

Research Article

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The nonlinear vibration and dispersive wave systems with extended homoclinic breather wave solutions

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Abstract: This article investigates the extended homoclinic (heteroclinic) breather wave solutions and interaction periodic and dark soliton solutions to the nonlinear vibration and dispersive wave systems. The solutions include periodic, breather, and soliton solutions. The bilinear form is considered in terms of Hirota derivatives. Accordingly, we utilize the Cole–Hopf algorithm to obtain the exact solutions of the $(2 + 1)$ -dimensional modified dispersive water-wave system. The analytical treatment of extended homoclinic breather wave solutions and interaction periodic and dark soliton solutions are studied and plotted in four forms of density plots. A nonlinear vibration system will be studied. Employing appropriate mathematical assumptions, the novel kinds of the extended homoclinic breather wave solutions and interaction periodic and dark soliton solutions are derived and constructed in view of the combination of kink, periodic, and soliton for an extended homoclinic breather and also a combination of

two kinks, periodic and dark soliton in terms of exponential, trigonometric, hyperbolic functions for interaction periodic and dark soliton of the governing equation. To achieve this, the illustrative example of the $(2 + 1)$ -D modified dispersive water-wave system is furnished to demonstrate the feasibility and reliability of the procedure applied in this research. The trajectory solutions of the traveling waves are offered explicitly and graphically. The effect of the free parameters on the behavior of designed figures of a few obtained solutions for two nonlinear rational exact cases was also considered. By comparing the suggested scheme with the other existing methods, the results state that the execution of this technique is succinct, extensive, and straightforward.

Keywords: Hirota bilinear operator technique, extended homoclinic breather, periodic wave, nonlinear vibration and dispersive wave systems

1 Introduction

The advent of the concept of nonlinear partial differential equations (NLPDEs) has attracted the interest of many researchers due to their importance in accurately demonstrating the dynamics of abundant real-world systems in various topics of sciences such as physics, diffusion, biology, chaos theory, chemistry, engineering, economics, commerce, and many others [1–3].

In particular, more and much attention has been paid to constructing exact and approximate solutions, for example, the csch-function method [4], an improved Hungarian algorithm [5], a social group optimization algorithm [6], a new solitary periodic wave solution [7], a mathematical performance framework for Hadoop data locality [8], the multiple Exp-function technique [9], the reduced chi-square and root-mean-square error [10], the Hirota's bilinear scheme [11,12], a Pearson's product moment correlation [13], solving absolute value problems [14], the

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inverse scattering transformation method [15], multiple soliton solutions and fusion interaction phenomena [16], the truncated Painlevé series [17], an optical flow approach [18], the modified Pfaffian technique [19], the conserved quantities method [20], and so forth.

In ref. [21], the N -soliton solutions, soliton molecules, and asymmetric solitons of the Korteweg–de Vries–Caudrey–Dodd–Gibbon equation were acquired by utilizing velocity resonance scheme by Ma *et al.* Researchers analyzed the higher-order algebraic soliton solutions of the Gerdjikov–Ivanov equation by utilizing the Darboux transformation and some limit technique, and according to the asymptotic balance between different algebraic terms, they obtained the asymptotic expressions of algebraic soliton solutions [22]. Wang studied the multi-soliton solutions of the $(2 + 1)$ -dimensional PT-symmetric couplers with varying coefficients by using homogeneous balance scheme [23]. The N -soliton solutions, M -breather solutions, and hybrid ones composed of solitons and breathers were established as a time-dependent KP equation by Wu [24].

Here, consider $(2 + 1)$ -dimensional modified dispersive water-wave system as follows:

$$\Sigma_{yt} + \Sigma_{xy} - 2\Gamma_{xx} - (\Sigma^2)_{xy} = 0, \quad \Gamma_t - \Gamma_{xx} - 2(\Sigma\Gamma)_x = 0, \quad (1.1)$$

where Σ and Γ are the unknown functions. The $(2 + 1)$ -dimensional modified dispersive water-wave system has been made in a model nonlinear and dispersive long gravity waves traveling in two horizontal directions on shallow waters of uniform depth [25]. In refs [26,27], the propagating localized excitations by using the Painlevé–Bäcklund transformation and a variable separation technique have been investigated for some $(2 + 1)$ -dimensional integrable systems. The powerful scholars have studied on nontravelling wave solutions [28]; abundant propagating, and nonpropagating solitons such as dromion, ring, peakon, and compacton [29]; solitary wave solutions, periodic wave solutions and rational function solutions [30]; the traveling wave solutions by the improved $\tan(\phi/2)$ -expansion technique [31]; and periodic folded waves solutions [32]. In ref. [33], a pair of quartic-linear forms for the modified dispersive water-wave system was constructed by choosing an appropriate seed solution in the truncated Painlevé series. Also, the multiple soliton solutions and fusion interaction phenomena were derived with the help of the Bäcklund transformation and bilinear form technique [34]. The nonlinear vibration system behaviors was investigated by obtaining a breather type of rogue wave solution, which contains two peaks and also rogue wave by Li [35].

By reducing NLPDE with the help of Hirota's bilinear method to nonlinear ordinary differential equation (NODE), an algebraic equation system was obtained by Maple. By solving this system, plenty wave solutions including lump and other interactions have been found therein [36–39]. Many researchers used various methods to study the nonlinear models by using Hirota bilinear technique (HBT), such as a $(3 + 1)$ -dimensional nonlinear evolution equation [40], generalized variable-coefficient Kadomtsev–Petviashvili equation [41], the new $(3 + 1)$ -dimensional generalized Kadomtsev–Petviashvili equation [42], and a generalized $(3 + 1)$ -D VC NLW equation [43]. Through these methods, some exact solutions of the nonlinear models of equations were obtained. To really understand these physical phenomena, it is of immense importance to solve nonlinear partial differential equations (NLPDEs), which govern these aforementioned phenomena. However, there is no general systematic theory that can be applied to NLPDEs, so that their analytic solutions can be obtained. Nevertheless, in recent times, scientists have developed effective techniques to obtain viable analytical solutions to NLPDEs, such as some nonlinear equations, for example, an analytical analysis to solve the fractional differential equations [44], an efficient alternating direction explicit method for solving a PDE [45], the semi-analytical solutions of the positive Gardner–Kadomtsev–Petviashvili equation [46], and so on. In ref. [47], M -lump solution and N -soliton solution of the $(2 + 1)$ -dimensional variable-coefficient Caudrey–Dodd–Gibbon–Kotera–Sawada equation were studied. Also, N -lump and interaction solutions of localized waves to the $(2 + 1)$ -dimensional generalized KDKK equation were investigated and obtained [48]. In particular, more and much attention has been paid to constructing exact lump solutions to the third-order evolution equation [49] and triple-soliton kind solutions to the modified HSI equation. In ref. [50], the vibration mode was acquired by finite element technology to explore the vibration response of the GIS busbar enclosure in a strong electric field. The authors of ref. [51] studied the Biswas–Arshed model in birefringent fibers for chirp-free solitons with the aid of sub-ordinary differential equations method. The new three-dimensional modified Benjamin–Bona–Mahony equations were analyzed with the introduction of the spatial and temporal fractional order derivatives using conformable fractional derivative [52]. The Hirota bilinear form has used the $(3 + 1)$ -dimensional Vakhnenko–Parkes equation to obtain the types of multi-waves, breather wave, lump-kink, lump-periodic solutions, and interaction solutions [53]. By implementing modified

analytical and numerical methods, the construction of the analytical and numerical wave solutions for the Ito integro-differential dynamical equation were reached in ref. [54]. The extended form of two methods, auxiliary equation mapping and direct algebraic methods, were used to find the families of new exact travelling wave solutions of the system of equations for the ion sound and Langmuir waves [55]. N-soliton solutions have been studied recently and extensively, by the Hirota bilinear method and the Riemann–Hilbert problems, for local integrable equations in (1 + 1)-dimensions involving [56] and in (2 + 1)-dimensions involving [57] and also for nonlocal integrable equations containing Riemann–Hilbert problems [58,59].

Motivated by the aforementioned studies, we apply the proposed analytical method presented by ref. [35] to solve the one problem mentioned earlier. The advantage of the suggested technique is that it can be applied to the integrable model and plenty of the cross-kink and solitary wave solutions with three kinds of plotted graphs will be obtained.

The rest of this article is organized as follows: Section 2 presents the definition of the binary Bell polynomials and their properties. Section 3 proposes the method based on the given algorithm transformation, for obtaining the extended homoclinic breather wave solutions and interaction periodic and dark soliton of system (1.1). Finally, Section 4 briefly summarizes and discusses the results in a conclusion.

2 Binary Bell polynomials

Through ref. [60], take $\lambda = \lambda(x_1, x_2, \dots, x_n)$ is a C^∞ function with multiparameters, and the general form can be written as follows:

$$Y_{n_1 x_1, \dots, n_j x_j}(\lambda) \equiv Y_{n_1, \dots, n_j}(\lambda_{d_1 x_1, \dots, d_j x_j}) = e^{-\lambda} \partial_{x_1}^{n_1} \dots \partial_{x_j}^{n_j} e^\lambda, \quad (2.1)$$

which are named the multi-D Bell polynomials as follows:

$$\begin{aligned} \lambda_{d_1 x_1, \dots, d_j x_j} &= \partial_{x_1}^{d_1} \dots \partial_{x_j}^{d_j} \lambda, \quad \lambda_{0 x_i} \equiv \lambda, \\ d_1 &= 0, \dots, n_1; \dots; d_j = 0, \dots, n_j, \end{aligned}$$

and we have

$$\begin{aligned} Y_1(\lambda) &= \lambda_x, \\ Y_2(\lambda) &= \lambda_{2x} + \lambda_x^2, \quad Y_3(\lambda) = \lambda_{3x} + 3\lambda_x \lambda_{2x} + \lambda_x^3, \dots, \\ \lambda &= \lambda(x, t), \\ Y_{x,t}(\lambda) &= \lambda_{x,t} + \lambda_x \lambda_t, \\ Y_{2x,t}(\lambda) &= \lambda_{2x,t} + \lambda_{2x} \lambda_t + 2\lambda_{x,t} \lambda_x + \lambda_x^2 \lambda_t, \dots \end{aligned} \quad (2.2)$$

The multidimensional binary Bell polynomials can be expressed as follows:

$$\begin{aligned} \Sigma_{n_1 x_1, \dots, n_j x_j}(\mu_1, \mu_2) \\ = Y_{n_1, \dots, n_j}(\lambda) \Big|_{\lambda_{d_1 x_1, \dots, d_j x_j} = \begin{cases} \mu_{1 d_1 x_1, \dots, d_j x_j}, & d_1 + d_2 + \dots + d_j, \text{ is odd} \\ \mu_{2 d_1 x_1, \dots, d_j x_j}, & d_1 + d_2 + \dots + d_j, \text{ is even} \end{cases}} \end{aligned} \quad (2.3)$$

The following issues can be expressed as follows:

$$\begin{aligned} \Sigma_x(\mu_1) &= \mu_{1x}, \quad \Sigma_{2x}(\mu_1, \mu_2) = \mu_{22x} + \mu_{1x}^2, \\ \Sigma_{x,t}(\mu_1, \mu_2) &= \mu_{2x,t} + \mu_{1x} \mu_{1t}, \dots \end{aligned} \quad (2.4)$$

Proposition 2.1. Let $\mu_1 = \ln(\Omega_1/\Omega_2)$, $\mu_2 = \ln(\Omega_1\Omega_2)$, then the relations between binary Bell polynomials and Hirota D-operator read:

$$\begin{aligned} \Sigma_{n_1 x_1, \dots, n_j x_j}(\mu_1, \mu_2) \Big|_{\mu_1 = \ln(\Omega_1/\Omega_2), \mu_2 = \ln(\Omega_1\Omega_2)} \\ = (\Omega_1\Omega_2)^{-1} D_{x_1}^{n_1} \dots D_{x_j}^{n_j} \Omega_1\Omega_2, \end{aligned} \quad (2.5)$$

via Hirota operator

$$\begin{aligned} \prod_{i=1}^j D_{x_i}^{n_i} g \cdot \eta \\ = \prod_{i=1}^j \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x'_i} \right)^{n_i} \Omega_1(x_1, \dots, x_j) \Omega_2(x'_1, \dots, x'_j) \Big|_{x_1=x'_1, \dots, x_j=x'_j}. \end{aligned} \quad (2.6)$$

Proposition 2.2. Take $\Xi(\gamma) = \sum_i \delta_i \mathfrak{P}_{d_1 x_1, \dots, d_j x_j} = 0$ and $\mu_1 = \ln(\Omega_1/\Omega_2)$, $\mu_2 = \ln(\Omega_1\Omega_2)$, we have

$$\begin{cases} \sum_i \delta_i Y_{n_1 x_1, \dots, n_j x_j}(\mu_1, \mu_2) = 0, \\ \sum_i \delta_i Y_{d_1 x_1, \dots, d_j x_j}(\mu_1, \mu_2) = 0, \end{cases} \quad (2.7)$$

which need to satisfy

$$\begin{aligned} \mathfrak{R}(\gamma', \gamma) &= \mathfrak{R}(\gamma') - \mathfrak{R}(\gamma) = \mathfrak{R}(\mu_2 + \mu_1) - \mathfrak{R}(\mu_2 - \mu_1) \\ &= 0. \end{aligned} \quad (2.8)$$

The generalized Bell polynomials $Y_{n_1 x_1, \dots, n_j x_j}(\xi)$ is expressed as follows:

$$\begin{aligned} (\Omega_1\Omega_2)^{-1} D_{x_1}^{n_1} \dots D_{x_j}^{n_j} \Omega_1\Omega_2 \\ = \Sigma_{n_1 x_1, \dots, n_j x_j}(\mu_1, \mu_2) \Big|_{\mu_1 = \ln(\Omega_1/\Omega_2), \mu_2 = \ln(\Omega_1\Omega_2)} \\ = \Sigma_{n_1 x_1, \dots, n_j x_j}(\mu_1, \mu_1 + \gamma) \Big|_{\mu_1 = \ln(\Omega_1/\Omega_2), \gamma = \ln(\Omega_1\Omega_2)} \\ = \sum_{k_1}^{n_1} \dots \sum_{k_j}^{n_j} \prod_{i=1}^j \binom{n_i}{k_i} \mathfrak{P}_{k_1 x_1, \dots, k_j x_j}(\gamma) Y_{(n_1-k_1)x_1, \dots, (n_j-k_j)x_j}(\mu_1). \end{aligned} \quad (2.9)$$

The Cole–Hopf relation is expressed as follows:

$$Y_{k_1 x_1, \dots, k_j x_j}(\mu_1 = \ln(\varphi)) = \frac{\varphi_{n_1 x_1, \dots, n_j x_j}}{\varphi}, \quad (2.10)$$

$$\begin{aligned}
& (\Omega_1 \Omega_2)^{-1} D_{x_1}^{n_1} \dots D_{x_j}^{n_j} \Omega_1 \Omega_2 |_{\Omega_2 = \exp(y/2), \Omega_1/\Omega_2 = \varphi} \\
& = \varphi^{-1} \sum_{k_1}^{n_1} \dots \sum_{k_j}^{n_j} \prod_{l=1}^j \binom{n_l}{k_l} \mathfrak{P}_{k_1 x_1, \dots, k_j x_j}(y) \varphi_{(n_1-k_1)x_1, \dots, (n_d-k_d)x_d},
\end{aligned} \quad (2.11)$$

with

$$\begin{aligned}
Y_l(\mu_1) &= \frac{\varphi_l}{\varphi}, \quad Y_{2x}(\mu_1, \beta) = Y_{2x} + \frac{\varphi_{2x}}{\varphi}, \\
Y_{2x,y}(\mu_1, \mu_2) &= \frac{Y_{2x}\varphi_y}{\varphi} + \frac{2Y_{x,y}\varphi_x}{\varphi} + \frac{\varphi_{2x,y}}{\varphi}.
\end{aligned} \quad (2.12)$$

Theorem 2.3. According to ref. [35], the used dependent variable transformation with the logarithm of function $f(x, y, t)$

$$\Sigma = \frac{\partial}{\partial x}(\ln h) + \Sigma_0, \quad \Gamma = \frac{\partial^2}{\partial x^2}(\ln h) + \Gamma_0, \quad (2.13)$$

where $\Sigma_0 = \Sigma_0(x, y, t)$ and $\Gamma_0 = \Gamma_0(x, y, t)$ are known solutions of system stated by system (1.1). By substituting $\Sigma_0 = 0$ and $\Gamma_0 = 0$ in (2.13) and inserting concluded solution in (1.1), we obtain the below nonlinear PDE based on h :

$$\begin{aligned}
& \frac{d}{dy} \left(\frac{1}{y^2} (hh_{xt} - h_x h_t + h_x h_{xx} - hh_{xxx}) \right) = 0, \\
& h = h(x, y, t).
\end{aligned} \quad (2.14)$$

Then, one obtains

$$hh_{xt} - h_x h_t + h_x h_{xx} - hh_{xxx} = 0. \quad (2.15)$$

3 Two types of solutions for MDWW system

Lump-three soliton solutions and multiwave solutions are investigated in the following sections.

