved in ref. [21]. The Sardar subequation method is used to gain the optical and some other wave solutions in ref. [22]. Three new types of wave solutions have been gained with the help of the modified $\exp(-\Omega)$-expansion method in ref. [23]. Different types of optical soliton solutions have been collected by using the collective variable method in ref. [24]. Similarly, other methods have been applied; generalized exponential rational function method [25–27], Liu's extended trial function method [28], generalized unified method [29], sine-Gordon expansion method [30], enhanced modified simple equation method [31], unified method [32], extended tanh function method [33], Lie symmetry method [34], symbolic computational method, Hirota's simple method and long wave method [35], Jacobi elliptic function expansion method [36], Elzaki transform decomposition method [37], $\left(m+\frac{1}{G}\right)$-expansion method and adomian decomposition method [38], extended modified auxiliary equation mapping method [39], simplest equation method and Kudryashov's new function method [40], modified simple equation method [41], modified Kudryashov simple equation method [42], first integral method [43], Bäcklund transformation method [44], extended Jacobi elliptic function expansion method [45],

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improved (G/G)-expansion method, improved (G'/G)-expansion method [46], and many more [47–51].

Our concerning model is time-fractional integrable generalized (2+1)-dimensional nonlinear Schrödinger system. Different types of exact solitons have been found by various techniques as follows: optical soliton solutions have been calculated by using the extended modified auxiliary equation mapping method in ref. [39], some travelling wave solutions of the integrable generalized NLS system have been obtained in ref. [52], and various optical wave solutions have been achieved of this system with the help of Kudryashov method and it is modified form given in ref. [53].

In addition to these methods, there are three other methods: Exp_a function method, extended (G'/G)-expansion method, and extended sinh-Gordon equation expansion method (ShGEEM). These methods have been used to explain the many different models: the Tzitzéica like equations are investigated for their exact solitons by using the Exp_a function method [54]. New kind of optical wave solutions of two nonlinear Schrödinger equations were searched by utilizing two analytical methods [55]. Similarly, this is used to investigate the roots of the other many nonlinear Schrödinger equations [56,57]. By using the extended (G'/G)-expansion method, different optical solitons of the Biswas–Milovic equation are generated in ref. [58]; bell-shaped, kink-shaped, and periodic type solitons of the Pochhammer–Chree equations are derived with the help of this method [59]; and discrete and periodic type solitons of the Ablowitz–Ladik lattice system are found [60]. Similarly, the extended ShGEEM has been applied to determine the various wave solutions of different models in refs [61–64].

The main task of this study is to research some new exact soliton solutions of the truncated M-fractional integrable generalized (2+1)-dimensional NLS system based on the Exp_afunction method, the extended (G'/G)-expansion method, and extended ShGEEM.

This paper is organized as follows: Section 2 describes the truncated M-fractional derivative and its characteristics. Section 3 presents the demonstration and mathematically treatment of model. Section 4 gives the mathematical analysis of the time-fractional integrable generalized NLS system. Section 5 obtains a large number of exact wave solutions. Section 6 makes a conclusion.

2 Truncated M-fractional derivative

Definition. Let $h(t) : [0, \infty) \to \Re$, then the truncated M-fractional derivative of h of order α is shown [65]:

$$D_{M,t}^{\alpha,\gamma}h(t) = \lim_{\tau\to 0}\frac{h(tE_\gamma(\tau t^{1-\alpha})) - h(t)}{\tau}, \quad 0 < \alpha < 1,\ \gamma > 0, \tag{1}$$

where $E_\gamma(\cdot)$ is a truncated Mittag-Leffler function of one parameter that is defined as ref. [66]:

$$E_\gamma(z) = \sum_{j=0}^{i}\frac{z^j}{\Gamma(\gamma j + 1)}, \quad \gamma > 0 \quad \text{and } z \in \mathbf{C}. \tag{2}$$

Characteristics: Let $0 < \alpha \le 1, \gamma > 0, a, b \in \Re$, and g, f, α-differentiable at a point $t > 0$, then by ref. [65]:

$$\text{(i)}\quad D_{M,t}^{\alpha,\gamma}(ag(t) + bf(t)) = aD_{M,t}^{\alpha}, g(t) + bD_{M,t}^{\alpha,\gamma}f(t), \tag{3}$$

$$\text{(ii)}\quad D_{M,t}^{\alpha,\gamma}(g(t).\, f(t)) = g(t)D_{M,t}^{\alpha,\gamma}f(t) + f(t)D_{M,t}^{\alpha,\gamma}g(t), \tag{4}$$

$$\text{(iii)}\quad D_{M,t}^{\alpha,\gamma}\left(\frac{g(t)}{f(t)}\right) = \frac{f(t)D_{M,t}^{\alpha,\gamma}g(t) - g(t)D_{M,t}^{\alpha,\gamma}f(t)}{(f(t))^2}, \tag{5}$$

$$\text{(iv)}\quad D_{M,t}^{\alpha,\gamma}(C) = 0, \quad \text{where } h(t) = C \quad \text{is a constant.} \tag{6}$$

$$\text{(v)}\quad D_{M,t}^{\alpha,\gamma}g(t) = \frac{t^{1-\alpha}}{\Gamma(\gamma + 1)}\frac{\mathrm{d}g(t)}{\mathrm{d}t}. \tag{7}$$

3 Description of strategies

3.1 Summary of Exp_a function method

In this section, we demonstrate this method.

