

Research Article

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Analysis of parametric effects in the wave profile of the variant Boussinesq equation through two analytical approaches

<https://doi.org/10.1515/phys-2022-0071>

received March 08, 2021; accepted November 29, 2021

Abstract: The variant Boussinesq equation has significant application in propagating long waves on the surface of the liquid layer under gravity action. In this article, the improved Bernoulli subequation function (IBSEF) method and the new auxiliary equation (NAE) technique are introduced to establish general solutions, some fundamental soliton solutions accessible in the literature, and some archetypal solitary wave solutions that are extracted from the broad-ranging solution to the variant Boussinesq wave equation. The established soliton solutions are knowledgeable and obtained as a combination of hyperbolic, exponential, rational, and trigonometric functions, and the physical significance of the attained solutions is speculated for the definite values of the included parameters by depicting the 3D profiles and interpreting the physical incidents. The wave profile represents different types of waves associated with the free parameters that are related to the wave number and velocity of the solutions. The obtained solutions and graphical representations visualize the dynamics of the phenomena and build up the mathematical foundation of the wave process in dissipative and

dispersive media. It turns out that the IBSEF method and the NAE are powerful and might be used in further works to find novel solutions for other types of nonlinear evolution equations ascending in physical sciences and engineering.

Keywords: improved Bernoulli subequation function method, the new auxiliary equation method, the variant Boussinesq equation, exact solutions, soliton

1 Introduction

Since many processes in science, technology, and engineering are modeled through nonlinear evolution equations (NLEEs), the investigation of closed-form analytical solutions of NLEEs is very important. Closed-form solutions provide further physical information and help us to understand the processes of the associated physical systems. Consequently, their studies are of fundamental importance due to the effective application of the analytical solutions in various fields, such as plasma physics, solid-state physics, neural physics, chaos, diffusion process, reaction process, optical fibers, nonlinear optics, quantum mechanics, mathematical biology, propagation of shallow water waves, and electromagnetic theory [1–11]. On account of this, researchers established several techniques and tried to examine various types of NLEEs with the help of computer algebra like Maple, MATLAB, and Mathematica. However, not all models are solvable by a single method. Consequently, several methods [12–31] have been established by mathematicians, physicists, and engineers.

Among the approaches, the improved Bernoulli subequation function (IBSEF) method [32,33] and the new auxiliary equation (NAE) [34] method are effective, direct, and compatible algebraic methods for establishing exact soliton solutions to NLEEs. In 1871, the Boussinesq equation was derived to describe certain physical processes. Since then, several generalizations and some variants of this model have been established in the literature. To the

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best of our understanding, through different schemes, several researchers have investigated the variant Boussinesq equation, namely: Gao and Tian [35] examined it through the generalized tanh method; Yan and Zhang [36] used an improved sine–cosine method and Wu elimination method; Jabbari *et al.* [37] investigated it by the homotopy analysis method; Ul-Hassan [38] applied exp-function method; Manafian *et al.* [39] implemented the improved tanh ($\phi(\xi)/2$) method and the improved (G'/G) -expansion method; Osman *et al.* [40] searched based on the generalized unified method; and Alharbi and Almatrafi [41] exploited the improved $\exp(-\phi(\eta))$ -expansion method. Mohapatra and Guedes Soares [42] used the hyperbolic tangent method to investigate the solitary wave solutions of the one-dimensional coupled nonlinear Boussinesq equations related to shallow water. They also use the first integral approach to search for exact solutions to the generalized nonlinear Boussinesq equations. Mohapatra *et al.* [43] studied the long nonlinear internal waves between analytical and numerical wave models in shallow water. The (G'/G) -expansion procedure was used to achieve the exact and explicit solitary wave solutions to the model in the presence of dispersive coefficient.

Nevertheless, the stated model has not been investigated through the IBSEF method and the NAE approach. Therefore, the purpose of this article is to accomplish broad-ranging and adequate standard soliton solutions to the variant Boussinesq equations by putting into use the proposed methods. Besides, we analyze various types of waves for different values of the free parameters of the obtained solutions illustrated in the 3D plot via MATLAB and highlight the significance of wave number, velocity, and speed of the solutions in changing the nature of the wave profile.

2 Description of two proposed methods

This section introduces and analyzes the IBSEF method and the NAE method in detail.

2.1 The IBSEF method

To illustrate the IBSEF method, we take into consideration the NLEE associated with two independent variables, x and t , of the form [39]:

$$R\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x \partial t}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial^2 u}{\partial x^2}, \dots\right) = 0, \quad (2.1.1)$$

where $u = u(x, t)$ is the elevation of waves; R is a polynomial of the wave variable $u(x, t)$ and $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \dots$; t is the temporal variable; and x is the spatial variable.

Under the traveling wave variable

$$u(x, t) = u(\eta) \text{ and } \eta = \alpha x - ct, \quad (2.1.2)$$

where η is the traveling wave variable, α is the wave number, and c is the speed of the traveling wave, the NLEE (2.1.1) is reduced by the action of the wave variable to the subsequent nonlinear differential equation:

$$L\left(u, \frac{du}{d\eta}, \frac{d^2u}{d\eta^2}, \dots\right) = 0, \quad (2.1.3)$$

where L is a polynomial of the variable $u(\eta)$ and its derivatives. In accordance with the IBSEF method, the solution of Eq. (2.1.3) can be formulated as

$$\begin{aligned} u(\eta) &= \frac{\sum_{i=0}^q a_i H^i}{\sum_{j=0}^p b_j H^j} \\ &= \frac{a_0 + a_1 H + a_2 H^2 + \dots + a_q H^q}{b_0 + b_1 H + b_2 H^2 + b_3 H^3 + \dots + b_p H^p}, \end{aligned} \quad (2.1.4)$$

where a_i ($i = 0, 1, 2, 3, \dots, q$) and b_j ($j = 0, 1, 2, 3, \dots, p$) are indefinite constants to be determined and $H = H(\eta)$ satisfies the improved Bernoulli equation

$$\begin{aligned} H' &= \sigma H + dH^M, \quad \sigma \neq 0, d \neq 0, \\ M &\in \mathbb{R} - \{0, 1, 2\}. \end{aligned} \quad (2.1.5)$$

The values of p and q of undefined parameters can be determined by considering the balancing principle of the highest order linear term with the maximum order nonlinear term. This technique provides the values of q , p , and M . Introducing solution (2.1.4) along with the improved Bernoulli equation into Eq. (2.1.3) yields a polynomial equation $\Omega(H)$ of H :

$$\Omega(H) = \rho_s H^s + \dots + \rho_1 H + \rho_0 = 0. \quad (2.1.6)$$

Equalizing the coefficients of $\Omega(H)$ to zero will provide a system of algebraic equations:

$$\rho_k = 0, \quad k = 0, \dots \quad (2.1.7)$$

With Mathematica software, we can unravel the system of algebraic equations to determine the values of $a_0, a_1, \dots, a_q$ and $b_0, b_1, \dots, b_p$. The solutions of the improved nonlinear Bernoulli equation depend on the values of the parameters σ and d . The two types of solutions to the improved Bernoulli's equation are as follows:

$$H(\eta) = \left\{ -\frac{d}{\sigma} + \frac{\tau}{e^{\sigma(M-1)\eta}} \right\}^{\frac{1}{1-M}}, \quad \sigma \neq d, \quad (2.1.8)$$

$$H(\eta) = \left\{ \frac{(\tau - 1) + (\tau + 1)\tanh\left(\frac{\sigma(1-M)\eta}{2}\right)}{1 - \tanh\left(\frac{\sigma(1-M)\eta}{2}\right)} \right\}^{\frac{1}{1-M}}, \quad (2.1.9)$$

$$\sigma = d, \quad \tau \in \mathbb{R}.$$

Thus, by embedding the values of the parameters a_i , ($i = 0, 1, 2, \dots, p$), b_j , ($j = 0, 1, 2, \dots, q$), and the solutions of the improved Bernoulli equation (2.1.5) together with wave variable, further general and some acknowledged solutions accessible in the literature for the definite values of the subjective parameters to the NLEE (2.1.1) can be ascertained.

2.2 The NAE method

Suppose the NLEE [38]

$$\mathcal{L}(q, q_t, q_x, q_y, q_z, q_{xx}, \dots) = 0, \quad (2.2.1)$$

where $\mathcal{L}$ is a function of the $(3 + 1)$ -dimensional wave function $q(x, y, z, t)$ and its derivatives.

