Research Article

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Abundant exact solutions of higher-order dispersion variable coefficient KdV equation

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Abstract: In this article, various exact solutions of the fifth-order variable coefficient KdV equation with higher-order dispersion term are studied. Because of the complexity of the exact solution of the variable coefficient t, it has a certain influence on the tension waves at the fluid interface on the gravity surface. First, the bilinear KdV equation is derived by using the Hirota bilinear method, and four mixed solutions consisting of positive quartic function, quadratic function, exponential function, and hyperbolic function are constructed. Second, the linear superposition principle is used to obtain the resonance multisoliton solution, and two cases are taken as examples to illustrate the study of resonance multi soliton solution. In addition, 3D images and contour images are drawn by mathematical symbol calculation and appropriate parameters, and the process of tension fluctuation is vividly explained by physical phenomena. The results obtained greatly expand the exact solution of the KdV equation in the existing literature and enable us to understand nonlinear dynamical systems more deeply.

Keywords: tension waves, fifth order variable coefficient KdV equation, Hirota bilinear method, linear superposition principle, interaction solution, resonant multi soliton solution

1 Introduction

Onlinear partial differential equations [1–5] have been widely used in natural and social sciences, especially in the fields of plasma physics, ocean dynamics, lattice dynamics, and hydrodynamics [6]. The most famous non linear variable coefficient partial differential equation is the variable coefficient KdV equation. Until now, many scholars have done a lot of research on the complex non linear waves of nonlinear constant coefficient and variable coefficient KdV equations, and put forward various effective methods to solve their exact solutions and numerical solutions, for example, Bäcklund transform method [7,8], Hirota bilinear method [9–11], (G'/G) expansion method [12], multisoliton solution [13], Wronskian method [14], *Painlevé* analysis [15], and other methods.

At the same time, it is found that the physical properties described by various types of KdV equations are different. For example, in ref. [16], the propagation characteristics of nonlinear solitary waves in the ocean were described, and the effects of dissipation term and perturbation term contained in KdV on the elastic collision of nonlinear solitary waves in two oceans were given. In ref. [17], unstable drift waves in plasma physics were described. In ref. [18], the propagation of solitons in inhomogeneous propagation media in the fields of quantum mechanics and nonlinear mechanics was described. In ref. [19], nonlinear solitary wave fluctuations in the atmosphere and ocean with large amplitude were described. Reference [20] showed that by simulating the propagation depth and width of small amplitude surface waves in large channels and straits, the transformation was slow, and the vorticity did not disappear. After more in-depth study of nonlinear partial differential equations with variable coefficients, scholars found that the coefficient of the dissipation term of the variable coefficient KdV equation changes with time and space, and the variable coefficient KdV equation can better describe the physical phenomena and physical properties behind it. Therefore, the research on the variable coefficient KdV equation has increased in recent years, and it is found that the aforementioned methods for studying the constant coefficient KdV equation can also be used to study the variable coefficient KdV equation, and some progress has been made. For example, in ref. [21], the author used pfaffians form to represent the multi soliton solution of the variable

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coefficient coupled KdV equation and further analyzed the dynamic characteristics of the soliton solution. In ref. [22], the author used the improved sine cosine method to construct the exact periodic solutions and soliton solutions of two groups of the fifth-order KdV equations with variable coefficients and linear damping term. In ref. [23], the author first obtained the N-soliton solution of the (2 + 1)-dimensional KdV equation with variable coefficients by using the bell polynomial method and analyzed the influence of soliton fission, fission and rear end collision on the coefficients, obtained the Bäcklund transformation of the equation, and then obtained the periodic wave solution of the equation by using the Riemann function method.

In this article, the generalized variable coefficient KdV equation with higher-order dissipative term is studied:

$$\begin{aligned} &u_t + a(t)uu_x + b(t)u_{xxx} + c(t)u^2u_x + d(t)u_xu_{2x} \\ &+ e(t)uu_{xxx} + f(t)u_{xxxxx} + g(t)u_x\partial_x^{-1}u_y \\ &+ h(t)u_{xxy} + k(t)uu_y = 0, \end{aligned} \tag{1}$$

or

$$\begin{aligned} &u_t + a(t)uu_x + b(t)u_{xxx} + c(t)u^2u_x + d(t)u_xu_{2x} \\ &+ e(t)uu_{xxx} + f(t)u_{xxxxx} + g(t)u_xv + h(t)v_{xxx} \\ &+ k(t)uv_x = 0, \\ &u_y = v_x, \end{aligned}$$

where $u = u(x, y, t)$ in Eq. (1) is the function space of x, y, t. $a(t)$, $c(t)$, $d(t)$, $e(t)$, $g(t)$, and $k(t)$ are nonlinear term of time t, $b(t)$, and $h(t)$ represent linear dispersion of time t, and $f(t)$ represents higher-order dispersion on time t. Eq. (1) shows tension waves on a gravitational surface at a fluid interface, where surface tension, gravity, and fluid inertia are affected.

Eq. (1) is the interaction between solitons, and the interaction between solitons can be divided into elastic and inelastic collisions. The variable coefficients $f(t)$ and $l(t)$ of the union Eq. (1) can lead to fusion and fission of solitons.

This article is organized as follows. In Section 2, the bilinear form of the KdV equation with a variable coefficients is obtained by using the Hirota bilinear method. In Section 3, by constructing the combined solutions of positive quartic function, quadratic function, exponential function, and hyperbolic function, the three-dimensional space corresponding to the interaction solutions of high-order Lump solution and periodic cross kink solution is obtained, and the forms of accurate solutions are more abundant. The shape and trajectory of waves are vividly displayed through Maple. In Section 4, the resonant multi soliton solution of the KdV equation with variable coefficients is obtained by linear superposition principle, and the physical structure of tension waves is illustrated by $N = 2$, $N = 3$, and $N = 6$ as examples. The symbolic calculation is carried out by using mathematical software, and the high-order lumps solution, the interaction solution with periodic cross kink solution, and resonant multiple solitons are obtained, which supplements the research on the exact solution of the generalized variable coefficient fifth-order variable coefficient KdV equation. The fifth part is the summary.

2 Preparatory theory

1) When $c(t) = \delta$, $d(t) = \gamma$, $e(t) = \beta$, $f(t) = \alpha$, and $a(t) = g(t) = h(t) = k(t) = 0$, Eq. (1) transforms into the generalized fifth-order KdV equation [24] with constant coefficients,

$$u_t + \alpha u_{xxxxx} + \beta uu_{xxx} + \gamma u_xu_{xx} + \delta u^2u_x = 0. \tag{2}$$

By using the Ansatz and Jacobi elliptic function expansion method, the elliptic cosine wave solution of the equation is constructed.

2) When $a(t) = b(t) = g(t) = h(t) = k(t) = 0$, Eq. (1) transforms into variable coefficient Sawada-Kotera equation [25],

$$u_t + f(t)u^2u_x + g(t)u_xu_{x^2} + h(t)uu_{x^3} + k(t)u_{x^5} = 0. \tag{3}$$

The auto-Bäcklund transform, soliton solution, and random soliton solution are obtained by using Hermite transform in Kondrativ distribution space.

3) When $k(t) = 4h(t) = 2g(t)$, $a(t) = 6b(t)$, and $c(t) = d(t) = e(t) = f(t) = 0$, Eq. (1) is transformed into a generalized (2 + 1)-dimensional equation [26],

$$u_t - h_1(4uu_y + 2u_x\partial_x^{-1}u_y + u_{xxy}) - h_2(6uu_x + u_{xxx}) = 0. \tag{4}$$

The bilinear form, bilinear Bäcklund transformation, Lax pair, and Darboux transformation are constructed by using binary Bell polynomials. The equation is simplified into integrable equation, and the infinite conservation law of the equation is obtained by using binary Bell polynomials.

In this section, the Hirota bilinear method is used to obtain the bilinear form of Eq. (1), and then the form of solution is constructed to obtain the interaction solution of the equation.