Suppose a nonlinear partial differential equation (NLPDE):

$$G(g, g^2g_t, g_x, g_{tt}, g_{xx}, g_{xt}, \ldots) = 0. \tag{8}$$

This NLPDE shown in Eq. (8) changed in to nonlinear ordinary differential equation (ODE):

$$\Lambda(G, G', G'', \ldots) = 0, \tag{9}$$

with the use of following wave transformations:

$$g(x, y, t) = G(\tau), \tau = ax + by + rt, \tag{10}$$

Let's consider a root of Eq. (9) is given in refs [54,56,67,68]:

$$G(\tau) = \frac{\alpha_0 + \alpha_1 d^\tau + \ldots + \alpha_m d^{m\tau}}{\beta_0 + \beta_1 d^\tau + \ldots + \beta_m d^{m\tau}}, \quad d \ne 0, 1, \tag{11}$$

where α_i and $\beta_i (0 \le i \le m)$ are unknowns that are found later. Natural number m is calculated with the help of the homogeneous balance scheme into Eq. (9). By inserting Eq. (11) into nonlinear Eq. (9), we obtain

$$\wp(d^\tau) = \ell_0 + \ell_1 d^\tau + \ldots + \ell_t d^{t\tau} = 0. \tag{12}$$

Considering $\ell_i (0 \le i \le t)$ in Eq. (12) to be equal to 0, a system of algebraic equations is achieved as follows:

$$\ell_i = 0, \quad \text{where } i = 0, \ldots, t \tag{13}$$

with the aforementioned achieved results, we gain the nontrivial solitons of Eq. (8).

3.2 Explanation of the extended (G'/G)-expansion scheme

This portion is about the key steps of the extended (G'/G)-expansion scheme [46].

Step 1: Let's assume the below NLPDE:

$$G(q, D^{\alpha,\gamma}_{M,t}q, q^2q_x, q_y, q_{yy}, q_{xx}, q_{xy}, \ldots) = 0, \tag{14}$$

where q is a wave profile and depend on x and y and t. Let the following travelling wave transformations:

Step 2:

$$q(x, y, t) = Q(\tau), \quad \tau = x - \nu y + \frac{\Gamma(\gamma + 1)}{\alpha}(\kappa t^{\alpha}). \tag{15}$$

By substituting Eq. (15) into Eq. (14), we obtain the nonlinear ODE:

$$\Lambda(Q(\tau), Q^2(\tau)Q'(\tau), Q''(\tau), \ldots) = 0, \tag{16}$$

Step 3: Suppose the solutions of the Eq. (16) is of the structure:

$$Q(\tau) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'(\tau)}{G(\tau)}\right)^i, \tag{17}$$

In Eq. (17), α_0 and α_i, ($i = \pm1, \pm2, \pm3, \ldots, \pm m$) are undetermined that are found later. Note that $\alpha_i \neq 0$. By applying the homogenous balance principle into Eq. (16), we find the value of m. The function $G = G(\tau)$ satisfy the Riccati differential equation given as follows:

$$dGG'' - aG^2 - bGG' - c(G')^2 = 0, \tag{18}$$

with a, b, c, and d are constants.

Step 4: Let Eq. (17) have solutions in the form given as follows:

Case 1: If $b \neq 0$ and $b^2 + 4ad - 4ac > 0$, then

$$\left(\frac{G'(\tau)}{G(\tau)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{-4ac + 4ad + b^2}}{2(d-c)}\left(\frac{C_1 \sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right) + C_2 \cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1 \cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right) + C_2 \sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right). \tag{19}$$

Case 2: If $b \neq 0$ and $b^2 + 4ad - 4ac < 0$, then

$$\left(\frac{G'(\tau)}{G(\tau)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{4ac - 4ad - b^2}}{2(d-c)}\left(\frac{C_2 \cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right) - C_1 \sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1 \cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right) + C_2 \sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right). \tag{20}$$

Case 3: If $b \neq 0$ and $b^2 + 4ad - 4ac = 0$, then

$$\left(\frac{G'(\tau)}{G(\tau)}\right) = \frac{b}{2(d-c)} + \frac{dD}{(d-c)(C - D\tau)}. \tag{21}$$

Case 4: If $b = 0$ and $ad - ac > 0$, then

$$\left(\frac{G'(\tau)}{G(\tau)}\right) = \frac{\sqrt{ad - ac}}{(d-c)}\left(\frac{C_1 \sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right) + C_2 \cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1 \cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right) + C_2 \sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right). \tag{22}$$

Case 5: If $b = 0$ and $ad - ac < 0$, then

$$\left(\frac{G'(\tau)}{G(\tau)}\right) = \frac{\sqrt{ac - ad}}{d-c}\left(\frac{C_2 \cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) - C_1 \sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1 \cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) + C_2 \sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right), \tag{23}$$

where a, b, c, d, C_1, and C_2 are the constants.

Step 5: Eq. (17) with Eq. (18) is inserted into Eq. (16) and summed up the coefficients of the same power of $\left(\frac{G'(\tau)}{G(\tau)}\right)$. By taking each coefficient equal to 0, we gain the system of algebraic equations involving ν, κ, α_i, ($i = 0, \pm1, \pm2, \ldots, \pm m$) and other parameters.