Step 1: To format into an ordinary differential equation of partial differential equation (2.2.1), we need to choose a wave variable as

$$q(x, y, z, t) = Q(\mu), \mu = \alpha x + \beta y + \gamma z - \delta t, \quad (2.2.2)$$

where α , β , and γ are wave numbers and δ is the wave speed. The wave variable transformed Eq. (2.2.1) into a nonlinear equation as follows:

$$\mathcal{M}(Q, Q', Q'', Q''', \dots) = 0, \quad (2.2.3)$$

where prime means the derivative with respect to μ .

Step 2: In harmony with the NAE method, the exact soliton solution of Eq. (2.2.3) is assigned to be

$$Q(\mu) = \sum_{k=0}^L S_k a^{kg(\mu)}, \quad (2.2.4)$$

where $S_0, S_1, S_2, \dots, S_L$ are unknown parameters to be calculated, wherein $S_L \neq 0$, and $g(\mu)$ is the solution of the nonlinear equation

$$g'(\mu) \ln a = \{pa^{-g(\mu)} + q + ra^{g(\mu)}\}. \quad (2.2.5)$$

Step 3: The balancing principle needs to be applied to find the value of the positive integer L in Eq. (2.2.4).

Step 4: Eq. (2.2.3) provides a polynomial of $a^{kg(\mu)}$, $k = 0, 1, 2, \dots$, by substituting the solution (2.2.4) together with (2.2.5). Furthermore, we attain a system of algebraic equations by conveying each coefficient of the resulting polynomials to zero.

The values of S_k , p , q , and r obtained from Step 4 and the solution $g(\mu)$, which is attained in (2.2.5) by making use of various cases in Eq. (2.2.4), provide ample soliton solution to the evolution Eq. (2.2.1).

3 Solution analysis

In this section, we have established some standard and broad-ranging explicit wave solutions through the aforementioned methods of the variant Boussinesq equation, which has the following form [39]:

$$\begin{aligned} u_t + H_x + uu_x &= 0, \\ H_t + (uH)_x + u_{xxx} &= 0. \end{aligned} \quad (3.1)$$

To convert the nonlinear model (3.1), we used the wave transformation as follows:

$$\begin{aligned} u(x, t) &= u(\eta) \quad \text{and} \quad H(x, t) = H(\eta), \quad \text{where} \\ \eta &= \lambda x - ct. \end{aligned} \quad (3.2)$$

Using Eq. (3.2) into Eq. (3.1) and integrating, we obtain

$$-cu + \lambda H + \frac{\lambda u^2}{2} = 0, \quad (3.3)$$

$$-cH + \lambda(uH) + \lambda^3 u'' = 0. \quad (3.4)$$

Eq. (3.3) gives

$$H = \frac{cu}{\lambda} - \frac{u^2}{2}. \quad (3.5)$$

By replacing the value of H in (3.4), we accomplish the subsequent nonlinear equation as follows:

$$2\lambda^4 u'' - 2c^2 u + 3c\lambda u^2 - \lambda^2 u^3 = 0. \quad (3.6)$$

3.1 Analysis of solutions through the IBSEF method

To apply the procedure of the IBSEF method, we gain the correspondence for q , p , and M using the balancing principle between u'' and u^3 in Eq. (3.6) as follows:

$$p + M = q + 1.$$

Picking $p = 1$ and $M = 3$ provides $q = 3$. Thus, the trial solution to Eq. (2.1.4) becomes

$$u(\eta) = \frac{a_0 + a_1 H(\eta) + a_2 H^2(\eta) + a_3 H^3(\eta)}{b_0 + b_1 H(\eta)}, \quad (3.1.1)$$

where $H(\xi)$ is the solution to the improved Bernoulli Eq. (2.1.5).

The substitution of Eq. (3.1.1) together with Eq. (2.1.5) in Eq. (3.6) yields a polynomial in H . Solving the system of the algebraic equations, we secure the subsequent set of solutions of the parameters as follows:

$$\text{Set1: } c = \pm 2\lambda^2\sigma, a_0 = \pm 4\lambda\sigma b_0, a_1 = \pm 4\lambda\sigma b_1, \\ a_2 = \pm 4\lambda db_0, a_3 = \pm 4\lambda db_1, \quad (3.1.2)$$

$$\text{Set2: } c = \mp 2\lambda^2\sigma, a_0 = 0, a_1 = 0, \\ a_2 = \pm 4\lambda db_0, a_3 = \pm 4\lambda db_1. \quad (3.1.3)$$

For $\sigma \neq d$,

Case 1(a): By making use of the values stated in (3.1.2) along with (2.1.8) in Eq. (3.1.1), the subsequent exponential function solution to the variant Boussinesq equation is established as follows:

$$u_1(x, t) = \mp \frac{4\lambda\sigma^2\tau}{de^{2\sigma\eta} - \sigma\tau}, \quad (3.1.4)$$

where $\sigma \neq d$ and .

Rewrite the solution function (3.1.4) into hyperbolic form

$$u_{1_0}(x, t) = \pm \frac{4\lambda\sigma^2\tau(\sinh \psi - \cosh \psi)}{(d - \sigma\tau) \cosh \psi + (d + \sigma\tau) \sinh \psi}, \quad (3.1.5)$$

where $\psi = \sigma(\lambda x - ct)$.

Let us choose $d = 2\sigma\tau$, then the solution (3.1.5) becomes the rational function solution of the hyperbolic function as follows:

$$u_{1_1}(x, t) = \pm \frac{4\lambda\sigma(\sinh \psi - \cosh \psi)}{\cosh \psi + 3 \sinh \psi}, \quad (3.1.6)$$

and we obtain a part of the solution of (3.1) by putting $u_{1_1}(x, t)$ in Eq. (3.5), which provides

$$H_{1_1}(x, t) \\ = \pm \frac{8\sigma\left((c - 2\lambda^2\sigma)(\cosh^2 \psi - \sinh \psi \cosh \psi) + \lambda^2\sigma - \frac{3c}{2}\right)}{(10 \cosh^2 \psi + 6 \sinh \psi \cosh \psi - 9)}. \quad (3.1.7)$$

For $d = \sigma\tau$, we bring out the hyperbolic function solution to Eq. (3.1) of the mode:

$$u_{1_2}(x, t) = \pm 2\lambda\sigma(1 - \coth \psi), \quad (3.1.8)$$

and we undertake from (3.5)

$$H_{1_2}(x, t) = \pm 2\sigma(1 - \coth \psi)(-\sigma(1 - \coth \psi)\lambda^2 + c), \quad (3.1.9)$$

where $\psi = \sigma(\lambda x - ct)$.

On the other hand, by setting $d = -\sigma\tau$ in (3.1.5), we accomplish the subsequent solution of the hyperbolic function:

$$u_{1_3}(x, t) = \pm 2\lambda\sigma(1 - \tanh \psi), \quad (3.1.10)$$

and based on Eq. (3.5), we obtain

$$H_{1_3}(x, t) = \pm 2\sigma(1 - \tanh \psi)(-\sigma(1 - \tanh \psi)\lambda^2 + c), \quad (3.1.11)$$

where $\psi = \sigma(\lambda x - ct)$.

Case 2(a): Considering the value of the parameter arranged in (3.1.3) with the help of Eq. (2.1.8) in Eq. (3.1.1), we accomplish the following solution:

$$u_2(x, t) = \mp \frac{4\sigma d\lambda(e^{2\sigma\eta}\sigma\tau + e^{4\sigma\eta}d)}{(de^{2\sigma\eta} - \sigma\tau)^2}, \quad (3.1.12)$$

which can be converted to hyperbolic form as follows:

$$u_{2_0}(x, t) \\ = \mp \frac{4d\sigma\lambda(\sigma\tau + d(\cosh 2\psi + \sinh 2\psi))}{\{(d - \sigma\tau) \cosh \psi + (d + \sigma\tau) \sinh \psi\}^2}, \quad (3.1.13)$$

where $\psi = \sigma(\lambda x - ct)$.

