Research Article

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Approximate solution of linear integral equations by Taylor ordering method: Applied mathematical approach

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Abstract: Since obtaining an analytic solution to some mathematical and physical problems is often very difficult, academics in recent years have focused their efforts on treating these problems using numerical methods. In science and engineering, systems of integral differential equations and their solutions are extremely important. The Taylor collocation method is described as a matrix approach for solving numerically Linear Differential Equations (LDE) by using truncated Taylor series. Integral equations are used to solve problems such as radiative transmission and the oscillation of a string, membrane, or axle. Differential equations can be used to tackle oscillating difficulties. To discover approximate solutions for linear systems of integral differential equations with variable coefficients in terms of Taylor polynomials, the collocation approach, which is offered for differential and integral equation solutions, will be developed. A system of LDE will be translated into matrix equations, and a new matrix equation will be generated in terms of the Taylor coefficients matrix by employing Taylor collocation points. The needed system will be converted to a linear algebraic equation system. Finding the Taylor coefficients will lead to the Taylor series technique.

Keywords: system of integral differential equation, collocation points, Taylor polynomial, Taylor series

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1 Introduction

Indeed, many occurrences in practically all engineering analysis and applied science concerns, such as physical implementations, potential theory, and electrodynamics, may be simplified to solving integral equations. Because these equations are rarely solved openly, approximate solutions must be obtained. There are several numerical approaches that have been developed to solve integral equations. According to their bounds, integral equations are divided into two types: Fredholm and Volterra. The distinction between these two equations is that the integral bounds in the Fredholm equation are constant, whereas the upper bound in the Volterra equation is changeable. The Taylor sorting method will be discussed in this part to acquire the Fredholm integral equation system's finite Taylor series solutions.

High-order linear systems in science and engineering. The F–V integro-differential equations and their solution are extremely important. Many system components, such as biomedical application in genetics and population dynamics, wherein impulses occur naturally, systems of integro-differential equations can be used to model systems of integro-differential equations that are formed *via* control. Integral systems are also linked to some controllability difficulties for impulsive systems and integro-differential theory conclusions, but they are yet to be thoroughly investigated. As a result, solutions to these and many other equations are critical in a variety of applications. Infinite series techniques can be used to solve a variety of physical problems. These are used to approximate various functions, solve the system of integro-differential equations and evaluate challenging integrals. Taylor's expansion is the most often used infinite series expansion method for this purpose. It is a strategy for modeling a function near to a start or initial point that is used locally. It does not, however, work effectively when a significant number of control factors must always be taken into account. The Taylor collocation method is the emphasis of this study, and it is applied to a variety of fascinating scenarios, including systems of integro-differential equations. Integral equations were also influenced by mathematical physics models such as diffraction issues, quantum mechanics scattering, conformal mapping, and water waves. Integral equations are used to solve problems like radiative transfer and the oscillation of a string, membrane, or axle. Differential equations may mathematically characterize engineering difficulties and as a result, differential equations play a critical part in the solution of practical problems. Newton's law, which states that the rate of change of a particle's momentum is equal to the magnitude acting on it, can be expressed mathematically as a differential equation. Issues occurring in electric circuits, chemical kinetics, and heat transfer in a medium can all be theoretically expressed as differential equations. These differential equations can be turned into Volterra and Fredholm-type integral equations.

Kanwal and Liu studied Taylor expansion approaches for resolving integral equations that have already been given for Volterra integral equation [1]. Sezer probed Taylor's polynomial solution of Volterra integral equations [2]. Sezer probed it for some differential equations [3], Nas *et al.* for integro-differential equations [4], Karamete and Sezer for linear integro-differential equations [5], and Yalçınbas and Sezer for F–V linear integro-differential equations [6]. Akyüz and Sezer studied Chebyshev methods that have been used to solve high-order linear differential equation systems [7]. Akyüz-Daşcıoğlu explained the linear integral equations [8], which is defined as

$$\sum_{i=1}^{k} P_{ji}(x) y_i(x) = f_j(x) + \int_a^b K_{ji}(x,t) y_i(t)\,dt; \quad j = 1, 2, \ldots k. \tag{1}$$

Yusufoğlu probed the numerical solution of Duffing equation by the Laplace decomposition algorithm [9]. Gülsu and Sezer explained Taylor polynomials, a Taylor collocation approach is provided for numerically solving the system of high-order linear Fredholm–Volterra integro-differential equations [10]. Yuldashev mixed the value problem for the nonlinear integro-differential equation with a parabolic operator of higher power [11]. Toutounian *et al.* solved high-order linear differential and difference equations with constant coefficients, and Fourier operational matrices of differentiation and transmission are introduced. They also used Legendre–Gauss collocation nodes to enhance our approaches for extended Pantograph equations with variable coefficients [12].