Step 6:

By manipulating the aforementioned achieved system with the help of MATHEMATICA tool.

Step 7:

By inserting the aforementioned achieved results into Eq. (17), we obtain solitons of the nonlinear Eq. (14).

3.3 Demonstration of the extended ShGEEM

In this section, we brief the basic steps of the extended ShGEEM:

Step 1:

Consider the NLPDE given as follows:

$$G(q, D_{M,t}^{\alpha,\gamma}q, q^2q_x, q_y, q_{yy}, q_{xx}, q_{xy}, \ldots) = 0, \tag{24}$$

where q is a wave profile and depends on x and y and t.

Let the following be travelling wave transformations:

$$q(x, y, t) = Q(\tau), \quad \tau = x - vy + \frac{\Gamma(\gamma+1)}{\alpha}(\kappa t^{\alpha}). \tag{25}$$

By substituting Eq. (25) into Eq. (24), we obtain the following nonlinear ODE given as follows:

$$\Lambda(Q(\tau), Q^2(\tau)Q'(\tau), Q''(\tau), \ldots) = 0. \tag{26}$$

Set 2:

Consider the solution of Eq. (26) follows:

$$Q(p) = \alpha_0 + \sum_{i=1}^{m}(\beta_i \sinh(p) + \alpha_i \cosh(p))^i, \tag{27}$$

where $\alpha_0,\ \alpha_i,\ \beta_i\ (i = 1, 2, 3, \ldots, m)$ are unknown parameters that are found later and p is a new function of τ that satisfy the below equations:

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \sinh(w). \tag{28}$$

By using the homogeneous balance scheme into Eq. (26), we achieve the value of m. Eq. (28) is obtained from the following sinh-Gordon equation:

$$q_{xt} = \kappa \sinh(v). \tag{29}$$

Indisputable to ref. [63], we obtain the results from Eq. (28) follows:

$$\sinh p(\tau) = \pm\operatorname{csch}(\tau) \quad \text{or} \quad \cosh p(\tau) = \pm\coth(\tau) \tag{30}$$

and

$$\sinh p(\tau) = \pm\iota\operatorname{sech}(\tau) \quad \text{or} \quad \cosh p(\tau) = \pm\tanh(\tau), \tag{31}$$

where $\iota = \sqrt{-1}$,

Step 3:

Inserting Eq. (27) along natural number m with Eq. (28) into Eq. (26) to achieve the algebraic expressions in $p'^{k}(\tau)\sinh^{l}p(\tau)\cosh^{m}p(\tau) \quad (k = 0, 1;\ l = 0, 1;\ m = 0, 1, 2, \ldots)$. Now take the each coefficient of $p'^{k}(\tau)\sinh^{l}p(\tau)\cosh^{m}p(\tau)$ equal to zero, to gain the set of algebraic equations containing v, κ, α_0, α_i and $\beta_i (i = 1, 2, 3, \ldots, m)$.

Step 4:

By solving the achieved set of algebraic equations with the use of MATHEMATICA tool, we can obtain the values of parameters, v, κ, α_0, α_i, and β_i.

Step 5:

By using the gained results and Eqs. (30) and (31), we may obtain solutions of Eq. (24) follows:

$$Q(\tau) = \alpha_0 + \sum_{i=1}^{m}(\pm\iota\beta_i\operatorname{sech}(\tau) \pm \alpha_i\tanh(\tau))^i \tag{32}$$

and

$$Q(\tau) = \alpha_0 + \sum_{i=1}^{m}(\pm\beta_i\operatorname{csch}(\tau) \pm \alpha_i\coth(\tau))^i. \tag{33}$$

4 Description and mathematical analysis of the model

Consider the following time-fractional integrable generalized NLS system given in ref. [52]:

$$\iota D_{M,t}^{\alpha,\gamma}g + b_1g_{xy} + b_2gh = 0, \quad b_3h_x + b_4(|g|^2)_y = 0, \tag{34}$$

where $g = g(x, y, t)$ is the complex-valued wave function and $h = h(x, y, t)$ is the real-valued wave function. In Eq. (34), $b_i(i = 1, 2, 3, 4)$ are the parameters.

Let's assume the following travelling wave transformation:

$$\begin{aligned} &g(x, y, t) = G(\tau) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\\ &h(x, y, t) = H(\tau), \text{ where } \tau = \kappa_1 x + \lambda_1 y + \theta_1\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}, \end{aligned} \tag{35}$$

where κ_i and λ_i, $i = 1, 2$ represent the speed of soliton and wave number, respectively, while θ_1 and θ_2 show the frequency of the soliton.

By substituting Eq. (35) into Eq. (34), we obtain the imaginary and real parts given as follows:

$$(\theta_1 + b_1(\kappa_1\lambda_2 + \kappa_2\lambda_1))G' = 0, \tag{36}$$

$$-\theta_2G - b_1\kappa_2\lambda_2G + b_1\kappa_1\lambda_1G'' + b_2GH = 0, \tag{37}$$

and

$$b_3\kappa_1H' + b_4\lambda_1(G^2)' = 0. \tag{38}$$

By integrating Eq. (38) once and taking constant of integration equal to zero, we obtain:

$$H(\tau) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}G(\tau)^2, \tag{39}$$

and from Eq. (36), we obtain

$$\theta_1 = -b_1(\kappa_1\lambda_2 + \kappa_2\lambda_1). \tag{40}$$

By substituting Eq. (39) into Eq. (37), we obtain

$$(\theta_2 + b_1\kappa_2\lambda_2)G - b_1\kappa_1\lambda_1 G'' + \frac{\lambda_1 b_2 b_4}{\kappa_1 b_3}G^3 = 0. \tag{41}$$

By using the homogenous balance scheme into Eq. (41), we obtain $m = 1$.