Setting $d = 2\sigma\tau$ in Eq. (3.1.13) represents the solution of (3.1) as follows:

$$u_{2_1}(x, t) = \mp \frac{8\sigma\lambda(1 + 2(\cosh 2\psi + \sinh 2\psi))}{(\cosh \psi + 2 \sinh \psi)^2}, \quad (3.1.14)$$

and on behalf of Eq. (3.5),

$$H_{2_1}(x, t) = \mp \left(\frac{8c\sigma(1 + 2 \sinh(2\psi) + 2 \cosh(2\psi))}{(\cosh \psi + 2 \sinh \psi)^2} \right. \\ \left. - \frac{32\lambda^2\sigma^2(1 + 2 \sinh(2\psi) + 2 \cosh(2\psi))^2}{(\cosh \psi + 2 \sinh \psi)^4} \right). \quad (3.1.17)$$

Choose $d = \sigma\tau$, then Eq. (3.1.13) reshapes as follows:

$$u_{2_2}(x, t) = \mp 2\sigma\lambda(\coth \psi + \coth^2 \psi) \quad (3.1.16)$$

and Eq. (3.5) yields

$$H_{2_2}(x, t) = \mp 2\sigma \coth \psi (\coth \psi + 1)(-\lambda^2\sigma(\coth^2 \psi \\ + \coth \psi) + c) \quad (3.1.17)$$

When $d = -\sigma\tau$, the solution function (3.1.13) can be rewritten as:

$$u_{2_3}(x, t) = \mp 2\sigma\lambda(\tanh^2 \psi - \tanh \psi) \quad (3.1.18)$$

and from Eq. (3.5),

$$H_{2_3}(x, t) = \mp 2\sigma \tanh \psi (\tanh \psi - 1) \\ \times (-\lambda^2\sigma(\tanh^2 \psi - \tanh \psi) + c), \quad (3.1.19)$$

where $\psi = \sigma(\lambda x - ct)$.

For $\sigma = d$,

Case 1(b): By substituting the values of the parameters arranged in (3.1.2) along with Eqs. (2.1.2) and (2.1.9) in Eq. (3.1.1), we obtain the exponential function solution to the variant Boussinesq equation of the exponential form:

$$u_3(x, t) = \pm \frac{4\sigma\lambda\tau(\tanh \psi - 1)}{(\tau + 1)\tanh \psi - (\tau - 1)}, \quad (3.1.20)$$

where $\psi = \sigma(\lambda x - ct)$.

Eq. (3.1.20) can be written as

$$u_{30}(x, t) = \pm \frac{4\sigma\lambda\tau(\sinh \psi - \cosh \psi)}{(\tau + 1)\sinh \psi - (\tau - 1)\cosh \psi}. \quad (3.1.21)$$

Choosing $\tau = 2$, the solution (3.1.21) yields the solution of (3.1) as follows:

$$u_{31}(x, t) = \pm \frac{4\sigma\lambda(\sinh \psi - \cosh \psi)}{2\sinh \psi - \cosh \psi}, \quad (3.1.22)$$

and another solution of Eq. (3.1) obtained by captivating (3.5) is

$$H_{31}(x, t) = \pm \frac{12\sigma\left(\left(-\frac{4\lambda^2\sigma}{3} + c\right)(\cosh^2 \psi - \sinh \psi \cosh \psi) + \left(\frac{2\lambda^2\sigma}{3} - \frac{2c}{3}\right)\right)}{5\cosh^2 \psi - 4\cosh \psi \sinh \psi - 4}. \quad (3.1.23)$$

If we pick $\tau = 1$, then Eq. (3.1.15) becomes

$$u_{32}(x, t) = \pm 2\sigma\lambda(1 - \coth \psi), \quad (3.1.24)$$

and Eq. (3.5) gives

$$H_{32}(x, t) = \pm 2\sigma(1 - \coth \psi)(-\lambda(1 - \coth \psi)\sigma\lambda + c). \quad (3.1.25)$$

When $\tau = -1$, then Eq. (3.1.15) becomes

$$u_{33}(x, t) = \pm 2\sigma\lambda(1 - \tanh \psi), \quad (3.1.26)$$

and using Eq. (3.5) gives

$$H_{33}(x, t) = \pm 2\sigma(1 - \tanh \psi)(-\lambda(1 - \tanh \psi)\sigma\lambda + c), \quad (3.1.27)$$

where $\psi = \sigma(\lambda x - ct)$.

Case 2(b): By using Eq. (3.1.3) with the help of Eqs. (2.1.2) and (2.1.9) from Eq. (3.1.1), we accomplish the following solution:

$$u_4(x, t) = \mp \frac{4\sigma\lambda(1 + \tanh \psi)}{(\tau + 1)\tanh \psi - (\tau - 1)},$$

where $\psi = \sigma(\lambda x - ct)$.

Calculating the result $u_4(x, t)$, we then find

$$u_{40}(x, t) = \mp \frac{4\sigma\lambda(\cosh \psi + \sinh \psi)}{(\tau + 1)\sinh \psi - (\tau - 1)\cosh \psi}. \quad (3.1.28)$$

When $\tau = 1$, $u_{40}(x, t)$ can be rewritten as

$$u_{41}(x, t) = \mp 2\sigma\lambda(\coth \psi + 1), \quad (3.1.29)$$

and

$$H_{41}(x, t) = \mp 2\sigma(1 + \coth \psi)(-\lambda(1 + \coth \psi)\sigma\lambda + c). \quad (3.1.30)$$

Applying the value $\tau = -1$, the solution $u_{40}(x, t)$ turns out to be

$$u_{42}(x, t) = \mp 2\sigma\lambda(\tanh \psi + 1), \quad (3.1.31)$$

and

$$H_{41}(x, t) = \mp 2\sigma(1 + \tanh \psi)(-\lambda(1 + \tanh \psi)\sigma\lambda + c), \quad (3.1.32)$$

where $\psi = \sigma(\lambda x - ct)$.

It is important to note that the wave solutions of the variant Boussinesq equation found here are functional and useful and were not proven in the earlier research. The solutions derived earlier might be fruitful in investigating unidirectional wave propagation in nonlinear media and dispersive relativistic one-particle theory, *etc.*

3.2 Analysis of solutions through the NAE method

To find the solutions to the stated equation through the NAE method, we gain the value of $L = 1$ based on the balancing principle between u'' and u^3 in Eq. (3.6).

Using the NAE approach, substitute the value of L into Eq. (2.2.4), which is the auxiliary solution of Eq. (3.6) of the form:

$$u(x, t) = a_0 + a_1 a^{g(\eta)}, \quad (3.2.1)$$

where $g(\eta)$ is the root of the nonlinear Eq. (2.2.5).

Based on the solution (3.2.1) with help of (2.2.5) from Eq. (3.6), we assert

$$\begin{aligned} & (4\lambda^4 n^2 A_1 - \lambda^2 A_1^3) \{a^{g(\eta)}\}^3 + (6\lambda^4 m n A_1 - 3\lambda^2 A_0 A_1^2 \\ & + 3c\lambda A_1^2) \{a^{g(\eta)}\}^2 + (4\lambda^4 n A_1 + 2\lambda^4 m^2 A_1 \\ & - 3\lambda^2 A_0^2 A_1 + 6c\lambda A_0 A_1 - 2c^2 A_1) \{a^{g(\eta)}\} \\ & + 2\lambda^4 m A_1 - \lambda^2 A_0^3 + 3c\lambda A_0^2 - 2c^2 A_0 = 0. \end{aligned} \quad (3.2.2)$$

Equalizing the like power of $a^{g(\eta)}$ from Eq. (3.2.2) yields a system of algebraic equations, which when solved provides the following results:

Set 1:

$$\begin{aligned} c &= \pm \frac{\lambda^2(2ln\sqrt{m^2 - 4ln} - m^2\sqrt{m^2 - 4ln} + 4lmn - m^3)}{2ln - m^2 - m\sqrt{m^2 - 4ln}}, \\ A_0 &= \pm(m + \sqrt{m^2 - 4ln})\lambda, \quad A_1 = 2\lambda n. \end{aligned} \quad (3.2.3)$$

Set 2:

$$c = \pm \frac{\lambda^2(2ln\sqrt{m^2 - 4ln} - m^2\sqrt{m^2 - 4ln} - 4lmn + m^3)}{m\sqrt{m^2 - 4ln} + 2ln - m^2}, \quad (3.2.4)$$

$$A_0 = \pm(-m + \sqrt{m^2 - 4ln})\lambda, \quad A_1 = 2ln.$$

Based on the above values of the parameters in Eq. (3.2.4), as follows:

When $m^2 - 4ln < 0$ and $n \neq 0$.

