Research Article

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A new modified technique to study the dynamics of fractional hyperbolic-telegraph equations

https://doi.org/10.1515/phys-2022-0072
received January 31, 2022; accepted June 25, 2022

Abstract: Usually, to find the analytical and numerical solution of the boundary value problems of fractional partial differential equations is not an easy task; however, the researchers devoted their sincere attempt to find the solutions of various equations by using either analytical or numerical procedures. In this article, a very accurate and prominent method is developed to find the analytical solution of hyperbolic-telegraph equations with initial and boundary conditions within the Caputo operator, which has very simple calculations. This method is called a new technique of Adomian decomposition method. The obtained results are described by plots to confirm the accuracy of the suggested technique. Plots are drawn for both fractional and integer order solutions to confirm the accuracy and validity of the proposed method. Solutions are obtained at different fractional orders to discuss the useful dynamics of the targeted problems. Moreover, the suggested technique has provided the highest accuracy with a small number of calculations. The suggested technique gives results in the form of a series of solutions with easily computable and convergent components. The method is simple and straightforward and therefore preferred for the solutions of other problems with both initial and boundary conditions.

Keywords: Adomian decomposition method, initial-boundary value problems, fractional hyperbolic-telegraph equations

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1 Introduction

Fractional calculus (FC) is an important branch of mathematics that studies the derivatives and integrals of fractional orders. Its history started with a question asked by L'Hospital in 1695. Since then, FC has gained much attention from researchers working in different fields. FC has various applications in science and engineering, such as optics, biological models, field theory, variational calculus, optimal control, quantum mechanics, nonlinear biological systems, fluid dynamics, stochastic dynamical systems, astrophysics, image processing, turbulence, signal analysis, pollution control, social systems, biomedicine, financial systems, controlled thermonuclear fusion, landscape evolution, bioengineering, elasticity, plasma physics [1–8] and so on.

In recent decades, fractional partial differential equations (FPDEs) have attracted researchers because of their important applications and uses in applied sciences [9–11]. FPDEs are very effective in the modeling of physical and engineering events. Some significant applications of FPDEs are image deionization [12], fractional dynamics, control theory and signal processing, fluid flow, system identification [4,13], diffusive transport, rheology, electrical network, probability [14,15], climate, social sciences like food supplements, the mechanics of materials, plasma physics [16,17], electromagnetic, controlled thermonuclear fusion, astrophysics, stochastic dynamical system, image processing, scattering, turbulent flow, chaotic dynamics, diffusion processes, electrical, and rheological materials [18].

Finding the solution of FPDEs is a hard and challenging task, with higher efforts required to perform for the harder mathematical solutions. Because the exact solutions of FPDEs are difficult to calculate, we need an easy and effective numerical and analytical algorithms. Many researchers have contributed their work to find the solutions of FPDEs and, therefore, different techniques have been developed. Some novel methods are the (G'/G) method [19], the EXP method [20], Bäcklund transformation method, Kudryashov method [21], fractional sub-equation method [22], the simplest equation method [23], Laplace transform [24], the Laplace Adomian decomposition method [25], the Elzaki transform decomposition method [26], the natural transform decomposition method [27], the Chebyshev wavelet method [28], the He's variational iteration method [29], the homotopy perturbation method [30], q-homotopy analysis transformed method [31], the extended rational sinh–cosh method and modified Khater method [32], the reduced differential transform method [33], the meshless Kansa method [34], optimal axillary function method [35], the variable separation method [36], the tanh method [37], the sine–cosine method [38], the spectral collocation method [39], and the residual power series method [40]. Some of the authors implemented the most time discretization scheme for solving time-fractional partial differential equations [41–43].

Many methods are introduced by the researchers to solve fractional-order hyperbolic-telegraph equations (FHTEs). Mohanty *et al.* [44] used an unconditional iterative scheme, and Lakestani *et al.* [45] used an interpolating scaling function technique to solve the 1D hyperbolic telegraph equation. Jiwari *et al.* [46] and Tezer–Sezgin *et al.* [47] used the differential quadrature method, the homotopy analysis method [48], the fictitious time integration method [49], the Chebyshev tau method [50], the hybrid meshless method [51], and the Houbolt method [52].

In this research article, we will use a new method of ADM for the solution of FHTEs. The method was introduced in the 1980s by Adomian to solve some functional equations [53,54]. After that other researchers have shown their keen interest and several modifications to the existing methods were also introduced. For example, Hosseini *et al.* applied it to linear and nonlinear differential equations [55]. Fractional integro-differential equations were solved by Hamoud *et al.* [56]. Pue-on and Viriyapong modified third-order ordinary differential equations [57]. Then, the Klein–Gordon equations were solved by Saelao and Yokchoo [58]. Other ADM modifications can be seen in refs [59–63].

This modification of ADM implemented in the current work was introduced by Elaf Jaafer Ali in ref. [64]. Furthermore, all of the aforementioned existing techniques attempt to solve fractional problems with either initial or boundary conditions, but in this work, we used both initial and boundary conditions to solve FHTEs using the current technique [64]. In ref. [65], the homotopy perturbation method is used to solve for the same problems. The same procedure is used in ref. [66] to solve initial-boundary value problems using the variational iteration method. We extended the idea to fractional initial-boundary value problems in ref. [67]. The proposed method has a higher rate of convergence toward the exact solution because of the new initial approximate solution for each term. The present method is recommended for other higher-order nonlinear problems in science and engineering.

2 Preliminaries

In this section, a few definitions related to our work are taken into consideration.

2.1 Definition

The integral operator of Reimann–Liouville having order ϑ is given by ref. [39]

$$(I_{\zeta}^{\vartheta}h)(\zeta) = \begin{cases} \dfrac{1}{\Gamma(\vartheta)}\displaystyle\int_0^{\zeta} \dfrac{h(\tau)}{(\zeta-\tau)}^{1-\vartheta}\mathrm{d}\tau, & \vartheta > 0, \\ h(\zeta), & \vartheta = 0, \end{cases}$$

and its fractional derivative for $\vartheta \geq 0$ is

$$(D_{\zeta}^{\vartheta}h)(\zeta) = \left(\frac{\mathrm{d}}{\mathrm{d}\zeta}\right)^{m}(I^{m-\vartheta}h)(\zeta), \quad (\vartheta > 0, m < \vartheta < m-1),$$

where m is an integer.

2.2 Definition

Using Reimann–Liouville [39] definition, we have

$$I_{\zeta}^{\vartheta}(\zeta^{\nu}) = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1+\vartheta)}\zeta^{\nu+\vartheta} \quad (\vartheta > 0, m < \vartheta \leq m-1).$$

2.3 Definition

The Mittag–Leffler function [68] $E_\eta(\rho)$ for $\eta > 0$ is

$$E_\eta(\rho) = \sum_{n=0}^{\infty}\left(\frac{\rho^n}{\Gamma(n\eta + 1)}\right), \quad \eta > 0, \ \rho \in C.$$

3 ADM [64]

The present technique was discovered by Adomian (1994) to solve linear, nonlinear differential, and integro differential equations. To understand the method, let us consider an equation of the following form:

$$F(\omega(\zeta)) = g(\zeta), \tag{1}$$

where F is a nonlinear differential operator and g is the known function. We will split the linear term in $F(\omega(\zeta))$ into the form $Lu + Ru$, where L is the invertible operator, chosen as the highest order derivative, R represents the linear operator, then Eq. (1) has the representation as follows:

$$L\omega + R\omega + N\omega = g, \tag{2}$$

where $N\omega$ is the nonlinear term of $F(\omega(\zeta))$. Apply the inverse operator, L^{-1}, to both sides of Eq. (2)

$$\omega = \phi + L^{-1}(g) - L^{-1}(R\omega) - L^{-1}(N\omega),$$

where ϕ is the constant of integration and $L\phi = 0$.