5 Exact wave solutions

5.1 Solution to the Exp$_a$ function method

For $m = 1$, Eq. (11) reduces into:

$$G(\tau) = \frac{\alpha_0 + \alpha_1 d^\tau}{\beta_0 + \beta_1 d^\tau}. \tag{42}$$

By using Eq. (42) into Eq. (41) and solving the system of equations, we achieve the following solution sets:

Set 1:

$$\left\{\alpha_0 = \frac{\sqrt{b_1 b_3}\beta_0\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}, \alpha_1 = \frac{-\sqrt{b_1 b_3}\beta_1\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\right.$$
$$\left.\beta_0 = \beta_0, \beta_1 = \beta_1, \theta_2 = \frac{1}{2}(b_1\kappa_1\lambda_1(-\log^2(d)) - 2b_1\kappa_2\lambda_2)\right\}. \tag{43}$$

From Eqs. (43), (42), and (35), we obtain

$$g(x,y,t) = \frac{\sqrt{b_1}\sqrt{b_3}\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}\left(\frac{\beta_0 - \beta_1 d^\tau}{\beta_0 + \beta_1 d^\tau}\right) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{44}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}\left(\frac{\beta_0 - \beta_1 d^\tau}{\beta_0 + \beta_1 d^\tau}\right)\right)^2. \tag{45}$$

As an explanation, the dynamic properties of Eq. (45) are demonstrated in Figure 1.

Set 2:

$$\left\{\alpha_0 = \frac{-\sqrt{b_1 b_3}\beta_0\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}, \alpha_1 = \frac{\sqrt{b_1 b_3}\beta_1\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\right.$$
$$\left.\beta_0 = \beta_0, \beta_1 = \beta_1, \theta_2 = \frac{1}{2}(b_1\kappa_1\lambda_1(-\log^2(d)) - 2b_1\kappa_2\lambda_2)\right\}, \tag{46}$$

From Eqs. (35), (42), and (46), we obtain

$$g(x,y,t) = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}\left(\frac{\beta_0 - \beta_1 d^\tau}{\beta_0 + \beta_1 d^\tau}\right) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{47}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1\log(d)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}\left(\frac{\beta_0 - \beta_1 d^\tau}{\beta_0 + \beta_1 d^\tau}\right)\right)^2. \tag{48}$$

5.2 Solutions to the extended (G'/G)-expansion method

For $m = 1$, Eq. (17) becomes:

$$G(\tau) = \alpha_{-1}\left(\frac{G'(\tau)}{G(\tau)}\right)^{-1} + \alpha_0 + \alpha_1\left(\frac{G'(\tau)}{G(\tau)}\right), \tag{49}$$

where α_{-1}, α_0, and α_1 are undetermined parameters.

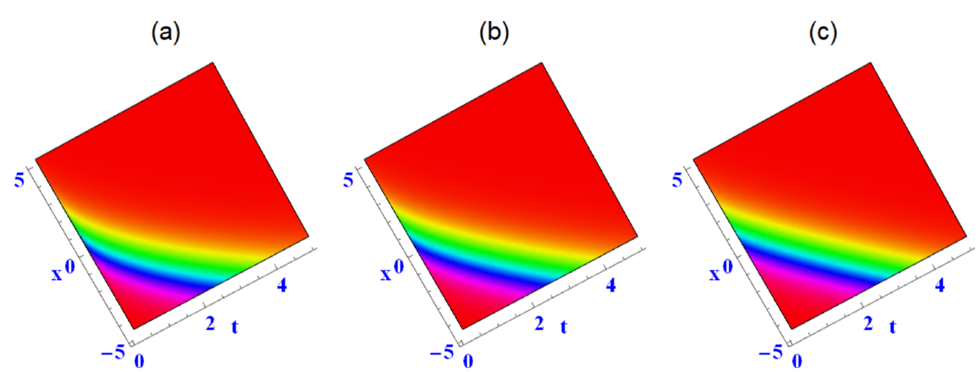

Figure 1: Solution (45) with , $\theta_1 = \beta_0 = 2$, $d = e$, $y = 0$, (a) $\alpha = 0.7$, (b) $\alpha = 0.8$, and (c) $\alpha = 0.9$.

Substituting Eq. (49) with Eq. (18) into Eq. (41) and solving the system for α_{-1}, α_0, α_1, and other parameters, we obtain the following solution sets:

Set 1:

$$\left\{\alpha_{-1} = -\frac{\sqrt{2}a\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}d}, \alpha_0 = -\frac{b\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}, \alpha_1 = 0, \theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c)+b^2)+2d^2\kappa_2\lambda_2)}{2d^2}\right\}. \tag{50}$$

Case 1:

From Eqs. (19), (35), (49), and (50), we obtain

$$g(x,y,t) = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b + 2a\left(\frac{b}{2(d-c)} + \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)^{-1}\right) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{51}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b + 2a\left(\frac{b}{2(d-c)} + \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)^{-1}\right)\right)^2, \tag{52}$$

where θ_2 is given in Eq. (50). As an explanation, the dynamic properties of Eq. (52) are demonstrated in Figure 2.