By substituting the values of the constants arranged in (3.2.3) and (2.2.5) into Eq. (3.2.1), we obtain the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{5_1}(x, t) = \lambda \left(\sqrt{4ln - m^2} \tan \left(\frac{\sqrt{4ln - m^2}\eta}{2} \right) + \sqrt{-4ln + m^2} \right), \quad (3.2.5)$$

and

$$H_{5_1}(x, t) = \frac{\theta\lambda^2 \tan^2 \left(\frac{\sqrt{-\theta}\eta}{2} \right)}{2} + \sqrt{-\theta}(-\lambda^2\sqrt{\theta} + c) \times \tan \left(\frac{\sqrt{-\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2}, \quad (3.2.6)$$

where and $\theta = m^2 - 4ln$,

or

$$u_{5_2}(x, t) = \lambda \left(-\sqrt{4ln - m^2} \cot \left(\frac{\sqrt{4ln - m^2}\eta}{2} \right) + \sqrt{-4ln + m^2} \right), \quad (3.2.7)$$

and

$$H_{5_2}(x, t) = \frac{\theta\lambda^2 \cot^2 \left(\frac{\sqrt{-\theta}\eta}{2} \right)}{2} - \sqrt{-\theta}(-\lambda^2\sqrt{\theta} + c) \times \cot \left(\frac{\sqrt{-\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2}, \quad (3.2.8)$$

where $\eta = \lambda x - ct$ and $\theta = m^2 - 4ln$.

By substituting the values of the constant arranged in (3.2.4) and (2.2.5) into Eq. (3.2.1), we obtain the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{6_1}(x, t) = \lambda \left(-\sqrt{4ln - m^2} \tan \left(\frac{\sqrt{4ln - m^2}\eta}{2} \right) + \sqrt{-4ln + m^2} \right), \quad (3.2.9)$$

and

$$H_{6_1}(x, t) = \frac{\theta\lambda^2 \tan^2 \left(\frac{\sqrt{-\theta}\eta}{2} \right)}{2} - \sqrt{-\theta}(-\lambda^2\sqrt{\theta} + c) \times \tan \left(\frac{\sqrt{-\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2}, \quad (3.2.10)$$

or

$$u_{6_2}(x, t) = \lambda \left(\sqrt{4ln - m^2} \cot \left(\frac{\sqrt{4ln - m^2}\eta}{2} \right) + \sqrt{-4ln + m^2} \right), \quad (3.2.11)$$

$$H_{6_2}(x, t) = \frac{\theta\lambda^2 \cot^2 \left(\frac{\sqrt{-\theta}\eta}{2} \right)}{2} + \sqrt{-\theta}(-\lambda^2\sqrt{\theta} + c) \times \cot \left(\frac{\sqrt{-\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2}, \quad (3.2.12)$$

where $\theta = m^2 - 4ln$ and $\eta = \lambda x - ct$.

When $m^2 - 4ln > 0$ and $n \neq 0$.

By combining (3.2.3) and (2.2.5) in Eq. (3.2.1), we accomplish the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{7_1}(x, t) = \lambda \sqrt{-4ln + m^2} \left(1 - \tanh \left(\frac{\sqrt{-4ln + m^2}\eta}{2} \right) \right), \quad (3.2.13)$$

and

$$H_{7_1}(x, t) = \left(1 - \tanh \left(\frac{\sqrt{\theta}\eta}{2} \right) \right) \left(\frac{\theta\lambda^2}{2} \tanh \left(\frac{\sqrt{\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2} \right), \quad (3.2.14)$$

or

$$u_{7_2}(x, t) = \lambda \sqrt{-4ln + m^2} \left(1 - \coth \left(\frac{\sqrt{-4ln + m^2}\eta}{2} \right) \right), \quad (3.2.15)$$

and

$$H_{7_2}(x, t) = \left(1 - \coth \left(\frac{\sqrt{\theta}\eta}{2} \right) \right) \left(\frac{\theta\lambda^2}{2} \coth \left(\frac{\sqrt{\theta}\eta}{2} \right) + c\sqrt{\theta} - \frac{\lambda^2\theta}{2} \right), \quad (3.2.16)$$

where $\theta = m^2 - 4ln$ and $\eta = \lambda x - ct$.

Using (3.2.4) and (2.2.5) into Eq. (3.2.1), we achieve the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{81}(x, t) = \lambda \sqrt{-4ln + m^2} \left(1 + \tanh \left(\frac{\sqrt{-4ln + m^2} \eta}{2} \right) \right), \quad (3.2.17)$$

and

$$H_{81}(x, t) = \left(1 + \tanh \left(\frac{\sqrt{\theta} \eta}{2} \right) \right) \times \left(\frac{-\theta \lambda^2}{2} \tanh \left(\frac{\sqrt{\theta} \eta}{2} \right) + c \sqrt{\theta} - \frac{\lambda^2 \theta}{2} \right), \quad (3.2.18)$$

or

$$u_{82}(x, t) = \lambda \sqrt{-4ln + m^2} \left(1 + \coth \left(\frac{\sqrt{-4ln + m^2} \eta}{2} \right) \right), \quad (3.2.19)$$

and

$$H_{82}(x, t) = \left(1 + \coth \left(\frac{\sqrt{\theta} \eta}{2} \right) \right) \times \left(\frac{-\theta \lambda^2}{2} \coth \left(\frac{\sqrt{\theta} \eta}{2} \right) + c \sqrt{\theta} - \frac{\lambda^2 \theta}{2} \right), \quad (3.2.20)$$

where $\theta = m^2 - 4ln$ and $\eta = \lambda x - ct$.

When $m^2 + 4l^2 < 0$, $n \neq 0$, and $n = -l$.

Employing (3.2.3) together with (2.2.5) into Eq. (3.2.1), we achieve the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{91}(x, t) = \lambda \left(\sqrt{-4l^2 - m^2} \tan \left(\frac{\sqrt{-4l^2 - m^2} \eta}{2} \right) + \sqrt{4l^2 + m^2} \right), \quad (3.2.21)$$

and

$$H_{91}(x, t) = \frac{\lambda^2 \phi}{2} \left(\tan^2 \left(\frac{\sqrt{-\phi} \eta}{2} \right) - 1 \right) + \sqrt{-\phi} (c - \sqrt{\phi}) \tan \left(\frac{\sqrt{-\phi} \eta}{2} \right) + c \sqrt{\phi}, \quad (3.2.22)$$

or

$$u_{92}(x, t) = \lambda \left(-\sqrt{-4l^2 - m^2} \cot \left(\frac{\sqrt{-4l^2 - m^2} \eta}{2} \right) + \sqrt{4l^2 + m^2} \right), \quad (3.2.23)$$

and

$$H_{92}(x, t) = \frac{\lambda^2 \phi}{2} \left(\cot^2 \left(\frac{\sqrt{-\phi} \eta}{2} \right) - 1 \right) - \sqrt{-\phi} (c - \sqrt{\phi}) \cot \left(\frac{\sqrt{-\phi} \eta}{2} \right) + c \sqrt{\phi}, \quad (3.2.24)$$

where $\phi = 4l^2 + m^2$ and $\eta = \lambda x - ct$.

Considering (3.2.4) together with (2.2.5) into Eq. (3.2.1), we attain the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) in the subsequent form

$$u_{101}(x, t) = \lambda \left(-\sqrt{-4l^2 - m^2} \tan \left(\frac{\sqrt{-4l^2 - m^2} \eta}{2} \right) + \sqrt{4l^2 + m^2} \right), \quad (3.2.25)$$

and

$$H_{101}(x, t) = \frac{\lambda^2 \phi}{2} \left(\tan^2 \left(\frac{\sqrt{-\phi} \eta}{2} \right) - 1 \right) - \sqrt{-\phi} (c - \sqrt{\phi}) \tan \left(\frac{\sqrt{-\phi} \eta}{2} \right) + c \sqrt{\phi}, \quad (3.2.26)$$

or

$$u_{102}(x, t) = \lambda \left(\sqrt{-4l^2 - m^2} \cot \left(\frac{\sqrt{-4l^2 - m^2} \eta}{2} \right) + \sqrt{4l^2 + m^2} \right), \quad (3.2.27)$$

and

$$H_{102}(x, t) = \frac{\lambda^2 \phi}{2} \left(\cot^2 \left(\frac{\sqrt{-\phi} \eta}{2} \right) - 1 \right) + \sqrt{-\phi} (c - \sqrt{\phi}) \cot \left(\frac{\sqrt{-\phi} \eta}{2} \right) + c \sqrt{\phi}, \quad (3.2.28)$$

where $\phi = 4l^2 + m^2$ and $\eta = \lambda x - ct$.

When $m^2 + 4l^2 > 0$, $n \neq 0$, and $n = -l$.