The ADM solution can be represented in the form of infinite series as

$$\omega = \sum_{n=0}^{\infty}\zeta_n.$$

The nonlinear term Nu is denoted by A_n, and is defined as follows:

$$N\omega = \sum_{n=0}^{\infty}A_n, \tag{3}$$

and we can calculate A_n using the following formula:

$$A_n = \frac{1}{n!}\left[\frac{\mathrm{d}^n}{\mathrm{d}\psi^n}N\left[\sum_{n=0}^{\infty}(\lambda^n\omega_n)\right]\right]_{\lambda=0}, \quad n = 0, 1, \ldots$$

The series has the following relation to represent the solution of Eq. (1),

$$\begin{cases}\omega_0 = \mu + L^{-1}(g), & n = 0.\\ \omega_{n+1} = L^{-1}(Ru_n) - L^{-1}(A_n), & n \geq 0.\end{cases}$$

4 Modification of ADM

To understand the main idea of the proposed technique, we will take the following one-dimensional equation [64].

$$\frac{\partial^\vartheta\omega}{\partial\tau^\vartheta} + \gamma\frac{\partial\omega}{\partial\tau} + \eta\omega = \frac{\partial^2\omega}{\partial\zeta^2} + f(\zeta, \tau), \quad 1 < \vartheta \leq 2. \tag{4}$$

With the following initial and boundary conditions as

$$\begin{cases}\omega(\zeta, \tau) = f_0(\zeta), \quad \dfrac{\partial\omega(\zeta, 0)}{\partial\tau} = f_1(\zeta), \quad 0 \leq \zeta \leq 1,\\ \omega(0, \tau) = g_0(\tau), \quad \omega(1, \tau) = g_1(\tau), \quad \tau > 0,\end{cases}$$

where $f(\zeta, \tau)$ represents the source term.

The new initial solution (ω_n^*) is calculated using the proposed technique and the following iteration for Eq. (4):

$$\omega_n^* = \omega_n(\zeta, \tau) + (1 - \zeta)[g_0(\tau) - \omega_n(0, \tau)] + \zeta[g_1(\tau) - \omega_n(1, \tau)].$$

Using ADM, the operator form of Eq. (4) is

$$Lu = \frac{\partial^2\omega(\zeta, \tau)}{\partial\zeta^2} + f(\zeta, \tau) - \gamma\frac{\partial\omega(\zeta, \tau)}{\partial\tau} - \eta\omega(\zeta, \tau), \quad 1 < \vartheta \leq 2, \tag{5}$$

where the differential operator L is defined as

$$L = \frac{\partial^\vartheta}{\partial\tau^\vartheta}.$$

Hence L^{-1} is defined as

$$L^{-1}(\cdot) = I^\vartheta(\cdot)\mathrm{d}\tau.$$

Applying L^{-1} to Eq. (5), we have

$$\omega(\zeta, \tau) = \omega(\zeta, 0) + L^{-1}\left(\frac{\partial^2\omega(\zeta, \tau)}{\partial\zeta^2} + f(\zeta, \tau) - \gamma\frac{\partial\omega(\zeta, \tau)}{\partial\tau} - \eta\omega(\zeta, \tau)\right), \ 1 < \vartheta \leq 2,$$

where $n = 0, 1, \ldots$.

Using the ADM solution, the initial approximation becomes

$$\omega_0(\zeta, \tau) = \omega(\zeta, 0) + \tau(\omega_\tau(\zeta, \tau)) + L^{-1}(f(\zeta, \tau)),$$

and using the new ADM technique, the iteration formula becomes

$$\omega_{n+1}(\zeta, \tau) = L^{-1}\left(\frac{\partial^2\omega(\zeta, \tau)}{\partial\zeta^2} - \gamma\frac{\partial\omega(\zeta, \tau)}{\partial\tau} - \eta\omega(\zeta, \tau)\right),$$
$$1 < \vartheta \leq 2.$$

It is obvious that initial solutions ω_n^* of Eq. (4) satisfy both the initial and boundary conditions, as given below:

$$\begin{aligned} \text{at } \tau = 0, \quad & \omega_n^*(\zeta, 0) = \omega_n(\zeta, 0), \\ \zeta = 0, \quad & \omega_n^*(0, \tau) = g_0(\tau), \\ \zeta = 1, \quad & \omega_n^*(1, \tau) = g_1(\tau). \end{aligned}$$

The proposed technique work effectively for the two-dimensional problems.

5 Numerical results

In this section, we will present the solution of some illustrative examples by using the new technique based on ADM.

5.1 Example

Consider the case of Eq. (4), when $\gamma = 4$, $\eta = 2$ [69]

$$\frac{\partial^{\vartheta}\omega(\zeta, \tau)}{\partial\tau^{\vartheta}} + 4\frac{\partial\omega(\zeta, \tau)}{\partial\tau} + 2\omega(\zeta, \tau) = \frac{\partial^2\omega(\zeta, \tau)}{\partial\tau^2}, \quad 0 < \zeta < \pi, \quad 1 < \vartheta \leq 2, \tag{6}$$

with the initial and boundary conditions as follows:

$$\begin{aligned} \omega(\zeta, 0) &= \sin(\zeta), \\ \omega_{\tau}(\zeta, 0) &= -\sin(\zeta), \\ \omega(0, \tau) &= 0, \\ \omega(\pi, \tau) &= 0. \end{aligned}$$

The problem has the exact solution at $\vartheta = 1$ as follows:

$$\omega(\zeta, \tau) = \mathrm{e}^{-\tau}\sin(\zeta).$$

Applying the new technique based on ADM to Eq. (6), we obtain the following result:

$$\begin{aligned} \omega_n^*(\zeta, \tau) = \omega_n(\zeta, \tau) + (1 - \zeta)[0 - \omega_n(0, \tau)] \\ + \zeta[0 - \omega_n(\pi, \tau)], \end{aligned} \tag{7}$$

where $n = 0, 1, \ldots$

Applying L to Eq. (6), we have

$$L\omega = \frac{\partial^2\omega(\zeta, \tau)}{\partial\tau^2} - 4\frac{\partial\omega(\zeta, \tau)}{\partial\tau} - 2\omega(\zeta, \tau), \tag{8}$$

where $L = \frac{\partial^{\vartheta}}{\partial\tau^{\vartheta}}$ and L^{-1} is defined as

$$L^{-1}(\cdot) = I^{\vartheta}(\cdot)\mathrm{d}t.$$

Operating Eq. (6) by L^{-1}, we have

$$\omega(\zeta, \tau) = \omega(\zeta, 0) + L^{-1}\left(\frac{\partial^2\omega(\zeta, \tau)}{\partial\tau^2} - 4\frac{\partial\omega(\zeta, \tau)}{\partial\tau} - 2\omega(\zeta, \tau)\right),$$

and using the ADM solution, the initial approximation becomes

$$\begin{aligned} \omega_0(\zeta, \tau) &= \omega(\zeta, 0) + \tau(\omega_{\tau}(\zeta, 0)), \\ \omega_0(\zeta, \tau) &= (1 - \tau)\sin(\zeta). \end{aligned}$$

Using the new technique of initial approximation ω_n^*, the iteration formula becomes

$$\omega_{n+1}(\zeta, \tau) = L^{-1}\left(\frac{\partial^2\omega_n^*(\zeta, \tau)}{\partial\tau^2} - 4\frac{\partial\omega_n^*(\zeta, \tau)}{\partial\tau} - 2\omega_n^*(\zeta, \tau)\right). \tag{9}$$

By putting initial and boundary conditions in Eq. (7), for $n = 0$,

$$\begin{aligned} \omega_0^*(\zeta, \tau) &= \omega_0(\zeta, \tau) + (1 - \zeta)(0 - \omega_0(0, \tau)) \\ &\quad + \zeta(0 - \omega_0(\pi, \tau)), \\ \omega_0^*(\zeta, \tau) &= (1 - \tau)\sin(\zeta). \end{aligned}$$