3.1 Extended homoclinic (heteroclinic) breather wave solutions

Here, we utilize to formulate the new exact solutions to the $(2+1)$ -dimensional MDWW system. Consider the below function for studying the extended homoclinic (heteroclinic) breather wave solutions as follows:

$$\begin{aligned}
h &= \tau_1(y)e^{a_1} + \tau_2(y)\cos(d(y)(x + e(y))) \\
&+ \tau_3(y)\sin(d(y)(x + e(y))) + \tau_4(y)e^{a_2},
\end{aligned} \quad (3.1)$$

$$a_l = m_l x + n_l(y) + p_l(y)t, \quad l = 1, 2.$$

Afterward, the values m_l , $n_l(y)$, $p_l(y)$, and $(l = 1:2)$ will be found. By substituting Eq. (3.1) into (2.15) and taking the coefficients of power of $\exp(a_l)$, $l = 1, 2$ and $\cos(d(y)(x + e(y)))$, $\sin(d(y)(x + e(y)))$ to be zero, we yield a system of equations (algebraic) (these are not collected here for minimalist) for m_l , $n_l(y)$, $p_l(y)$, $(l = 1:2)$. These algebraic equations with the help of emblematic computation software like, Maple, give the following solutions by using $\Sigma = (\ln h)_x$ and $\Gamma = (\ln h)_{xy}$:

$$\Sigma = \frac{\tau_1(y)m_1 e^{a_1} - \tau_2(y)\sin(d(y)(x + e(y)))d(y) + \tau_3(y)\cos(d(y)(x + e(y)))d(y) + \tau_4(y)m_2 e^{a_2}}{\tau_1(y)e^{a_1} + \tau_2(y)\cos(d(y)(x + e(y))) + \tau_3(y)\sin(d(y)(x + e(y))) + \tau_4(y)e^{a_2}},$$

$$\begin{aligned}
\Gamma &= \frac{M_4 + M_1 + M_2 + M_3 + M_5}{M_6} \\
&- \frac{N_1(N_5 + N_6 + N_4 + N_3 + N_2)}{M_6^2},
\end{aligned}$$

$$S_1 = \sin(d(y)(x + e(y))), \quad S_2 = \cos(d(y)(x + e(y))),$$

$$\begin{aligned}
M_1 &= -S_1 \left(\left(\frac{d}{dy} \tau_2(y) \right) d(y) + \tau_2(y) \frac{d}{dy} d(y) \right) \\
&+ S_2 \left(\tau_3(y) \frac{d}{dy} d(y) + d(y) \frac{d}{dy} \tau_3(y) \right),
\end{aligned}$$

$$M_2 = -\tau_2(y)S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_3 = -\tau_3(y)S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_4 = e^{a_1} m_1 \left(\tau_1(y) \left(\frac{d}{dy} p_1(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$M_5 = e^{a_2} m_2 \left(\left(\frac{d}{dy} p_2(y) \right) \tau_4(y) t + \tau_4(y) \frac{d}{dy} n_2(y) + \frac{d}{dy} \tau_4(y) \right),$$

$$M_6 = \tau_1(y)e^{a_1} + \tau_2(y)S_2 + \tau_3(y)S_1 + \tau_4(y)e^{a_2},$$

$$N_1 = \tau_1(y)m_1 e^{a_1} - \tau_2(y)S_1 d(y) + \tau_3(y)S_2 d(y) + \tau_4(y)m_2 e^{a_2},$$

$$N_2 = e^{a_2} \left(\left(\frac{d}{dy} p_2(y) \right) \tau_4(y) t + \tau_4(y) \frac{d}{dy} n_2(y) + \frac{d}{dy} \tau_4(y) \right),$$

$$N_3 = \tau_3(y)S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_4 = -\tau_2(y)S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_5 = e^{a_1} \left(\tau_1(y) \left(\frac{d}{dy} p_1(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$N_6 = \left(\frac{d}{dy} \tau_2(y) \right) S_2 + \left(\frac{d}{dy} \tau_3(y) \right) S_1.$$

3.1.1 Set I solutions

$$m_l = m_l, \quad l = 1, 2, \quad p_2(y) = p_2(y), \quad p_1(y) = m_1^2 + d(y)^2, \quad \tau_4(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 2, 3, \quad (3.2)$$

where $m_l, l = 1, 2$, and $n_l(y), l = 1, 2$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_1 = \frac{\tau_1(y)m_1 e^{m_1 x + n_1(y) + ((d(y))^2 + m_1^2)t} - \tau_2(y) \sin(d(y)(x + e(y)))d(y) + \tau_3(y) \cos(d(y)(x + e(y)))d(y)}{\tau_1(y)e^{m_1 x + n_1(y) + ((d(y))^2 + m_1^2)t} + \tau_2(y) \cos(d(y)(x + e(y))) + \tau_3(y) \sin(d(y)(x + e(y)))}, \quad (3.3)$$

$$\Gamma_1 = \frac{M_4 + M_1 + M_2 + M_3}{M_6} - \frac{N_1(N_5 + N_6 + N_4 + N_3)}{M_6^2}, \quad (3.4)$$

$$S_1 = \sin(d(y)(x + e(y))), \quad S_2 = \cos(d(y)(x + e(y))),$$

$$M_1 = -S_1 \left(\left(\frac{d}{dy} \tau_2(y) \right) d(y) + \tau_2(y) \frac{d}{dy} d(y) \right) + S_2 \left(\tau_3(y) \frac{d}{dy} d(y) + d(y) \frac{d}{dy} \tau_3(y) \right),$$

$$M_2 = -\tau_2(y) S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_3 = -\tau_3(y) S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_4 = e^{a_1} m_1 \left(q_1(y) \left(\frac{d}{dy} p_1(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$M_6 = \tau_1(y) e^{a_1} + \tau_2(y) S_2 + \tau_3(y) S_1,$$

$$N_1 = \tau_1(y) m_1 e^{a_1} - \tau_2(y) S_1 d(y) + \tau_3(y) S_2 d(y),$$

$$N_3 = \tau_3(y) S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_4 = -\tau_2(y) S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_5 = e^{a_1} \left(q_1(y) \left(\frac{d}{dy} p_1(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$N_6 = \left(\frac{d}{dy} \tau_2(y) \right) S_2 + \left(\frac{d}{dy} \tau_3(y) \right) S_1,$$

$$a_1 = m_1 x + n_1(y) + (m_1^2 + d(y)^2)t, \quad a_2 = 0.$$

Figures 1–3 show the analysis of the treatment of extended homoclinic (heteroclinic) breather wave solution with two-wave solution, where graphs of Γ_1 are given with the following selected parameters:

$$m_1 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad n_1(y) = y, \quad d(y) = y, \quad e(y) = y, \quad (3.5)$$

$$m_1 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad n_1(y) = \frac{y}{3}, \quad d(y) = 3, \quad e(y) = y, \quad (3.6)$$

$$m_1 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad n_1(y) = y^2, \quad d(y) = y, \quad e(y) = y^3, \quad (3.7)$$

given in Eq. (3.4). By the aforementioned parameters, the structural property homoclinic breather wave solutions as presented in Figures 1–3 with density plot with different times. It shows a type of interaction solutions between exponential and trigonometric waves.

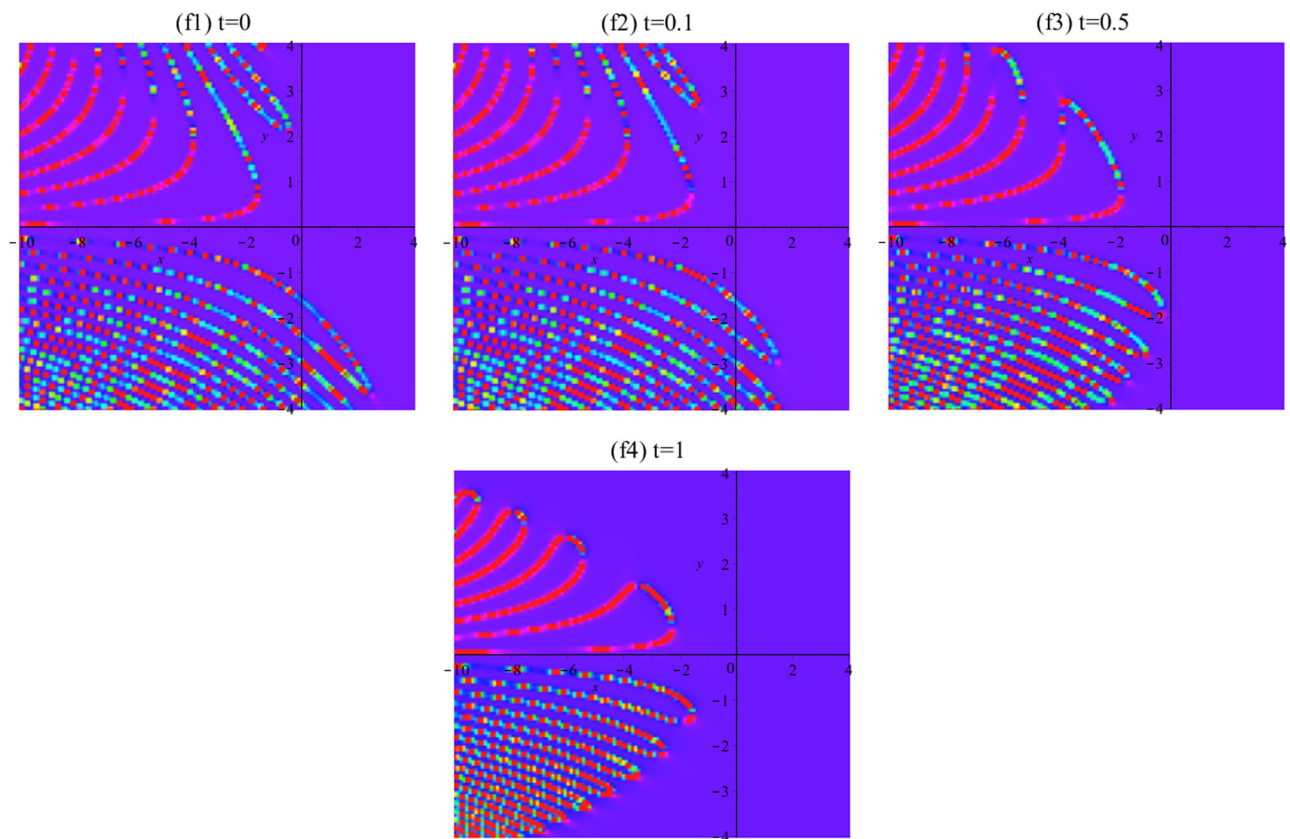


Figure 1: Plot of the extended homoclinic breather wave solution (3.5) (f_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

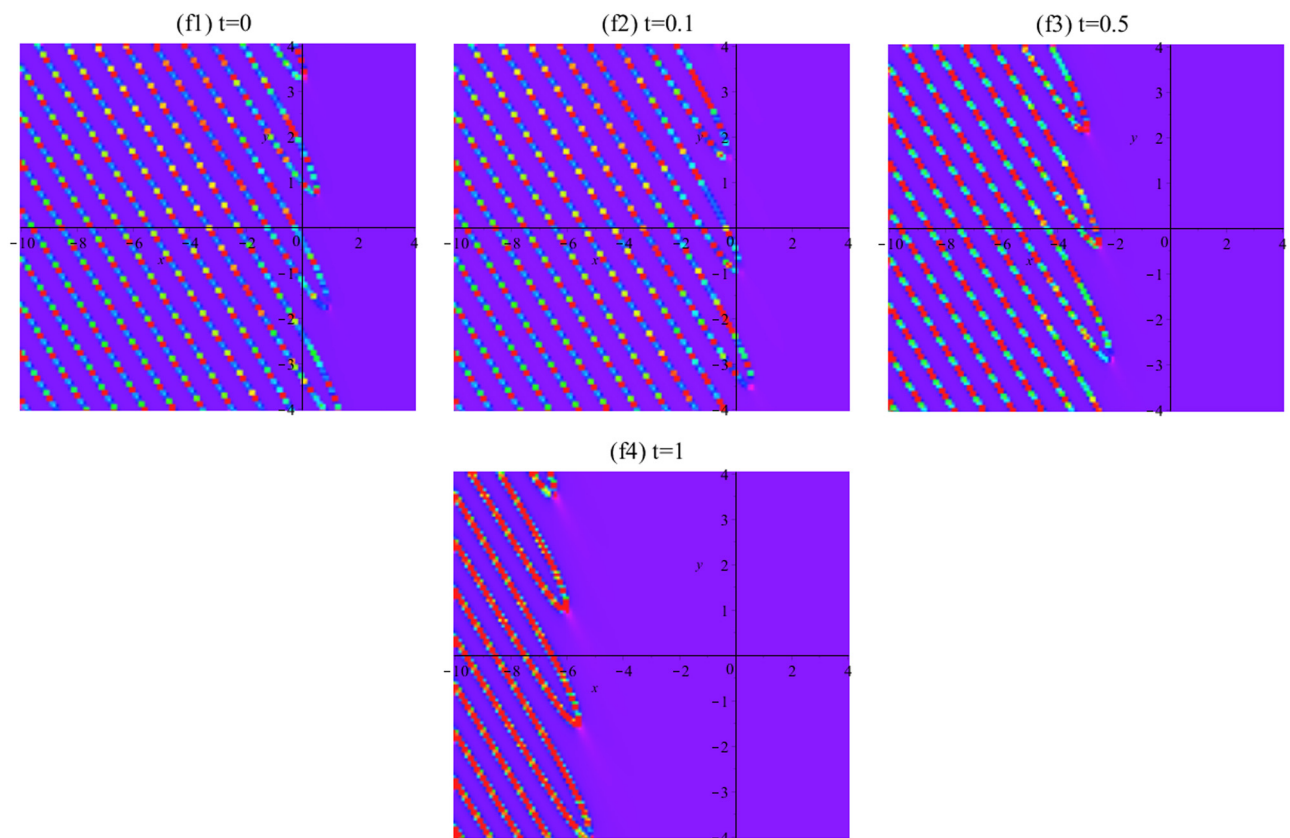


Figure 2: Plot of the extended homoclinic breather wave solution (3.6) (f_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

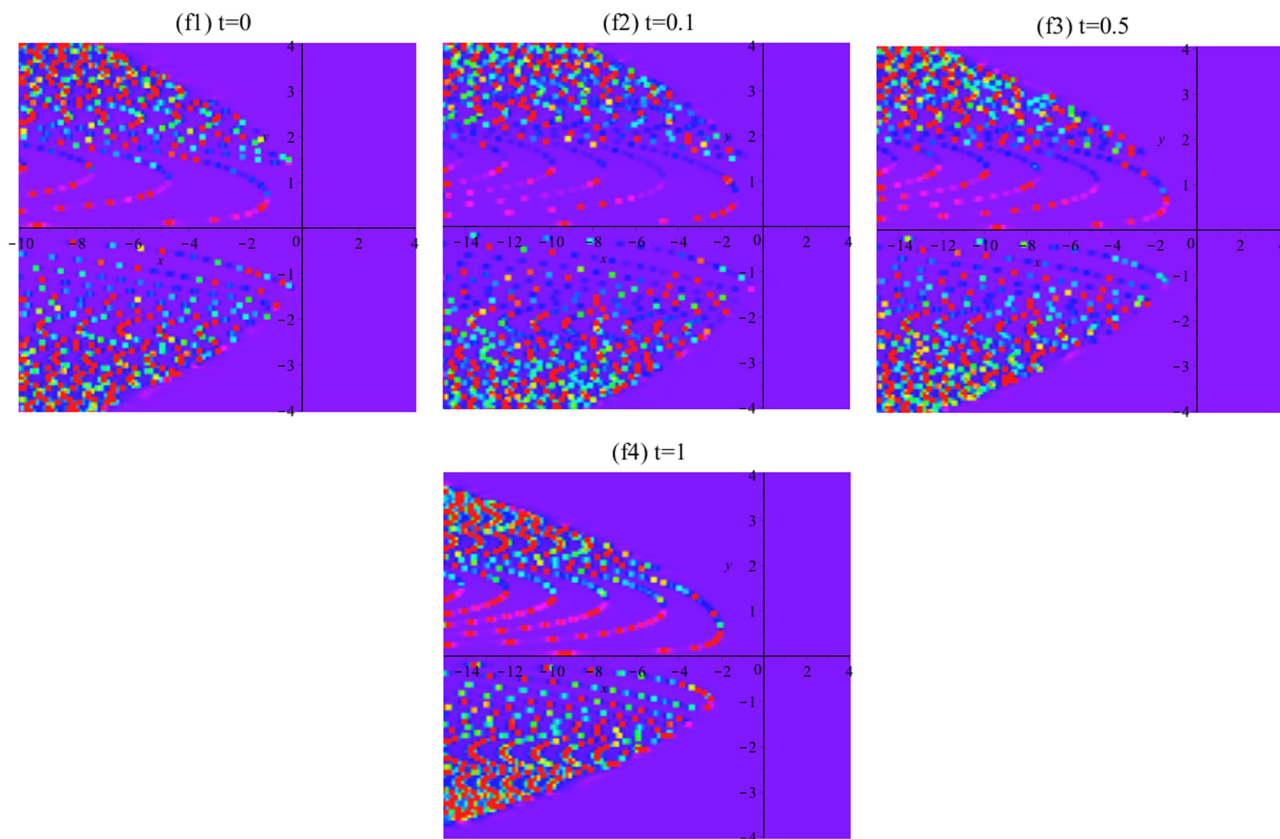


Figure 3: Plot of the extended homoclinic breather wave solution (3.7) (Γ_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

3.1.2 Set II solutions

$$m_l = m_l, \quad l = 1, 2, \quad p_2(y) = m_2^2 + d(y)^2 + p_2, \quad p_1(y) = p_1(y), \quad \tau_1(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 2, 3, 4, \quad (3.8)$$

where $m_l, l = 1, 2$, and $n_l(y), l = 1, 2$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_2 = \frac{-\tau_2(y) \sin(d(y)(x + e(y)))d(y) + \tau_3(y) \cos(d(y)(x + e(y)))d(y) + \tau_4(y)m_2 e^{m_2 x + n_2(y) + ((d(y))^2 + m_2^2)t}}{\tau_2(y) \cos(d(y)(x + e(y))) + \tau_3(y) \sin(d(y)(x + e(y))) + \tau_4(y)e^{m_2 x + n_2(y) + ((d(y))^2 + m_2^2)t}}, \quad (3.9)$$

$$\Gamma_2 = \frac{d}{dy} \left(\frac{-\tau_2(y) \sin(d(y)(x + e(y)))d(y) + \tau_3(y) \cos(d(y)(x + e(y)))d(y) + \tau_4(y)m_2 e^{m_2 x + n_2(y) + ((d(y))^2 + m_2^2)t}}{\tau_2(y) \cos(d(y)(x + e(y))) + \tau_3(y) \sin(d(y)(x + e(y))) + \tau_4(y)e^{m_2 x + n_2(y) + ((d(y))^2 + m_2^2)t}} \right). \quad (3.10)$$

3.1.3 Set III solutions

$$m_l = m_l, \quad l = 1, 2, \quad p_2(y) = p_2(y), \quad p_1(y) = m_2^2 + d(y)^2, \quad \tau_4(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 2, \quad \tau_3(y) = \frac{m_2 \tau_2(y)}{d(y)}, \quad (3.11)$$

where $m_l, l = 1, 2$, and $n_l(y), l = 1, 2$, are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_3 = \frac{\tau_1(y)m_2 e^{m_2 x + n_1(y) + ((d(y))^2 + m_2^2)t} - \tau_2(y) \sin(d(y)(x + e(y)))d(y) + m_2 \tau_2(y) \cos(d(y)(x + e(y)))}{\tau_1(y)e^{m_2 x + n_1(y) + ((d(y))^2 + m_2^2)t} + \tau_2(y) \cos(d(y)(x + e(y))) + \frac{m_2 \tau_2(y) \sin(d(y)(x + e(y)))}{d(y)}}, \quad (3.12)$$

$$\Gamma_3 = \frac{d}{dy} \left(\frac{\tau_1(y)m_2 e^{m_2 x + n_1(y) + ((d(y))^2 + m_2^2)t} - \tau_2(y) \sin(d(y)(x + e(y)))d(y) + m_2 \tau_2(y) \cos(d(y)(x + e(y)))}{\tau_1(y)e^{m_2 x + n_1(y) + ((d(y))^2 + m_2^2)t} + \tau_2(y) \cos(d(y)(x + e(y))) + \frac{m_2 \tau_2(y) \sin(d(y)(x + e(y)))}{d(y)}} \right). \quad (3.13)$$

Figures 4–6 show the analysis of the treatment of extended homoclinic (heteroclinic) breather wave solution with two-wave solution, where graphs of Γ_3 are given with the following selected parameters:

$$m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad n_1(y) = y, \quad d(y) = 1, \quad e(y) = -y, \quad (3.14)$$

$$m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad n_1(y) = y, \quad d(y) = \frac{2}{1 + y^2}, \quad e(y) = y + y^3, \quad (3.15)$$

$$m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad n_1(y) = y, \quad d(y) = \cos(y), \quad e(y) = y + y^3, \quad (3.16)$$

given in Eq. (3.13). By the aforementioned parameters, the structural property homoclinic breather wave solutions are presented in Figures 4–6 with density plot with different times. It presents a type of interaction solutions between exponential and trigonometric waves.