Inserting (3.2.3) together with (2.2.5) into Eq. (3.2.1), we secure the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{111}(x, t) = \lambda \sqrt{4l^2 + m^2} \left(1 - \tanh \left(\frac{\sqrt{4l^2 + m^2} \eta}{2} \right) \right), \quad (3.2.29)$$

and

$$H_{11_1}(x, t) = c\sqrt{\phi} \left(1 - \tanh \left(\frac{\sqrt{\phi}\eta}{2} \right) \right) - \frac{\lambda^2\phi}{2} \left(1 - \tanh \left(\frac{\sqrt{\phi}\eta}{2} \right) \right)^2, \quad (3.2.30)$$

and

or

$$u_{11_2}(x, t) = \lambda\sqrt{4l^2 + m^2} \left(1 - \coth \left(\frac{\sqrt{4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.31)$$

and

$$H_{11_2}(x, t) = c\sqrt{\phi} \left(1 - \coth \left(\frac{\sqrt{\phi}\eta}{2} \right) \right) - \frac{\lambda^2\phi}{2} \left(1 - \coth \left(\frac{\sqrt{\phi}\eta}{2} \right) \right)^2, \quad (3.2.32)$$

or

$$u_{13_1}(x, t) = \lambda \left(\sqrt{4l^2 - m^2} \tan \left(\frac{\sqrt{4l^2 - m^2}\eta}{2} \right) + \sqrt{-4l^2 + m^2} \right), \quad (3.2.37)$$

$$H_{13_1}(x, t) = c\sqrt{\omega} \left(1 + i \tan \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right) - \frac{\lambda^2\omega}{2} \left(1 - \tan \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right), \quad (3.2.38)$$

$$u_{13_2}(x, t) = \lambda \left(-\sqrt{4l^2 - m^2} \cot \left(\frac{\sqrt{4l^2 - m^2}\eta}{2} \right) + \sqrt{-4l^2 + m^2} \right), \quad (3.2.39)$$

where $\phi = 4l^2 + m^2$ and $\eta = \lambda x - ct$.

By involving (3.2.4) along with (2.2.5) into Eq. (3.2.1), we derive the solution of stated Eqs. (3.1)–(3.2.1) and (3.5) as follows:

$$u_{12_1}(x, t) = \lambda\sqrt{4l^2 + m^2} \left(1 + \tanh \left(\frac{\sqrt{4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.33)$$

and

$$H_{12_1}(x, t) = c\sqrt{\phi} \left(1 + \tanh \left(\frac{\sqrt{\phi}\eta}{2} \right) \right) - \frac{\lambda^2\phi}{2} \left(1 + \tanh \left(\frac{\sqrt{\phi}\eta}{2} \right) \right)^2, \quad (3.2.34)$$

or

$$u_{12_2}(x, t) = \lambda\sqrt{4l^2 + m^2} \left(1 + \coth \left(\frac{\sqrt{4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.35)$$

and

$$H_{12_2}(x, t) = c\sqrt{\phi} \left(1 + \coth \left(\frac{\sqrt{\phi}\eta}{2} \right) \right) - \frac{\lambda^2\phi}{2} \left(1 + \coth \left(\frac{\sqrt{\phi}\eta}{2} \right) \right)^2, \quad (3.2.36)$$

where $\phi = m^2 + 4l^2$ and $\eta = \lambda x - ct$.

When $m^2 - 4l^2 < 0$ and $n = l$.

Applying (3.2.3) along with (2.2.5) into Eqs. (3.2.1) and (3.5) gives

and

$$H_{13_2}(x, t) = c\sqrt{\omega} \left(1 - i \cot \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right) - \frac{\lambda^2\omega}{2} \left(1 + \cot \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right). \quad (3.2.40)$$

Substituting (3.2.4) along with (2.2.5) into Eqs. (3.2.1) and (3.5) gives

$$u_{14_1}(x, t) = \lambda \left(-\sqrt{4l^2 - m^2} \tan \left(\frac{\sqrt{4l^2 - m^2}\eta}{2} \right) + \sqrt{-4l^2 + m^2} \right), \quad (3.2.41)$$

and

$$H_{14_1}(x, t) = c\sqrt{\omega} \left(1 - i \tan \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right) - \frac{\lambda^2\omega}{2} \left(1 + \tan \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right), \quad (3.2.42)$$

or

$$u_{14_2}(x, t) = \lambda \left(\sqrt{4l^2 - m^2} \cot \left(\frac{\sqrt{4l^2 - m^2}\eta}{2} \right) + \sqrt{-4l^2 + m^2} \right), \quad (3.2.43)$$

and

$$H_{14_2}(x, t) = c\sqrt{\omega} \left(1 + i \cot \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right) - \frac{\lambda^2 \omega}{2} \left(1 - \cot \left(\frac{\sqrt{-\omega}\eta}{2} \right) \right), \quad (3.2.44)$$

where $m^2 - 4l^2 = \omega$ and $\eta = \lambda x - ct$.

When $m^2 - 4l^2 > 0$, $n \neq 0$ and $n = l$.

Setting (3.2.3) together with (2.2.5) into Eq. (3.2.1), we derive the solutions of (3.1) from Eqs. (3.2.1) and (3.5), which provides

$$u_{15_1}(x, t) = \lambda \sqrt{-4l^2 + m^2} \left(1 - \tanh \left(\frac{\sqrt{-4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.45)$$

and

$$H_{15_1}(x, t) = \left(1 - \tanh \left(\frac{\sqrt{\omega}\eta}{2} \right) \right) \times \left(\frac{\lambda^2 \omega}{2} \left(\tanh \left(\frac{\sqrt{\omega}\eta}{2} \right) - 1 \right) + c\sqrt{\omega} \right), \quad (3.2.46)$$

or

$$u_{15_2}(x, t) = \lambda \sqrt{-4l^2 + m^2} \left(1 - \coth \left(\frac{\sqrt{-4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.47)$$

and

$$H_{15_2}(x, t) = \left(1 - \coth \left(\frac{\sqrt{\omega}\eta}{2} \right) \right) \times \left(\frac{\lambda^2 \omega}{2} \left(\coth \left(\frac{\sqrt{\omega}\eta}{2} \right) - 1 \right) + c\sqrt{\omega} \right), \quad (3.2.48)$$

By means of (3.2.4) together with (2.2.5) from Eq. (3.2.1), we derive the solutions of (3.1) in the subsequent form

$$u_{16_1}(x, t) = \lambda \sqrt{-4l^2 + m^2} \left(1 + \tanh \left(\frac{\sqrt{-4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.49)$$

and

$$H_{16_1}(x, t) = \left(1 + \tanh \left(\frac{\sqrt{\omega}\eta}{2} \right) \right) \times \left(c\sqrt{\omega} - \frac{\lambda^2 \omega}{2} \left(\tanh \left(\frac{\sqrt{\omega}\eta}{2} \right) + 1 \right) \right), \quad (3.2.50)$$

or

$$u_{16_2}(x, t) = \lambda \sqrt{-4l^2 + m^2} \left(1 + \coth \left(\frac{\sqrt{-4l^2 + m^2}\eta}{2} \right) \right), \quad (3.2.51)$$

and

$$H_{16_2}(x, t) = \left(1 - \coth \left(\frac{\sqrt{\omega}\eta}{2} \right) \right) \times \left(c\sqrt{\omega} - \frac{\lambda^2 \omega}{2} \left(\coth \left(\frac{\sqrt{\omega}\eta}{2} \right) + 1 \right) \right), \quad (3.2.52)$$

where $m^2 - 4l^2 = \omega$ and $\eta = \lambda x - ct$. When

$$m^2 = 4ln.$$

By means of (3.2.3) together with (2.2.5) into Eq. (3.2.1), we gain the solutions of (3.1) from Eqs. (3.2.1) and (3.5), which gives

$$u_{17_1}(x, t) = -\frac{2\lambda}{\eta}, \quad (3.2.53)$$

and

$$H_{17_1}(x, t) = \frac{-2c\eta - 2\lambda^2}{\eta^2}. \quad (3.2.54)$$

Using (3.2.4) along with (2.2.5) into Eq. (3.2.1), we derive the solutions of (3.1) from Eqs. (3.2.1) and (3.5), which provides

$$u_{18_1}(x, t) = \frac{2\lambda}{\eta}, \quad (3.2.55)$$

and

$$H_{18_1}(x, t) = \frac{2c\eta - 2\lambda^2}{\eta^2}, \quad (3.2.56)$$

where $\eta = \lambda x - ct$.