Case 2:

From Eqs. (20), (35), (49), and (50), and we obtain

$$g(x,y,t) = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b + 2a\left(\frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)^{-1}\right) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{53}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b + 2a\left(\frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)^{-1}\right)\right)^2, \tag{54}$$

where θ_2 is given in Eq. (50). As an explanation, the dynamic properties of Eq. (56) are demonstrated in Figure 3.

Case 3:

From Eqs. (22), (35), (49), and (50), we obtain

$$g(x,y,t) = \frac{-a\sqrt{2b_1b_3}\kappa_1}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1} \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{55}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(-\frac{\sqrt{2}a\sqrt{b_1b_3}\kappa_1}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1}\right)^2, \tag{56}$$

where $\theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

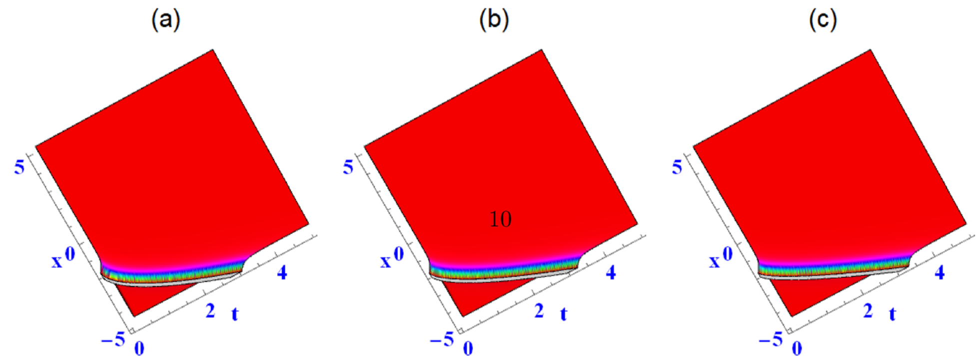

Figure 2: Solution (52) with $b_1 = b_2 = b_3 = b_4 = \theta_1 = \kappa_1 = \lambda_1 = \gamma = a = b = c = 1$, $y = 0$, $C_1 = d = 2$, (a) $\alpha = 0.6$, (b) $\alpha = 0.7$, and (c) $\alpha = 0.8$.

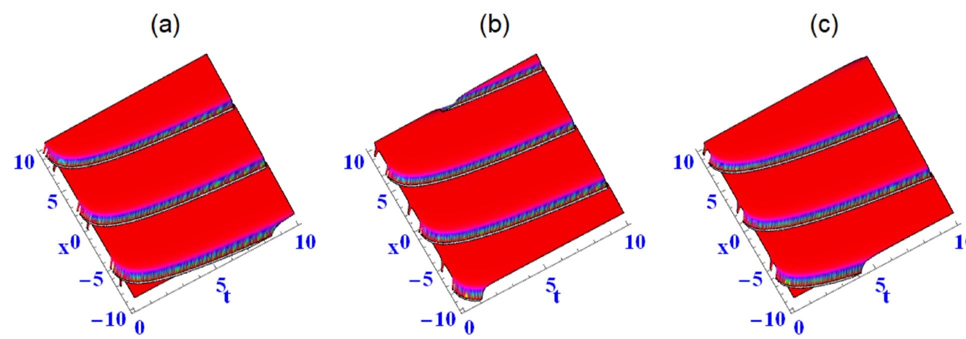

Figure 3: Solution (56) with $b_1 = b_2 = b_3 = b_4 = \theta_1 = \kappa_1 = \lambda_1 = \gamma = a = b = C_2 = 1$, $y = 0$, $c = 3$, $C_1 = d = 2$, (a) $\alpha = 0.1$, (b) $\alpha = 0.2$, and (c) $\alpha = 0.3$.

Case 4:

From Eqs. (23), (35), (49), and (50), we obtain

$$g(x,y,t) = \frac{-a\sqrt{2b_1b_3}\kappa_1}{\sqrt{b_2b_4}\,d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) - C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) + C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1} \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{57}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(-\frac{\sqrt{2}a\sqrt{b_1b_3}\kappa_1}{\sqrt{b_2b_4}\,d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) - C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right) + C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1}\right)^2, \tag{58}$$

where $\theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c)) + 2d^2\kappa_2\lambda_2)}{2d^2}$.

Set 2:

$$\left\{\alpha_{-1} = 0,\ \alpha_0 = \frac{-b\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}\,d},\ \alpha_1 = \frac{-\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}\,d},\ \theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c)+b^2) + 2d^2\kappa_2\lambda_2)}{2d^2}\right\}, \tag{59}$$

Case 1:

From Eqs. (19), (35), (49), and (59), and we obtain

$$g(x,y,t)=-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)\right)\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(y+1)}{\alpha}t^\alpha\right)\right),\tag{60}$$

$$h(x,y,t)=-\frac{\lambda_1b_4}{\kappa_1b_3}\left(-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)\right)\right)^2,\tag{61}$$

where θ_2 is given in Eq. (59).