Tohidi and Kılıçman discussed how to solve a variety of one-dimensional parabolic partial differential equations (PDEs) with given initial and nonlocal boundary conditions. The central concept is based on direct collocation and translating the PDEs under consideration into corresponding algebraic equations [13]. Mirzaee *et al.* presented a unique method that is under mixed beginning conditions, a method for solving linear complex differential equations is developed by using complex operational matrices of Euler functions and their fascinating properties [14]. Tohidi presented an effective numerical approach with piecewise intervals for solving linear two-dimensional Fredholm integral equations that produce an approximate polynomial solution [15]. Zogheib and Tohidi explained numerical solution of nonlinear one- and two-dimensional heat transfer problems with provided beginning conditions and linear Robin boundary conditions is considered [16]. Zogheib and Tohidi presented a computational solution for multi-dimensional hyperbolic telegraph formulas with suitable initial time and boundary space conditions. In both the spatial and temporal variables, the truncated Hermite series with unknown coefficients are employed to approximate the answer [17]. Zézé *et al.* found multi-point Taylor series to solve differential equations [18].

Hadadian and Yousefi derived a new and efficient numerical approach for solving advection-diffusion equations (ADE). The method is based on the combination of the ADE under consideration. The ADE is transformed into a corresponding integral equation *via* integration, and the integral equation comprises initial and boundary conditions [19]. Yuldashev examined a boundary-value problem for Boussinesq type nonlinear integro-differential equation with reflecting argument [20]. Singh *et al.* proposed an efficient matrix approach for numerically solving the two-dimensional diffusion and telegraph equations subject to Dirichlet boundary conditions using Euler approximation [21]. Yang *et al.* contributed to the development of a highly accurate computational approach for solving two-dimensional mass transport equations during apple slice convective air drying. Numerical technique is based on interpolating the solution of the above equations in a nodal form across the roots of orthogonal Jacobi polynomials [22].

The collocation approach, which is provided for the solution of differential and integral equations, is expanded to find approximate solutions for linear systems of integral differential equations with variable coefficients in terms of Taylor polynomials. A set of linear differential equations is converted into matrix equations, and by using Taylor collocation points, a new matrix equation is created in terms of the Taylor coefficient matrix. The required system is transformed into a system of linear algebraic equations. The Taylor series method is reached by locating the Taylor coefficients. Integrals are utilized in a variety of industries, including engineering, where engineers use integrals to determine the geometry of a building. In physics, it is utilized in the center of gravity, for example. Three-dimensional designs are exhibited in the realm of schematic diagram.

2 Fundamental relations

Linear Fredholm integral equation consisting of k equations with k unknown systems is defined as

$$\begin{aligned} &P_{11}(x)y_1(x) + P_{12}(x)y_2(x)+\ldots+ P_{1k}(x)y_k(x) \\ &= f_1(x) + \int_a^b (K_{11}(x,t)y_1(t) + K_{12}(x,t)y_2(t)+\ldots \\ &\quad + K_{1k}(x,t)y_k(t))\mathrm{d}t, \\ &P_{21}(x)y_1(x) + P_{22}(x)y_2(x)+\ldots+ P_{1k}(x)y_k(x) \\ &= f_2(x) + \int (K_{21}(x,t)y_1(t) + K_{22}(x,t)y_2(t) \\ &\quad + \ldots + K_{2k}(x,t)y_k(t))\mathrm{d}t, \\ &P_{k1}(x)y_1(x) + P_{k2}(x)y_2(x)+\ldots+ P_{kk}(x)y_k(x) \\ &= f_k(x) + \int (K_{k1}(x,t)y_1(t) + K_{k2}(x,t)y_2(t) \\ &\quad + \ldots + K_{kk}(x,t)y_k(t))\mathrm{d}t, \end{aligned}$$

In short, this system is

$$\sum_{i=1}^{k} P_{ji}(x)y_i(x) = f_j(x)+\int_a^b K_{ji}(x,t)y_i(t)\mathrm{d}t; \quad j = 1, 2, \ldots k,$$

where kernel functions $K(x, t)$, $P(x)$, and $f(x)$ are known functions in Eq. (1) problem.