First, the transformation of related variables is used:

$$u = 2(\ln g)_{xx}. \tag{5}$$

Convert Eq. (1) into the bilinear equation as follows:

$$\begin{aligned} B_{\mathrm{KdV}}(g) &= (D_xD_t + b(t)D_x^4 + f(t)D_x^6 + h(t)D_x^3D_y)g\cdot g \\ &= -g_tg_x + gg_{xt} + 3b(t)g_{xx}^2 - 4b(t)g_xg_{xxx} \\ &\quad - 10f(t)g_{xxx}^2 + b(t)gg_{xxxx} + 15f(t)g_{xx}g_{xxxx} \\ &\quad - 6f(t)g_xg_{xxxxx} + f(t)gg_{xxxxxx} + h(t)gg_{xxxy} \\ &\quad - 3h(t)g_xg_{xxy} + 3h(t)g_{xx}g_{xy} - h(t)g_{xxx}g_y \\ &= 0, \end{aligned} \tag{6}$$

where $g = g(x, y, t)$, and D-operator is defined by ref. [27]:

$$D_x^mD_y^nD_t^kfg = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^n\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^k f(x, y, t)g(x', y', t')|_{x'=x,y'=y,t'=t}. \tag{7}$$

2.1 Interaction solution

To obtain the interaction solution between the high-order Lump solution and the periodic cross kinking solution of the fifth-order variable coefficient KdV equation, the following positive quartic function, quadratic function, and the test function of the combination of exponential function and hyperbolic function are assumed,

$$\begin{aligned} g &= e^{-p_1\xi_1} + k_1e^{p_1\xi_2} + k_2\cos(p_2\xi_2) + k_3\cosh(\xi_3) \\ &\quad + k_4\sinh(\xi_4) + k_5\sin(\xi_5) + a_{21}, \end{aligned} \tag{8}$$

where

$$\begin{aligned} \xi_1 &= a_1x + a_2y + a_3(t) + a_4, \\ \xi_2 &= a_5x + a_6y + a_7(t) + a_8, \\ \xi_3 &= a_9x + a_{10}y + a_{11}(t) + a_{12}, \\ \xi_4 &= a_{13}x + a_{14}y + a_{15}(t) + a_{16}, \\ \xi_5 &= a_{17}x + a_{18}y + a_{19}(t) + a_{20}. \end{aligned}$$

where a_i $(2 \le i \le 21)$, k_j $(1 \le j \le 5)$ are constant. A set of algebraic equations about $a_i(1 \le i \le 21,$ and $i \ne 3, 7, 9, 11, 15, 19)$ are obtained by substituting Eq. (8) into Eq. (1) and making the coefficients of x and y zero. By solving the algebraic equations, we can find the following groups of rich interaction solutions.

Case 1:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_9 = 0, a_{13} = 0, \\ a_{15}(t) = \int_a^t a_{17}^2a_{14}h(s)\mathrm{d}s, p_1 = 0, f(t) = f(t), \\ a_{19}(t) = \int_a^t 4a_{17}^2f(s)\mathrm{d}s, h(t) = h(t), k_1 = 0, \\ a_{11}(t) = \int_a^t a_{17}^2a_{10}h(s)\mathrm{d}s, k_2 = 0. \end{cases} \tag{9}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, k_3, k_4, k_5, and p_2 are all constants. a_{17} is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = \frac{5a_{17}^3f(t) - a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_1(x, y, t) = \frac{\Phi_1}{\Psi_1}, \tag{10}$$

where

$$\begin{aligned} \Phi_1 &= -2a_{17}^2k_5(\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &\quad \times (k_4\sinh(a_{15}(t) + a_{14}y + a_{16}) + k_3\cosh(a_{11}(t) + a_{10}y \\ &\quad + a_{12}) + a_{21} + 1) + k_5), \\ \Psi_1 &= 2k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20})(k_4\sinh(a_{15}(t) \\ &\quad + a_{14}y + a_{16}) + k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} + 1) \\ &\quad + 2(a_{21} + 1)k_4\sinh(a_{15}(t) + a_{14}y + a_{16}) \\ &\quad + k_3^2\cosh(a_{11}(t) + a_{10}y + a_{12})^2 \\ &\quad + k_4^2\cosh(a_{15}(t) + a_{14}y + a_{16})^2 + (a_{21} + 1)^2 \\ &\quad + 2k_3\cosh(a_{11}(t) + a_{10}y + a_{12})(a_{21} + 1 \\ &\quad + k_4\sinh(a_{15}(t) + a_{14}y + a_{16})) \\ &\quad - k_5^2\cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 - k_4^2 + k_5^2. \end{aligned}$$

Case 2:

$$\begin{cases} a_1 = 0, a_3(t) = \int_a^t a_{17}^2a_2h(s)\mathrm{d}s, a_5 = 0, \\ a_7(t) = a_7(t), a_{11}(t) = \int_a^t a_{17}^2a_{10}h(s)\mathrm{d}s, \\ a_{13} = 0, a_{15}(t) = \int_a^t a_{17}^2a_{14}h(s)\mathrm{d}s, k_2 = 0, \\ a_{19}(t) = \int_a^t 4a_{17}^5f(s)\mathrm{d}s, a_9 = 0, f(t) = f(t), \\ k_1 = 0, h(t) = h(t). \end{cases} \tag{11}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, k_i (3 \le i \le 5)$, p_1, and p_2 are all constants. a_{17} is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = \frac{5a_{17}^3 f(t) - a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_2(x, y, t) = \frac{\Phi_2}{\Psi_2}, \tag{12}$$

where

$$\begin{aligned}\Phi_2 = &-2a_{17}^2 k_5 (\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20})\big(k_4 \sinh(a_{15}(t) \\ &+ a_{14}y + a_{16}) + \sin(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20})e^{-p_1(a_3(t)+a_2y+a_4)} + k_5 + k_3 \cosh(a_{11}(t) + a_{10}y \\ &+ a_{12}) + a_{21}\big)),\\ \Psi_2 = &\ e^{-p_1(a_3(t)+a_2y+a_4)}(2k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ 2k_4 \sinh(a_{15}(t) + a_{14}y + a_{16}) \\ &+ 2k_3 \cosh(a_{11}(t) + a_{10}y + a_{12}) + 2a_{21}) \\ &- k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20})(a_{21}^2 - k_4^2 \\ &+ 2k_4k_5 \sinh(a_{15}(t) + a_{14}y + a_{16}) + k_5^2 + 2a_{21}k_5 \\ &+ 2k_3k_5 \cosh(a_{11}(t) + a_{10}y + a_{12})) \\ &+ 2a_{21}k_4 \sinh(a_{15}(t) + a_{14}y + a_{16}) \\ &+ k_3^2 \cosh(a_{11}(t) + a_{10}y + a_{12})^2 \\ &+ k_4^2 \cosh(a_{15}(t) + a_{14}y + a_{16})^2(2k_4k_3 \sinh(a_{15}(t) \\ &+ a_{14}y + a_{16}) + 2a_{21}k_3) + \cosh(a_{11}(t) + a_{10}y + a_{12}) \\ &+ e^{-2p_1(a_3(t)+a_2y+a_4)}.\end{aligned}$$

Case 3:

$$\begin{cases} a_1 = 0,\ a_3(t) = a_3(t),\ a_5 = 0,\ a_7(t) = a_7(t), \\ a_9 = 0,\ k_2 = 0,\ a_{11}(t) = -\int_a^t a_{13}^2 a_{10} h(s)\mathrm{d}s, \\ a_{15}(t) = 4\int_a^t a_{13}^5 f(s)\mathrm{d}s,\ a_{17} = 0,\ f(t) = f(t), \\ a_{19}(t) = -\int_a^t a_{13}^2 a_{18} h(s)\mathrm{d}s,\ h(t) = h(t), \\ p_1 = 0,\ k_1 = 0. \end{cases} \tag{13}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{18}, k_i (3 \le i \le 5)$, and p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1), and the higher-order dispersion term is $b(t) = -\frac{5a_{13}^3 f(t) + a_{14}h(t)}{a_{13}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_3(x, y, t) = \frac{\Phi_3}{\Psi_3}, \tag{14}$$

where

$$\begin{aligned}\Phi_3 = &\ 2a_{13}^2 k_4 (k_4 \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &- k_4 \cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &+ \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})(k_5 \sin(a_{19}(t) \\ &+ a_{18}y + a_{20}) + k_3 \cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} \\ &+ 1)),\\ \Psi_3 = &\ (k_4 \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) \\ &+ k_5 \sin(a_{19}(t) + a_{18}y + a_{20}) \\ &+ k_3 \cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} + 1)^2.\end{aligned}$$