Case 2:

From Eqs. (20), (35), (49), and (59), we obtain

$$g(x,y,t)=-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)\right)\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(y+1)}{\alpha}t^\alpha\right)\right),\tag{62}$$

$$h(x,y,t)=-\frac{\lambda_1b_4}{\kappa_1b_3}\left(-\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)\right)\right)^2,\tag{63}$$

where θ_2 is given in Eq. (59).

Case 3:

From Eqs. (22), (35), (49), and (59), we obtain

$$g(x,y,t)=\frac{-\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(y+1)}{\alpha}t^\alpha\right)\right),\tag{64}$$

$$h(x,y,t)=-\frac{\lambda_1b_4}{\kappa_1b_3}\left(\frac{-\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)\right)^2,\tag{65}$$

where $\theta_2=-\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

Case 4:

From Eqs. (23), (35), (49), and (59), we obtain

$$g(x,y,t)=\frac{-\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(y+1)}{\alpha}t^\alpha\right)\right),\tag{66}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{-\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)\right)^2, \tag{67}$$

where $\theta_2=-\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

Set 3:

$$\left\{\alpha_{-1}=\frac{\sqrt{2}a\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}d},\ \alpha_0=\frac{b\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d},\ \alpha_1=0,\ \theta_2=-\frac{b_1(\kappa_1\lambda_1(4a(d-c)+b^2)+2d^2\kappa_2\lambda_2)}{2d^2}\right\}, \tag{68}$$

Case 1:

From Eqs. (19), (35), (49), and (68), we obtain

$$g(x,y,t)=\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2a\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)^{-1}\right)$$
$$\times\exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{69}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2a\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)^{-1}\right)\right)^2, \tag{70}$$

where θ_2 is given in Eq. (68).

Case 2:

From Eqs. (20), (35), (49), and (68), we obtain

$$g(x,y,t)=\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2a\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)^{-1}\right)$$
$$\times\exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{71}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2a\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)^{-1}\right)\right)^2, \tag{72}$$

where θ_2 is given in Eq. (68).

Case 3:

From Eqs. (22), (35), (49), and (68), we obtain

$$g(x,y,t)=\frac{a\sqrt{2b_1b_3}\kappa_1}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1}$$
$$\times\exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{73}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{2}a\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1}\right)^2, \tag{74}$$

where $\theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

Case 4:
From Eqs. (23), (35), (49), and (68), we obtain

$$g(x,y,t) = \frac{a\sqrt{2b_1b_3}\kappa_1}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1} \times \exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{75}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{2}a\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1}\right)^2, \tag{76}$$

where $\theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

Set 4:

$$\left\{\alpha_{-1}=0, \alpha_0 = \frac{b\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}, \alpha_1 = \frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}, \theta_2 = -\frac{b_1(\kappa_1\lambda_1(4a(d-c)+b^2)+2d^2\kappa_2\lambda_2)}{2d^2}\right\}. \tag{77}$$

Case 1: From Eqs. (19), (35), (49), and (77), we obtain

$$g(x,y,t) = \frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)\right) \times \exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{78}$$

$$h(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)+C_2\sinh\left(\frac{\tau\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right)\right)\right)\right)^2, \tag{79}$$

where θ_2 is given in Eq. (77).

Case 2:
From Eqs. (20), (35), (49), and (77), we obtain

$$g(x,y,t) = \frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)\right) \times \exp\left(\iota\left(\kappa_2 x+\lambda_2 y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \tag{80}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}d}\left(b+2(c-d)\left(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\tau\sqrt{4ac-4ad-b^2}}{2d}\right)}\right)\right)\right)\right)^2, \tag{81}$$

where θ_2 is given in Eq. (77).

Case 3:

From Eqs. (22), (35), (49), and (77), we obtain

$$\begin{aligned} g(x,y,t) &= \frac{\sqrt{2b_1b_3}\kappa_1(c-d)}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)\\ &\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \end{aligned} \tag{82}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\tau\sqrt{ad-ac}}{d}\right)}\right)\right)\right)^2, \tag{83}$$

where $\theta_2=-\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$,

Case 4:

From Eqs. (23), (35), (49), and (77), we obtain

$$\begin{aligned} g(x,y,t) &= \frac{\sqrt{2b_1b_3}\kappa_1(c-d)}{\sqrt{b_2b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)\\ &\times\exp\left(\iota\left(\kappa_2x+\lambda_2y+\theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^\alpha\right)\right), \end{aligned} \tag{84}$$

$$h(x,y,t)=-\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{\sqrt{2b_1b_3}\kappa_1(c-d)}{\sqrt{b_2}\sqrt{b_4}d}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\tau\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\tau\sqrt{ac-ad}}{d}\right)}\right)\right)\right)^2, \tag{85}$$

where $\theta_2=-\frac{b_1(\kappa_1\lambda_1(4a(d-c))+2d^2\kappa_2\lambda_2)}{2d^2}$.