$$y_i(t) = \sum_{j=0}^{N} \frac{y_i^{(j)}(c)}{j!}(t-c)^j; \quad i = 1, 2, \ldots, k. \tag{2}$$

The finite Taylor approach will be examined in the form. Taylor series expansions will be presumed for the time being of $K(x,t)$ functions that exist. Writing the system of Eq. (1) in matrix form:

$$p(x)y(x) = f(x) + \int_a^b K(x,t)y(x)\mathrm{d}t. \tag{3}$$

The matrices are

$$P_i(t) = \begin{bmatrix} P_{11}(t) & P_{12}(t) & \ldots & P_{1k}(t) \\ P_{21}(t) & P_{22}(t) & \ldots & P_{2k}(t) \\ P_{31}(t) & P_{31}(t) & \ldots & P_{31}(t) \\ \vdots & \vdots & \ldots & \vdots \\ P_{k1}(t) & P_{k2}(t) & \ldots & P_{kk}(t) \end{bmatrix}, \quad y^{(i)}(t) = \begin{bmatrix} y^{(i)}(t) \\ y^{(i)}(t) \\ y^{(i)}(t) \\ \vdots \\ y_k^{(i)}(t) \end{bmatrix},$$

$$f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ f_3(t) \\ \vdots \\ f_k(t) \end{bmatrix},$$

$$K = \begin{bmatrix} k_{11}(x,t) & k_{12}(x,t) & \ldots & k_{1k}(x,t) \\ k_{21}(x,t) & k_{22}(x,t) & \ldots & k_{2k}(x,t) \\ \vdots & \vdots & \ddots & \vdots \\ k_{k1}(x,t) & k_{k2}(x,t) & \ldots & k_{kk}(x,t) \end{bmatrix}.$$

Now to an integral part of this system

$$I(x) = \int_a^b K(x,t)y(x)\mathrm{d}t.$$

If it is said that system (3) takes shape.

$$p(x)y(x) = f(x) + I(x).$$

Now, we add Taylor sorting points defined to this system

$$p(x_0)y(x_0) = f(x_0) + I(x_0),$$

$$p(x_1)y(x_1) = f(x_1) + I(x_1),$$

$$p(x_N)y(x_N) = f(x_N) + I(x_N).$$

Matrices form

$$P_k = \begin{bmatrix} P(x_0) & 0 & . & . & . & 0 \\ 0 & P(x_1) & . & . & . & 0 \\ . & . & P(x_2) & . & . & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & . & . & . & P(x_N) \end{bmatrix},$$

$$F = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix}, \quad Y^{(i)} = \begin{bmatrix} y^{(k)}(x_0) \\ y^{(k)}(x_1) \\ y^{(k)}(x_2) \\ \vdots \\ y_k^{(k)}(x_N) \end{bmatrix}, \quad I = \begin{bmatrix} I(x_0) \\ I(x_1) \\ I(x_2) \\ \vdots \\ I(x_N) \end{bmatrix}.$$

Including

$$PY = F + I, \quad (4)$$

can be written in this format

$$PTM_0 = F + I. \quad (5)$$

Let us try to write the matrix I here in terms of A. The rows of the matrix $I(x)$ for the system (3) is defined as

$$I_j(x) = \int_a^c \sum_{i=1}^k K_{ji}(x, t) y_i(t) \mathrm{d}t, \quad j = 1, 2, \ldots, k.$$

Now, Taylor sort order x, $s = 0, 1, \ldots, N$

Let us place the points:

$$I_j(x_s) = \int_a^b \sum_{i=1}^k K_{ji}(x_s, t) y_i(t) \mathrm{d}t, \; j = 1, 2, \ldots, k, \quad (6)$$

where $K(x, t)$ function now depends only on t. Therefore,

$$K_{ji}(x_s, t) = \sum_{m=0}^N \frac{K_{ji}^{(m)}(x_s, t)(c)}{m!}(t - c); \; m = 1, 2, \ldots, k, \quad (7)$$

can open a univariate Taylor expression. Thus, in the matrix form of Eq. (7) spelling

$$K_{ji}(x_s, t) = K_{ji}(x) T(t)^T, \quad (8)$$

included

$$K_{ji}(x_s) = [k_0^{ji}(x_s) k_1^{ji}(x_s) k_2^{ji}(x_s) \ldots k_N^{ji}(x_s)],$$

is in the form. If K is (x_s) elements,

$$k_r^{ji}(x_s) = \frac{1}{r!} \frac{\mathrm{d}K_{ji}^r}{\mathrm{d}t^r} |_{x=x_s}, \quad x = 0, 1, 2, \ldots N.$$