3.1.4 Set IV solutions

$$m_l = m_l, \quad l = 1, 2, \quad p_1(y) = m_1^2 + d(y)^2, \quad p_2(y) = m_2^2 + d(y)^2, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 2, 3, 4, \quad (3.17)$$

where m_l , $l = 1, 2$, and $n_l(y)$, $l = 1, 2$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_4 = \frac{\tau_1(y)m_1 e^{a_1} - \tau_2(y) \sin(d(y)(x + e(y)))d(y) + \tau_3(y) \cos(d(y)(x + e(y)))d(y) + \tau_4(y)m_2 e^{a_2}}{\tau_1(y)e^{a_1} + \tau_2(y) \cos(d(y)(x + e(y))) + \tau_3(y) \sin(d(y)(x + e(y))) + \tau_4(y)e^{a_2}}, \quad (3.18)$$

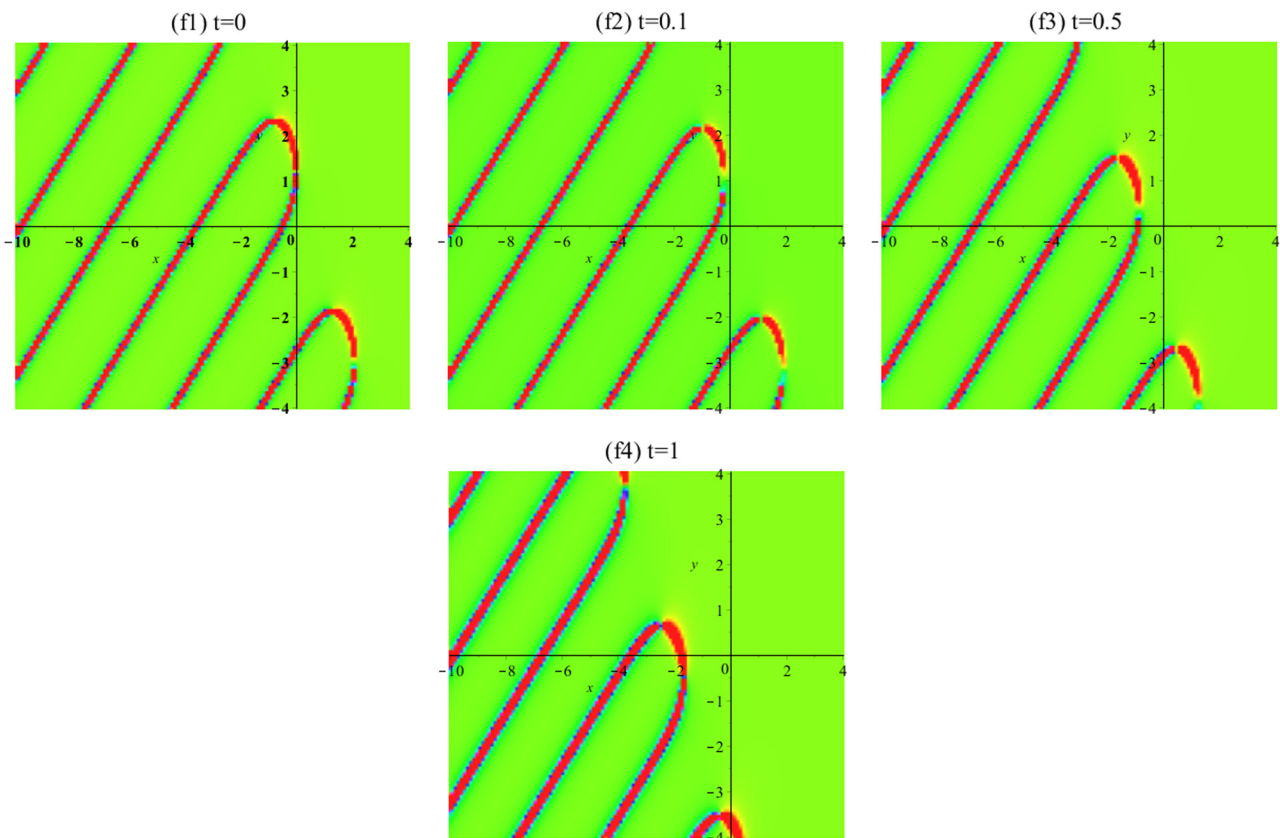


Figure 4: Plot of the extended homoclinic breather wave solution (3.14) (Γ_3) such as density plot for four cases $t = 0$, $t = 0.1$, 0.5 , 1 , respectively.

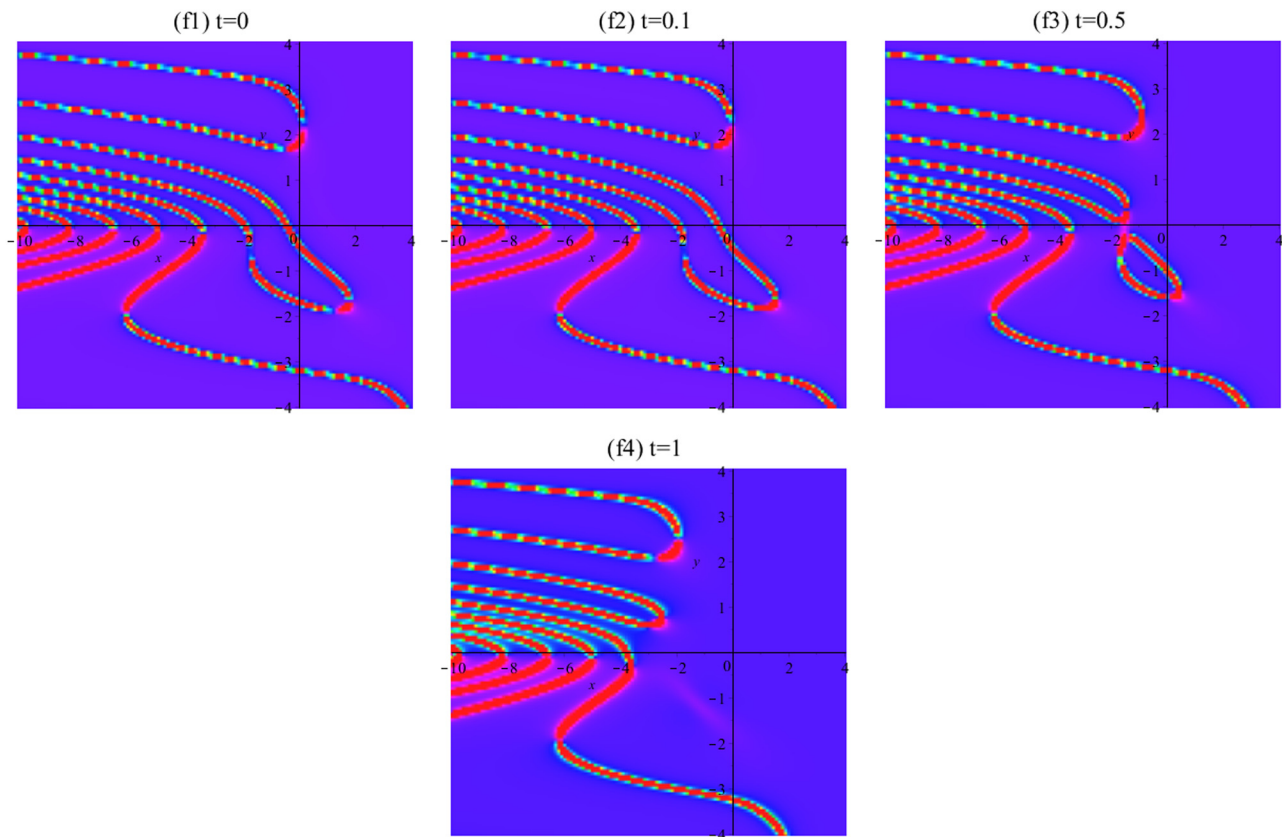


Figure 5: Plot of the extended homoclinic breather wave solution (3.15) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

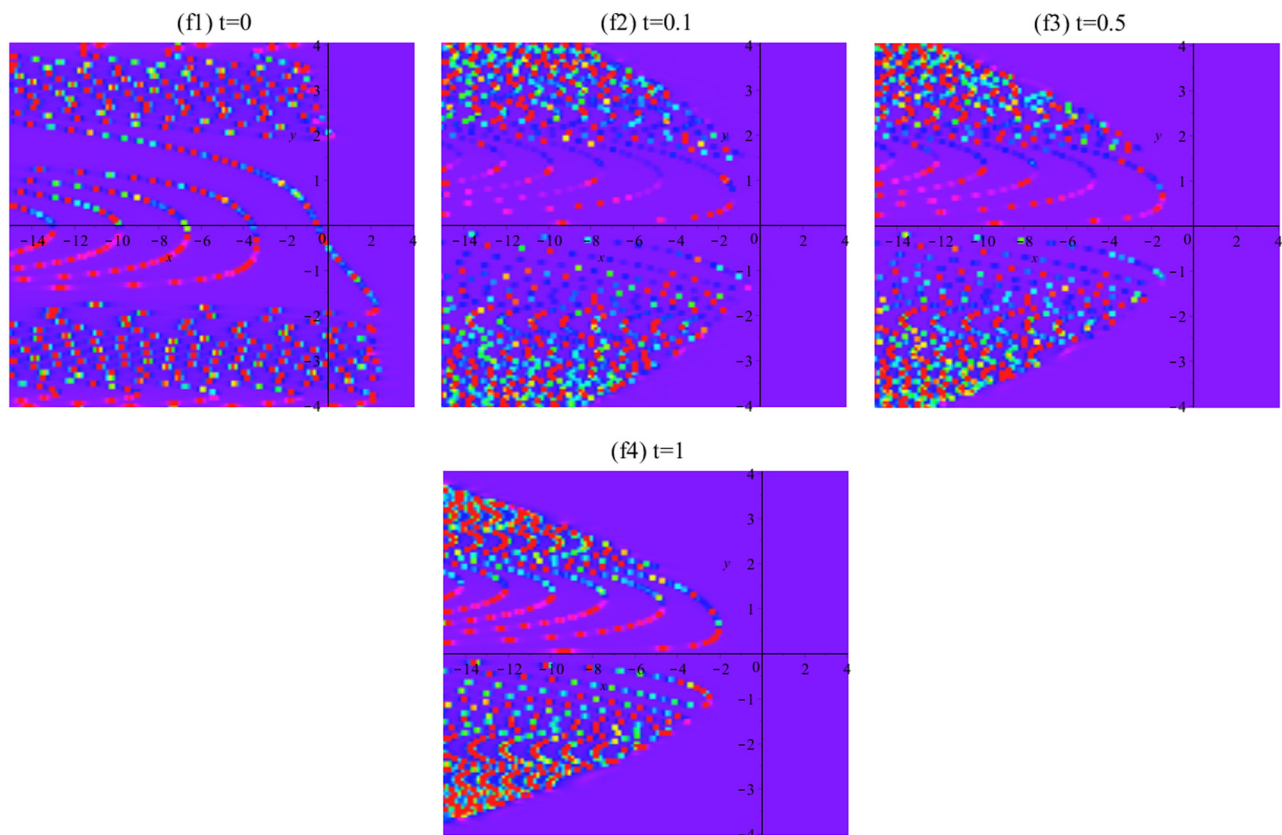


Figure 6: Plot of the extended homoclinic breather wave solution (3.7) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

$$\Gamma_4 = \frac{M_4 + M_1 + M_2 + M_3 + M_5}{M_6} - \frac{N_1(N_5 + N_6 + N_4 + N_3 + N_2)}{M_6^2}, \quad (3.19)$$

$$S_1 = \sin(d(y)(x + e(y))), \quad S_2 = \cos(d(y)(x + e(y))),$$

$$M_1 = -S_1 \left(\left(\frac{d}{dy} \tau_2(y) \right) d(y) + \tau_2(y) \frac{d}{dy} d(y) \right) + S_2 \left(\tau_3(y) \frac{d}{dy} d(y) + d(y) \frac{d}{dy} \tau_3(y) \right),$$

$$M_2 = -\tau_2(y) S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_3 = -\tau_3(y) S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right) d(y),$$

$$M_4 = e^{a_1} m_1 \left(\tau_1(y) \left(2d(y) \frac{d}{dy} d(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$M_5 = e^{a_2} m_2 \left(\left(2d(y) \frac{d}{dy} d(y) \right) \tau_4(y) t + \tau_4(y) \frac{d}{dy} n_2(y) + \frac{d}{dy} \tau_4(y) \right),$$

$$M_6 = \tau_1(y) e^{a_1} + \tau_2(y) S_2 + \tau_3(y) S_1 + \tau_4(y) e^{a_2},$$

$$N_1 = \tau_1(y) m_1 e^{a_1} - \tau_2(y) S_1 d(y) + \tau_3(y) S_2 d(y) + \tau_4(y) m_2 e^{a_2},$$

$$N_2 = e^{a_2} \left(\left(2d(y) \frac{d}{dy} d(y) \right) \tau_4(y) t + \tau_4(y) \frac{d}{dy} n_2(y) + \frac{d}{dy} \tau_4(y) \right),$$

$$N_3 = \tau_3(y) S_2 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_4 = -\tau_2(y) S_1 \left(\left(\frac{d}{dy} d(y) \right) (x + e(y)) + d(y) \frac{d}{dy} e(y) \right),$$

$$N_5 = e^{a_1} \left(\tau_1(y) \left(2d(y) \frac{d}{dy} d(y) \right) t + \tau_1(y) \frac{d}{dy} n_1(y) + \frac{d}{dy} \tau_1(y) \right),$$

$$N_6 = \left(\frac{d}{dy} \tau_2(y) \right) S_2 + \left(\frac{d}{dy} \tau_3(y) \right) S_1.$$

$$a_l = m_l x + n_l(y) + ((d(y))^2 + m_l^2) t, \quad l = 1, 2.$$

Figures 7–9 show the analysis of the treatment of extended homoclinic (heteroclinic) breather wave solution with two-wave solution, where graphs of Γ_4 are given with the following selected parameters:

$$m_1 = 1, \quad m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad \tau_4(y) = 2, \quad n_1(y) = y, \quad n_2(y) = 3y, \quad d(y) = 1, \quad e(y) = \ln(1 + y^2), \quad (3.20)$$

$$m_1 = 1, \quad m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad \tau_4(y) = 2, \quad n_1(y) = y, \quad n_2(y) = 3y, \quad d(y) = \frac{2}{1 + y^2}, \quad e(y) = y + y^3, \quad (3.21)$$