Case 4:

$$\begin{cases} a_1 = 0,\ a_3(t) = -\int_a^t a_{13}^2 a_2 h(s)\mathrm{d}s,\ a_5 = 0, \\ a_9 = 0,\ a_{11}(t) = -\int_a^t a_{13}^2 a_{10} h(s)\mathrm{d}s,\ h(t) = h(t), \\ a_{15}(t) = 4\int_a^t a_{13}^5 f(s)\mathrm{d}s,\ a_{17} = 0,\ f(t) = f(t), \\ a_{19}(t) = -\int_a^t a_{13}^2 a_{18} h(s)\mathrm{d}s,\ k_1 = 0,\ k_2 = 0. \end{cases} \tag{15}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{18}, k_i (3 \le i \le 5)$, p_1, and p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = -\frac{5a_{13}^3 f(t) + a_{14}h(t)}{a_{13}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_4(x, y, t) = \frac{\Phi_4}{\Psi_4}, \tag{16}$$

where

$$\begin{aligned}\Phi_4 = &\ 2a_{13}^2 k_4 \big(k_4 \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &- k_4 \cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &+ \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})(k_5 \sin(a_{19}(t) \\ &+ a_{18}y + a_{20}) + k_3 \cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21}) \\ &+ e^{-p_1(a_3(t)+a_2y+a_4)} \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})\big),\\ \Psi_4 = &\ (k_4 \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) + k_5 \sin(a_{19}(t) \\ &+ a_{18}y + a_{20}) + k_3 \cosh(a_{11}(t) + a_{10}y + a_{12}) \\ &+ a_{21} + e^{-p_1(a_3(t)+a_2y+a_4)})^2.\end{aligned}$$

Case 5:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), \\ a_9 = 0, a_{10} = 0, a_{11}(t) = 0, a_{14} = \dfrac{a_{13}a_{18}}{a_{17}}, \\ a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, k_2 = 0, p_1 = 0, \\ f(t) = 0, h(t) = h(t). \end{cases} \tag{17}$$

$a_2, a_4, a_6, a_8, a_{12}, a_{13}, a_{16}, a_{18}, a_{20}, a_{21}, k_i (3 \le i \le 5)$, and p_2 are all constants. a_{17} is not equal to zero for the constant, the high-order dispersion term of Eq. (1) is zero, and the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_5(x, y, t) = \frac{\Phi_5}{\Psi_5}, \tag{18}$$

where

$$\begin{aligned} \Phi_5 = & -2a_{17}^2 k_5 (k_3 \cosh(a_{11}(t) + a_{12}) + a_{21} + 1) \sin(a_{19}(t) \\ & + a_{17}x + a_{18}y + a_{20}) \\ & - \left((a_{17}^2 - a_{13}^2) k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y \right. \\ & + a_{20}) 2k_4 \sinh\left(a_{15}(t) + (a_{13}x + a_{16}) + \frac{a_{13}a_{18}y}{a_{17}} \right) \\ & \left. - a_{13}^2 (k_3 \cosh(a_{11}(t) + a_{12}) + a_{21} + 1) \right) \\ & - 4a_{13}a_{17}k_4k_5 \cos(a_{19}(t) - 2a_{17}^2k_5^2 - 2a_{13}^2k_4^2 + a_{17}x \\ & + a_{18}y + a_{20}) \cosh\left(a_{15}(t) + (a_{13}x + a_{16}) + \frac{a_{13}a_{18}y}{a_{17}} \right), \\ \Psi_5 = & \, 2k_5 (k_3 \cosh(a_{11}(t) + a_{12}) + a_{21} + 1) \sin(a_{19}(t) + a_{17}x \\ & + a_{18}y + a_{20}) + 2k_4 \sinh(a_{15}(t) + (a_{13}x + a_{16}) \\ & + \frac{a_{13}a_{18}y}{a_{17}} (k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ & + k_3 \cosh(a_{11}(t) + a_{12}) + a_{21} + 1) + (a_{21} + 1)^2 \\ & + 2(a_{21} + 1) k_3 \cosh(a_{11}(t) + a_{12}) \\ & + k_3^2 \cosh(a_{11}(t) + a_{12})^2 - k_4^2 + k_5^2 \\ & + k_4^2 \cosh\left(a_{15}(t) + (a_{13}x + a_{16}) + \frac{a_{13}a_{18}y}{a_{17}} \right)^2 \\ & - k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2. \end{aligned}$$

Case 6:

$$\begin{cases} a_1 = 0, a_2 = 0, a_3(t) = 0, a_5 = 0, a_7(t) = a_7(t), \\ a_9 = 0, a_{10} = 0, a_{11}(t) = 0, a_{14} = \dfrac{a_{13}a_{18}}{a_{17}}, \\ a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, k_2 = 0, f(t) = 0, \\ h(t) = h(t). \end{cases} \tag{19}$$

$a_4, a_6, a_8, a_{12}, a_{13}, a_{16}, a_{18}, a_{20}, a_{21}, k_i (3 \le i \le 5)$, p_1, and p_2 are all constants. a_{17} is not equal to zero for the constant, the high-order dispersion term of Eq. (1) is zero, and the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_6(x, y, t) = \frac{\Phi_6}{\Psi_6}, \tag{20}$$

where

$$\begin{aligned} \Phi_6 = & -2a_{17}^2 k_5 \Big(k_3 \cosh(a_{11}(t) + a_{12}) + e^{-p_1(a_3(t) + a_4)} \\ & + a_{21} \Big) \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ & + (a_{13}^2 - a_{17}^2) k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ & + 2k_4 \sinh(a_{15}(t) + (a_{13}x + a_{16}) \\ & + \frac{a_{13}a_{18}y}{a_{17}}) \Big(e^{-p_1(a_3(t) + a_4)} + a_{13}^2 (k_3 \cosh(a_{11}(t) + a_{12}) \\ & + a_{21}) \Big) - 4a_{13}a_{17}k_4k_5 \cos(a_{19}(t) + a_{17}x + a_{18}y \\ & + a_{20}) \cosh\left(a_{15}(t) + (a_{13}x + a_{16}) + \frac{a_{13}a_{18}y}{a_{17}} \right) \\ & - 2a_{17}^2 k_5^2 - 2a_{13}^2 k_4^2, \\ \Psi_6 = & \, 2k_5 \Big(k_3 \cosh(a_{11}(t) + a_{12}) + e^{-p_1(a_3(t) + a_4)} \\ & + a_{21} \Big) \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ & + 2k_4 \sinh\left(a_{15}(t) + (a_{13}x + a_{16}) \right. \\ & \left. + \frac{a_{13}a_{18}y}{a_{17}} \right) \Big(k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ & + k_3 \cosh(a_{11}(t) + a_{12}) + e^{-p_1(a_3(t) + a_4)} + a_{21} \Big) \\ & + e^{-p_1(a_3(t) + a_4)} (2k_3 \cosh(a_{11}(t) + a_{12}) + 2a_{21}) \\ & - k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ & + 2a_{21}k_3 \cosh(a_{11}(t) + a_{12}) + k_3^2 \cosh(a_{11}(t) + a_{12})^2 \\ & + e^{-2p_1(a_3(t) + a_4)} + a_{21}^2 - k_4^2 + k_5^2 \\ & + k_4^2 \cosh\left(a_{15}(t) + (a_{13}x + a_{16}) + \frac{a_{13}a_{18}y}{a_{17}} \right)^2. \end{aligned}$$