If we use the previously determined matrix notation for $y(t)$, we get from Eq. (6)

$$I_J(x_s) = \int_a^b \sum_{i=1}^k K_{ji}(x_s) T^t T M_0 A_i \mathrm{d}t, \quad j = 0, 1, 2, \ldots k.$$

If the integral is included in the above sum, then

$$I_J(x_s) = \sum_{i=1}^k K_{ji}(x_s) \int_a^b T^t T M_0 A_i \mathrm{d}t, \quad j = 0, 1, 2, \ldots k.$$

The integral here

$$H = \int_a^b T^t T \mathrm{d}t = \left[\int_a^b T_i T_j \right] = [h_{ij}], \quad i, j = 0, 1, 2, \ldots N.$$

The elements of this matrix are

$$h_{ij} = \frac{(b - c)^{i+j+1} - (a - c)^{i+j+1}}{i + j + 1}.$$

Thus, the Eq. (6) is calculated as

$$I_J(x_s) = \sum_{i=1}^k K_{ji}(x_s) H M_0 A_i \mathrm{d}t, \quad j = 0, 1, 2, \ldots k. \quad (9)$$

Let us write this clearly:

$$I_1(x_s) = K_{11}(x_s) H M_0 A_1 + K_{12}(x_s) H M_0 A_2 + , \ldots + K_{1k}(x_s) H M_0 A_k,$$

$$I_2(x_s) = K_{21}(x_s) H M_0 A_1 + K_{22}(x_s) H M_0 A_2 + , \ldots + K_{2k}(x_s) H M_0 A_k,$$

....

$$I_k(x_s) = K_{k1}(x_s) H M_0 A_1 + K_{k2}(x_s) H M_0 A_2 + , \ldots + K_{kk}(x_s) H M_0 A_k.$$

Thus, the matrix representation of Eq. (9) is possible.

$$I(x_s) = K(x_s) H M_0 A. \quad (10)$$

The matrices are

$$K(x_s) = \begin{bmatrix} k_{11}(x_s) & k_{12}(x_s) & \ldots & k_{1k}(x_s) \\ k_{21}(x_s) & k_{22}(x_s) & \ldots & k_{2k}(x_s) \\ \vdots & \vdots & \ddots & \vdots \\ k_{k1}(x_s) & k_{k2}(x_s) & \ldots & k_{kk}(x_s) \end{bmatrix}, \quad I(x_s) = \begin{bmatrix} I(x_s) \\ I(x_s) \\ I(x_s) \\ \vdots \\ I(x_s) \end{bmatrix},$$

$$M_0 = \begin{bmatrix} M_0 & 0 & 0 & \ldots & 0 \\ 0 & M_0 & 0 & \ldots & . \\ 0 & 0 & M_0 & \ldots & 0 \\ . & . & . & \ldots & 0 \\ . & . & . & \ldots & 0 \\ . & . & . & \ldots & 0 \\ 0 & 0 & 0 & \ldots & M_0 \end{bmatrix}_{k \times k},$$

$$H_0 = \begin{bmatrix} H_0 & 0 & 0 & \ldots & 0 \\ 0 & H_0 & 0 & \ldots & . \\ 0 & 0 & H_0 & \ldots & 0 \\ . & . & . & \ldots & 0 \\ . & . & . & \ldots & 0 \\ . & . & . & \ldots & 0 \\ 0 & 0 & 0 & \ldots & H_0 \end{bmatrix}_{k \times k}.$$

The matrix I in Eq. (4) can specified as the terms of Taylor coefficient matrix.

To find the expression, Eq. (10) is substituted in the matrix I;

$$I = K H M_0 A,$$

and if the derived expression is placed in Eq. (5), the result is obtained as

$$PTM_0 A = F + K H M_0 A.$$

When this expression is edited

$$(PTM_0 - K H M_0) A = F. \quad (11)$$

The matrix values in the brackets in Eq. (11) are known, and as a result of the operations, a square matrix of size $k(N+1)$ is obtained as

$$WA = F. \tag{12}$$

This is a linear system consisting of $k(N+1)$ with $k(N+1)$ unknowns. The Taylor coefficients are calculated by solving this system. When these coefficients are placed in Eq. (2), the Fredholm integral system's approximate solution in terms of Taylor polynomials is found.