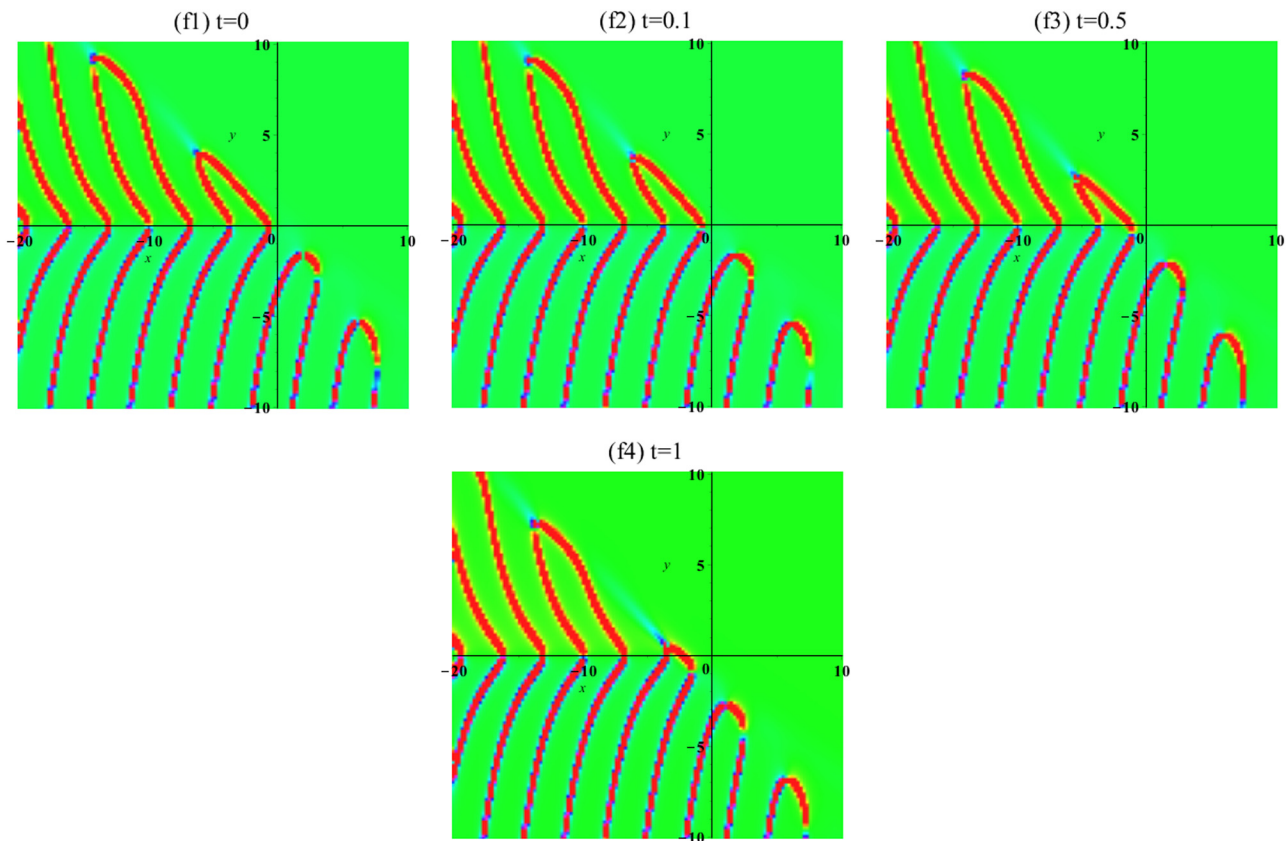


Figure 7: Plot of the extended homoclinic breather wave solution (3.20) (Γ_4) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

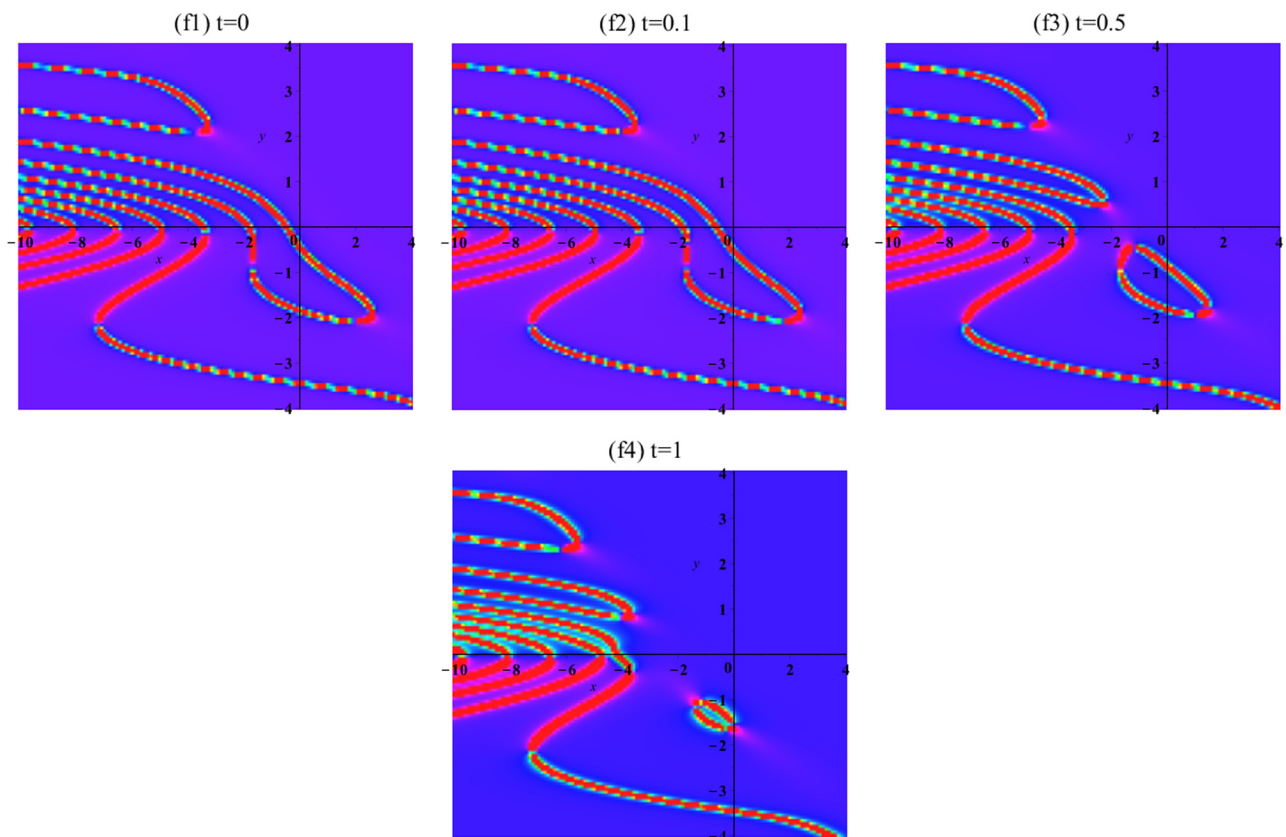


Figure 8: Plot of the extended homoclinic breather wave solution (3.21) (Γ_4) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

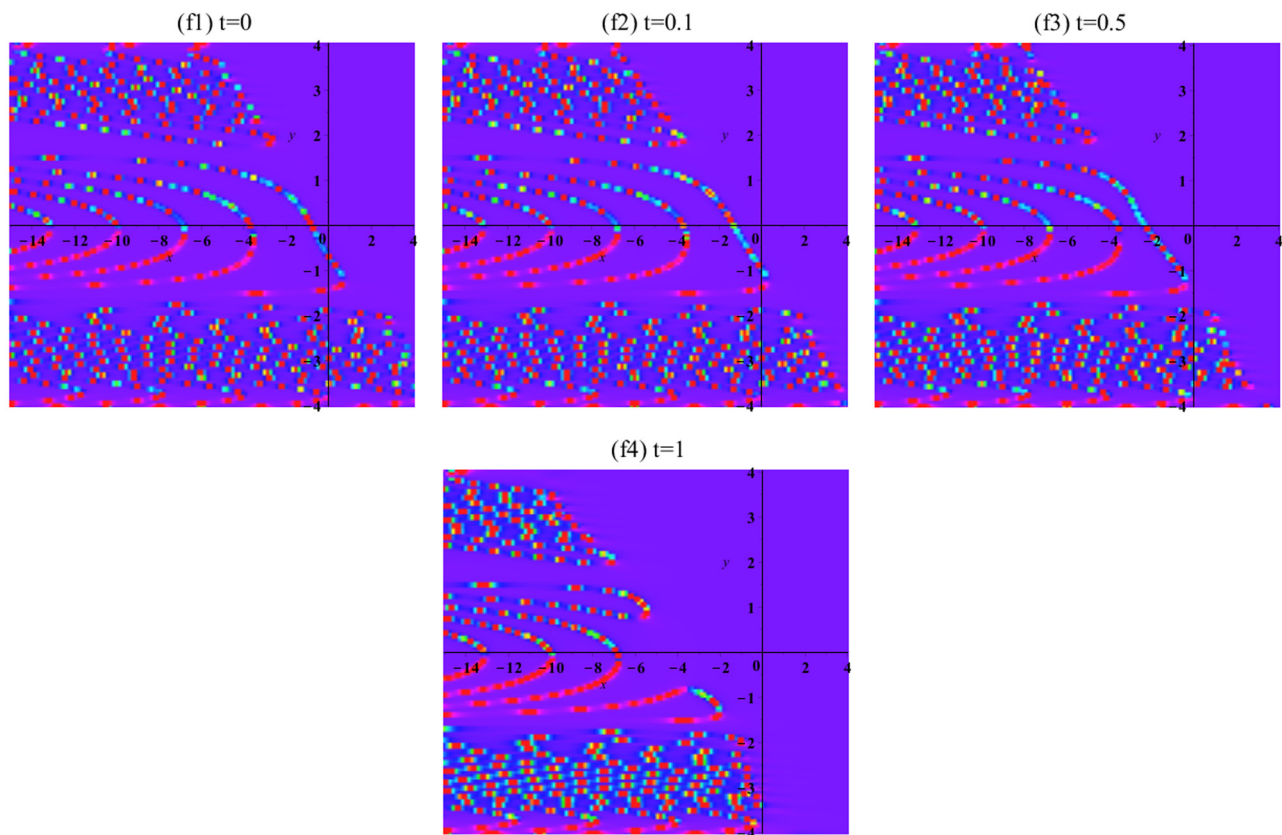


Figure 9: Plot of the extended homoclinic breather wave solution (3.22) (Γ_4) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

$$m_1 = 1, \quad m_2 = 2, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad \tau_4(y) = 2, \quad n_1(y) = y, \quad n_2(y) = 3y, \quad (3.22)$$

$$d(y) = \cos(y), \quad e(y) = y + y^3,$$

given in Eq. (3.4). By the aforementioned parameters, the structural property homoclinic breather wave solutions are presented in Figures 7–9 with the density plot with different times. It shows a type of interaction solutions between exponential and trigonometric waves.

3.2 Interaction periodic and dark soliton

Here, we utilize to formulate the new exact solutions to the $(2 + 1)$ -dimensional MDWW system. Consider the following function for studying the interaction periodic and dark soliton solution:

$$h = \tau_1(y)e^{a_1} + \tau_2(y)\tan(a_2) + \tau_3(y)\tanh(a_3) + \tau_4(y)e^{a_4}, \quad (3.23)$$

$$a_l = m_l x + n_l(y) + p_l(y)t, \quad l = 1, 2, 3, 4.$$

Afterward, the values m_l , $n_l(y)$, $p_l(y)$, and $(l = 1, 2, 3, 4)$ will be found. By substituting Eq. (3.23) into (2.15) and taking the coefficients of the power of each $\exp(a_l)$, $l = 1, 2$ and $\tan(a_2)$, $\tanh(a_3)$ to be zero, we yield a system of equations (algebraic) (these are not collected here for minimalist) for m_l , $n_l(y)$, $p_l(y)$, $(l = 1, 2, 3, 4)$. These algebraic equations with the help of the emblematic computation software like, Maple, give the solutions by using $\Sigma = (\ln h)_x$ and $\Gamma = (\ln h)_{xy}$:

$$\Sigma = \frac{\tau_1(y)m_1 e^{a_1} + \tau_2(y)(1 + (\tan(a_2))^2)m_2 + \tau_3(y)(1 - (\tanh(a_3))^2)m_3 + \tau_4(y)m_4 e^{a_4}}{\tau_1(y)e^{a_1} + \tau_2(y)\tan(a_2) + \tau_3(y)\tanh(a_3) + \tau_4(y)e^{a_4}},$$

$$\begin{aligned}
\Gamma &= \frac{M_1 + M_3 + M_4 + M_2}{M_5} - \frac{N_1}{M_5^2} \left(\frac{M_1}{m_1} + N_2 + N_3 + \frac{M_2}{m_4} \right), \\
M_1 &= \left(\frac{d}{dy} \tau_1(y) \right) m_1 e^{a_1} + \tau_1(y) m_1 \left(\frac{d}{dy} n_1(y) + \left(\frac{d}{dy} p_1(y) \right) t \right) e^{a_1}, \\
M_2 &= \left(\frac{d}{dy} \tau_4(y) \right) m_4 e^{a_4} + \tau_4(y) m_4 \left(\frac{d}{dy} n_4(y) + \left(\frac{d}{dy} p_4(y) \right) t \right) e^{a_4}, \\
M_3 &= \left(\frac{d}{dy} \tau_2(y) \right) (1 + (\tan(a_2))^2) m_2 + 2 \tau_2(y) \tan(a_2) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) + \left(\frac{d}{dy} p_2(y) \right) t \right) m_2, \\
M_4 &= \left(\frac{d}{dy} \tau_3(y) \right) (1 - (\tanh(a_3))^2) m_3 - 2 \tau_3(y) \tanh(a_3) (1 - (\tanh(a_3))^2) \left(\frac{d}{dy} n_3(y) + \left(\frac{d}{dy} p_3(y) \right) t \right) m_3, \\
M_5 &= \tau_1(y) e^{a_1} + \tau_2(y) \tan(a_2) + \tau_3(y) \tanh(a_3) + \tau_4(y) e^{a_4}, \\
N_1 &= \tau_1(y) m_1 e^{a_1} + \tau_2(y) (1 + (\tan(a_2))^2) m_2 + \tau_3(y) (1 - (\tanh(a_3))^2) m_3 + \tau_4(y) m_4 e^{a_4}, \\
N_2 &= \left(\frac{d}{dy} \tau_2(y) \right) \tan(a_2) + \tau_2(y) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) + \left(\frac{d}{dy} p_2(y) \right) t \right), \\
N_3 &= \left(\frac{d}{dy} \tau_3(y) \right) \tanh(a_3) + \tau_3(y) (1 - (\tanh(a_3))^2) \left(\frac{d}{dy} n_3(y) + \left(\frac{d}{dy} p_3(y) \right) t \right), \quad a_l = m_l x + n_l(y) + p_l(y) t, \quad l = 1:4.
\end{aligned}$$

3.2.1 Set I solutions

$$\begin{aligned}
m_l &= m_l, \quad l = 1, 2, 4, \quad m_3 = 0, \quad p_1(y) = m_1^2, \quad p_2(y) = p_2(y), \quad p_3(y) = 0 \\
p_4(y) &= m_4^2, \quad \tau_2(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 3, 4,
\end{aligned} \tag{3.24}$$

where m_l , $l = 1, 2$, and $n_l(y)$, $l = 1, 2, 3, 4$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_1 = \frac{\tau_1(y) m_1 e^{m_1 x + n_1(y) + m_1^2 t} + q_4(y) m_4 e^{m_4 x + n_4(y) + m_4^2 t}}{\tau_1(y) e^{m_1 x + n_1(y) + m_1^2 t} + \tau_3(y) \tanh(n_3(y)) + \tau_4(y) e^{m_4 x + n_4(y) + m_4^2 t}}, \tag{3.25}$$

$$\Gamma_1 = \frac{M_1 + M_3 + M_2}{M_5} - \frac{N_1}{M_5^2} \left(\frac{M_1}{m_1} + N_2 + N_3 + \frac{M_2}{m_4} \right), \tag{3.26}$$

$$\begin{aligned}
M_1 &= \left(\frac{d}{dy} \tau_1(y) \right) m_1 e^{a_1} + \tau_1(y) m_1 \left(\frac{d}{dy} n_1(y) + \left(\frac{d}{dy} p_1(y) \right) t \right) e^{a_1}, \\
M_2 &= \left(\frac{d}{dy} \tau_4(y) \right) m_4 e^{a_4} + \tau_4(y) m_4 \left(\frac{d}{dy} n_4(y) + \left(\frac{d}{dy} p_4(y) \right) t \right) e^{a_4}, \\
M_3 &= \left(\frac{d}{dy} \tau_2(y) \right) (1 + (\tan(a_2))^2) m_2 + 2 \tau_2(y) \tan(a_2) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) + \left(\frac{d}{dy} p_2(y) \right) t \right) m_2, \\
M_5 &= \tau_1(y) e^{a_1} + \tau_2(y) \tan(a_2) + \tau_3(y) \tanh(a_3) + \tau_4(y) e^{a_4}, \\
N_1 &= \tau_1(y) m_1 e^{a_1} + \tau_2(y) (1 + (\tan(a_2))^2) m_2 + \tau_4(y) m_4 e^{a_4}, \\
N_2 &= \left(\frac{d}{dy} \tau_2(y) \right) \tan(a_2) + \tau_2(y) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) + \left(\frac{d}{dy} p_2(y) \right) t \right),
\end{aligned}$$

$$N_3 = \left(\frac{d}{dy} \tau_3(y) \right) \tanh(a_3) + \tau_3(y)(1 - (\tanh(a_3))^2) \left(\frac{d}{dy} n_3(y) + \left(\frac{d}{dy} p_3(y) \right) t \right), \quad a_l = m_l x + n_l(y) + p_l(y)t, \quad l = 1:4.$$

$$a_1 = m_1 x + n_1(y) + (m_1^2 + d(y)^2)t, \quad a_2 = 0.$$

Figures 10–13 show the analysis of the treatment of interaction periodic and dark soliton solutions with two-wave solution, where graphs of Γ_1 are provided with the following selected parameters:

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y, \quad n_3(y) = y, \quad n_4(y) = y, \quad (3.27)$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y, \quad n_3(y) = y^2, \quad n_4(y) = y, \quad (3.28)$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y^2, \quad n_3(y) = y^2, \quad n_4(y) = y^2, \quad (3.29)$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y^2, \quad n_3(y) = \cos(y), \quad n_4(y) = y^2, \quad (3.30)$$

given in Eq. (3.26). By the aforementioned parameters, the structural property interaction periodic and dark soliton solutions are presented in Figures 10–13 with the density plot with different times. It shows a type of interaction solutions among exponential, hyperbolic, and trigonometric waves.