From Eq. (9), we have

$$\begin{aligned} \omega_1(\zeta, \tau) &= L^{-1}\left(\frac{\partial^2\omega_0^*(\zeta, \tau)}{\partial\zeta^2} - 4\frac{\partial\omega_0^*(\zeta, \tau)}{\partial\tau} - 2\omega_0^*(\zeta, \tau)\right), \\ &= L^{-1}(-\sin(\zeta)(1 - \tau) - 4(-1)\sin(\zeta) \\ &\quad - 2(1 - \tau)\sin(\zeta)), \\ &= L^{-1}(\sin(\zeta)(-1 + \tau - 2 + 2\tau + 4)), \\ &= L^{-1}(\sin(\zeta)(1 + 3\tau)), \\ \omega_1(\zeta, \tau) &= \sin(\zeta)\left[\frac{\tau^{\vartheta}}{\Gamma(\vartheta + 1)} + \frac{3\tau^{\vartheta+1}}{\Gamma(\vartheta + 2)}\right]. \end{aligned}$$

For $n = 1$, Eq. (7) becomes

$$\begin{aligned} \omega_1^*(\zeta, \tau) &= \omega_1(\zeta, \tau) + (1 - \zeta)[0 - \omega_1(0, \tau)] \\ &\quad + \zeta[0 - \omega_1(\pi, \tau)], \\ \omega_1^*(\zeta, \tau) &= \sin(\zeta)\left[\frac{\tau^{\vartheta}}{\Gamma(\vartheta + 1)} + \frac{3\tau^{\vartheta+1}}{\Gamma(\vartheta + 2)}\right]. \end{aligned}$$

From Eq. (9), we have

$$\begin{aligned} \omega_2(\zeta, \tau) &= L^{-1}\left(\frac{\partial^2\omega_1^*(\zeta, \tau)}{\partial\zeta^2} - 4\frac{\partial\omega_1^*}{\partial\tau} - 2\omega_1^*\right), \\ \omega_2(\zeta, \tau) &= -3\sin(\zeta)\left[\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta + 1)} + \frac{3\tau^{2\vartheta+1}}{\Gamma(2\vartheta + 2)}\right] \\ &\quad - 4\sin(\zeta)\left[\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{3\tau^{2\vartheta}}{\Gamma(2\vartheta + 1)}\right]. \end{aligned}$$

For $n = 2$, Eq. (7), becomes

$$\omega_2^*(\zeta,\tau) = \omega_2(\zeta,\tau) + (1-\zeta)[0-\omega_2(0,\tau)] + \zeta[0-\omega_2(\pi,\tau)],$$

$$\omega_2^*(\zeta,\tau) = -3\sin(\zeta)\left[\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{3\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)}\right] - 4\sin(\zeta)\left[\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{3\tau^{2\vartheta}}{\Gamma(2\vartheta+1)}\right].$$

From Eq. (9), we have

$$\omega_3(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_2^*(\zeta,\tau)}{\partial\zeta^2} - 4\frac{\partial^2\omega_2^*}{\partial\tau} - 2\omega_2^*\right),$$

$$\omega_3(\zeta,\tau) = 9\sin(\zeta)\left[\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + \frac{3\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)}\right] + 24\sin(\zeta)\left[\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + \frac{3\tau^{3\vartheta}}{\Gamma(3\vartheta+1)}\right] + 16\sin(\zeta)\left[\frac{\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} + \frac{3\tau^{3\vartheta-1}}{\Gamma(3\vartheta)}\right].$$

$$\vdots$$

Thus, the ADM solution in the series form is

$$\omega(\zeta,\tau) = \omega_0(\zeta,\tau) + \omega_1(\zeta,\tau) + \omega_2(\zeta,\tau) + \omega_3(\zeta,\tau) + \ldots$$

5.2 Example

Consider the case, of Eq. (4), when $\gamma = 6$ and $\eta = 2$ [69]

$$\frac{\partial^\vartheta\omega(\zeta,\tau)}{\partial\tau^\vartheta} + 6\frac{\partial\omega(\zeta,\tau)}{\partial\tau} + 2\omega(\zeta,\tau) = \frac{\partial^2\omega(\zeta,\tau)}{\partial\zeta^2} - 2e^{-\tau}\sin(\zeta) \quad 0<\zeta<\pi,\ 1<\vartheta\le 2, \tag{10}$$

with the initial and boundary conditions as follows:

$$\omega(\zeta,0) = \sin(\zeta),$$
$$\omega_\tau(\zeta,0) = -\sin(\zeta),$$
$$\omega(0,\tau) = 0,$$
$$\omega(\pi,\tau) = 0.$$

The problem has the exact solution at $\vartheta = 1$ as follows:

$$\omega(\zeta,\tau) = e^{-\tau}\sin(\zeta).$$

Applying the new technique based on ADM to Eq. (10), we obtain the following result:

$$\omega_n^*(\zeta,\tau) = \omega_n(\zeta,\tau) + (1-\zeta)[0-\omega_n(0,\tau)] + \zeta[0-\omega_n(\pi,\tau)], \tag{11}$$

where $n = 0, 1, \ldots$

Applying L to Eq. (10), we have

$$L\omega = \frac{\partial^2\omega(\zeta,\tau)}{\partial\tau^2} - 4\frac{\partial\omega(\zeta,\tau)}{\partial\tau} - 2\omega(\zeta,\tau) - 2e^{-\tau}\sin(\zeta), \tag{12}$$

where $L = \frac{\partial^\vartheta}{\partial\tau^\vartheta}$ and L^{-1} is defined as

$$L^{-1}(\cdot) = I^\vartheta(\cdot)dt.$$

Operating Eq. (10) by L^{-1}, we have

$$\omega(\zeta,\tau) = \omega(\zeta,0) + L^{-1}\left(\frac{\partial^2\omega(\zeta,\tau)}{\partial\tau^2} - 4\frac{\partial\omega(\zeta,\tau)}{\partial\tau} - 2\omega(\zeta,\tau)\right).$$

Using ADM solution, the initial approximation becomes

$$\omega_0(\zeta,\tau) = \omega(\zeta,0) + \tau(\omega_\tau(\zeta,0)) + L^{-1}(2e^{-\tau}\sin(\zeta)),$$

$$\omega_0(\zeta,\tau) = \left(1 - \tau - 2\frac{\tau^\vartheta}{\Gamma(\vartheta+1)} + \frac{2\tau^{\vartheta+1}}{\Gamma(\vartheta+2)} - \frac{4\tau^{\vartheta+2}}{\Gamma(\vartheta+3)}\right)\sin(\zeta).$$

Using the new technique of initial approximation ω_n^*, the iteration formula becomes

$$\omega_{n+1}(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_n^*(\zeta,\tau)}{\partial\tau^2} - 6\frac{\partial\omega_n^*(\zeta,\tau)}{\partial\tau} - 2\omega_n^*(\zeta,\tau)\right). \tag{13}$$

By putting initial and boundary conditions in Eq. (11), for $n = 0$,

$$\omega_0^*(\zeta,\tau) = \omega_0(\zeta,\tau) + (1-\zeta)(0-\omega_0(0,\tau)) + \zeta(0-\omega_0(\pi,\tau)),$$

$$\omega_0^*(\zeta,\tau) = \left(1 - \tau - 2\frac{\tau^\vartheta}{\Gamma(\vartheta+1)} + \frac{2\tau^{\vartheta+1}}{\Gamma(\vartheta+2)} - \frac{4\tau^{\vartheta+2}}{\Gamma(\vartheta+3)}\right)\sin(\zeta).$$