Case 7:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), \\ a_9 = 0, a_{11}(t) = 0, a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, \\ k_2 = 0, p_1 = 0, f(t) = 0, h(t) = 0, b(t) = 0. \end{cases} \tag{21}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_i (3 \le i \le 5)$, and p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq. (1) are zero. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_7(x, y, t) = \frac{\Phi_7}{\Psi_7}, \tag{22}$$

where

$$\begin{aligned}\Phi_7 = &-4a_{13}a_{17}k_4k_5\cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})\cosh(a_{15}(t) \\ &+ a_{13}x + a_{14}y + a_{16}) \\ &- 2a_{17}^2k_5(k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} \\ &+ 1)\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- (x + a_{14}y + a_{16}) - 2k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y \\ &+ a_{16})((a_{17}^2 - a_{13}^2)k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- a_{13}^2(k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} + 1)) \\ &- 2a_{17}^2k_5^2 - 2a_{13}^2k_4^2,\end{aligned}$$

$$\begin{aligned}\Psi_7 = &-k_5^2\cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ k_4^2\cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &+ 2k_5(k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21} \\ &+ 1)\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ 2k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})(k_5\sin(a_{19}(t) \\ &+ a_{17}x + a_{18}y + a_{20}) + k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) \\ &+ a_{21} + 1) + 2(a_{21} + 1)k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) \\ &+ k_3^2\cosh(a_{11}(t) + a_{10}y + a_{12})^2 + (a_{21} + 1)^2 - k_4^2 \\ &+ k_5^2.\end{aligned}$$

Case 8:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), \\ a_9 = 0, a_{11}(t) = 0, a_{15}(t) = 0, a_{19}(t) = 0, k_1 = 0, \\ k_2 = 0, f(t) = 0, h(t) = 0, b(t) = 0. \end{cases} \tag{23}$$

a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{13}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, $k_i (3 \le i \le 5)$, p_1, and p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq. (1) are zero. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_8(x, y, t) = \frac{\Phi_8}{\Psi_8}, \tag{24}$$

where

$$\begin{aligned}\Phi_8 = &e^{-p_1(a_3(t)+a_2y+a_4)}(2a_{13}^2k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) \\ &- 2a_{17}^2k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20})) \\ &- 4a_{13}a_{17}k_4k_5\cos(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20})\cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) \\ &- 2a_{17}^2k_5(k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21})\sin(a_{19}(t) \\ &+ a_{17}x + a_{18}y + a_{20}) + 2k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y \\ &+ a_{16})((a_{13}^2 - a_{17}^2)k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ a_{13}^2(k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21})) - 2a_{17}^2k_5^2 \\ &- 2a_{13}^2k_4^2,\end{aligned}$$

$$\begin{aligned}\Psi_8 = &e^{-p_1(a_3(t)+a_2y+a_4)}(2k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) + a_{21}^2 \\ &+ 2k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) \\ &+ 2k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + 2a_{21}) \\ &- k_5^2\cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ k_4^2\cosh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})^2 \\ &+ (2k_3k_5\cosh(a_{11}(t) + a_{10}y + a_{12}) + 2a_{21}k_5)\sin(a_{19}(t) \\ &+ a_{17}x + a_{18}y + a_{20}) + \sinh(a_{15}(t) + a_{13}x + a_{14}y \\ &+ a_{16})(2k_5k_4\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ 2k_3k_4\cosh(a_{11}(t) + a_{10}y + a_{12}) + 2a_{21}k_4) \\ &+ 2a_{21}k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) \\ &+ k_3^2\cosh(a_{11}(t) + a_{10}y + a_{12})^2 - k_4^2 + k_5^2 \\ &+ e^{-2p_1(a_3(t)+a_2y+a_4)}.\end{aligned}$$

Case 9:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), \\ a_{11}(t) = 0, a_{13} = 0, a_{15}(t) = a_{15}(t), a_{19}(t) = 0, \\ k_1 = 0, k_2 = 0, p_1 = 0, f(t) = 0, h(t) = 0, \\ b(t) = 0. \end{cases} \tag{25}$$

a_2, a_4, a_6, a_8, a_9, a_{10}, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, $k_i (3 \le i \le 5)$, and p_2 are all constants. Both the dispersion term and the higher-order dispersion term of Eq. (1) are zero. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_9(x, y, t) = \frac{\Phi_9}{\Psi_9}, \tag{26}$$

where

$$\begin{aligned}\Phi_9 = &-2(a_{21} + 1)a_{17}^2k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- 2a_{17}^2k_5^2 - 4a_9a_{17}k_3k_5\cos(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20})\sinh(a_{11}(t) + a_9x + a_{10}y + a_{12}) \\ &+ 2k_3\cosh(a_{11}(t) + a_9x + a_{10}y \\ &+ a_{12})((a_9^2 - a_{17}^2)k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ (a_{21} + 1)a_9^2) + 2a_9^2k_3^2,\end{aligned}$$

$$\begin{aligned}\Psi_9 = &2(a_{21} + 1)k_5\sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- k_5^2\cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ k_3^2\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12})^2 \\ &+ 2k_3\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12})(k_5\sin(a_{19}(t) \\ &+ a_{17}x + a_{18}y + a_{20}) + a_{21} + 1) + (a_{21} + 1)^2 + k_5^2.\end{aligned}$$

Case 10:

$$\begin{cases} a_1 = 0, a_2 = 0, a_3(t) = 0, a_5 = 0, a_{10} = \dfrac{a_9 a_{18}}{a_{17}}, \\ a_{11}(t) = 0, a_{13} = 0, a_{15}(t) = a_{15}(t), a_{19}(t) = 0, \\ k_1 = 0, k_2 = 0, k_4 = 0, f(t) = 0, h(t) = h(t). \end{cases} \quad (27)$$

$a_4, a_6, a_8, a_9, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, k_3, k_5, p_1$, and p_2 are all constants. a_{17} is not equal to zero for the constant, the high-order dispersion term of Eq. (1) is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5):

$$u_{10}(x, y, t) = \frac{\Phi_{10}}{\Psi_{10}}, \quad (28)$$

where

$$\begin{aligned} \Phi_{10} = &-2a_{17}^2 k_5 (e^{-p_1(a_3(t)+a_4)} + a_{21}) \sin(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20}) - 2k_3 \cosh\left(a_{11}(t) + (a_9 x + a_{12})\right. \\ &\left.+ \frac{a_9 a_{18} y}{a_{17}}\right)\left((a_{17}^2 - a_9^2) k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y\right. \\ &\left.+ a_{20}) - a_9^2\left(e^{-p_1(a_3(t)+a_4)} + a_{21}\right)\right) - 2a_{17}^2 k_5^2 + 2a_9^2 k_3^2 \\ &- 4a_9 a_{17} k_3 k_5 \cos(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20}) \sinh\left(a_{11}(t) + (a_9 x + a_{12}) + \frac{a_9 a_{18} y}{a_{17}}\right), \end{aligned}$$

$$\begin{aligned} \Psi_{10} = &\, 2k_5 (e^{-p_1(a_3(t)+a_4)} + a_{21}) \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &+ 2k_3 \cosh\left(a_{11}(t) + (a_9 x + a_{12})\right. \\ &\left.+ \frac{a_9 a_{18} y}{a_{17}}\right)\left(k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20})\right. \\ &+ e^{-p_1(a_3(t)+a_4)} \\ &\left.+ a_{21}\right) - k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ k_3^2 \cosh\left(a_{11}(t) + (a_9 x + a_{12}) + \frac{a_9 a_{18} y}{a_{17}}\right)^2 \\ &+ 2a_{21} e^{-p_1(a_3(t)+a_4)} + e^{-2p_1(a_3(t)+a_4)} + a_{21}^2 + k_5^2. \end{aligned}$$

Case 11:

$$\begin{cases} a_1 = 0, a_3(t) = a_3(t), a_5 = 0, a_7(t) = a_7(t), \\ a_{10} = \dfrac{a_9 a_{18}}{a_{17}}, a_{11}(t) = 0, a_{13} = 0, a_{15}(t) = a_{15}(t), \\ a_{19}(t) = 0, k_1 = 0, k_2 = 0, k_4 = 0, p_1 = 0, \\ f(t) = 0, h(t) = h(t). \end{cases} \quad (29)$$