3.2.2 Set II solutions

$$m_l = m_l, \quad l = 1, 4, \quad m_2 = m_3 = 0, \quad p_1(y) = p_1(y), \quad p_2(y) = p_3(y) = 0$$

$$p_4(y) = m_4^2, \quad \tau_1(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 2, 3, 4, \quad (3.31)$$

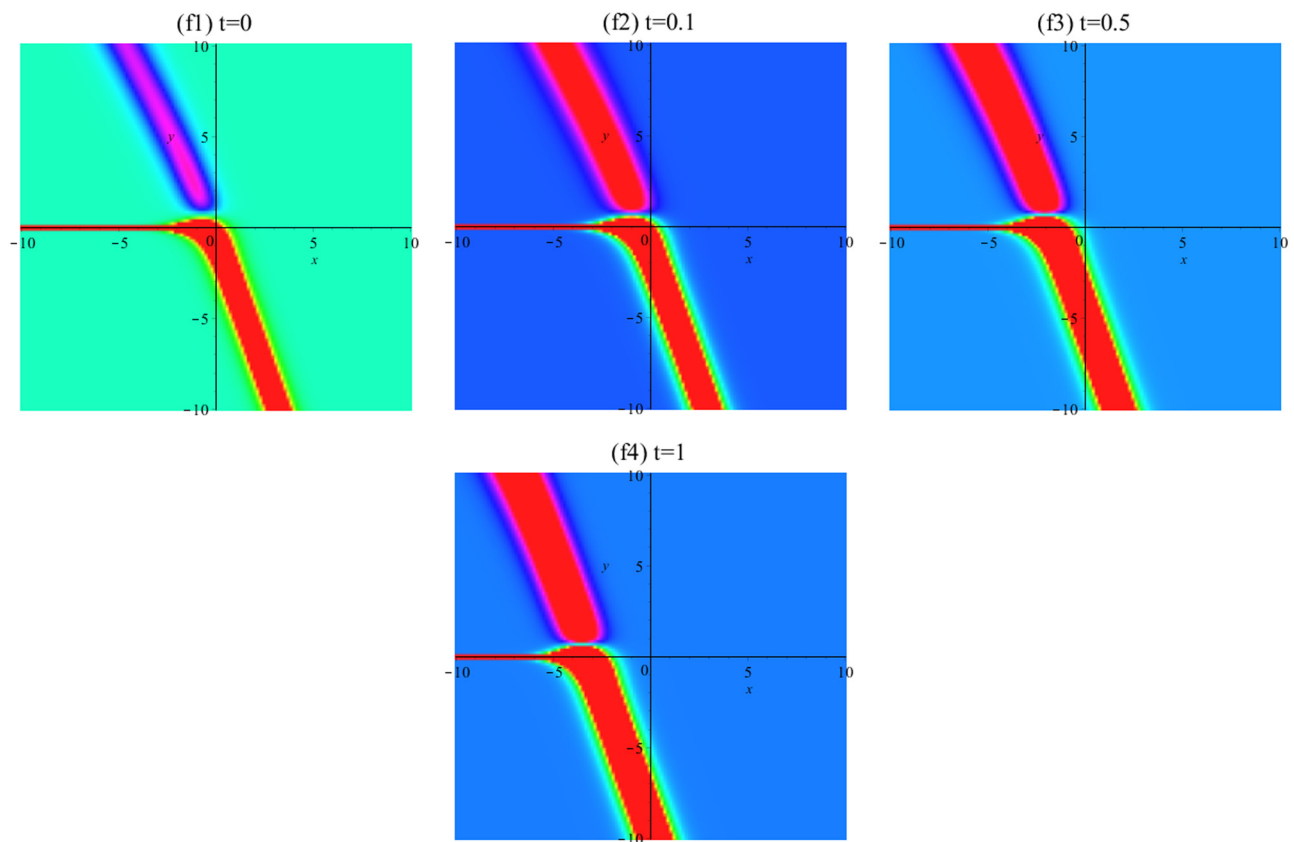


Figure 10: Plot of interaction periodic and dark soliton solutions (3.27) (Γ_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

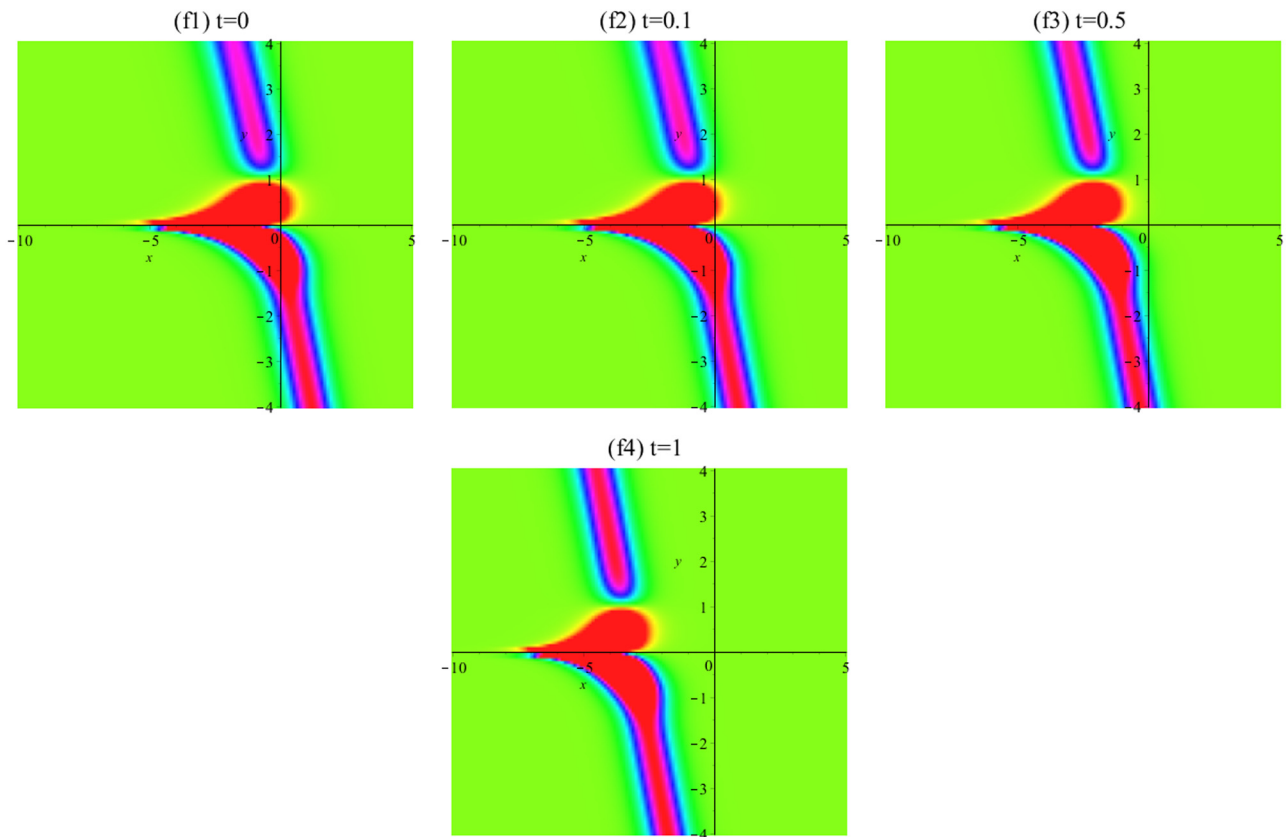


Figure 11: Plot of interaction periodic and dark soliton solutions (3.28) (Γ_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

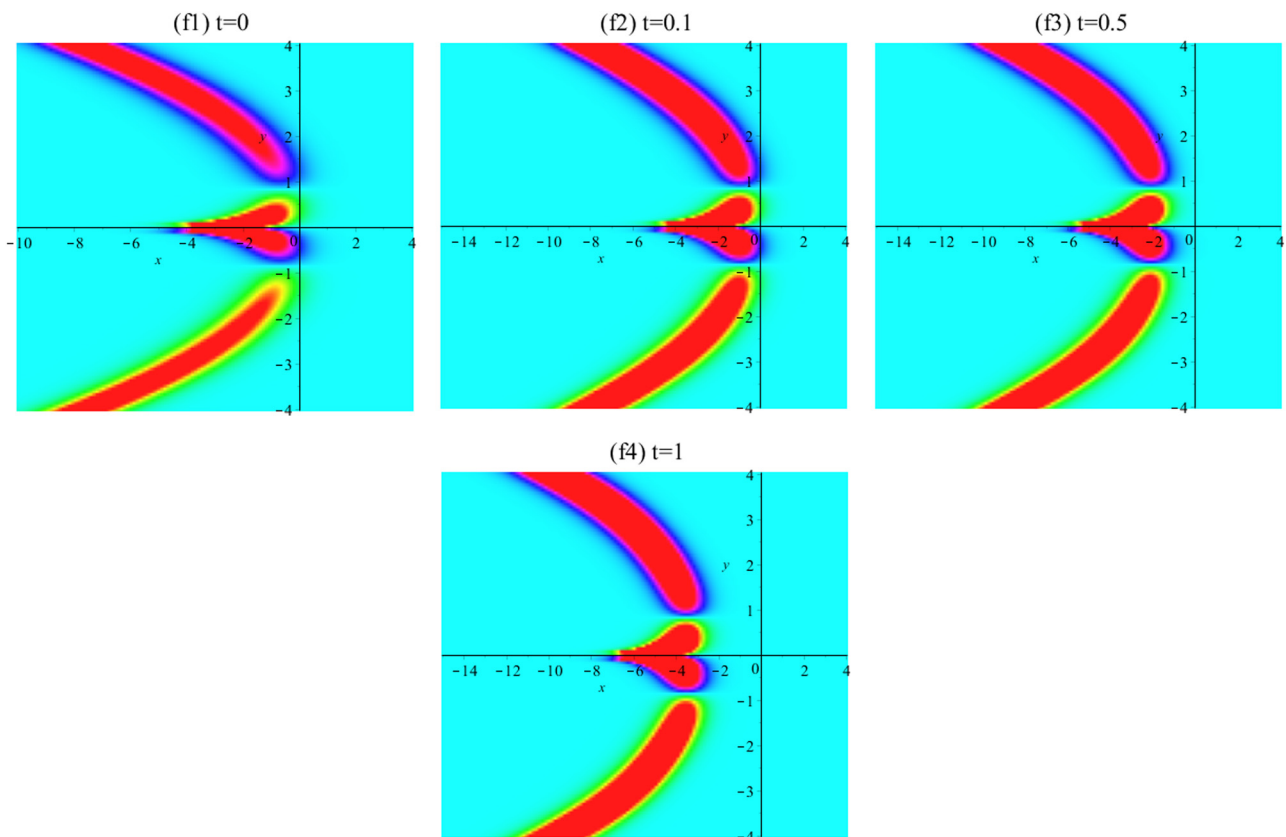


Figure 12: Plot of interaction periodic and dark soliton solutions (3.29) (Γ_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

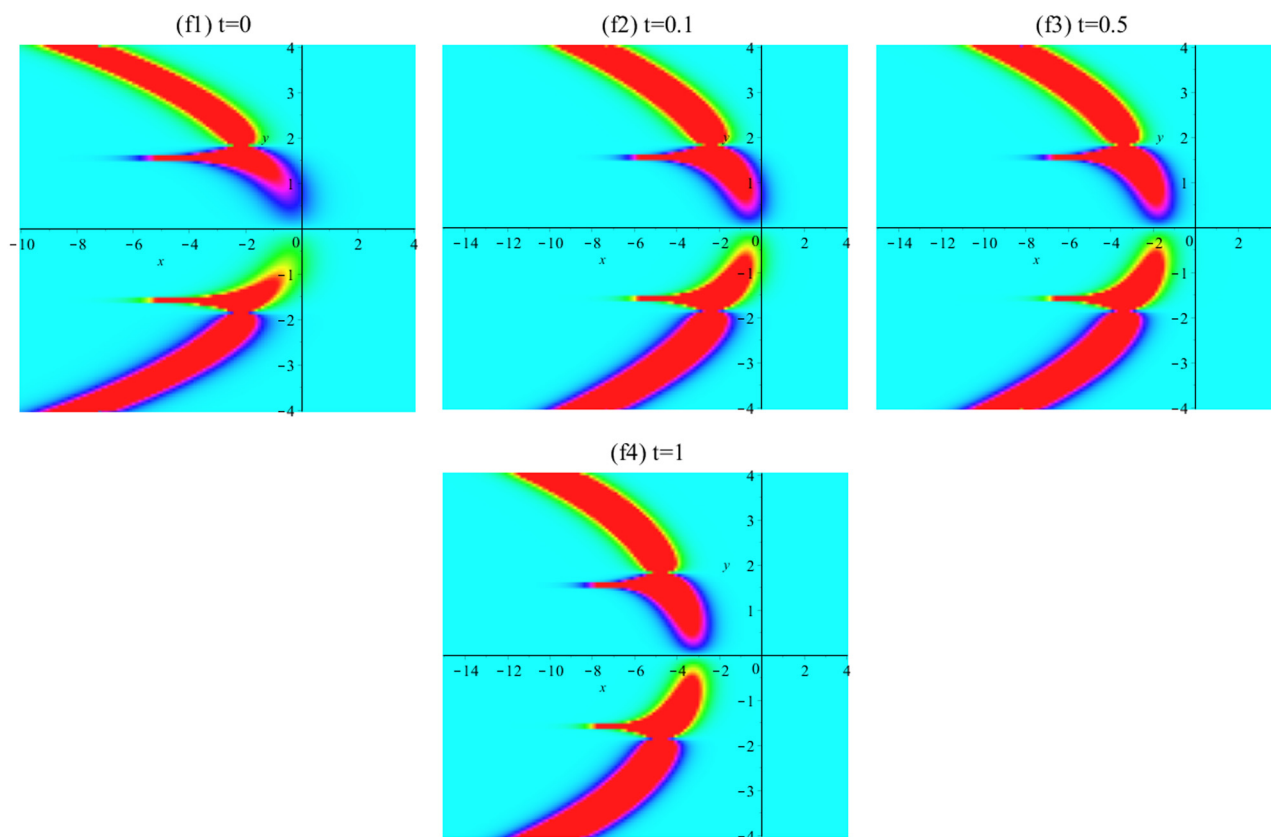


Figure 13: Plot of interaction periodic and dark soliton solutions (3.30) (Γ_1) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

where $m_l, l = 1, 2$, and $n_l(y), l = 1, 2, 3, 4$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_2 = \frac{\tau_4(y)m_4e^{m_4x+n_4(y)+m_4^2t}}{q_2(y)\tan(n_2(y)) + \tau_3(y)\tanh(n_3(y)) + \tau_4(y)e^{m_4x+n_4(y)+m_4^2t}}, \quad (3.32)$$

$$\Gamma_2 = \frac{M_1 + M_2}{M_5} - \frac{N_1}{M_5^2} \left(\frac{M_1}{m_1} + N_2 + N_3 + \frac{M_2}{m_4} \right), \quad (3.33)$$

$$M_1 = \left(\frac{d}{dy} \tau_1(y) \right) m_1 e^{a_1} + \tau_1(y) m_1 \left(\frac{d}{dy} n_1(y) + \left(\frac{d}{dy} p_1(y) \right) t \right) e^{a_1},$$

$$M_2 = \left(\frac{d}{dy} \tau_4(y) \right) m_4 e^{a_4} + \tau_4(y) m_4 \left(\frac{d}{dy} n_4(y) + \left(\frac{d}{dy} p_4(y) \right) t \right) e^{a_4},$$

$$M_5 = q_1(y)e^{a_1} + \tau_2(y)\tan(a_2) + \tau_3(y)\tanh(a_3) + \tau_4(y)e^{a_4},$$

$$N_1 = \tau_1(y)m_1e^{a_1} + \tau_4(y)m_4e^{a_4},$$

$$N_2 = \left(\frac{d}{dy} \tau_2(y) \right) \tan(a_2) + \tau_2(y)(1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) \right),$$

$$N_3 = \left(\frac{d}{dy} \tau_3(y) \right) \tanh(a_3) + \tau_3(y)(1 - (\tanh(a_3))^2) \left(\frac{d}{dy} n_3(y) \right),$$

$$a_l = m_l x + n_l(y) + p_l(y)t, \quad l = 1:4.$$

Figures 14–17 show the analysis of the treatment of interaction periodic and dark soliton solutions with two-wave solution, where graphs of Γ_2 are given with the following selected parameters:

$$m_4 = 3, \quad \tau_2(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_2(y) = y, \quad n_3(y) = y, \quad n_4(y) = y, \quad (3.34)$$

$$m_4 = 3, \quad \tau_2(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_2(y) = y^2, \quad n_3(y) = y, \quad n_4(y) = y, \quad (3.35)$$

$$m_4 = 3, \quad \tau_2(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_2(y) = y^2, \quad n_3(y) = y^2, \quad n_4(y) = y^2, \quad (3.36)$$

$$m_4 = 3, \quad \tau_2(y) = 1, \quad \tau_3(y) = 2, \quad \tau_4(y) = 3, \quad n_2(y) = y, \quad n_3(y) = \cos(y), \quad n_4(y) = y, \quad (3.37)$$

given in Eq. (3.33). By the aforementioned parameters, the structural property interaction periodic and dark soliton solutions are presented in Figures 14–17 with the density plot with different times. It shows a type of interaction solutions among exponential, hyperbolic, and trigonometric waves.