3 Illustrations

Example 1:

$$\int_{-1}^{1} y_1(t)\mathrm{d}t = 0,$$

$$\int_{-1}^{1} t\cdot y_1(t) - \int_{-1}^{1} y_2(t) = 0.$$

Solution:

$$\int_{-1}^{1}\begin{bmatrix}1 & 0\\ t & -1\end{bmatrix}\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}\mathrm{d}t = \begin{bmatrix}0\\0\end{bmatrix},$$

$$P(x) = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}, K(x,t) = \begin{bmatrix}1 & 0\\ t & -1\end{bmatrix}, f(x) = \begin{bmatrix}0\\0\end{bmatrix}.$$

Fundamental equation

$$(PTM_0 - KHM_0)A = F,$$

For $p(x) = 0$, then

$$KHM_0A = F.$$

For $N = 2$, then the points are given by

$$KHM_0A = F.$$

$$x_0 = -1,\ x_1 = 0,\ x_2 = 1,$$

$$K = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & -1 & 0 & 0\end{bmatrix},\ H = \begin{bmatrix}2 & 0 & \frac{2}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0\\ \frac{2}{3} & 0 & \frac{2}{3} & 0 & 0 & 0\\ 0 & 1 & 0 & 2 & 0 & \frac{2}{3}\\ 1 & 0 & 0 & 0 & \frac{2}{3} & 0\\ 0 & 1 & 0 & \frac{2}{3} & 0 & \frac{2}{3}\end{bmatrix},$$

$$W = \begin{bmatrix}2 & 0 & \frac{1}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & 0 & -2 & 0 & \frac{-1}{3}\\ 2 & 0 & \frac{1}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & 0 & -2 & 0 & \frac{-1}{3}\\ 2 & 0 & \frac{1}{3} & 0 & 0 & 0\\ 0 & \frac{2}{3} & 0 & -2 & 0 & \frac{-1}{3}\end{bmatrix},$$

$$2a_{10} + \frac{1}{3}a_{12} = 0$$

$$\frac{2}{3}a_{11} - 2a_{20} - \frac{1}{3}a_{22} = 0.$$

Now

$$a_{10} = c_1,\ a_{11} = c_2,\ a_{12} = -6c_1$$

$$a_{20} = c_3,\ a_{21} = c_4,\ a_{22} = 2c_2 - 6c_3.$$

The exact solution is given by

$$y_1(t) = c_1 + c_2t - 3c_1t^2$$

$$y_2(t) = c_3 + c_4t + (c_2 - 3c_3)t^2.$$

Example 2:

$$xy_1''' + y_1''' = x^2 + \frac{1}{3} + \int_{-1}^{1}[ty_1 - t^3y_2]\mathrm{d}t,$$

$$y_1''' - x^2 y_2''' = -x^2 + x - \frac{2}{3} + \int_{-1}^{1} [(t+2)y_1 - t y_2]\,dt.$$

Solution:

$$P_3 = \begin{bmatrix} x & 1 \\ 1 & -x^2 \end{bmatrix},\ f(x) = \begin{bmatrix} x^2 + \frac{1}{3} \\ -x^2 + x - \frac{2}{3} \end{bmatrix},$$

$$K(x,t) = \begin{bmatrix} t & -t^3 \\ t+2 & -t \end{bmatrix},$$

$$t_0 = -1,\ t_1 = \frac{-1}{3},\ t_2 = \frac{1}{3},\ t_3 = 1.$$

$$P = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{3} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{-1}{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{-1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix},\ F = \begin{bmatrix} \frac{4}{3} \\ \frac{-8}{3} \\ \frac{4}{9} \\ \frac{-10}{9} \\ \frac{4}{9} \\ \frac{-4}{9} \\ \frac{4}{3} \\ \frac{-2}{3} \end{bmatrix},$$

$$H = \begin{bmatrix} 2 & 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{2}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{2}{7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & \frac{2}{7} \end{bmatrix},\ M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2!} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3!} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2!} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3!} \end{bmatrix},$$