$a_2, a_4, a_6, a_8, a_9, a_{12}, a_{14}, a_{16}, a_{17}, a_{18}, a_{20}, a_{21}, k_3, k_5$, and p_2 are all constants. a_{17} is not equal to zero for the constant, the high-order dispersion term of Eq. (1) is zero, the relationship between a linear dispersion $b(t) = -\frac{a_{18}h(t)}{a_{17}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_{11}(x, y, t) = \frac{\Phi_{11}}{\Psi_{11}}, \quad (30)$$

where

$$\begin{aligned} \Phi_{11} = &-2(a_{21} + 1) a_{17}^2 k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- 2a_{17}^2 k_5^2 + 2a_9^2 k_3^2 - 4a_9 a_{17} k_3 k_5 \cos(a_{19}(t) + a_{17}x \\ &+ a_{18}y + a_{20}) \sinh(a_{11}(t) + (a_9 x + a_{12}) + \frac{a_9 a_{18} y}{a_{17}}) \\ &+ 2k_3 \cosh(a_{11}(t) + (a_9 x + a_{12}) \\ &+ \frac{a_9 a_{18} y}{a_{17}})((a_9^2 - a_{17}^2) k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y \\ &+ a_{20}) + (a_{21} + 1) a_9^2), \end{aligned}$$

$$\begin{aligned} \Psi_{11} = &\, 2(a_{21} + 1) k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) \\ &- k_5^2 \cos(a_{19}(t) + a_{17}x + a_{18}y + a_{20})^2 \\ &+ k_3^2 \cosh\left(a_{11}(t) + (a_9 x + a_{12}) + \frac{a_9 a_{18} y}{a_{17}}\right)^2 \\ &+ 2k_3 \cosh\left(a_{11}(t) + (a_9 x + a_{12})\right. \\ &\left.+ \frac{a_9 a_{18} y}{a_{17}}\right)(k_5 \sin(a_{19}(t) + a_{17}x + a_{18}y + a_{20}) + a_{21} \\ &+ 1) + (a_{21} + 1)^2 + k_5^2. \end{aligned}$$

Case 12:

$$\begin{cases} a_1 = 0, a_3(t) = -\displaystyle\int_a^t a_9^2 a_2 h(s) \mathrm{d}s, a_5 = 0, \\ a_7(t) = a_7(t), a_{11}(t) = 4\displaystyle\int_a^t a_9^5 f(s) \mathrm{d}s, a_{13} = 0, \\ a_{15}(t) = a_{15}(t), a_{19}(t) = -\displaystyle\int_a^t a_9^2 a_{18} h(s) \mathrm{d}s, \\ k_1 = 0, k_2 = 0, k_4 = 0, p_1 = 0, f(t) = 0, \\ h(t) = h(t), a_{17} = 0. \end{cases} \quad (31)$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_5$, and p_2 are all constants. a_9 is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = -\frac{2a_9^3 f(t) + a_{10}h(t)}{a_9}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_{12}(x, y, t) = \frac{\Phi_{12}}{\Psi_{12}}, \quad (32)$$

where

$$\Phi_{12} = 2a_9^2k_3\Big((k_5\sin(a_{19}(t) + a_{18}y + a_{20}) + a_{21})\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + \cosh(a_{11}(t) + a_9x + a_{10}y + a_{12})e^{-p_1(a_3(t)+a_2y+a_4)} + k_3\Big),$$

$$\Psi_{12} = e^{-p_1(a_3(t)+a_2y+a_4)}(2k_3\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + 2k_5\sin(a_{19}(t) + a_{18}y + a_{20}) + 2a_{21}) + k_3^2\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12})^2 + (2k_5k_3\sin(a_{19}(t) + a_{18}y + a_{20}) + 2a_{21}k_3)\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + 2a_{21}k_5\sin(a_{19}(t) + a_{18}y + a_{20}) - k_5^2\cos(a_{19}(t) + a_{18}y + a_{20})^2 + e^{-2p_1(a_3(t)+a_2y+a_4)} + a_{21}^2 + k_5^2.$$

Case 13:

$$\begin{cases} a_1 = 0,\ a_3(t) = a_3(t),\ a_5 = 0,\ a_7(t) = a_7(t), \\ a_{11}(t) = 4\int_a^t a_9^5 f(s)\mathrm{d}s,\ a_{13} = 0,\ a_{15}(t) = a_{15}(t), \\ a_{17} = 0,\ a_{19}(t) = -\int_a^t a_9^2 a_{18}h(s)\mathrm{d}s,\ k_1 = 0, \\ k_2 = 0,\ k_4 = 0,\ p_1 = 0, f(t) = 0,\ h(t) = h(t). \end{cases} \tag{33}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_5$, and p_2 are all constants. a_9 is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = -\frac{5a_9^3 f(t) + a_{10}h(t)}{a_9}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_{13}(x, y, t) = \frac{\Phi_{13}}{\Psi_{13}}, \tag{34}$$

where

$$\Phi_{13} = 2a_9^2k_3((k_5\sin(a_{19}(t) + a_{18}y + a_{20}) + a_{21} + 1)\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + k_3),$$

$$\Psi_{13} = k_3^2\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12})^2 + 2k_3t(k_5\sin(a_{19}(t) + a_{18}y + a_{20}) + a_{21} + 1)\cosh(a_{11}(t) + a_9x + a_{10}y + a_{12}) + k_5^2 + 2(a_{21} + 1)k_5\sin(a_{19}(t) + a_{18}y + a_{20}) - k_5^2\cos(a_{19}(t) + a_{18}y + a_{20})^2 + (a_{21} + 1)^2.$$

Case 14:

$$\begin{cases} a_1 = 0,\ a_3(t) = -\int_a^t a_{13}^2 a_2 h(s)\mathrm{d}s,\ a_5 = 0, \\ a_7(t) = a_7(t),\ a_{11}(t) = -\int_a^t a_{13}^2 a_{10}h(s)\mathrm{d}s, \\ a_{13} = 0,\ a_{15}(t) = 4\int_a^t a_{13}^5 a_2 f(s)\mathrm{d}s,\ a_{17} = 0, \\ a_9 = 0,\ a_{19}(t) = a_{19}(t),\ k_1 = 0,\ k_2 = 0,\ k_5 = 0, \\ f(t) = 0,\ h(t) = h(t). \end{cases} \tag{35}$$

$a_2, a_4, a_6, a_8, a_{10}, a_{12}, a_{14}, a_{16}, a_{18}, a_{20}, a_{21}, k_3, k_4, p_1$, and p_2 are all constants. a_{13} is a constant that is not equal to zero, and the relation between the dispersion term of Eq. (1) and the higher-order dispersion term is $b(t) = -\frac{5a_{13}^3 f(t) + a_{14}h(t)}{a_{13}}$. The interaction solution of Eq. (1) can be obtained by transformation Eq. (5).

$$u_{14}(x, y, t) = \frac{\Phi_{14}}{\Psi_{14}}, \tag{36}$$

where

$$\Phi_{14} = 2a_{13}^2k_4\Big((k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + a_{21})\sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) + \sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16})e^{-p_1(a_3(t)+a_2y+a_4)} - k_4\Big),$$

$$\Psi_{14} = \Big(k_4\sinh(a_{15}(t) + a_{13}x + a_{14}y + a_{16}) + k_3\cosh(a_{11}(t) + a_{10}y + a_{12}) + e^{-p_1(a_3(t)+a_2y+a_4)} + a_{21}\Big)^2.$$

According to the case of higher-order dispersion term, it can be divided into three types. On the basis of these three cases, the corresponding type of solitary wave images are drawn by mathematical software for analysis.

(1) Category 1:

The higher-order dispersion terms of $u_1(x, y, t)$, $u_2(x, y, t)$, $u_3(x, y, t)$, $u_4(x, y, t)$, and $u_{13}(x, y, t)$ are not zero. It can be obtained that there is a certain relationship between the higher-order dispersion term and the linear dispersion term.

Physical characteristics and dynamic structure of the equation when $y = 10$, $y = -5$, $y = 0$, $y = 4$, and $y = -10$ are selected. As shown in Figure 1, the parameters of

$u_1(x, y, t)$ is selected as the three-dimensional graphics and contour graphics of $a_2 - 1$, $a_3(t) = \sin(t)$, $a_4 = -2.3$, $a_9 = 1$, $a_{10} = 1$, $a_{11} = t$, $a_{12} = 1$, $a_{14} = 2.3$, $a_{15}(t) = \cos(t)$, $a_{16} = 1$, $a_{17} = 2.2$, $a_{18} = 1.5$, $a_{19}(t) = \sin(t)$, $a_{20} = 1.3$, and $a_{21} = -1.2$.

As can be seen from Figure 1(a)–(e), when different parameter values are selected for the spatial variable y and the time variable t, the solitary wave image presents a non smooth solitary spike wave, and its maximum peaks are equal. The solitary wave behavior from Figures 1(a) to 2(c) gradually moves toward $t \to +\infty$ and the maximum wave crest is upward. This solitary wave behavior of plasma is bright wave. The dynamic behavior of this nonlinear solitary wave in plasma is bright wave.