When $m = 0$ and $l = -n$, we secure the solutions of (3.1) by putting (3.2.3) and (2.2.5) into Eqs. (3.2.1) and (3.5) as follows:

$$u_{19_1}(x, t) = \frac{2(1 + e^{-2n\eta})\lambda n}{(-1 + e^{-2n\eta})} + \sqrt{4} \sqrt{n^2} \lambda, \quad (3.2.57)$$

$$= 2n\lambda(1 - \coth(n\eta)),$$

and

$$H_{19_1}(x, t) = 2n(1 - \coth(n\eta))(c - \lambda^2 n(1 - \coth(n\eta))). \quad (3.2.58)$$

Providing (3.2.4) along with (2.2.5) into Eq. (3.2.1), we attain the solution functions (3.2.1) and (3.5) as shown below

$$u_{20_1} = \frac{-2(1 + e^{-2n\eta})\lambda n}{(-1 + e^{-2n\eta})} + \sqrt{4} \sqrt{n^2} \lambda \quad (3.2.59)$$

$$= 2n\lambda(\coth(n\eta) - 1),$$

and

$$H_{20_1}(x, t) = 2n(\coth(n\eta) - 1)(c - \lambda^2 n(\coth(n\eta) - 1)), \quad (3.2.60)$$

where $\eta = \lambda x - ct$.

When $l = m = K$, $n = 0$.

Applying the parametric values stated in (3.2.3) into Eqs. (3.2.1) and (3.5) represent the solutions

$$u_{21_1}(x, t) = 2K\lambda, \quad (3.2.61)$$

and

$$H_{21_1}(x, t) = -2K^2\lambda^2 + 2cK. \quad (3.2.62)$$

When $m = -(l + n)$.

Substituting the value assigned in (3.2.3) and Eq. (2.2.5) into Eq. (3.2.1) yields the solution

$$u_{24_1}(x, t) = \frac{2(l - e^{(l-n)\eta})\lambda n}{(n - e^{(l-n)\eta})} - 2l\lambda. \quad (3.2.63)$$

Using the value $u_{24_1}(x, t)$ in Eq. (3.5), we secure

$$H_{24_1}(x, t) = \frac{2(l - n)e^{(l-n)\eta}(cn + (-l\lambda^2 + \lambda^2 n - c)e^{(l-n)\eta})}{(n - e^{(l-n)\eta})^2}, \quad (3.2.64)$$

where

$$\eta = \lambda x - ct.$$

Applying (3.2.4) and Eq. (2.2.5) to Eqs. (3.2.1) and (3.5) gives

$$u_{25_1}(x, t) = \frac{-2(l - e^{(l-n)\eta})\lambda n}{(n - e^{(l-n)\eta})} + 2l\lambda, \quad (3.2.65)$$

and

$$H_{25_1}(x, t) = \frac{-2(l - n)e^{(l-n)\eta}(cn + (l\lambda^2 - \lambda^2 n - c)e^{(l-n)\eta})}{(n - e^{(l-n)\eta})^2}, \quad (3.2.66)$$

respectively, where $\eta = \lambda x - ct$.

When $m = l + n$.

Selecting (3.2.3) and (2.2.5) for Eq. (3.2.1) and after simplification, we obtain

$$u_{22}(x, t) = \frac{-2\lambda n \left((1 - l) \cosh\left(\frac{(l-n)\eta}{2}\right) - (1 + l) \sinh\left(\frac{(l-n)\eta}{2}\right) \right)}{(1 - n) \cosh\left(\frac{(l-n)\eta}{2}\right) - (1 + n) \sinh\left(\frac{(l-n)\eta}{2}\right)} + 2l\lambda \quad (3.2.67)$$

Choosing $n = 1$ in $u_{22}(x, t)$ gives

$$u_{22_1}(x, t) = \lambda \left((1 - l) \coth\left(\frac{(l-1)\eta}{2}\right) - (1 + l) \right) + 2l\lambda, \quad (3.2.68)$$

and from Eq. (3.5),

$$H_{22_1}(x, t) = \frac{\lambda^2}{2}(l-1)^2 \left(1 - \coth\left(\frac{(l-1)\eta}{2}\right) \right) \times \left(\coth\left(\frac{(l-1)\eta}{2}\right) - 1 + \frac{c}{\lambda^2(l-1)} \right), \quad (3.2.69)$$

where $\eta = \lambda x - ct$.

If $n = -1$, $u_{22}(x, t)$ pays

$$u_{22_2}(x, t) = \lambda \left((1 - l) - (1 + l) \tanh\left(\frac{(l+1)\eta}{2}\right) \right) + 2l\lambda, \quad (3.2.70)$$

and (3.5) gives

$$H_{22_2}(x, t) = \frac{\lambda^2}{2}(l+1)^2 \left(1 - \tanh\left(\frac{(l+1)\eta}{2}\right) \right) \times \left(\tanh\left(\frac{(l+1)\eta}{2}\right) - 1 + \frac{c}{\lambda^2(l+1)} \right), \quad (3.2.71)$$

where

$$\eta = \lambda x - ct.$$

Based on (3.2.4) with the help of (2.2.5), after computation, the solution function (3.2.1) becomes

$$u_{23}(x, t) = \frac{2\lambda n \left((1 - l) \cosh\left(\frac{(l-n)\eta}{2}\right) - (1 + l) \sinh\left(\frac{(l-n)\eta}{2}\right) \right)}{(1 - n) \cosh\left(\frac{(l-n)\eta}{2}\right) - (1 + n) \sinh\left(\frac{(l-n)\eta}{2}\right)} - 2l\lambda. \quad (3.2.72)$$

For $n = 1$, the solution function $u_{23}(x, t)$ provides

$$u_{23_1}(x, t) = -\lambda \left((1 - l) \coth\left(\frac{(l-1)\eta}{2}\right) - (1 + l) \right) - 2l\lambda, \quad (3.2.73)$$

and from Eq. (3.5), we obtain

$$H_{23_1}(x, t) = \frac{\lambda^2}{2}(l-1)^2 \left(1 - \coth\left(\frac{(l-1)\eta}{2}\right) \right) \times \left(\coth\left(\frac{(l-1)\eta}{2}\right) - 1 - \frac{c}{\lambda^2(l-1)} \right), \quad (3.2.74)$$

where $\eta = \lambda x - ct$.

Selecting $n = -1$, we obtain from $u_{23}(x, t)$,

$$u_{23_2}(x, t) = -\lambda \left((1 - l) - (1 + l) \tanh\left(\frac{(l+1)\eta}{2}\right) \right) - 2l\lambda, \quad (3.2.75)$$

and obtain from Eq. (3.5) as

$$H_{23_2}(x, t) = \frac{\lambda^2}{2}(l+1)^2 \left(1 - \tanh\left(\frac{(l+1)\eta}{2}\right) \right) \times \left(\tanh\left(\frac{(l+1)\eta}{2}\right) - 1 - \frac{c}{\lambda^2(l+1)} \right), \quad (3.2.76)$$

where $\eta = \lambda x - ct$.

When $n = m = l \neq 0$.

We attain the required solution of (3.1) by considering (3.2.3) and (2.2.5) into (3.2.1) and (3.5) as follows:

$$u_{26_1}(x, t) = \sqrt{3} \lambda l \left(\tan\left(\frac{\sqrt{3}l\eta}{2}\right) - 1 \right), \quad (3.2.77)$$

and

$$H_{26_1}(x, t) = -\frac{l}{2} \left(1 + \tanh\left(\frac{\sqrt{3}l\eta}{2}\right) \right) \times \left(3\lambda^2 \tanh\left(\frac{\sqrt{3}l\eta}{2}\right) + 3\lambda^2 l - 2\sqrt{3}c \right), \quad (3.2.78)$$

where $\eta = \lambda x - ct$.

Again, by selecting the values of (3.2.4) and (2.2.5) and putting into Eqs. (3.2.1) and (3.5), we obtain

$$u_{27_1}(x, t) = -\sqrt{3} \lambda l \left(\tan\left(\frac{\sqrt{3}l\eta}{2}\right) + 1 \right), \quad (3.2.79)$$

and

$$H_{27_1}(x, t) = -\frac{l}{2} \left(1 + \tanh\left(\frac{\sqrt{3}l\eta}{2}\right) \right) \times \left(3\lambda^2 \tanh\left(\frac{\sqrt{3}l\eta}{2}\right) + 3\lambda^2 l + 2\sqrt{3}c \right), \quad (3.2.80)$$

where $\eta = \lambda x - ct$.