$$\mathrm{xM}_3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\ T = \begin{bmatrix} T(x_0) & 0 \\ 0 & T(x_0) \\ T(x_1) & 0 \\ 0 & T(x_1) \\ T(x_2) & 0 \\ 0 & T(x_2) \\ T(x_3) & 0 \\ 0 & T(x_3) \end{bmatrix},$$

$$T(x_0) = [1\ \ -1\ \ 1\ \ -1],\quad K_{11}(x_1) = [0\ \ 1\ \ 0\ \ 0],$$

$$T(x_1) = \left[1\ \ \frac{-1}{3}\ \ \left(\frac{-1}{3}\right)^2\ \ \left(\frac{-1}{3}\right)^3\right],\quad K_{12}(x_i) = [0\ \ 0\ \ 0\ \ -1],$$

$$T(x_2) = \left[1\ \ \frac{1}{3}\ \ \left(\frac{1}{3}\right)^2\ \ \left(\frac{1}{3}\right)^3\right],\quad K_{21}(x_i) = [2\ \ 1\ \ 0\ \ 0],$$

$$T(x_3) = [1\ \ 1\ \ 1\ \ 1],\quad K_{22}(x_i) = [0\ \ -1\ \ 0\ \ 0],$$

$$(PTM_3 - KHM_0)A = F,$$

$$W = \begin{bmatrix} -1 & -0.3333 & -0.5 & 0.1 & 1 & -0.6 & 0.5 & -0.119 \\ -3 & -1.6667 & -0.1667 & -0.12333 & -1 & 1.667 & -0.5 & 0.233 \\ -0.3333 & -0.5556 & -0.1199 & -0.065 & 1 & 0.067 & 0.056 & 0.041 \\ -3 & -1 & -0.6111 & -0.0073 & -0.11111 & 0.704 & -0.0061 & 0.067 \\ -0.3333 & -0.5556 & -0.0199 & -0.065 & 1 & 0.733 & 0.056 & 0.054 \\ -3 & -0.3333 & -0.6111 & 0.06 & -0.1111 & 0.63 & 0.0061 & 0.566 \\ 1 & -0.3333 & 0.5 & 0.666 & 0.76 & 0.654 & 0.76 & 0.0876 \\ -3 & -0.3333 & -0.1667 & 0.76 & 0.547 & 0.6535 & 0.7189 & 0.646 \end{bmatrix},$$

$$A = \begin{bmatrix} -8.549 \times 10^{-4} \\ 1 \\ 5.121 \times 10^{-3} \\ 8.5484 \times 10^{-3} \\ 1 \\ 8.639 \times 10^{-4} \\ -3.497 \times 10^{-4} \\ -0.015 \end{bmatrix}.$$

Taylor polynomial approach

$$y_1(t) = t, \; y_2(t) = 1.$$

Comparison tables

Similarly, the results are obtained for different values of N as presented in Figure 1 and Tables 1–3.

Example 3:

$$y_1(x) - 2xy_2(x) = 22x + 3 + 3\int (x + t)y_1(t)\mathrm{d}t + 3\int (x - t)y_2(t)\mathrm{d}t,$$

$$5y_1(x) + y_2(x) = -x + 9 + 3\int x^2 y_1(t)\mathrm{d}t + 3\int (xt - t^2)y_2(t)\mathrm{d}t.$$

Solution:

For $N = 2$

$$P = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 5 & 1 \end{bmatrix}, \; F = \begin{bmatrix} -19 \\ 10 \\ 3 \\ 9 \\ 25 \\ 8 \end{bmatrix},$$

$$T = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix},$$

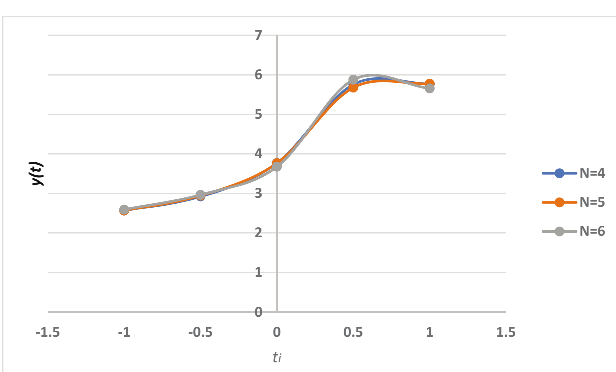

Figure 1: Numerical results of $y(t_i)$.