3.2.3 Set III solutions

$$m_l = m_l, \quad l = 1, 3, 4, \quad m_2 = 0, \quad p_1(y) = m_1^2, \quad p_2(y) = 0, \quad p_2(y) = p_2(y), \quad p_4(y) = m_4^2, \quad (3.38)$$

$$\tau_3(y) = 0, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 2, 4,$$

where m_l , $l = 1, 3, 4$, and $n_l(y)$, $l = 1, 2, 3, 4$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_3 = \frac{\tau_1(y)m_1 e^{m_1 x + n_1(y) + m_1^2 t} + q_4(y)m_4 e^{m_4 x + n_4(y) + m_4^2 t}}{\tau_1(y)e^{m_1 x + n_1(y) + m_1^2 t} + q_2(y)\tan(n_2(y)) + \tau_4(y)e^{m_4 x + n_4(y) + m_4^2 t}}, \quad (3.39)$$

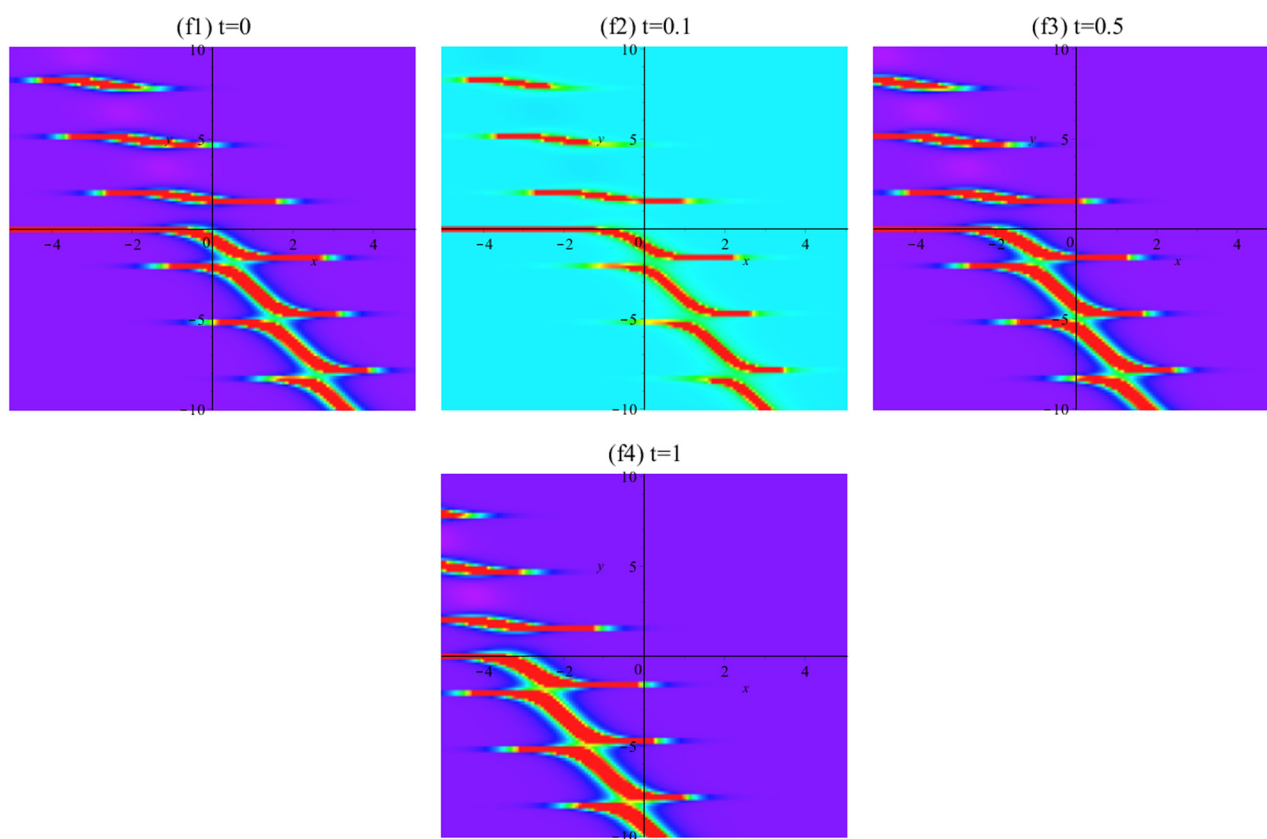


Figure 14: Plot of interaction periodic and dark soliton solutions (3.34) (Γ_2) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

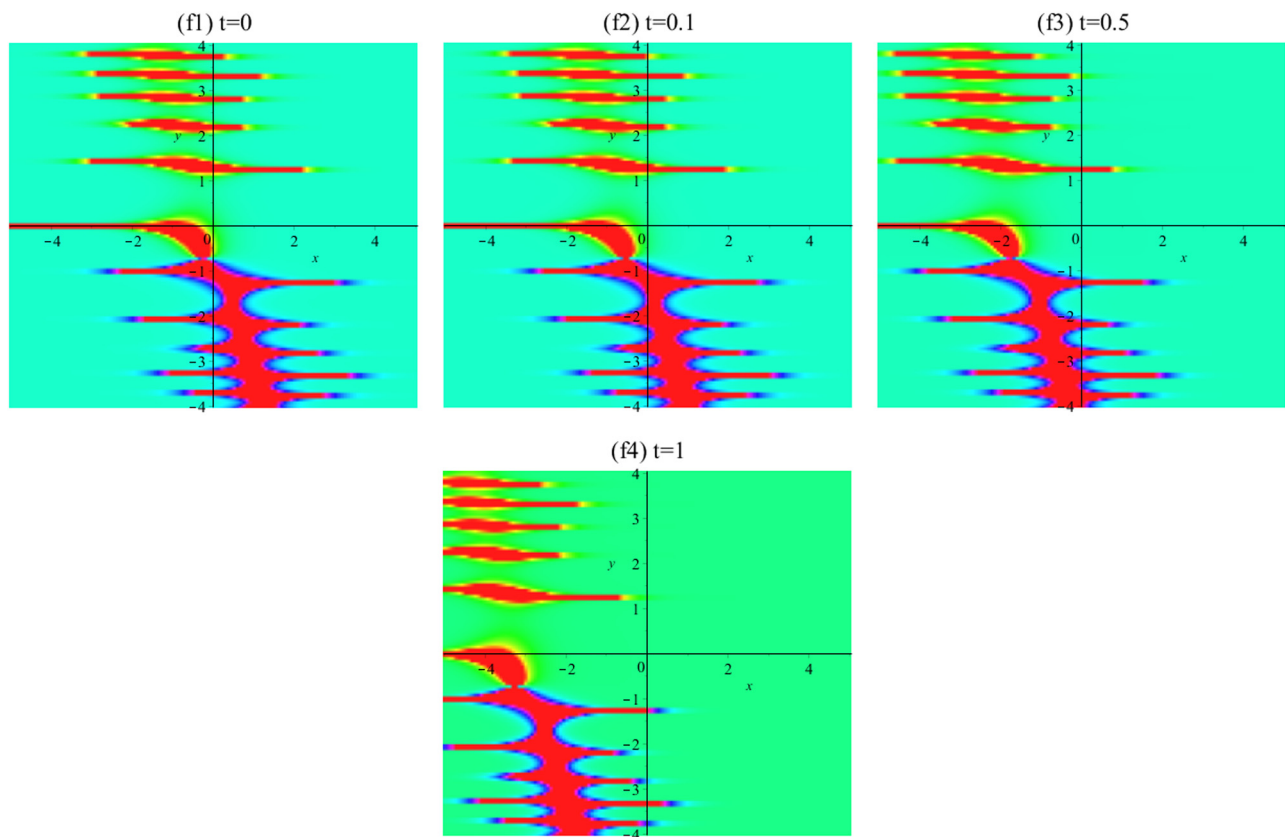


Figure 15: Plot of interaction periodic and dark soliton solutions (3.35) (Γ_2) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

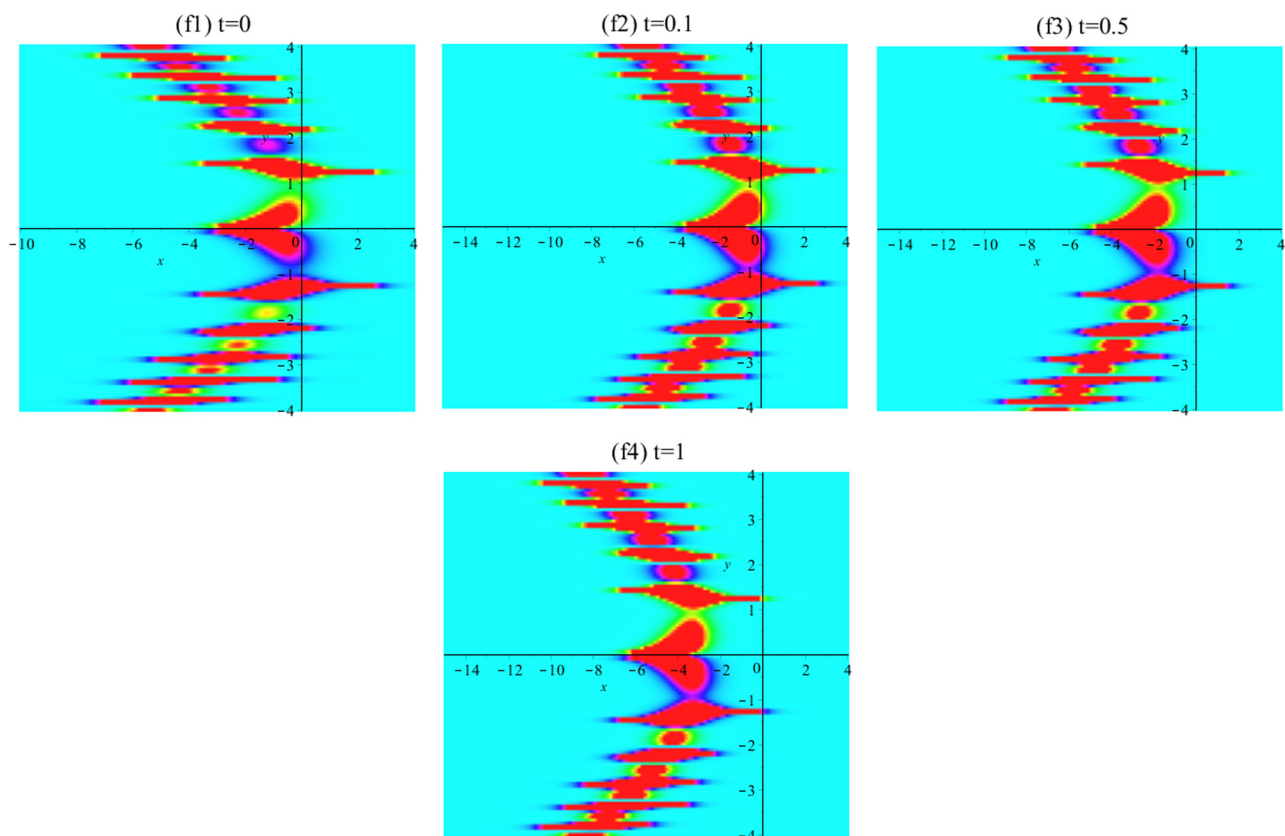


Figure 16: Plot of interaction periodic and dark soliton solutions (3.36) (Γ_2) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

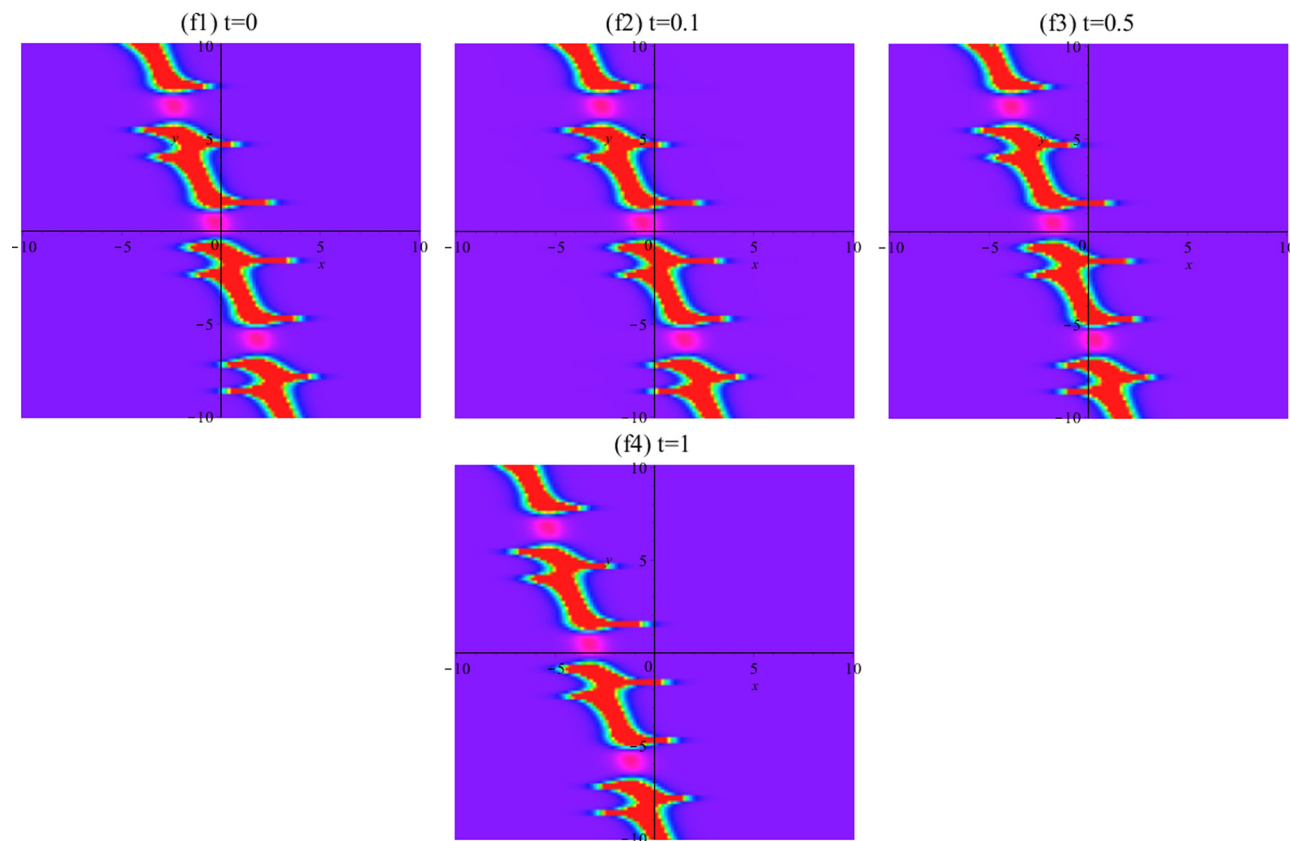


Figure 17: Plot of interaction periodic and dark soliton solutions (3.37) (Γ_2) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

$$\Gamma_3 = \frac{M_1 + M_2}{M_5} - \frac{N_1}{M_5^2} \left(\frac{M_1}{m_1} + N_2 + \frac{M_2}{m_4} \right), \quad (3.40)$$

$$M_1 = \left(\frac{d}{dy} \tau_1(y) \right) m_1 e^{a_1} + \tau_1(y) m_1 \left(\frac{d}{dy} n_1(y) \right) e^{a_1},$$

$$M_2 = \left(\frac{d}{dy} \tau_4(y) \right) m_4 e^{a_4} + \tau_4(y) m_4 \left(\frac{d}{dy} n_4(y) + \left(\frac{d}{dy} p_4(y) \right) t \right) e^{a_4},$$

$$M_5 = \tau_1(y) e^{a_1} + q_2(y) \tan(a_2) + \tau_4(y) e^{a_4},$$

$$N_1 = \tau_1(y) m_1 e^{a_1} + \tau_2(y) (1 + (\tan(a_2))^2) m_2 + \tau_4(y) m_4 e^{a_4},$$

$$N_2 = \left(\frac{d}{dy} \tau_2(y) \right) \tan(a_2) + \tau_2(y) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) \right),$$

$$a_l = m_l x + n_l(y) + p_l(y) t, \quad l = 1:4.$$

Figures 18–21 show the analysis of the treatment of interaction periodic and dark soliton solutions with two-wave solution, where graphs of Γ_2 are given with the following selected parameters:

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y, \quad n_2(y) = y, \quad n_4(y) = y, \quad (3.41)$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y, \quad n_2(y) = y^2, \quad n_4(y) = y, \quad (3.42)$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y^2, \quad n_2(y) = y^2, \quad n_4(y) = y^2, \quad (3.43)$$

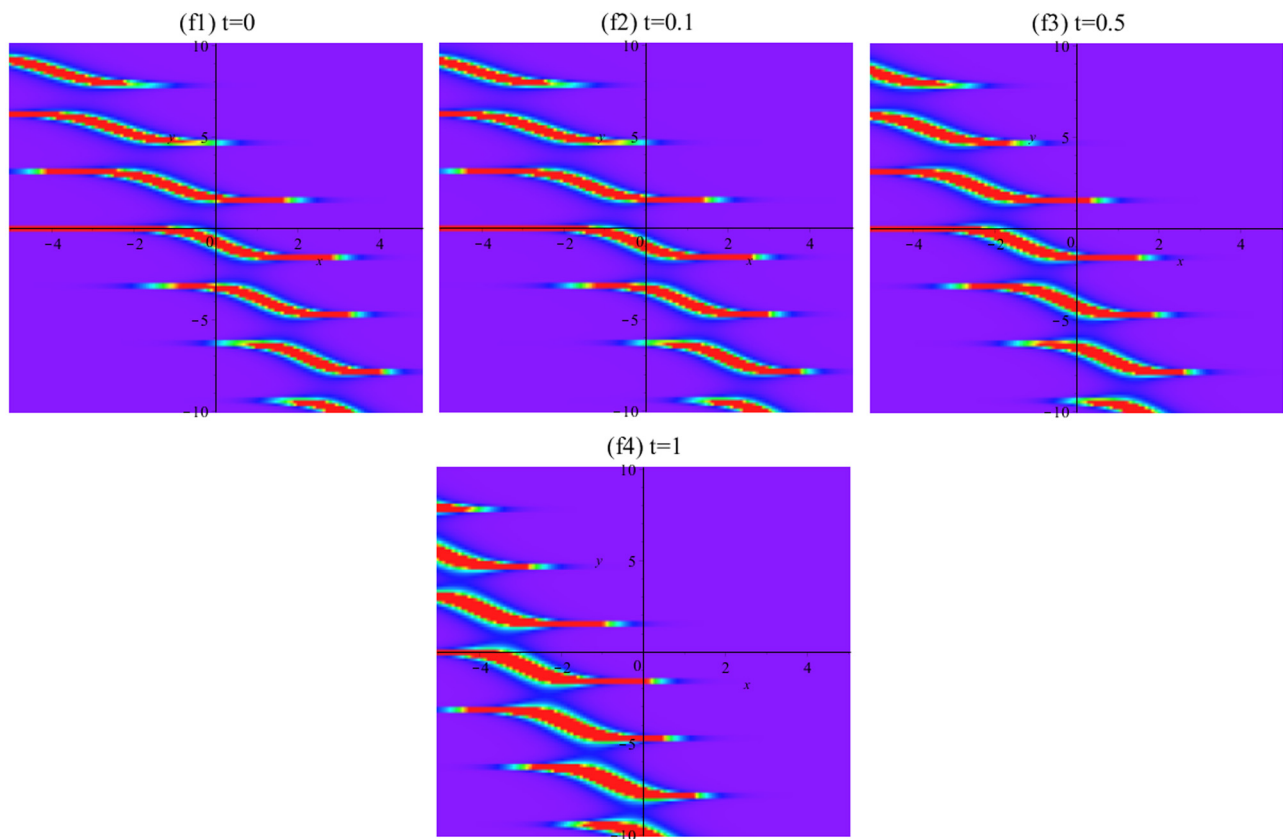


Figure 18: Plot of interaction periodic and dark soliton solutions (3.41) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