5.3 Exact soliton solutions to the extended ShGEEM

For $m = 1$, Eqs. (27), (32), and (33) become:

$$G(\tau)=\alpha_0\pm\iota\beta_1\operatorname{sech}(\tau)\pm\alpha_1\tanh(\tau), \tag{86}$$

$$G(\tau)=\alpha_0\pm\beta_1\operatorname{csch}(\tau)\pm\alpha_1\coth(\tau), \tag{87}$$

$$G(\tau)=\alpha_0+\beta_1\sinh(p)+\alpha_1\cosh(p), \tag{88}$$

where α_0, α_1, and β_1 are the unknowns. Inserting Eq. (88) into Eq. (41), we achieve the algebraic equations containing α_0, α_1, β_1, and other parameters. Now with the help of software, we obtain the following sets:

Set 1:

$$\left\{\begin{array}{l}\alpha_0 = 0,\ \alpha_1 = -\dfrac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = 0,\\ \theta_2 = -b_1(2\kappa_1\lambda_1 + \kappa_2\lambda_2)\end{array}\right\}. \tag{89}$$

From Eqs. (35), (86), (88), and (89), we obtain

$$\begin{aligned} g_1(x,y,t) &= \mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\tanh(\tau)\\ &\quad\times\exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\end{aligned} \tag{90}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\tanh(\tau)\right)^2. \tag{91}$$

As an explanation, the dynamic properties of Eq. (91) are demonstrated in Figure 4.

From Eqs. (35), (87), (88), and (89), we obtain

$$\begin{aligned} g_2(x,y,t) &= \mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\coth(\tau)\\ &\quad\times\exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\end{aligned} \tag{92}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\coth(\tau)\right)^2. \tag{93}$$

Set 2:

$$\left\{\alpha_0 = 0,\ \alpha_1 = \frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = 0,\ \theta_2 = -b_1(2\kappa_1\lambda_1 + \kappa_2\lambda_2)\right\}. \tag{94}$$

From Eqs. (35), (86), (88), and (94), we obtain

$$\begin{aligned} g_1(x,y,t) &= \pm\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\tanh(\tau)\\ &\quad\times\exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\end{aligned} \tag{95}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\pm\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\tanh(\tau)\right)^2. \tag{96}$$

From Eqs. (35), (87), (88), and (94), we obtain

$$\begin{aligned} g_2(x,y,t) &= \pm\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\coth(\tau)\\ &\quad\times\exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\end{aligned} \tag{97}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\pm\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\coth(\tau)\right)^2. \tag{98}$$

Set 3:

$$\left\{\alpha_0 = 0,\ \alpha_1 = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \theta_2 = -\frac{1}{2}b_1(\kappa_1\lambda_1 + 2\kappa_2\lambda_2)\right\}. \tag{99}$$

From Eqs. (35), (86), (88), and (99) we obtain

$$\begin{aligned} g_1(x,y,t) &= \mp\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\iota\,\mathrm{sech}(\tau) + \tanh(\tau))\\ &\quad\times\exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right),\end{aligned} \tag{100}$$

$$\begin{aligned} &h_1(x,y,t)\\ &= -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\iota\,\mathrm{sech}(\tau) + \tanh(\tau))\right)^2.\end{aligned} \tag{101}$$

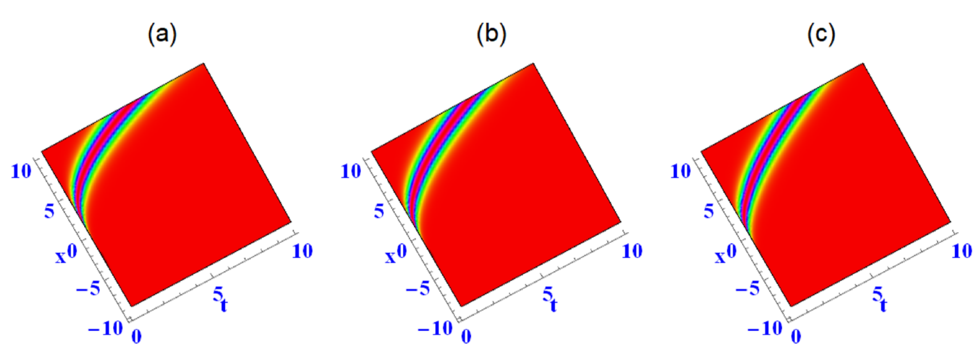

Figure 4: Solution (91) with $b_1 = b_2 = b_3 = b_4 = \theta_1 = \kappa_1 = \lambda_1 = \kappa_2 = \lambda_2 = \gamma = 1$, $y = 0$, (a) $\alpha = 0.4$, (b) $\alpha = 0.5$, and (c) $\alpha = 0.6$.

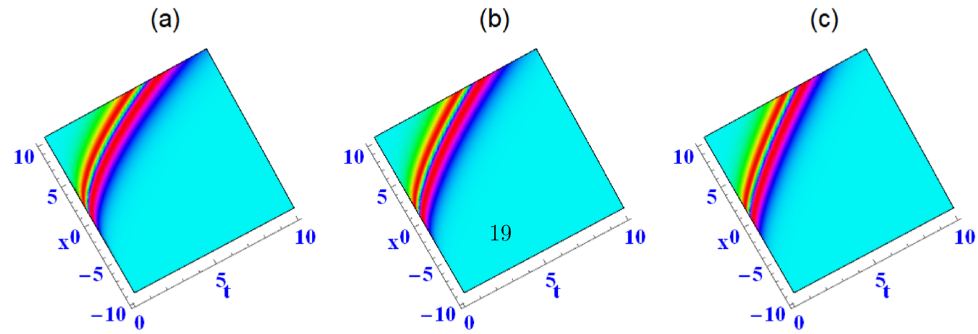

Figure 5: Solution (101) with $b_1 = b_2 = b_3 = b_4 = \theta_1 = \kappa_1 = \lambda_1 = \kappa_2 = \lambda_2 = \gamma = 1$, $y = 0$, $\iota = 1$, (a) $\alpha = 0.6$, (b) $\alpha = 0.7$, (c) $\alpha = 0.8$.