(2) Category 2:

The higher-order dispersion terms of $u_7(x, y, t)$, $u_8(x, y, t)$, and $u_9(x, y, t)$ are zero. It can be obtained that there is a certain relationship between the linear dispersion terms.

Selection in the variable $y = 10$, $y = 4$, $y = -1$, $y = -5$, and $y = -16$ equation when the physical characteristics of structure and dynamics. As shown in Figure 1, the parameters of $u_5(x, y, t)$ is selected as $a_3(t) = \sin(t)$, $a_9 = 1.2$, $a_{10} = 1$, $a_{11}(t) = \cos(t)$, $a_{12} = 1$, $a_{17} = 0.5$, $a_{18} = -0.06$, $a_{19}(t) = \sin(t)$, $a_{20} = 1$, and $a_{21} = -1.5$, the drawn three-dimensional graphics and contour graphics.

By looking at Figure 2, we can see that the dynamic image is distributed in $x - t$ flat level, higher-order Lump cross kink wave solutions and periodic solutions of interaction force from strong to weak to strong, and solitary wave solution, are mutual inelastic collision between amplitude and shape all happen very big change, peak shape isolated sharp wave tapering and reach maximum peak.

(3) Category 3:

Both the higher-order dispersion term and the linear dispersion term of $u_5(x, y, t)$, $u_6(x, y, t)$, $u_{10}(x, y, t)$, $u_{11}(x, y, t)$, $u_{12}(x, y, t)$, and $u_{14}(x, y, t)$ are zero.

3 Resonant multisoliton solution

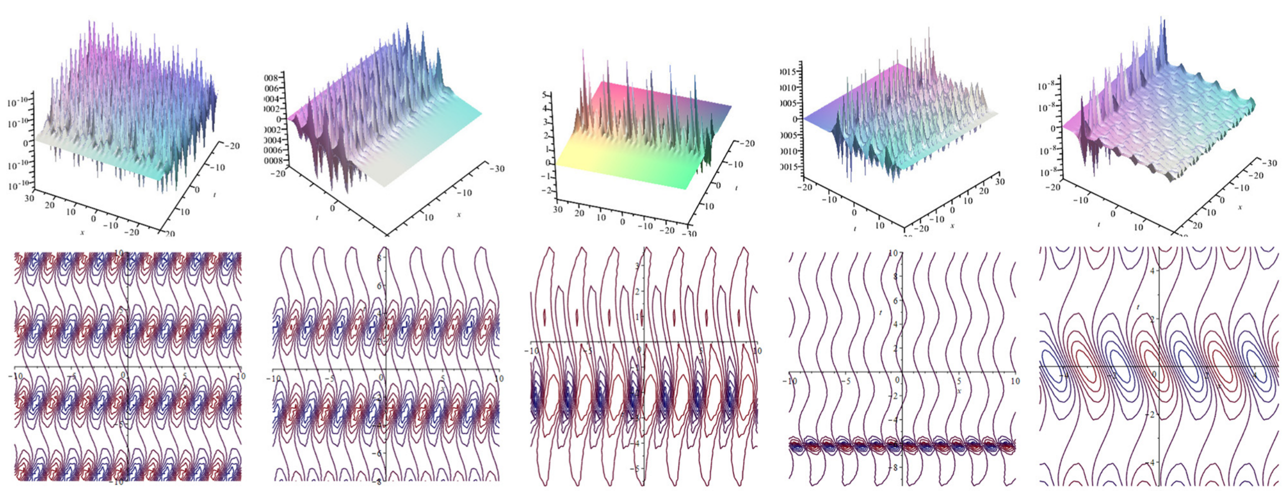
The principle of linear superposition is applied to assume that the bilinear generalized variable coefficient KdV equation has the form of the following solution.

$$g = \sum_{i=0}^{N} \varepsilon_i \widehat{g}_i = \sum_{i=0}^{N} \varepsilon_i \exp(\varrho_i), \tag{37}$$

where $\widehat{g}_i = \exp(\varrho_i) = \exp(\xi_i x + \lambda_i y + \tau_i t)$, $1 \le i \le N$, and ξ_i, λ_i, and τ_i are all constants. Substituting Eq. (37) in the bilinear Eq. (6) and by using the bilinear equation identities, we can obtain

$$P(D_x, D_y, D_t) = \sum_{i,j=1}^{N} \varepsilon_i \varepsilon_j P(\xi_i - \xi_j, \lambda_i - \lambda_j, \tau_i - \tau_j) \exp(\varrho_i + \varrho_j). \tag{38}$$

So we can draw the double linear equation solution of g if and only if $P(\xi_i - \xi_j, \lambda_i - \lambda_j, \tau_i - \tau_j) \exp(\varrho_i + \varrho_j) = 0$.



(a) (b) (c) (d) (e)

Figure 1: When $y = 10$, $y = 4$, $y = 0$, $y = -5$, and $y = -10$, the three-dimensional graphics and contour graphics of the interaction solution $u_1(x, y, t)$.

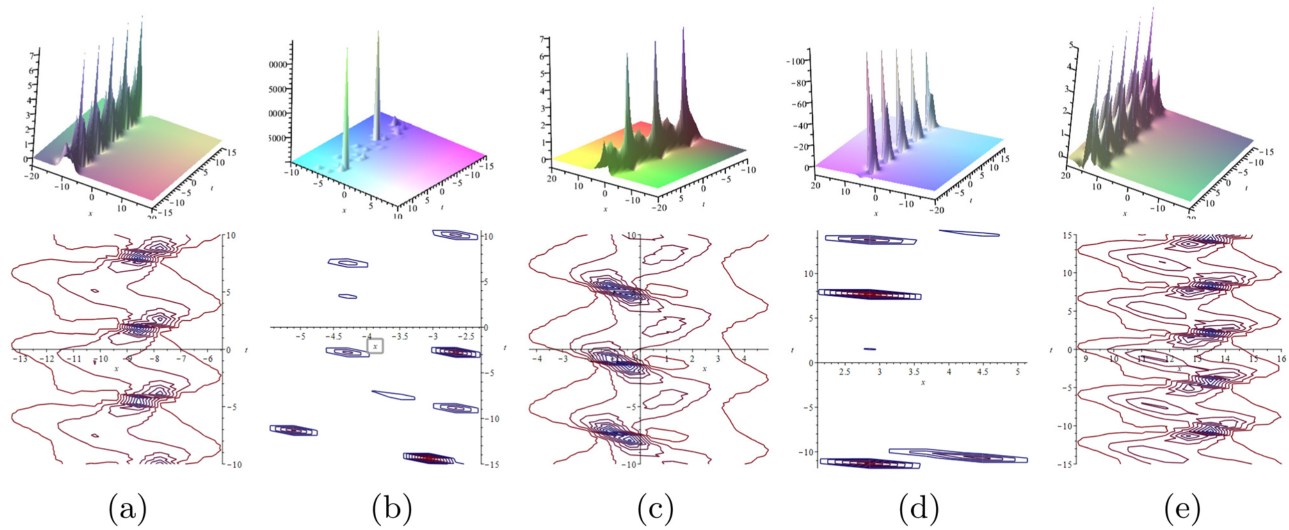


Figure 2: When $y = 10$, $y = 4$, $y = 0$, $y = -5$, and $y = -16$, the three-dimensional graphics and contour graphics of the interaction solution $u_7(x, y, t)$.