Table 1: Numerical results of $y_1(t_i)$ and $y_2(t_i)$

	***N* = 2**	***N* = 2**	***N* = 3**	***N* = 3**	**Chebyshev**	**Method**
t_i	$y_1(t_i)$	$y_2(t_i)$	$y_1(t_i)$	$y_2(t_i)$	$y_1(t_i)$	$y_2(t_i)$
0	-2.858×10^{-4}	1	-8.549×10^{-4}	1	0	1
0.1	0.1	1	0.0999	1	0.1	1
0.2	0.2	1	0.1999	1	0.2	1
0.3	0.3	1	0.2999	1	0.3	1
0.4	0.4	1	0.3999	1	0.4	1
0.5	0.5	1	0.5	1.001	0.5	1
0.6	0.6	1	0.6	1.001	0.6	1
0.7	0.7	1	0.7	1.001	0.7	1
0.8	0.8	1	0.8	1.001	0.8	1
0.9	0.9	1	0.9	1	0.9	1
1	1	1	1	1	1	1

Table 2: Numerical result of $y_1(x)$

t_i	**Present** $N = 6$	**Chebyshev** $N = 6$	**Exact** $N = 6$
−1	0.476	0.498	0.4987
−1/2	0.5987	0.5876	0.5321
0	0.8765	0.8543	0.86554
1/2	0.465	0.4987	0.4987
1	0.5876	0.567	0.5434

Table 3: Numerical result of $y_2(x)$

t_i	Present $N = 6$	Chebyshev $N = 6$	Exact $N = 6$
−1	0.6557	0.6574	0.6572
−1/2	0.70435	0.7314	0.7288
0	0.56543	0.64375	0.5743
1/2	0.5389	0.5763	0.5373
1	0.5876	0.567	0.5434

$$K = \begin{bmatrix} -3 & 3 & 0 & -3 & -3 & 0 \\ 3 & 0 & 0 & 0 & -3 & 3 \\ 0 & 3 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 3 & 0 & 3 & -3 & 0 \\ 3 & 0 & 0 & 0 & 3 & 3 \end{bmatrix},$$

$$M_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2!} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2!} \end{bmatrix},$$

$$(PTM_0 - KHM_0)A = F.$$

After simplification, the Taylor coefficient of the matrix is obtained

$$A = \begin{bmatrix} 1 \\ 0 \\ 4 \\ -4 \\ 1 \\ 3.1 \times 10^{-14} \end{bmatrix},$$

$$y_1(x) = 1 + 2x^2,$$

$$y_2(x) = -4 + x + 1.55 \times 10^{-14}x^2.$$

4 Conclusion

Many scientific and engineering issues use integral equations. Volterra or Fredholm integral equations can be used to solve a wide range of starting and boundary value issues. Potential theory assisted more than any other discipline to the development of integral equations. Integral equations were also influenced by mathematical physics models such as dispersion difficulties, scattered in quantum physics, conformal mapping, and water waves. Solving integral-differential equalities analytically is frequently difficult. Obtaining approximate solutions is usually required. For this purpose, the procedure mentioned can be recommended. The coefficients in Taylor expansion of the solution of a linear integro-differential equation are calculated using the approach outlined in this study, and it works whenever the function is a linear integro-differential equation. $P_k(x)$ is the kernel function of $K(x,t)$. In this range, there is a Taylor series expansion. Furthermore, it appears that the approach works best when the functions are combined. It is possible to extend the Taylor series of $K(x,t)$, $f(x)$, and $P_k(x)$, which converges quickly. We select several terms from the Taylor expansion to get the best approximation solution of the equation; that is, the truncation limit N must have been large enough. An estimate for N, the degree of the approximation polynomial, for computing efficiency (the truncation limit of the Taylor series) to $y(x)$ should also be available. The Taylor collocation method can be used to solve both differential and integral problems. This is demonstrated in example 3. Both differential and integral formulations are the foundation of algorithms for the numerical solution of continuous electromagnetic field issues. The former strategy, utilizing the finite element method, has been the most popular, at least for low-frequency issues. The distinctive benefits of integral equations over differential equations are explored, along with some of the associated challenges. It is suggested that the integral equation technique may be particularly successful when applied to more complex situations, such as moving systems and optimization.

This method can also be used to find an exact solution N or less than N if the equation has an exact solution which is a polynomial of degree. Partial integro-differential equations and ordinary differential equations with variable coefficients can also be solved using this method.

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