From Eq. (13), we have

$$\omega_1(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_0^*(\zeta,\tau)}{\partial\zeta^2} - 6\frac{\partial\omega_0^*(\zeta,\tau)}{\partial\tau} - 2\omega_0^*(\zeta,\tau)\right).$$

$$\omega_1(\zeta,\tau) = -3\sin(\zeta)\left(\frac{\tau^\vartheta}{\Gamma(\vartheta+1)} - \frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)} - 2\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{2\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)} - \frac{4\tau^{2\vartheta+2}}{\Gamma(2\vartheta+3)}\right) - 6\sin(\zeta)\left(-\frac{\tau^\vartheta}{\Gamma(\vartheta+1)} - \frac{2\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{2\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{4\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)}\right).$$

For $n = 1$, Eq. (11) becomes

$$\omega_1^*(\zeta,\tau) = \omega_1(\zeta,\tau) + (1-\zeta)[0-\omega_1(0,\tau)] + \zeta[0-\omega_1(\pi,\tau)],$$

$$\omega_1^*(\zeta,\tau) = -3\sin(\zeta)\left(\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)} - \frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)} - 2\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + \frac{2\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)} - \frac{4\tau^{2\vartheta+2}}{\Gamma(2\vartheta+3)}\right) - 6\sin(\zeta)\left(-\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)} - \frac{2\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \frac{2\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{4\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)}\right).$$

From Eq. (13), we have

$$\omega_2(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_1^*(\zeta,\tau)}{\partial\zeta^2} - 6\frac{\partial\omega_1^*}{\partial\tau} - 2\omega_1^*\right),$$

$$\omega_2(\zeta,\tau) = 9\sin(\zeta)\left(\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)} - 2\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + \frac{2\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)} - \frac{4\tau^{3\vartheta+2}}{\Gamma(3\vartheta+3)}\right) + 18\sin(\zeta)\left(-\frac{t^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{2t^{3\vartheta-1}}{\Gamma(3\vartheta)} + \frac{2\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{4\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)}\right)$$
$$+ 18\sin(\zeta)\left(\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} - \frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{2\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + \frac{2\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{4\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)}\right) + 36\sin(\zeta)\left(\frac{-\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} - \frac{2\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} + \frac{2\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{4\tau^{3\vartheta}}{\Gamma(3\vartheta+1)}\right).$$

For $n = 2$, Eq. (11) becomes

$$\omega_2^*(\zeta,\tau) = \omega_2(\zeta,\tau) + (1-\zeta)[0-\omega_2(0,\tau)] + \zeta[0-\omega_2(\pi,\tau)],$$

$$\omega_2^*(\zeta,\tau) = 9\sin(\zeta)\left(\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{\tau^{2\vartheta+1}}{\Gamma(2\vartheta+2)} - 2\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + \frac{2\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)} - \frac{4\tau^{3\vartheta+2}}{\Gamma(3\vartheta+3)}\right) + 18\sin(\zeta)\left(-\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{2\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + \frac{2\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{4\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)}\right)$$
$$+ 18\sin(\zeta)\left(\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} - \frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - \frac{2\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + \frac{2\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{4\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)}\right) + 36\sin(\zeta)\left(\frac{-\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} - \frac{2\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} + \frac{2\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{4\tau^{3\vartheta}}{\Gamma(3\vartheta+1)}\right).$$

From Eq. (13), we have

$$\omega_3(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_2^*(\zeta,\tau)}{\partial\zeta^2} - 6\frac{\partial^2\omega_2^*}{\partial\tau} - 2\omega_2^*\right),$$

$$\omega_3(\zeta,\tau) = -27\sin(\zeta)\left(\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{\tau^{3\vartheta+1}}{\Gamma(3\vartheta+2)} - 2\frac{\tau^{4\vartheta}}{\Gamma(4\vartheta+1)} + \frac{2\tau^{4\vartheta+1}}{\Gamma(4\vartheta+2)} - \frac{4\tau^{4\vartheta+2}}{\Gamma(4\vartheta+3)}\right)$$
$$- 54\sin(\zeta)\left(-\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{2\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} + \frac{2\tau^{4\vartheta}}{\Gamma(4\vartheta+1)} - \frac{4\tau^{4\vartheta+1}}{\Gamma(4\vartheta+2)}\right)$$
$$- 108\sin(\zeta)\left(\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} - \frac{2\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} + \frac{2\tau^{4\vartheta}}{\Gamma(4\vartheta+1)} - \frac{4\tau^{4\vartheta+1}}{\Gamma(4\vartheta+2)}\right)$$
$$- 108\sin(\zeta)\left(-\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{2\tau^{4\vartheta-2}}{\Gamma(4\vartheta-1)} + \frac{2\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} - \frac{4\tau^{4\vartheta}}{\Gamma(4\vartheta+1)}\right)$$
$$- 108\sin(\zeta)\left(\frac{-\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{2\tau^{4\vartheta}}{\Gamma(4\vartheta+1)} + \frac{2\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} - \frac{4\tau^{4\vartheta}}{\Gamma(4\vartheta+1)}\right)$$
$$- 108\sin(\zeta)\left(\frac{\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} - \frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - \frac{2\tau^{4\vartheta-2}}{\Gamma(4\vartheta-1)} + \frac{2\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} - \frac{4\tau^{4\vartheta}}{\Gamma(4\vartheta+1)}\right)$$
$$- 216\sin(\zeta)\left(\frac{-\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} - \frac{\tau^{4\vartheta-3}}{\Gamma(4\vartheta-2)} + \frac{2\tau^{4\vartheta-2}}{\Gamma(4\vartheta-1)} - \frac{4\tau^{4\vartheta-1}}{\Gamma(4\vartheta)}\right).$$
$$\vdots$$

Thus, the ADM solution in the series form is

$$\omega(\zeta,\tau) = \omega_0(\zeta,\tau) + \omega_1(\zeta,\tau) + \omega_2(\zeta,\tau) + \omega_3(\zeta,\tau) + \ldots$$
$$\omega(\zeta,\tau) = e^{-\tau}\sin(\zeta).$$

5.3 Example

Consider the case of Eq. (4), when $\gamma = 1$ and $\eta = 1$ [69]

$$\frac{\partial^{\vartheta}\omega(\zeta,\tau)}{\partial\tau^{\vartheta}} + \frac{\partial\omega(\zeta,\tau)}{\partial\tau} + \omega(\zeta,\tau) = \frac{\partial^2\omega(\zeta,\tau)}{\partial\zeta^2} + \zeta^2 + \tau - 1 \tag{14}$$
$$0 < \zeta < \pi, 1 < \vartheta \leq 2,$$

with the initial and boundary conditions as follows:

$$\omega(\zeta,0)=\zeta^2,$$
$$\omega_\tau(\zeta,0)=1,$$
$$\omega(0,\tau)=\tau,$$
$$\omega(1,\tau)=1+\tau.$$

The problem has the exact solution at $\vartheta = 1$ as follows:

$$\omega(\zeta,\tau)=\zeta^2+\tau.$$

Applying the new technique based on ADM to Eq. (14), we obtain the following result:

$$\omega_n^*(\zeta,\tau)=\omega_n(\zeta,\tau)+(1-\zeta)[\tau-\omega_n(0,\tau)] + \zeta[(1+\tau)-\omega_n(1,\tau)], \quad (15)$$

where $n = 0, 1, \ldots$

Applying L to Eq. (14), we have

$$L\omega=\frac{\partial^2\omega(\zeta,\tau)}{\partial\tau^2}-\frac{\partial\omega(\zeta,\tau)}{\partial\tau}-\omega(\zeta,\tau)+\zeta^2+\tau-1, \quad (16)$$

where $L=\frac{\partial^\vartheta}{\partial\tau^\vartheta}$ and L^{-1} is defined as

$$L^{-1}(\cdot)=I^\vartheta(\cdot)\mathrm{d}t.$$

Operating Eq. (14) by L^{-1}, we have

$$\omega(\zeta,\tau)=\omega(\zeta,0)+L^{-1}\left(\frac{\partial^2\omega(\zeta,\tau)}{\partial\tau^2}-\frac{\partial\omega(\zeta,\tau)}{\partial\tau} - \omega(\zeta,\tau)+\zeta^2+\tau-1\right).$$