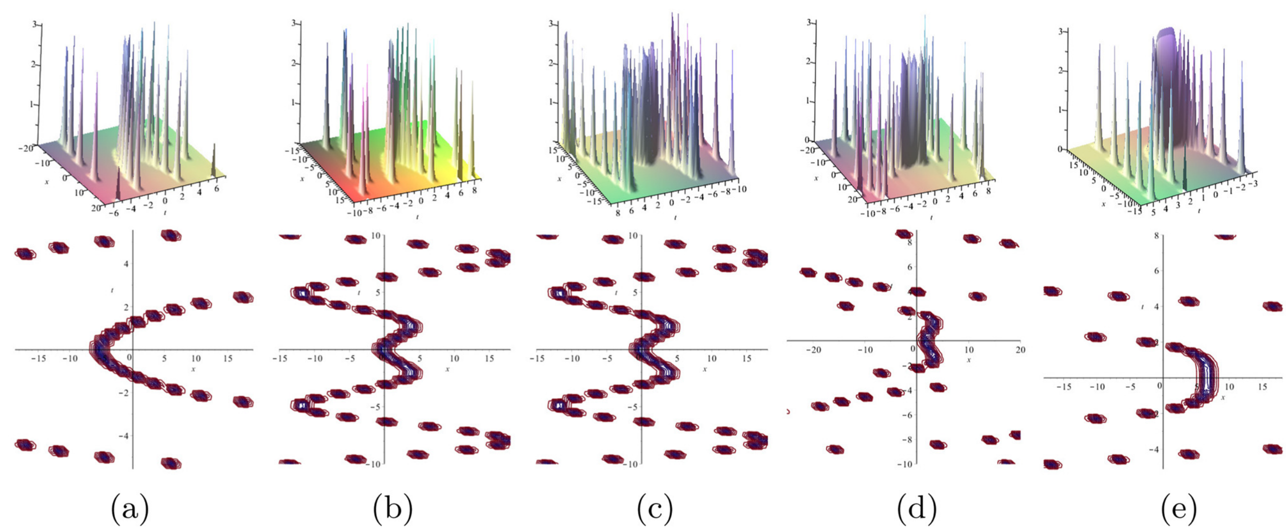


Figure 3: When $N = 3$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, three-dimensional graph and contour plot of resonant soliton solution $u_{15}(x, y, t)$.

According to the aforementioned linear superposition principle, the polynomial corresponding to bilinear Eq. (6) is expressed as follows:

$$P(x, y, t) = xt + b(t)x^4 + f(t)x^6 + h(t)x^3y. \tag{39}$$

Substituting Eq. (39) into Eq. (38), we obtain

$$\begin{aligned}(\xi_i - \xi_j)(\tau_i - \tau_j) + b(t)(\xi_i - \xi_j)^4 + f(t)(\xi_i - \xi_j)^6 \\ + h(t)(\xi_i - \xi_j)^3(\lambda_i - \lambda_j) = 0.\end{aligned} \tag{40}$$

By simplifying Eq. (40), we obtain the following sets of solutions. We choose two cases as examples:

Case 1: $\xi_i = \xi_i$, $\lambda_i = -\frac{b(t)}{h(t)}\xi_i$, $\tau_i = -f(t)\xi_i^5$. The resonant multisoliton solutions of Eq. (1) can be obtained by transformation Eq. (5):

$$\begin{aligned}u_{15}(x, y, t) = {} & 2\frac{\sum_{i=1}^{N}\xi_i^2 e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}}{\sum_{i=1}^{N} e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}} \\ & - 2\frac{\left(\sum_{i=1}^{N}\xi_i e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}\right)^2}{\left(\sum_{i=1}^{N} e^{\xi_i x - \frac{b(t)\xi_i y}{h(t)} - f(t)\xi_i^5 t}\right)^2}.\end{aligned} \tag{41}$$

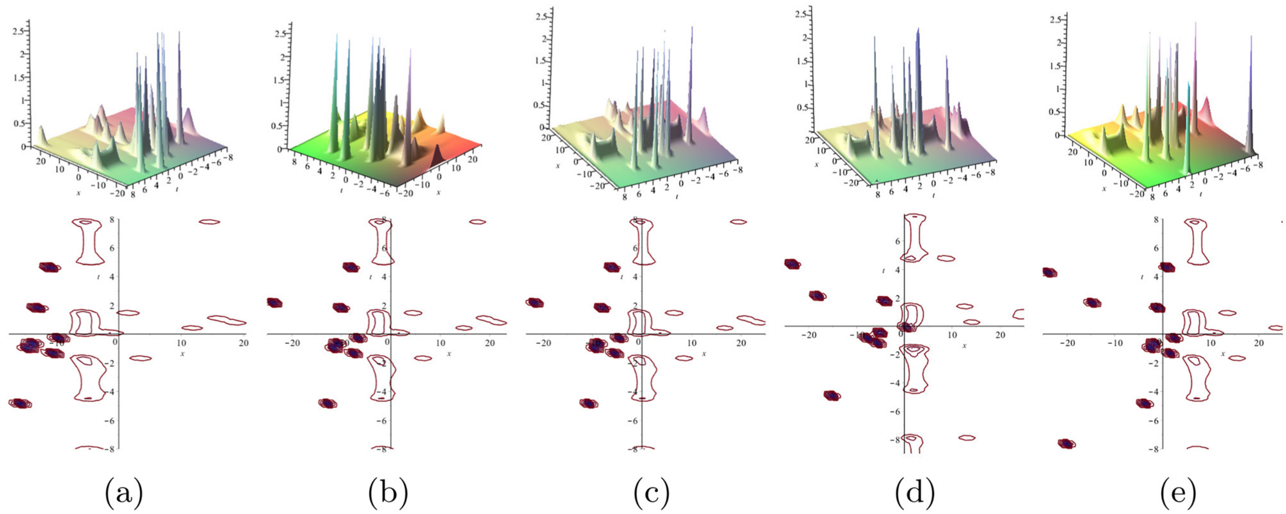


Figure 4: When $N = 6$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, three-dimensional graph and contour plot of resonant soliton solution $u_{16}(x, y, t)$.

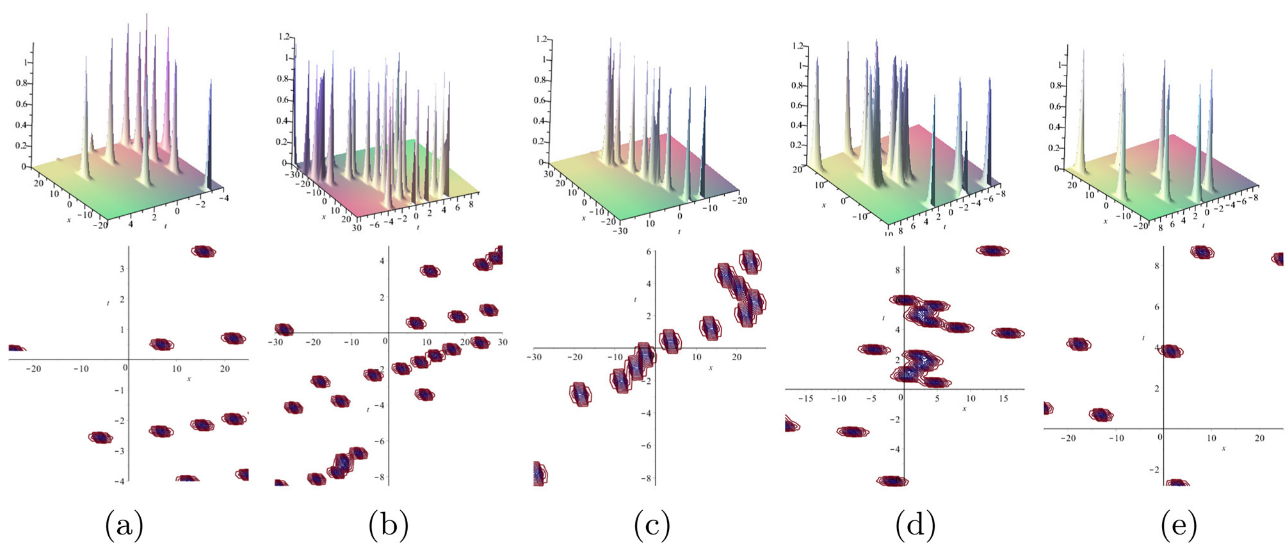
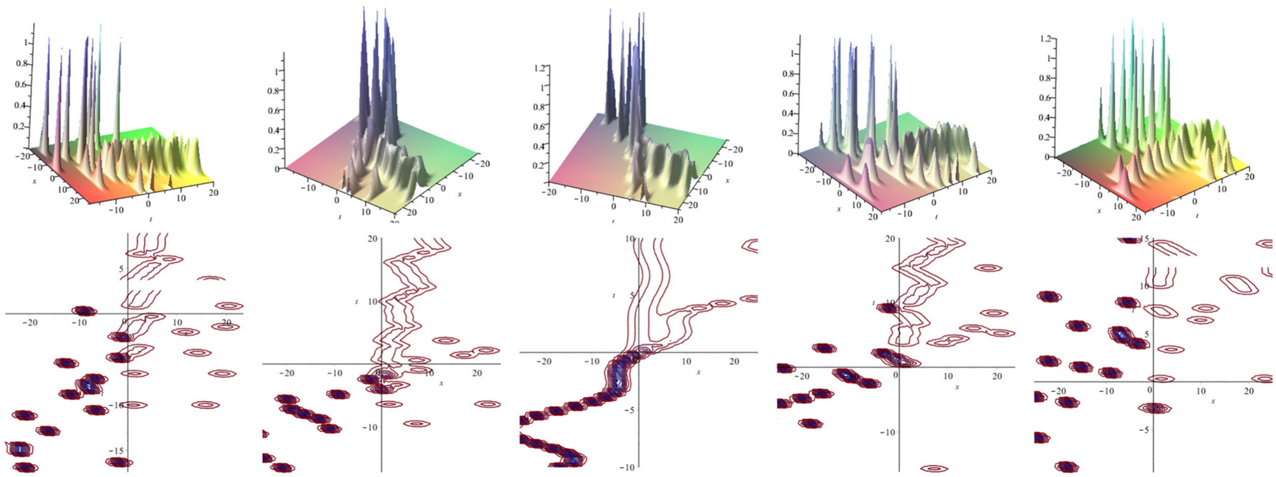