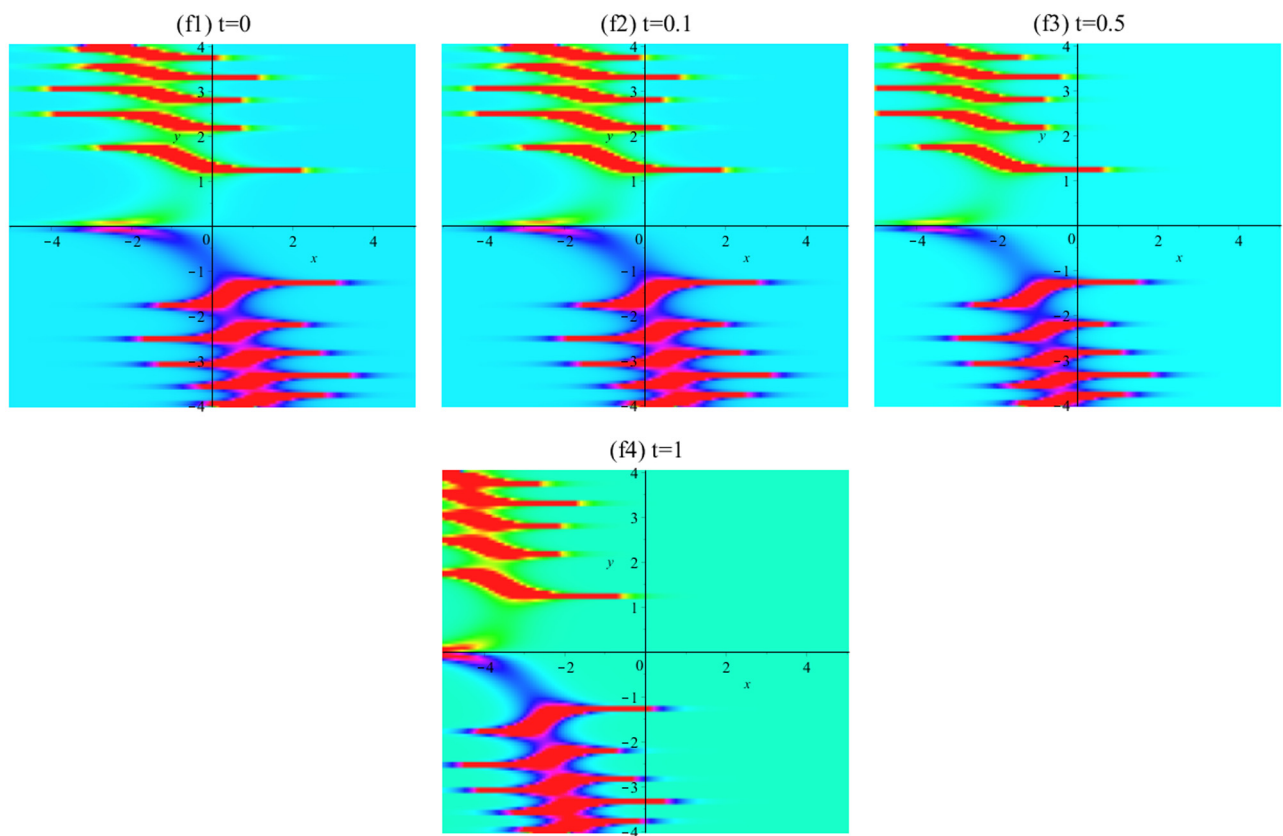


Figure 19: Plot of interaction periodic and dark soliton solutions (3.42) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

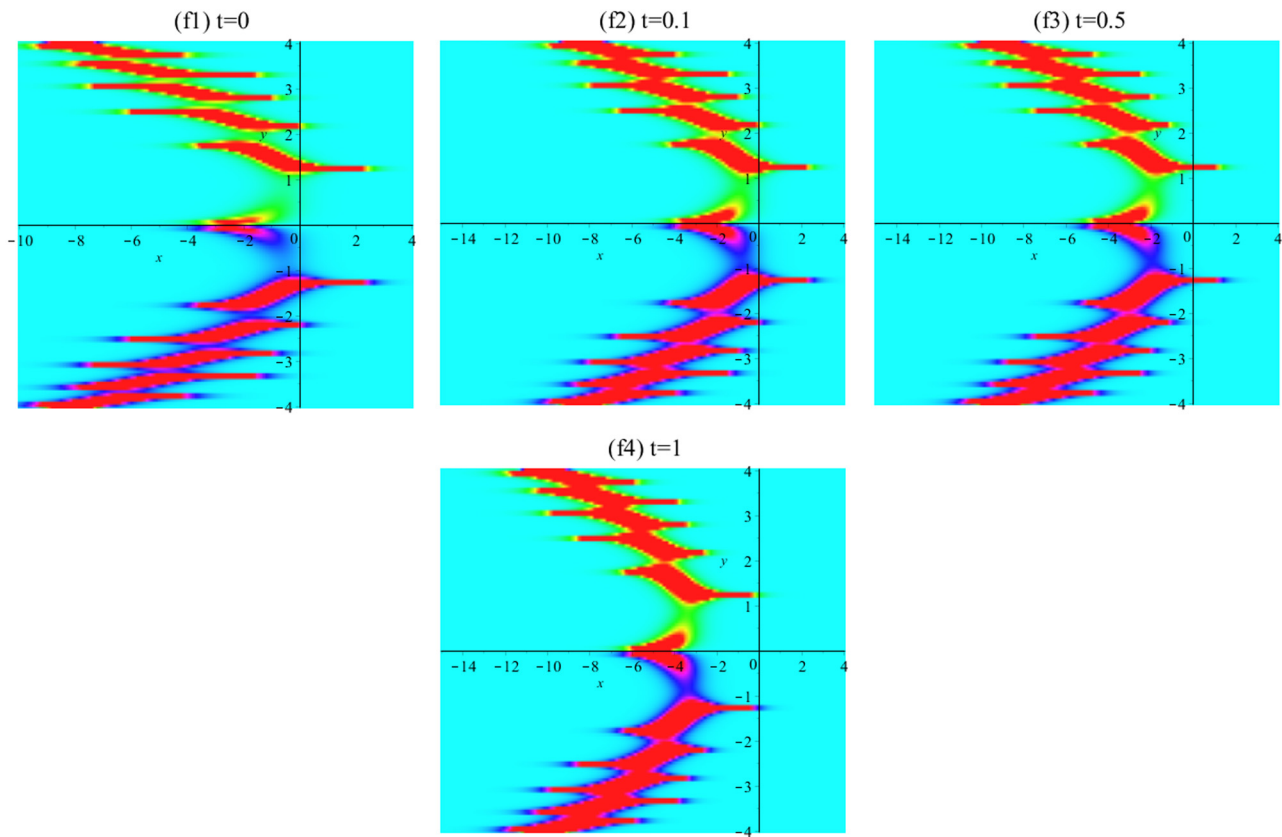


Figure 20: Plot of interaction periodic and dark soliton solutions (3.43) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

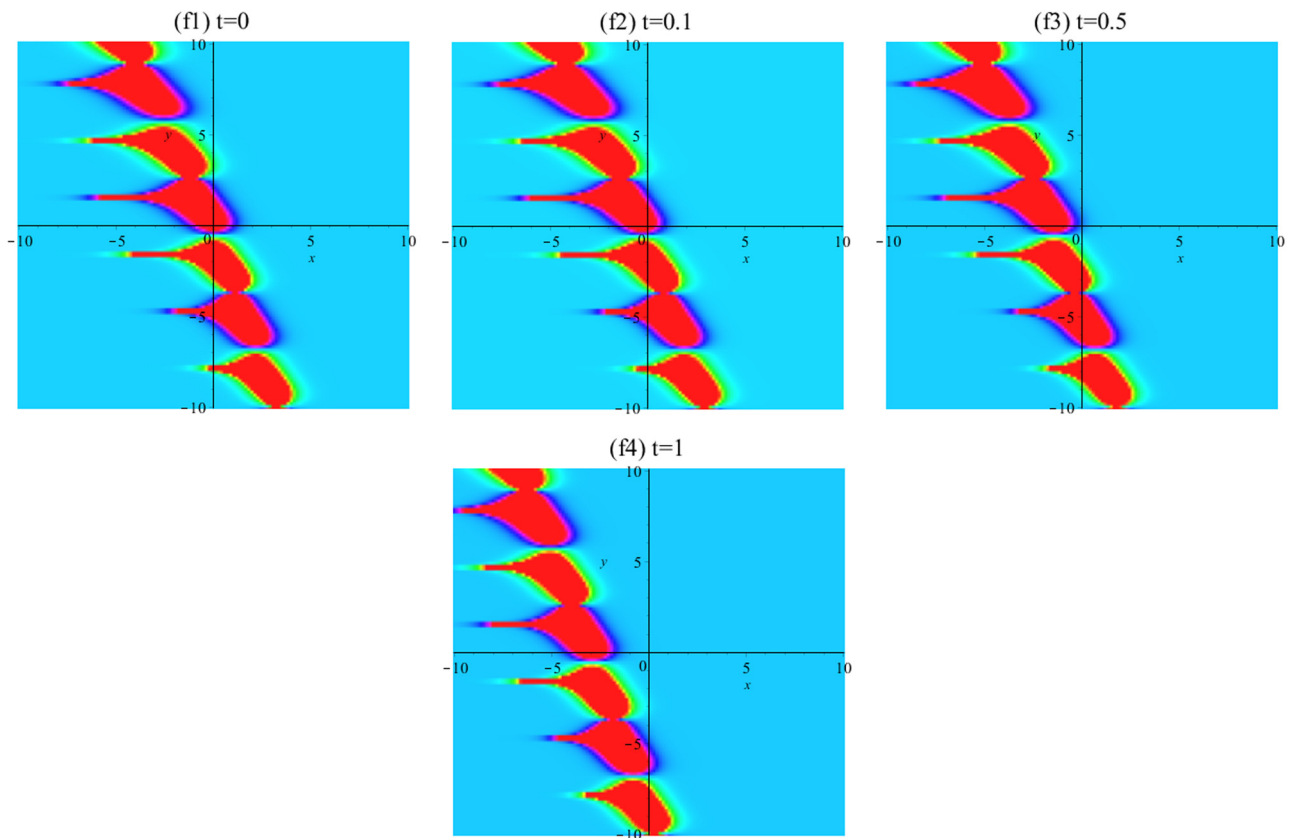


Figure 21: Plot of interaction periodic and dark soliton solutions (3.44) (Γ_3) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_4(y) = 3, \quad n_1(y) = y, \quad n_2(y) = \cos(y), \quad n_4(y) = y, \quad (3.44)$$

given in Eq. (3.40). By the aforementioned parameters, the structural property interaction periodic and dark soliton solutions are presented in Figures 18–21, with the density plot with different times. It shows a type of interaction solutions among exponential, hyperbolic and trigonometric waves.

3.2.4 Set IV solutions

$$\Sigma_4 = \frac{\tau_1(y)m_1 e^{m_1 x + n_1(y) + m_1^2 t}}{\tau_1(y)e^{m_1 x + n_1(y) + m_1^2 t} + \tau_2(y) \tan(n_2(y)) + \tau_3(y) \tanh(n_3(y))}, \quad (3.45)$$

$$\Gamma_4 = \frac{d}{dy} \left(\frac{\tau_1(y)m_1 e^{m_1 x + n_1(y) + m_1^2 t}}{\tau_1(y)e^{m_1 x + n_1(y) + m_1^2 t} + \tau_2(y) \tan(n_2(y)) + \tau_3(y) \tanh(n_3(y))} \right). \quad (3.46)$$

3.2.5 Set V solutions

$$m_l = m_i, \quad l = 1, 4, \quad m_2 = m_3 = 0, \quad p_1(y) = m_1^2, \quad p_2(y) = p_3(y) = 0 \quad (3.47)$$

$$p_4(y) = m_4^2, \quad \tau_i(y) = \tau_i(y), \quad i = 1, 2, 3, 4,$$

where m_i , $l = 1, 2$ and $n_l(y)$, and $l = 1, 2, 3, 4$ are free functions of y and unknown parameters. They are derived based on the bilinear frame, and the rational analytical solution can be introduced as follows:

$$\Sigma_5 = \frac{\tau_1(y)m_1 e^{m_1 x + n_1(y) + m_1^2 t} + \tau_4(y)m_4 e^{m_4 x + n_4(y) + m_4^2 t}}{\tau_1(y)e^{m_1 x + n_1(y) + m_1^2 t} + \tau_2(y) \tan(n_2(y)) + \tau_3(y) \tanh(n_3(y)) + \tau_4(y)e^{m_4 x + n_4(y) + m_4^2 t}}, \quad (3.48)$$

$$\Gamma_5 = \frac{M_1 + M_4}{M_5} - \frac{N_1}{M_5^2} \left(\frac{M_1}{m_1} + N_2 + N_3 + \frac{M_2}{m_4} \right), \quad (3.49)$$

$$M_1 = \left(\frac{d}{dy} \tau_1(y) \right) m_1 e^{a_1} + \tau_1(y) m_1 \left(\frac{d}{dy} n_1(y) \right) e^{a_1},$$

$$M_2 = \left(\frac{d}{dy} \tau_4(y) \right) m_4 e^{a_4} + \tau_4(y) m_4 \left(\frac{d}{dy} n_4(y) + \left(\frac{d}{dy} p_4(y) \right) t \right) e^{a_4},$$

$$M_5 = \tau_1(y) e^{a_1} + \tau_2(y) \tan(a_2) + \tau_3(y) \tanh(a_3) + \tau_4(y) e^{a_4},$$

$$N_1 = \tau_1(y) m_1 e^{a_1} + \tau_2(y) (1 + (\tan(a_2))^2) m_2 + \tau_4(y) m_4 e^{a_4},$$

$$N_2 = \left(\frac{d}{dy} \tau_2(y) \right) \tan(a_2) + \tau_2(y) (1 + (\tan(a_2))^2) \left(\frac{d}{dy} n_2(y) + \left(\frac{d}{dy} p_2(y) \right) t \right),$$

$$N_3 = \left(\frac{d}{dy} \tau_3(y) \right) \tanh(a_3) + \tau_3(y) (1 - (\tanh(a_3))^2) \left(\frac{d}{dy} n_3(y) \right),$$

$$a_l = m_l x + n_l(y) + p_l(y) t, \quad l = 1:4.$$

Figures 22–26 show the analysis of the treatment of interaction periodic and dark solitons solutions with two-wave solution, where graphs of Γ_5 are given with the following selected parameters:

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad (3.50)$$

$$\tau_4(y) = 4, \quad n_1(y) = y, \quad n_2(y) = y, \quad n_3(y) = y, \quad n_4(y) = y,$$

$$m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \quad (3.51)$$

$$\tau_4(y) = 4, \quad n_1(y) = y, \quad n_2(y) = y^2, \quad n_3(y) = y, \quad n_4(y) = y,$$

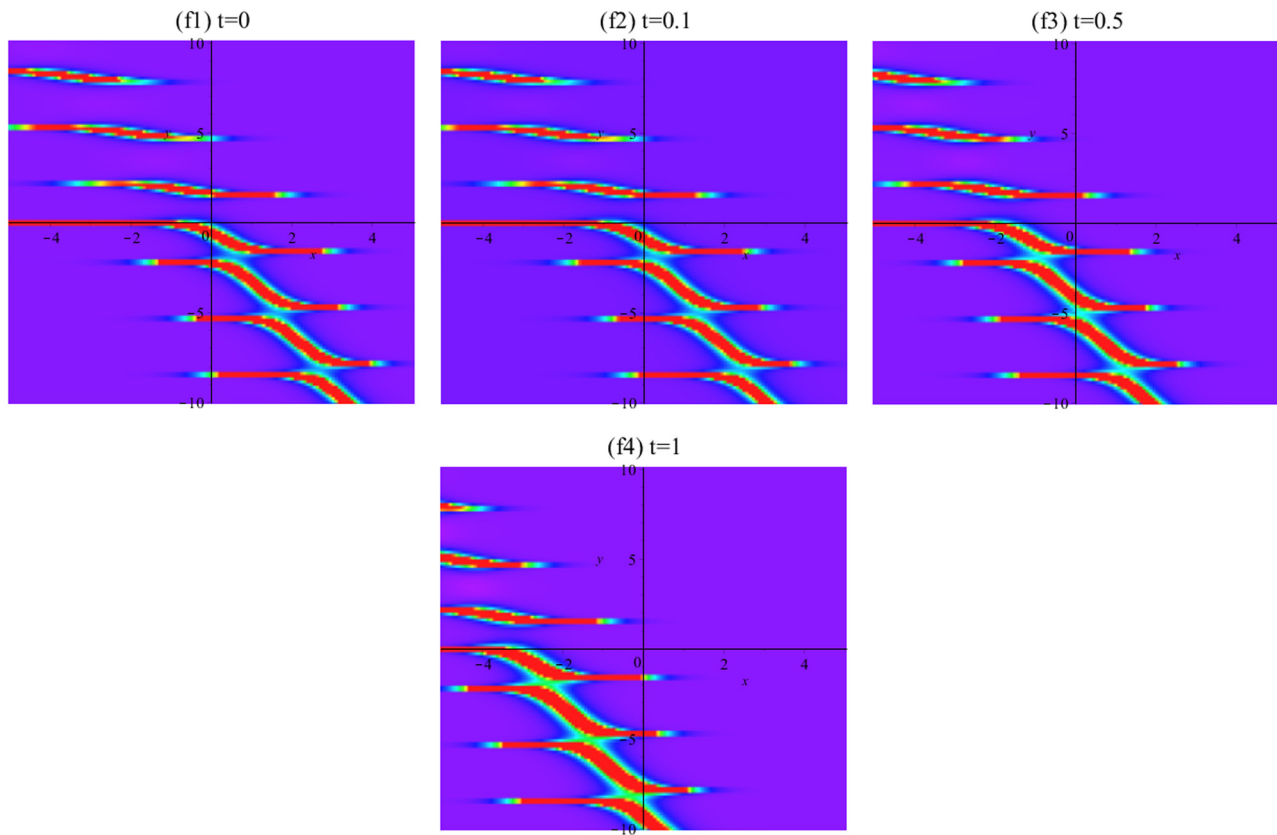


Figure 22: Plot of interaction periodic and dark soliton solutions (3.50) (Γ_5) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