Using ADM solution, the initial approximation becomes

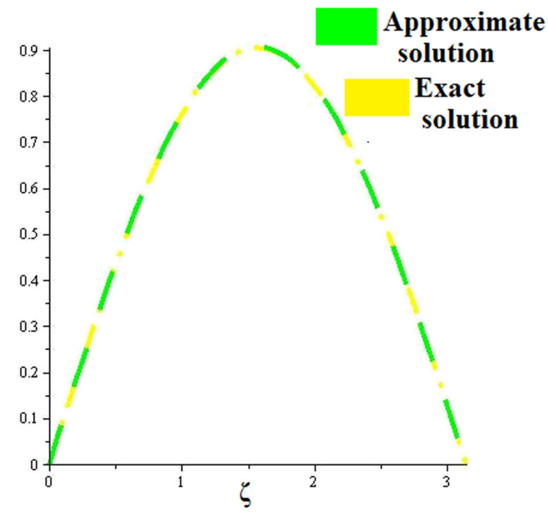

Figure 2: Two-dimensional plots for comparison between exact and approximate solution for $\vartheta = 2$ of Example 5.1.

$$\omega_0(\zeta,\tau)=\omega(\zeta,0)+\tau(\omega_\tau(\zeta,0))+L^{-1}(\zeta^2+\tau-1),$$
$$\omega_0(\zeta,\tau)=\zeta^2+\tau+(\zeta^2-1)\frac{\tau^\vartheta}{\Gamma(\vartheta+1)}+\frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)},$$

and using the new technique of initial approximation ω_n^*, the iteration formula becomes

$$\omega_{n+1}(\zeta,\tau)=L^{-1}\left(\frac{\partial^2\omega_n^*(\zeta,\tau)}{\partial\zeta^2}-\frac{\partial\omega_n^*(\zeta,\tau)}{\partial\tau}-2\omega_n^*(\zeta,\tau)\right). \quad (17)$$

By putting initial and boundary conditions in Eq. (15), for $n = 0$

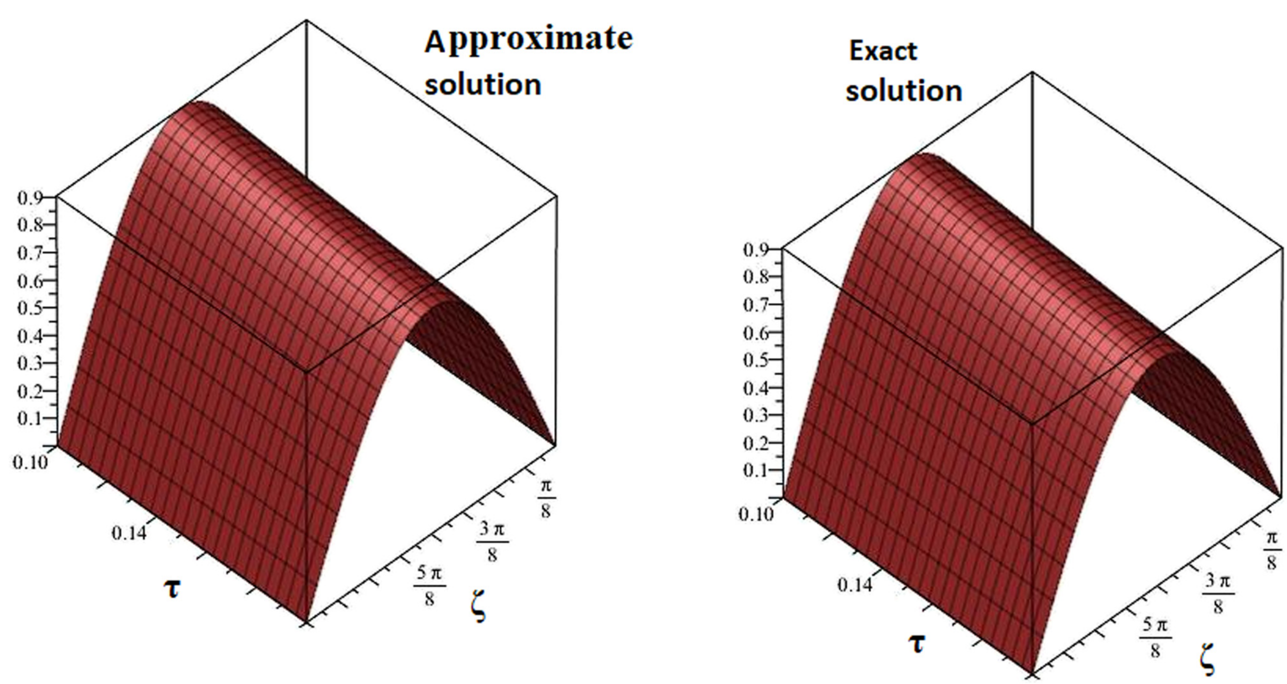

Figure 1: Three-dimensional plots of exact and approximate solutions for $\vartheta = 2$ of Example 5.1.

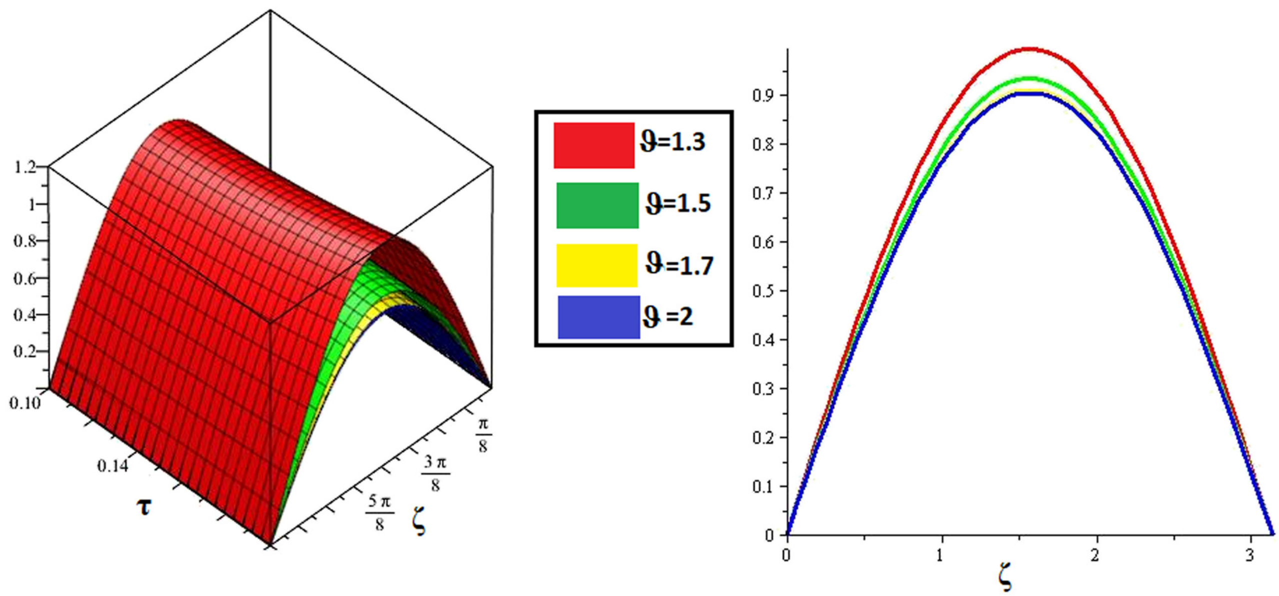

Figure 3: Plots for different values of ϑ for Example 5.1.