Figure 5: When $N = 2$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, three-dimensional graph and contour plot of resonant soliton solution $u_{17}(x, y, t)$.

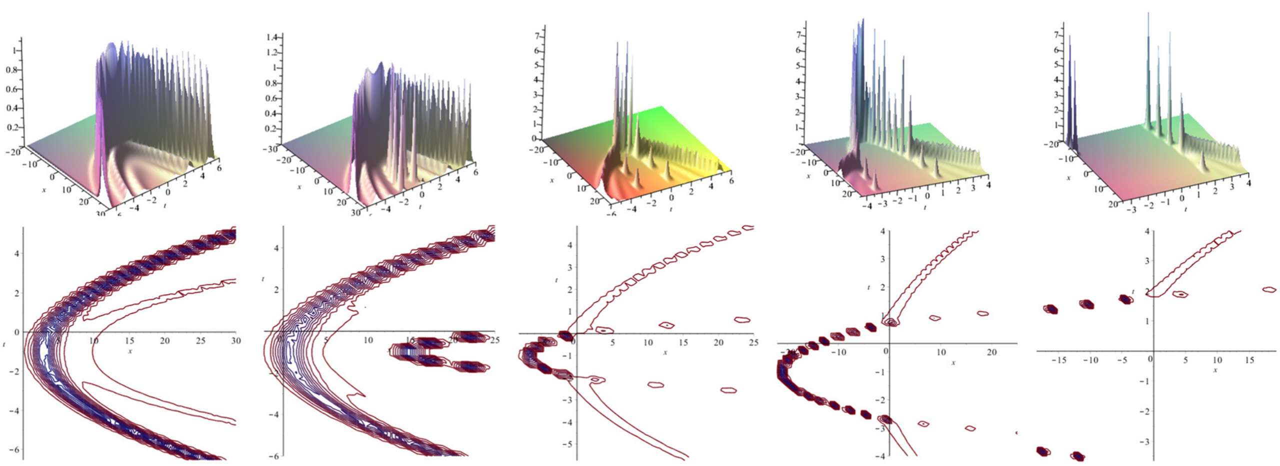
In the first case, $N = 3, 6$, and variable selection $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, the equation of the physical characteristics and dynamic structure. As shown in Figures 3 and 4, when $u_{15}(x, y, t)$ the parameter selection of $N = 3$, $f(t) = \sin t$, $h(t) = \tan t$, $b(t) = t$, $\xi_1 = -1.3$, $\xi_3 = 1.1$ and $N = 6$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \sin t$, $\xi_1 = 1$, $\xi_2 = 1.2$, $\xi_3 = 1.1$, $\xi_4 = 0.5$, $\xi_5 = -0.25$, $\xi_6 = 2.1$, the drawn three-dimensional graphics and contour graphics.

When $N = 3$ and 6, we can observe that the ratio of linear dispersion terms and the higher-order dispersion terms increases, with the increase of $t \to +\infty$. As can be seen from the contour diagram in Figure 3, the initial enlarged "W" image gradually shrinks and then gradually expands, and the isolated peak wave gradually increases and then decreases with the ratio of linear dispersion terms and the transformation of higher-order dispersion terms. The dynamic behavior is always bright. It can be observed from the contour line in Figure 4 that peak-like isolated sharp waves move slowly toward $t \to +\infty$ with the increase of the ratio of linear dispersion terms and the increase of higher-order dispersion terms, and elastic collision occurs. The amplitude and maximum peak value of isolated waves remain unchanged.



(a) (b) (c) (d) (e)

Figure 6: When $N = 3$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, three-dimensional graph and contour plot of resonant soliton solution $u_{18}(x, y, t)$.



(a) (b) (c) (d) (e)

Figure 7: When $N = 6$, $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$, three-dimensional graph and contour plot of resonant soliton solution $u_{19}(x, y, t)$.

Case 2: $\xi_i = \xi_i$, $\lambda_i = \left(2 - \frac{f(t)}{h(t)}\right)\xi_i^3$, $\tau_i = -(2 + b(t))\xi_i^3$. The resonant multi soliton solutions of Eq. (1) can be obtained by transformation Eq. (5),

$$u_{16}(x, y, t) = 2\frac{\sum_{i=1}^{N}\xi_i^2 e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}}{\sum_{i=1}^{N} e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}} - 2\frac{\left(\sum_{i=1}^{N}\xi_i e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}\right)^2}{\left(\sum_{i=1}^{N} e^{\xi_i x + 2\xi_i^3 y - \frac{f(t)\xi_i^3 y}{h(t)} - 2\xi_i^3 t - b(t)\xi_i^3 t}\right)^2}. \tag{42}$$

In the second case, $N = 2, 3, 6$, and $y = -5$, $y = -2$, $y = 0$, $y = 2$, $y = 7$ equation when the physical properties of structure and dynamics, as shown in Figures 5 and 6, when $u_{16}(x, y, t)$ parameters selection $N = 2$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \sin t$, $\xi_1 = 0.25$, $\xi_2 = 1.8$ and $N = 3$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \cos t$, $\xi_1 = \frac{1}{2}$, $\xi_2 = -0.25$, $\xi_3 = 1.3$ and $N = 6$, $f(t) = \cos t$, $h(t) = \sin t$, $b(t) = \cos t$, $\xi_1 = \frac{1}{2}$, $\xi_2 = -0.25$, $\xi_3 = 1.3$, $\xi_4 = 0.5$, $\xi_5 = -0.25$, $\xi_6 = 2.1$, the drawn three-dimensional graphics and contour graphics.

When $N = 2, 3, 6$, we can observe that the ratio of linear dispersion terms and the higher-order dispersion terms decrease with the increase of $t \to +\infty$. It can be observed from the contour diagram in Figures 5–7 that the soliton gradually decreases from the initial concentrated distribution to $x - t$ with a fast speed. The peak isolated sharp wave also gradually decreases with the ratio of linear dispersion terms and the transformation of higher-order dispersion terms, and inelastic collisions occur between multiple solitons.

4 Conclusion

In this article, two methods are used to solve the interaction solutions and resonant multi soliton solutions of the generalized variable coefficient fifth-order KdV equation with higher-order dispersion terms. First, the fifth-order KdV equation with variable coefficients is transformed into a bilinear KdV equation with variable coefficients by using the Hirota method, and then four mixed solutions containing positive quartic function, quadratic function, exponential function, and hyperbolic function are constructed. Second, the resonant soliton function solution form of bilinear variable coefficient KdV equation is constructed by using the linear superposition principle The specific interaction solution and resonant soliton solution are solved, respectively, and the relationship between the higher-order dispersion term and the linear dispersion term of the fifth-order variable coefficient KdV equation is obtained. A simple physical analysis is carried out according to the nonlinear solitary wave image.

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