When $l = m = 0$.

Proceeding in this way, for (3.2.3) we preserve

$$u_{28_1}(x, t) = 2\lambda l(\tan(l\eta) - 1), \quad (3.2.81)$$

and

$$H_{28_1}(x, t) = 2l(\tan(l\eta) - 1)(-\lambda^2 \tan(l\eta) + \lambda^2 + c), \quad (3.2.82)$$

where $\eta = \lambda x - ct$.

Also for (3.2.4), we gain the required solution of (3.1) as follows:

$$u_{29_1}(x, t) = 2\lambda l(-\tan(l\eta) - 1), \quad (3.2.83)$$

and

$$H_{29_1}(x, t) = -2l(\tan(l\eta) + 1)(\lambda^2 \tan(l\eta) + \lambda^2 + c), \quad (3.2.84)$$

where $\eta = \lambda x - ct$.

When $n = 0$, depending on (3.2.3), the solution functions (3.2.1) and (3.5) provide the results as

$$u_{30_1}(x, t) = 2m\lambda, \quad (3.2.85)$$

and

$$H_{30_1}(x, t) = -2m^2\lambda^2 + 2cm. \quad (3.2.86)$$

From the above investigation, we have found various types of solutions of the variant Boussinesq equation, such as hyperbolic function, trigonometric solution, exponential function solution, and rational solution.

4 Result discussions and wave profile description

In this section, the obtained traveling wave solutions of the stated equations are represented in the figures and reviewed the natures of these waves for dissimilar values of the free parameters with the aid of software MATLAB.

4.1 Wave profile interpretations of attained solutions using the IBSEF method

In this article, we have found 16 solutions (whereas Manafian *et al.* [39] established 15 solutions) using the IBSEF approach, which are illustrative and have rich structured wave profiles that might be helpful in describing the unidirectional wave propagation in non-linear media with dispersion relativistic one-particle theory. It is important to note that the wave solutions of the variant Boussinesq equation found here are functional and useful and were not proven in the earlier research.

For $\lambda = 0.001$ and $\sigma = 1$, the solution $u_{1_2}(x, t)$ represents a spike-type singular soliton as shown in Figure 1a, where the spike's amplitude is infinite near the origin and gradually decreases. Increasing the values of the free parameters proportionally, likewise $\lambda = 0.5$ and $\sigma = 0.5$, as well as descending the velocity speed c , the same solution is converted to breather soliton with singularity asserted as shown in Figure 1b. For the values $\lambda = 0.5$ and $\sigma = 1$, the exact solution delivers a singular soliton presented in Figure 1c. Furthermore, $u_{1_2}(x, t)$ demonstrates kink shape soliton with a singularity at $(0,0)$ traced in Figure 1(d) for $\lambda, \sigma \in \mathbb{R} - [-3, 3]$. The other part $H_{1_2}(x, t)$ of the solution provides a four petal-type wave graphed in Figure 1e, where two spikes are outer and the other two

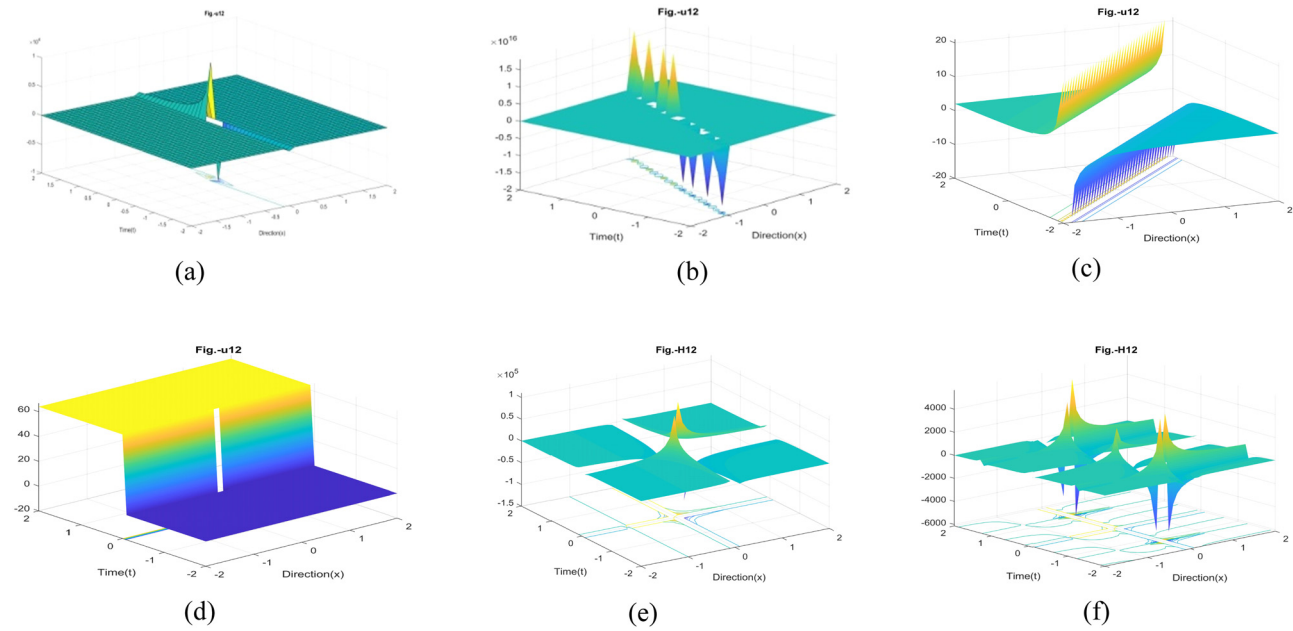


Figure 1: Plot of $u_{12}(x, t)$ and $H_{12}(x, t)$ for various values of λ and σ . (a) $\lambda = 0.001$ and $\sigma = 1$; (b) $\lambda = 0.5$ and $\sigma = 0.5$; (c) $\lambda = 0.5$ and $\sigma = 1$; (d) $\lambda = 4$ and $\sigma = 4$; (e) $\lambda = 0.001$ and $\sigma = 1$; and (f) $\lambda = 0.2$ and $\sigma = 0.2$.