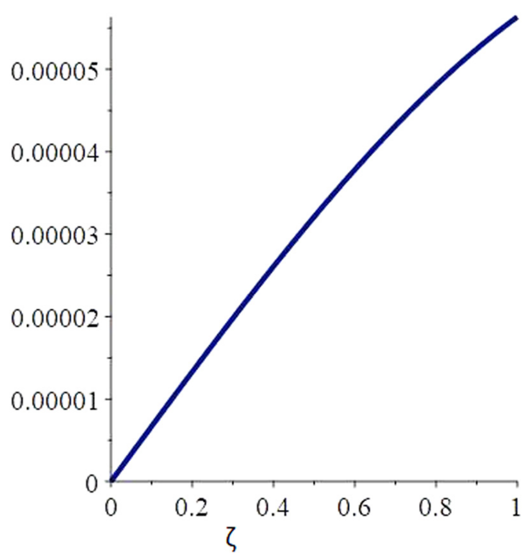

Figure 4: Plot of absolute error at $\vartheta = 2$ for Example 5.1.

$$\omega_0^*(\zeta,\tau) = \omega_0(\zeta,\tau) + (1-\zeta)(\tau - \omega_0(0,\tau)) + \zeta((1+\tau) - \omega_0(\pi,\tau)),$$

$$\omega_0^*(\zeta,\tau) = \zeta^2 + \tau + (\zeta^2 - \zeta)\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)}.$$

From Eq. (17), we have

$$\omega_1(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_0^*(\zeta,\tau)}{\partial\zeta^2} - \frac{\partial\omega_0^*(\zeta,\tau)}{\partial\tau} - \omega_0^*(\zeta,\tau)\right),$$

$$\omega_1(\zeta,\tau) = (1-\zeta^2)\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)} + (2-\zeta^2+\zeta)\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} - (\zeta^2-\zeta)\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} - \frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)}.$$

For $n = 1$, Eq. (15) becomes

$$\omega_1^*(\zeta,\tau) = \omega_1(\zeta,\tau) + (1-\zeta)[\tau - \omega_1(0,\tau)] + \zeta[(1+\tau) - \omega_1(1,\tau)],$$

$$\omega_1^*(\zeta,\tau) = (\zeta-\zeta^2)\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)} + (\zeta-\zeta^2)\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + (\zeta-\zeta^2)\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \tau + \zeta.$$

From Eq. (17), we have

$$\omega_2(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_1^*(\zeta,\tau)}{\partial\zeta^2} - \frac{\partial\omega_1^*}{\partial\tau} - \omega_1^*\right),$$

$$\omega_2(\zeta,\tau) = (-2-\zeta+\zeta^2)\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + (-2-\zeta+\zeta^2)\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + (-2-2\zeta+2\zeta^2)\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + (\zeta-\zeta^2)\frac{\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} - (\zeta-\zeta^2)\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + (-1-\zeta)\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)} - \frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)}.$$

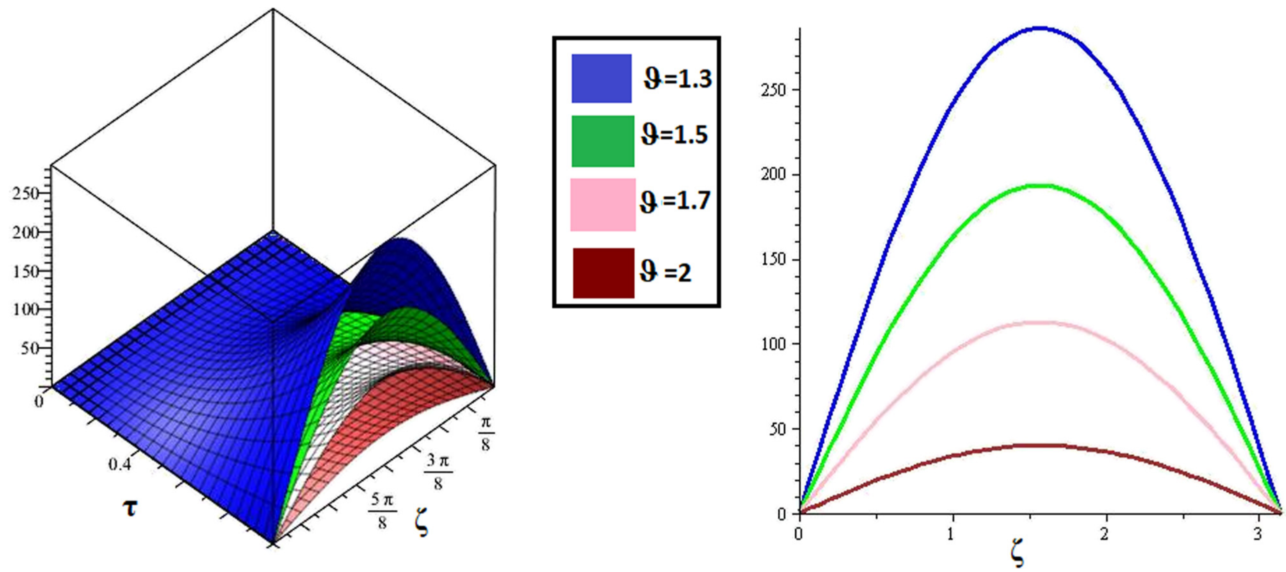

Figure 5: Three- and two-dimensional plots of approximate solution for different ϑ of Example 5.2.

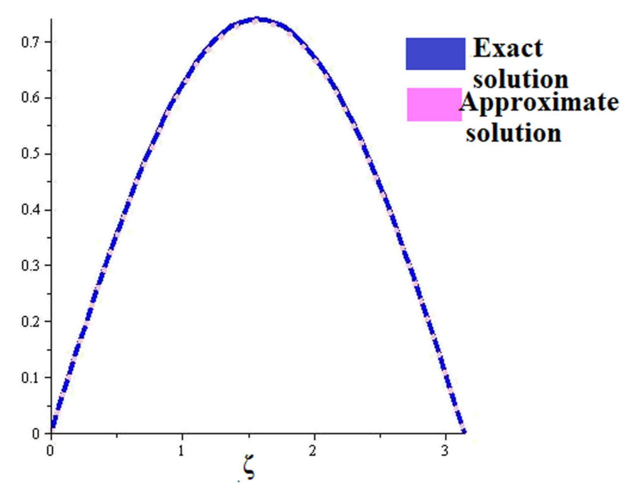

Figure 6: Two-dimensional plots for comparison between the exact and approximate solutions $\vartheta = 2$ of Example 5.2.

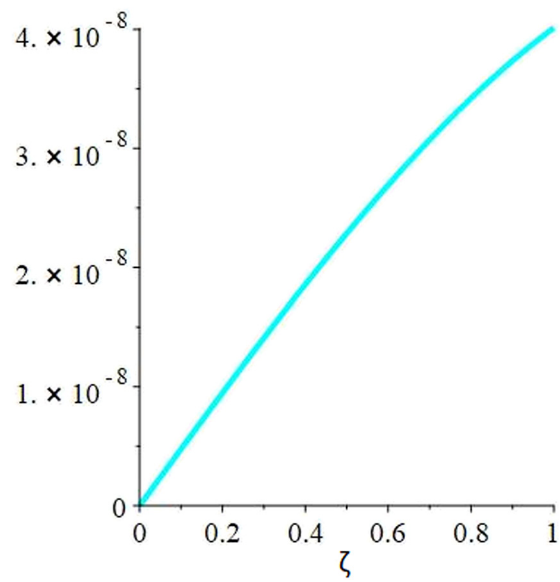

Figure 7: Plot of absolute error at $\vartheta = 2$ for Example 5.2.