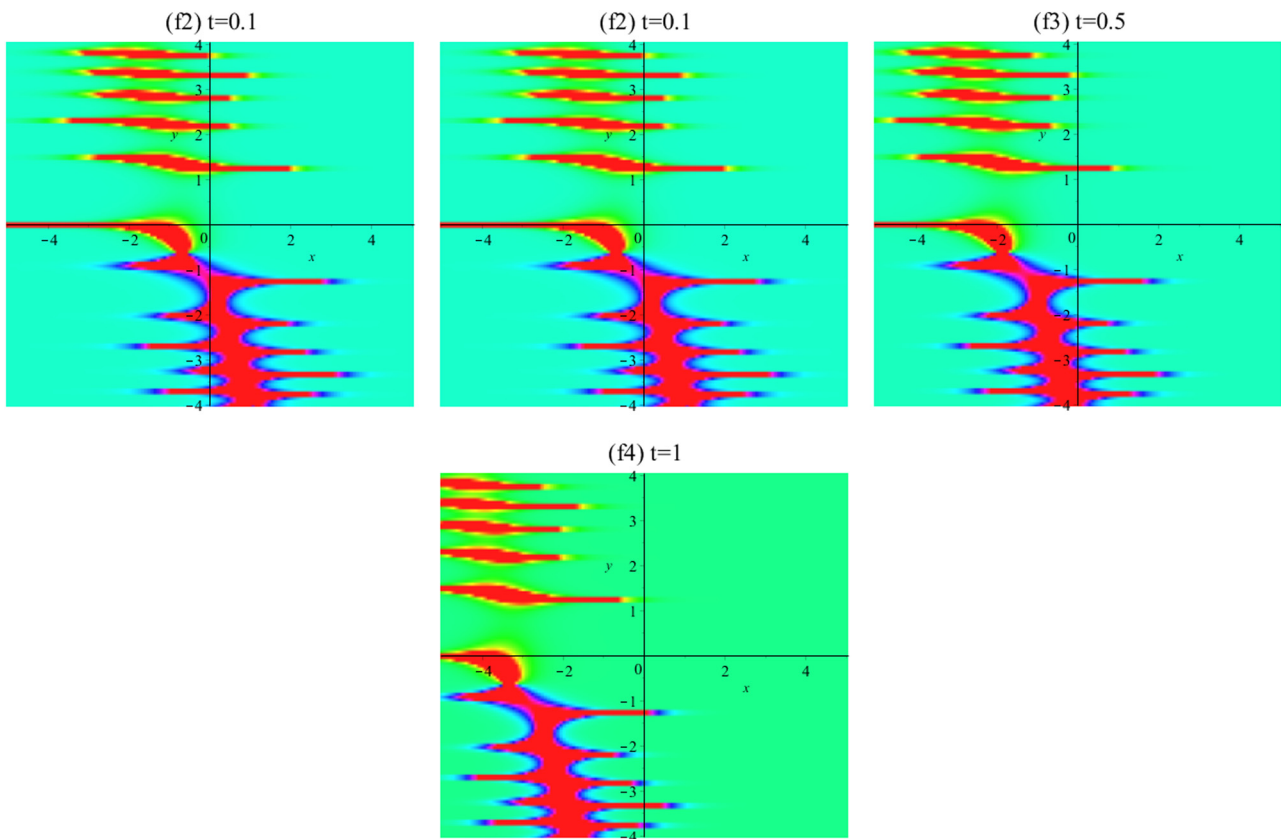


Figure 23: Plot of interaction periodic and dark soliton solutions (3.51) (Γ_5) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

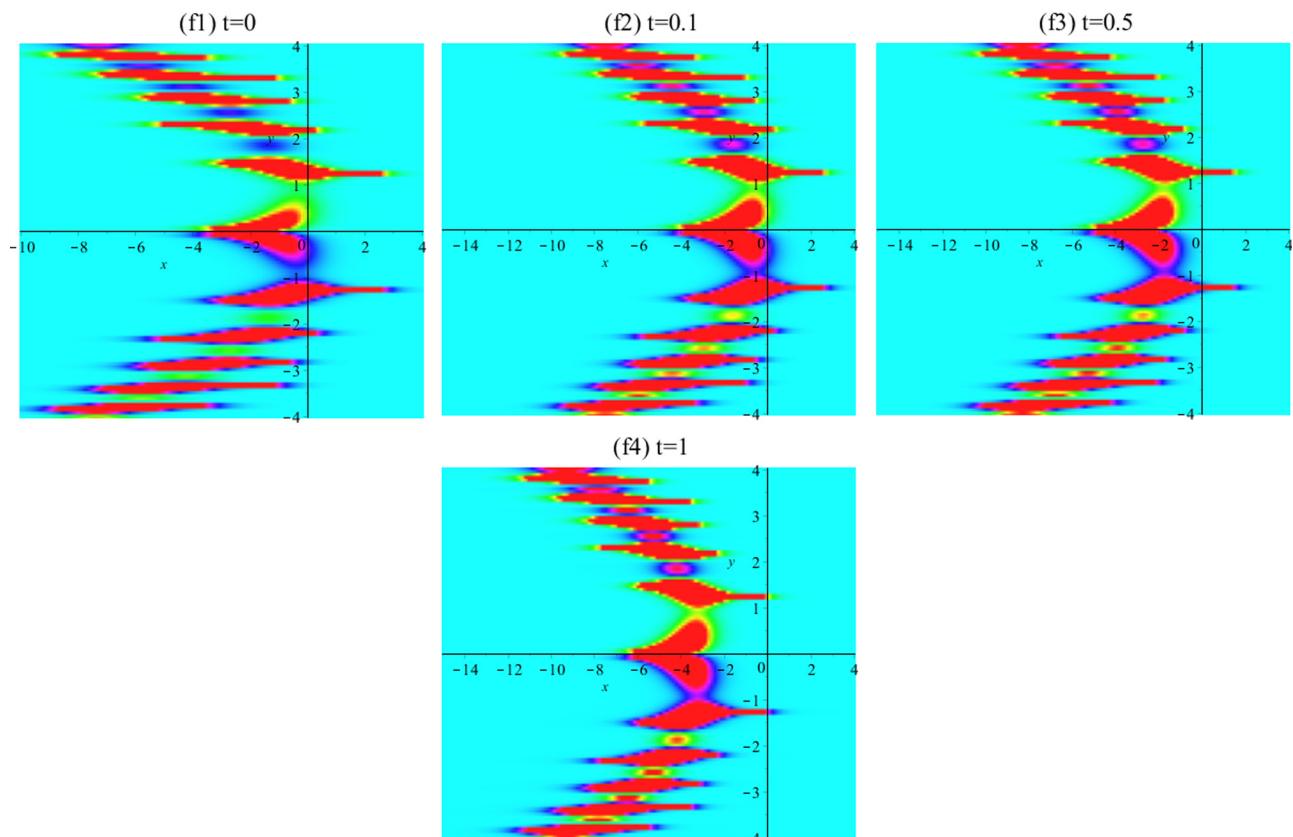


Figure 24: Plot of interaction periodic and dark soliton solutions (3.52) (Γ_5) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

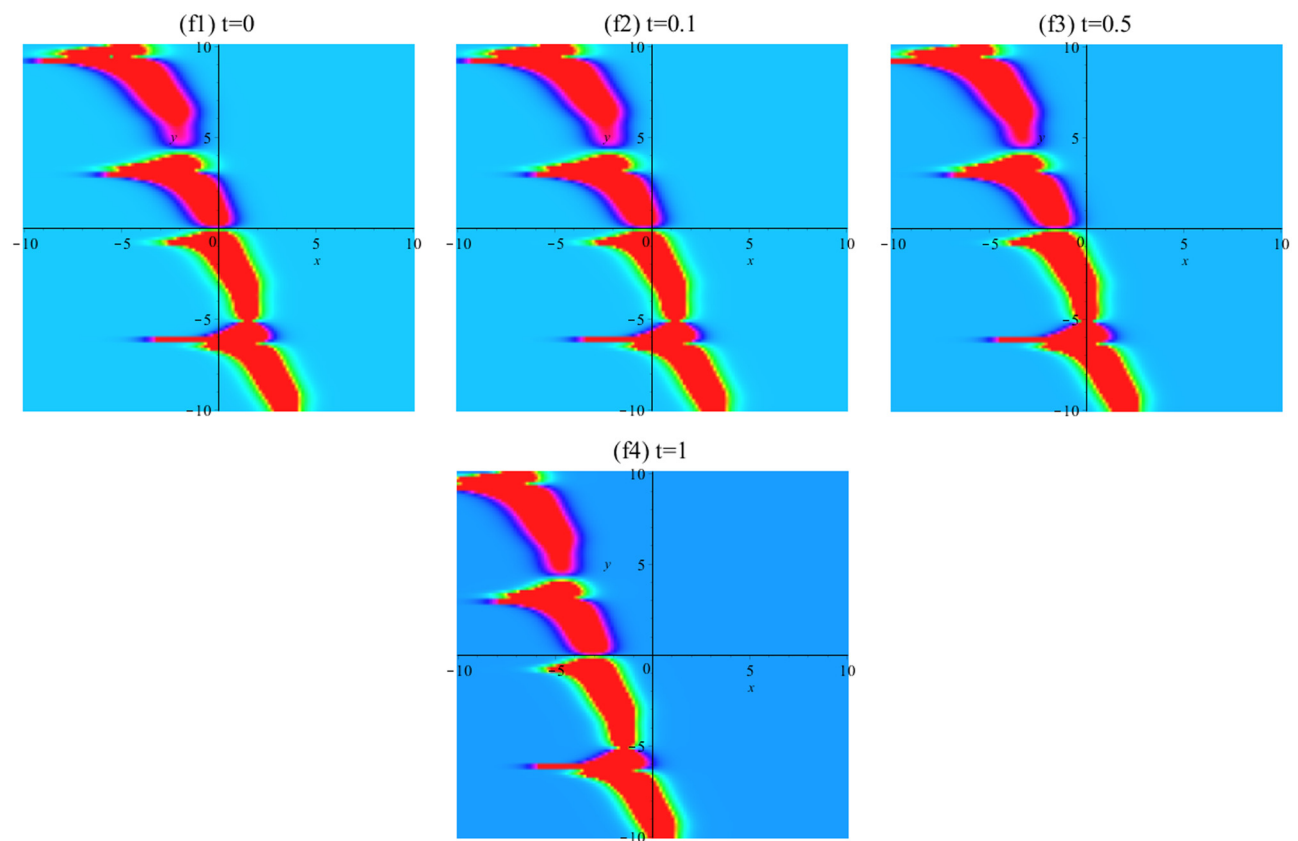


Figure 25: Plot of interaction periodic and dark soliton solutions (3.53) (Γ_5) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

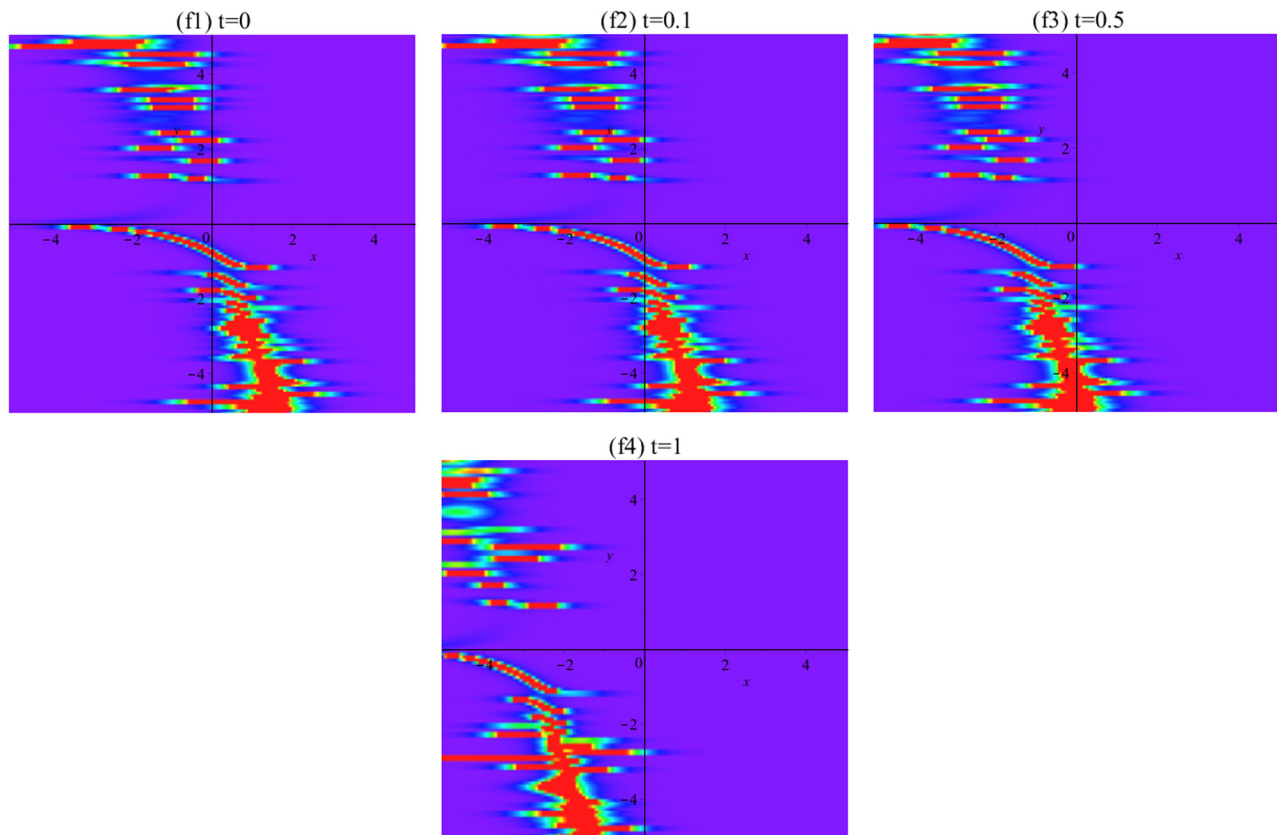


Figure 26: Plot of interaction periodic and dark soliton solutions (3.52) (Γ_5) such as density plot for four cases $t = 0, t = 0.1, 0.5, 1$, respectively.

$$\begin{aligned} m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \\ \tau_4(y) = 4, \quad n_1(y) = y^2, \quad n_2(y) = y^2, \quad n_3(y) = y^2, \quad n_4(y) = y^2, \end{aligned} \quad (3.52)$$

$$\begin{aligned} m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \\ \tau_4(y) = 4, \quad n_1(y) = y, \quad n_2(y) = \cos(y), \quad n_3(y) = y, \quad n_4(y) = y, \end{aligned} \quad (3.53)$$

$$\begin{aligned} m_1 = 2, \quad m_4 = 3, \quad \tau_1(y) = 1, \quad \tau_2(y) = 2, \quad \tau_3(y) = 3, \\ \tau_4(y) = 4, \quad n_1(y) = y, \quad n_2(y) = y^3, \quad n_3(y) = y^3, \quad n_4(y) = y, \end{aligned} \quad (3.54)$$

given in Eq. (3.49). By the aforementioned parameters, the structural property interaction periodic and dark soliton solutions are presented in Figures 22–26, with the density plot with different times. It shows a type of interaction solutions among exponential, hyperbolic, and trigonometric waves.

4 Conclusion

This article studied the extended homoclinic (heteroclinic) breather wave solutions and interaction periodic and dark soliton solutions to the nonlinear vibration and dispersive wave systems. The bilinear form of equation has been described by means of algorithm transformation. The governing equation is translated to nonlinear

ODE using Cole–Hopf algorithm. Extended homoclinic breather wave solutions containing four cases and interaction periodic and dark soliton solutions containing five cases of the nonlinear wave ordinary differential equation were obtained. We found the combination of two kinks and periodic solution for extended homoclinic breather and also combination of two kink, periodic, and dark soliton in terms of exponential, trigonometric, and hyperbolic functions for interaction periodic and dark soliton solutions of the governing equation. We present a kind of interaction solutions between trigonometrical waves and exponential waves or between trigonometrical waves and hyperbolic waves for extended homoclinic breather and also interaction solutions among two exponential, trigonometric, and hyperbolic waves for interaction periodic and dark soliton solutions. The dynamic features of different kinds of traveling waves are examined in detail through numerical simulation. Meanwhile, the profiles of the surface for the deduced solutions have been depicted in density plot for free parameters. From the acquired results, it can be concluded that the procedures followed in this analysis can be implemented in a simple and straightforward manner to create new exact

solutions of many other nonlinear partial differential equations in terms of Cole–Hopf algorithm.

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