As an explanation, the dynamic properties of Eq. (101) are demonstrated in Figure 5. From Eqs. (35), (87), (88), and (99), we obtain

$$g_2(x,y,t) = \mp\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\coth(\tau) + \operatorname{csch}(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{102}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\coth(\tau) + \operatorname{csch}(\tau))\right)^2. \tag{103}$$

Set 4:

$$\left\{\alpha_0 = 0,\ \alpha_1 = \frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \theta_2 = -\frac{1}{2}b_1(\kappa_1\lambda_1 + 2\kappa_2\lambda_2)\right\}. \tag{104}$$

From Eqs. (104), (88), (35), and (86), we obtain

$$g_1(x,y,t) = \frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\mp\iota\operatorname{sech}(\tau) \pm \tanh(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{105}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\mp\iota\operatorname{sech}(\tau) \pm \tanh(\tau))\right)^2. \tag{106}$$

From Eqs. (104), (88), (35), and (87), we obtain

$$g_2(x,y,t) = \frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\mp\operatorname{csch}(\tau) \pm \coth(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{107}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\mp\operatorname{csch}(\tau) \pm \coth(\tau))\right)^2. \tag{108}$$

Set 5:

$$\left\{\alpha_0 = 0,\ \alpha_1 = -\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = \frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \theta_2 = -\frac{1}{2}b_1(\kappa_1\lambda_1 + 2\kappa_2\lambda_2)\right\}. \tag{109}$$

From Eqs. (109), (88), (35), and (86), we obtain

$$g_1(x,y,t) = \frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\pm\iota\operatorname{sech}(\tau) \mp \tanh(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{110}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\pm\iota\operatorname{sech}(\tau) \mp \tanh(\tau))\right)^2. \tag{111}$$

From Eqs. (109), (88), (35) and (87), we obtain

$$g_2(x,y,t) = \frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\pm\operatorname{csch}(\tau) \mp \coth(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{112}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\frac{(\sqrt{b_1}\sqrt{b_3}\kappa_1)}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\pm\operatorname{csch}(\tau) \mp \coth(\tau))\right)^2, \tag{113}$$

Set 6:

$$\left\{\begin{array}{l}\alpha_0 = 0,\ \alpha_1 = \dfrac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\ \beta_1 = \dfrac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}},\\ \theta_2 = -\dfrac{1}{2}b_1(\kappa_1\lambda_1 + 2\kappa_2\lambda_2)\end{array}\right\}. \tag{114}$$

From Eqs. (114), (88), (35), and (86), we obtain

$$g_1(x,y,t) = \pm\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\iota\,\mathrm{sech}(\tau) + \tanh(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{115}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\pm\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\iota\,\mathrm{sech}(\tau) + \tanh(\tau))\right)^2. \tag{116}$$

From Eqs. (114), (88), (35), and (87), we obtain

$$g_2(x,y,t) = \pm\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\coth(\tau) + \mathrm{csch}(\tau)) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{117}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\pm\frac{\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{2}\sqrt{b_2}\sqrt{b_4}}(\coth(\tau) + \mathrm{csch}(\tau))\right)^2. \tag{118}$$

Set 7:

$$\left\{\begin{array}{l}\alpha_0 = 0,\ \alpha_1 = 0,\ \beta_1 = -\dfrac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}, \\ \theta_2 = b_1(\kappa_1\lambda_1 - \kappa_2\lambda_2)\end{array}\right\}. \tag{119}$$

From Eqs. (119), (88), (35), and (86), we obtain

$$g_1(x,y,t) = \mp\frac{\iota\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\mathrm{sech}(\tau) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{120}$$

$$h_1(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\iota\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\mathrm{sech}(\tau)\right)^2. \tag{121}$$

From Eqs. (119), (88), (35), and (87), we obtain

$$g_2(x,y,t) = \mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\mathrm{csch}(\tau) \times \exp\left(\iota\left(\kappa_2 x + \lambda_2 y + \theta_2\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}\right)\right), \tag{122}$$

$$h_2(x,y,t) = -\frac{\lambda_1 b_4}{\kappa_1 b_3}\left(\mp\frac{\sqrt{2}\sqrt{b_1}\sqrt{b_3}\kappa_1}{\sqrt{b_2}\sqrt{b_4}}\mathrm{csch}(\tau)\right)^2, \tag{123}$$

where $\tau = \kappa_1 x + \lambda_1 y - b_1(\kappa_1\lambda_2 + \kappa_2\lambda_1)\frac{\Gamma(\gamma+1)}{\alpha}t^{\alpha}$.

6 Conclusion

In this paper, we have gained the new optical solitons of the integrable generalized (2+1)-dimensional NLS system along with novel definition of derivative by applying the modified integration methods, Exp_a function, extended (G'/G)-expansion, and extended ShGEEM. The achieved solitons are including dark, bright and combo optical solitons. These solutions are also verified by using the MATHEMATICA software [69,70]. The obtained solutions are newer than the existing solutions of this model in the literature. The important thing of this paper is that truncated M-fractional derivative is first time used for this model. These optical solitons suggest that these methods are effective, straight forward, and reliable compared to other methods.

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