For $n = 2$, Eq. (15) becomes

$$\omega_2^*(\zeta,\tau) = \omega_2(\zeta,\tau) + (1-\zeta)[\tau - \omega_2(0,\tau)] + \zeta[(1+\tau) - \omega_2(1,\tau)],$$

$$\omega_2^*(\zeta,\tau) = (\zeta^2-\zeta)\frac{\tau^{2\vartheta}}{\Gamma(2\vartheta+1)} + (\zeta^2-\zeta)\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + (2\zeta^2-2\zeta)\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} - (\zeta-\zeta^2)\frac{\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} - (\zeta-\zeta^2)\frac{\tau^{2\vartheta-1}}{\Gamma(2\vartheta)} + \zeta + \tau.$$

From Eq. (17), we have

$$\omega_3(\zeta,\tau) = L^{-1}\left(\frac{\partial^2\omega_2^*(\zeta,\tau)}{\partial\zeta^2} - \frac{\partial\omega_2^*}{\partial\tau} - \omega_2^*\right),$$

$$\omega_3(\zeta,\tau) = (2+\zeta-\zeta^2)\frac{\tau^{3\vartheta}}{\Gamma(3\vartheta+1)} + (2+\zeta-\zeta^2)\frac{\tau^{4\vartheta}}{\Gamma(4\vartheta+1)} + (4+3\zeta-3\zeta^2)\frac{\tau^{4\vartheta-1}}{\Gamma(4\vartheta)} + (2+2\zeta-2\zeta^2)\frac{\tau^{4\vartheta-2}}{\Gamma(4\vartheta-1)} + (2+2\zeta-2\zeta^2)\frac{\tau^{3\vartheta-1}}{\Gamma(3\vartheta)} + (\zeta-\zeta^2)\frac{\tau^{3\vartheta-2}}{\Gamma(3\vartheta-1)} - (\zeta-\zeta^2)\frac{\tau^{4\vartheta-3}}{\Gamma(4\vartheta-2)} - (1+\zeta)\frac{\tau^{\vartheta}}{\vartheta+1} - \frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)} + \ldots$$

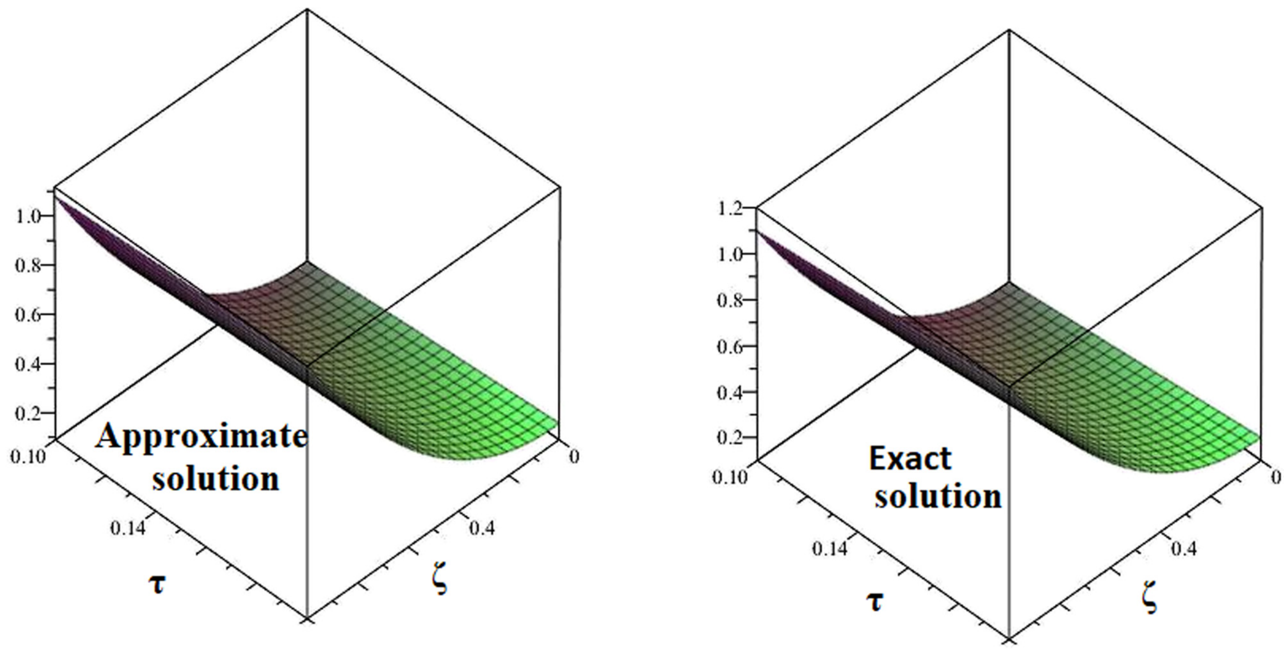

Figure 8: Three-dimensional plots for comparison between the exact and approximate solutions at $\vartheta = 2$ of Example 5.3.

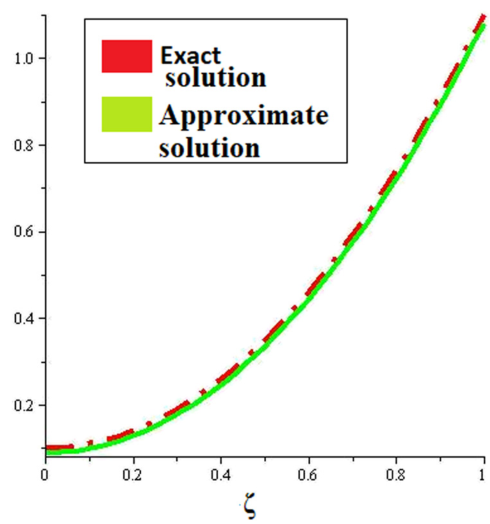

Figure 9: Two-dimensional plots of comparison between the exact and approximate solutions at $\vartheta = 2$ of Example 5.3.

Thus, the ADM solution in the series form is

$$\omega(\zeta,\tau) = \omega_0(\zeta,\tau) + \omega_1(\zeta,\tau) + \omega_2(\zeta,\tau) + \omega_3(\zeta,\tau) + \ldots$$
$$\omega(\zeta,\tau) = \zeta^2 + \tau.$$

6 Results and discussion

The solution plots show the accuracy of the method. Figure 1 shows the 3D plots of the exact and approximate solutions of Example 5.1 at $\vartheta = 2$. It is clear from both the graphs that there is a close contact between the exact and calculated solutions of the problem. Figure 2 shows the comparison between the exact and approximate solutions at an integer order of $\vartheta = 2$. In Figure 3, the 3D and 2D solution plots are drawn for different fractional orders of ϑ for Example 5.1, which shows that as the fractional orders varies, the accuracy of the method from the approximate to the exact solution increases. Figure 4 shows the plot for absolute error of Example 5.1. In Figure 5, the approximate solution at different fractional orders of ϑ is shown by 3D and 2D plots. Figure 6 shows a comparison between the exact and approximate solutions of Example 5.2. In Figure 7, the 2D plot for the absolute error of Example 5.2 is drawn. Figure 8 shows the 2D and 3D plots for the exact and approximate solutions of Example 5.3. Figure 9 shows a comparison between the exact and approximate solutions of Example 5.3. In Figure 10, the 2D and 3D plots for different fractional order of ϑ. Figure 11 is drawn for the absolute error of Example 5.3. All of the graphs provide very useful information about the dynamical behavior of the targeted problems. The solution calculated for each fractional order has shown that the fractional solutions are closely converging to the exact solution of each problem. An error table is drawn for Example 5.3, which shows the high accuracy of the proposed technique. In addition, the proposed method is an accurate way of finding solutions for FPDEs having initial and boundary

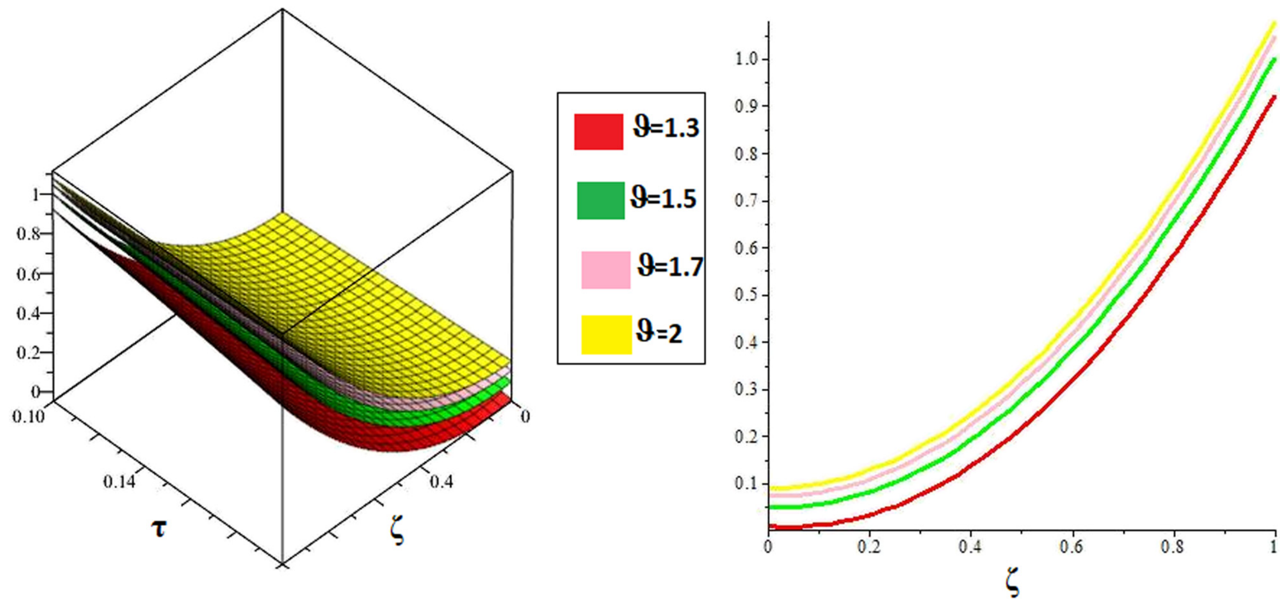

Figure 10: Three- and two-dimensional plots of approximate solution at different ϑ of Example 5.3.

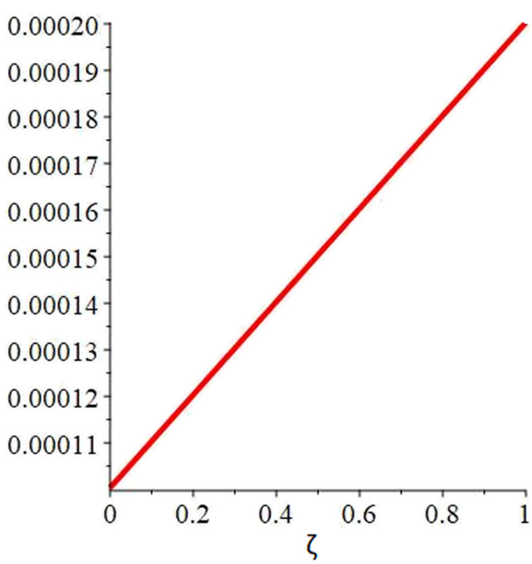

Figure 11: Plot of absolute error at $\vartheta = 2$ for Example 5.3.

conditions. The method is applied to linear problems, which gives better results. Therefore, the suggested method can be extended further to linear and nonlinear problems in future.

7 Conclusion

In this article, the modified ADM is developed to solve FPDEs with initial and boundary conditions. For this purpose, the Caputo operator is used to define the fractional derivative. The present method has a two-step representation. In the first step, the solutions are approximated by using the ADM iteration formula. In the second step, these approximate solutions are further refined by using another iteration formula that utilizes the boundary conditions and increases the accuracy of the proposed technique. To verify the accuracy of the method, the solutions of few numerical examples are discussed. The solutions for the targeted problems are calculated for both fractional and integer orders of the derivatives. Figures and tables are constructed to show the accuracy and applicability of the present method. In Figures 2, 6, and 9 show the comparison of exact and approximate solutions at $\vartheta = 2$ of Examples 5.1, 5.2, and 5.3, respectively. A very strong relation is observed between the exact and approximate solutions. Figures 3, 5, and 10 are constructed to show the 2D and 3D plots of problems 5.1, 5.2, and 5.3, respectively. The 3D solution plots for the

Table 1: Absolute error for different times and fractional order of Example 5.3

τ	ζ	Absolute error at $\vartheta = 1.3$	Absolute error at $\vartheta = 1.5$	Absolute error at $\vartheta = 1.7$	Absolute error at $\vartheta = 2$
0.01	0.2	$5.18455806 \times 10^{-2}$	$1.81143765 \times 10^{-2}$	6.2045458×10^{-4}	1.2033333×10^{-4}
	0.4	6.0444999×10^{-2}	2.1123280×10^{-2}	7.235455×10^{-4}	1.403333×10^{-4}
	0.6	6.9056642×10^{-2}	2.4132289×10^{-2}	8.266362×10^{-4}	1.603333×10^{-4}
	0.8	7.7680510×10^{-3}	2.7141404×10^{-3}	9.297269×10^{-4}	1.803333×10^{-4}
	1	8.631659×10^{-3}	3.015062×10^{-3}	1.032817×10^{-3}	2.00333×10^{-4}
0.03	0.2	$2.178994850 \times 10^{-3}$	$9.47609717 \times 10^{-3}$	$4.04099673 \times 10^{-3}$	$1.08900091 \times 10^{-3}$
	0.4	$2.53688321 \times 10^{-2}$	$1.10393772 \times 10^{-2}$	4.7083030×10^{-3}	1.2690010×10^{-3}
	0.6	$2.89602877 \times 10^{-2}$	$1.26028917 \times 10^{-2}$	5.3756102×10^{-3}	1.4490010×10^{-3}
	0.8	$3.25643152 \times 10^{-2}$	$1.41666408 \times 10^{-2}$	6.0429183×10^{-3}	1.6290010×10^{-3}
	1	$3.61809146 \times 10^{-2}$	1.5730624×10^{-2}	6.710228×10^{-3}	1.809000×10^{-3}
0.05	0.2	$4.267824285 \times 10^{-2}$	$2.052902812 \times 10^{-2}$	$9.68948097 \times 10^{-3}$	$3.04167829 \times 10^{-3}$
	0.4	$4.96171264 \times 10^{-2}$	$2.38921758 \times 10^{-2}$	$1.12797422 \times 10^{-2}$	3.5416791×10^{-3}
	0.6	$5.65920100 \times 10^{-2}$	$2.72562483 \times 10^{-2}$	$1.28700020 \times 10^{-2}$	4.0416791×10^{-3}
	0.8	$6.36028935 \times 10^{-2}$	$3.06212458 \times 10^{-2}$	$1.44602605 \times 10^{-2}$	4.5416783×10^{-3}
	1	$7.06497760 \times 10^{-2}$	3.3987169×10^{-2}	1.6050520×10^{-2}	5.041677×10^{-3}

exact and approximate solutions are shown in Figures 1, 4, and 8, respectively, for problems 5.1, 5.2, and 5.3, respectively. In Table 1, the absolute error associated with the present technique is given, which shows the higher accuracy of the suggested method. Because the current technique required fewer calculations, it can be extended for the solutions of other higher nonlinear fractional order problems.

Acknowledgments: This research was supported by Researchers Supporting Project number (RSP2022R440), King Saud University, Riyadh, Saudi Arabia.

Funding information: The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. This research project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2022 under project number FRB650048/0164.

Author contributions: Hassan Khan: supervision; Hajira: methodology; Qasim Khan: methodology, investigation; Fairouz Tchier: project administration; Poom Kumam: funding, draft writing; Gurpreet Singh: investigation; Kanokwan Sitthithakerngkiet: funding, draft writing. All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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