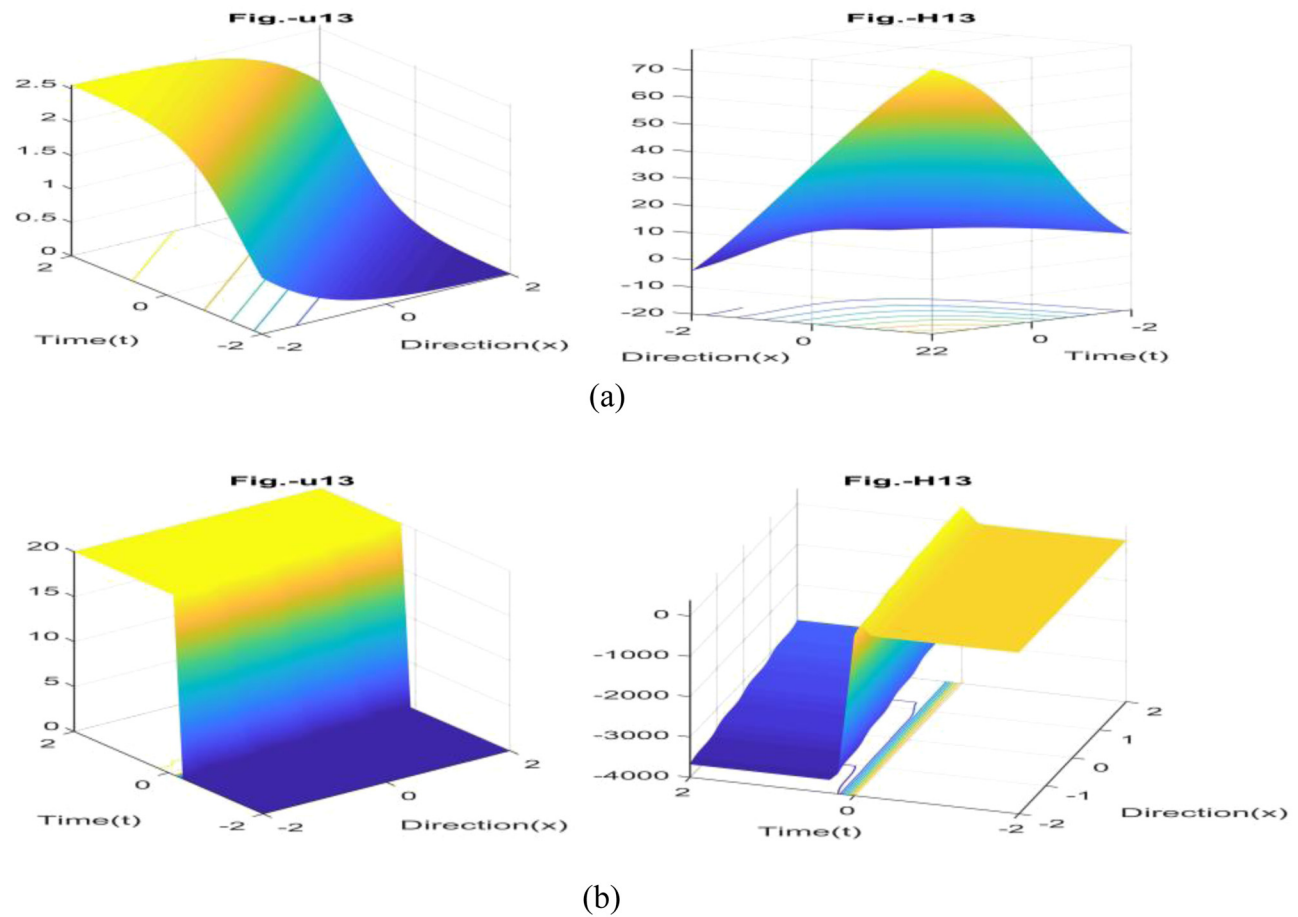


Figure 2: Plot of $u_{13}(x, t)$ and $H_{13}(x, t)$ for various values of λ and σ . (a) $\lambda = 0.8$ and $\sigma = 0.8$ and (b) $\lambda = 0.5$ and $\sigma = 10$.

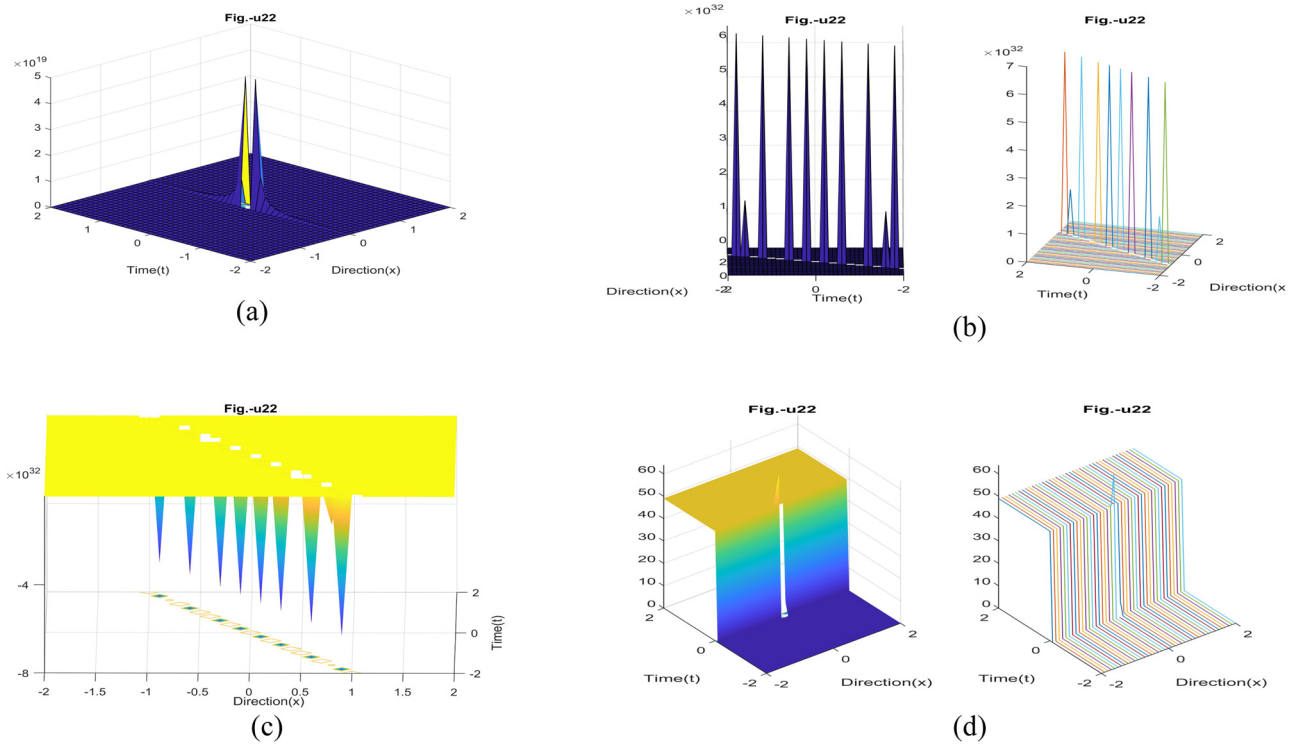


Figure 3: Plot of $u_{22}(x, t)$ for various values of λ and σ . (a) $\lambda = 0.001$ and $\sigma = 0.001$; (b) $\lambda = 0.5$ and $\sigma = 0.5$; (c) $\lambda = 0.5$ and $\sigma = -0.5$; and (d) $\lambda = 3.5$ and $\sigma = 3.5$.

spikes are inner for $\lambda = 0.001$ and $\sigma = 1$. By varying the values of the parameters equally, as $\lambda = 25$ and $\sigma = 0.2$, the same solution portrays double figures than the previous one that has been outlined in Figure 1(f).

Proceeding in this way, by increasing the value of those parameters, we achieve the same result.

The profile of the solution $u_3(x, t)$ shows a smooth kink shape soliton for the proportional value of λ and

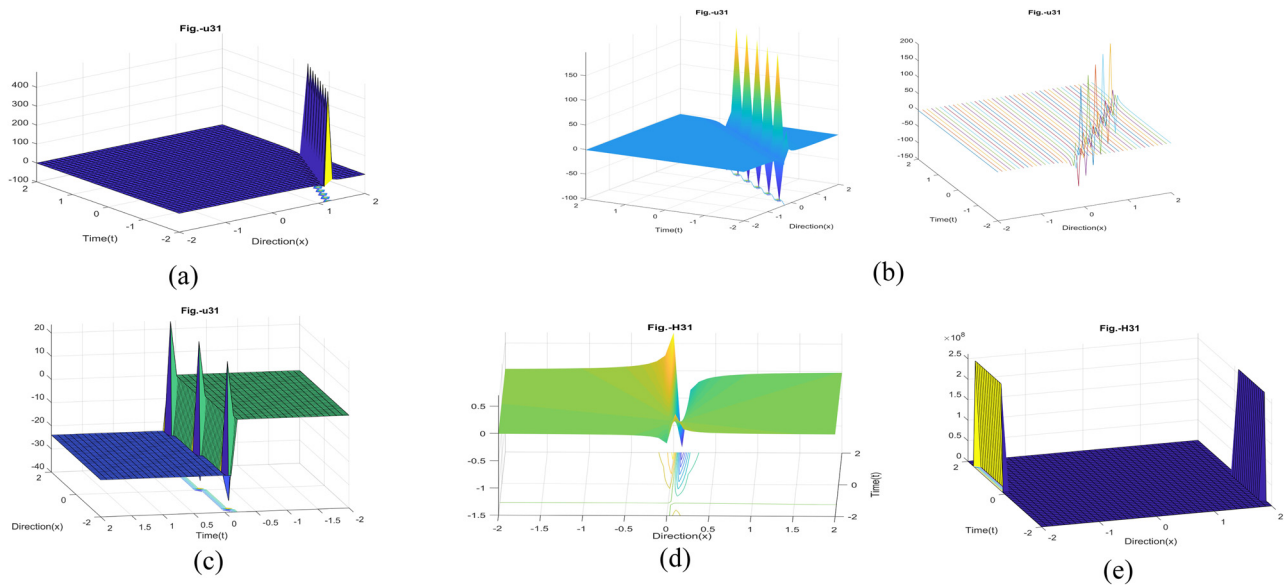


Figure 4: Plot of $u_{31}(x, t)$ and $H_{31}(x, t)$ for various values of λ and σ . (a) $\lambda = 0.5$ and $\sigma = 0.5$; (b) $\lambda = 1$ and $\sigma = 0.5$; (c) $\lambda = 3$ and $\sigma = 3$; (d) $\lambda = 0.3$ and $\sigma = 0.3$; and (e) $\lambda = 10$ and $\sigma = 1$.

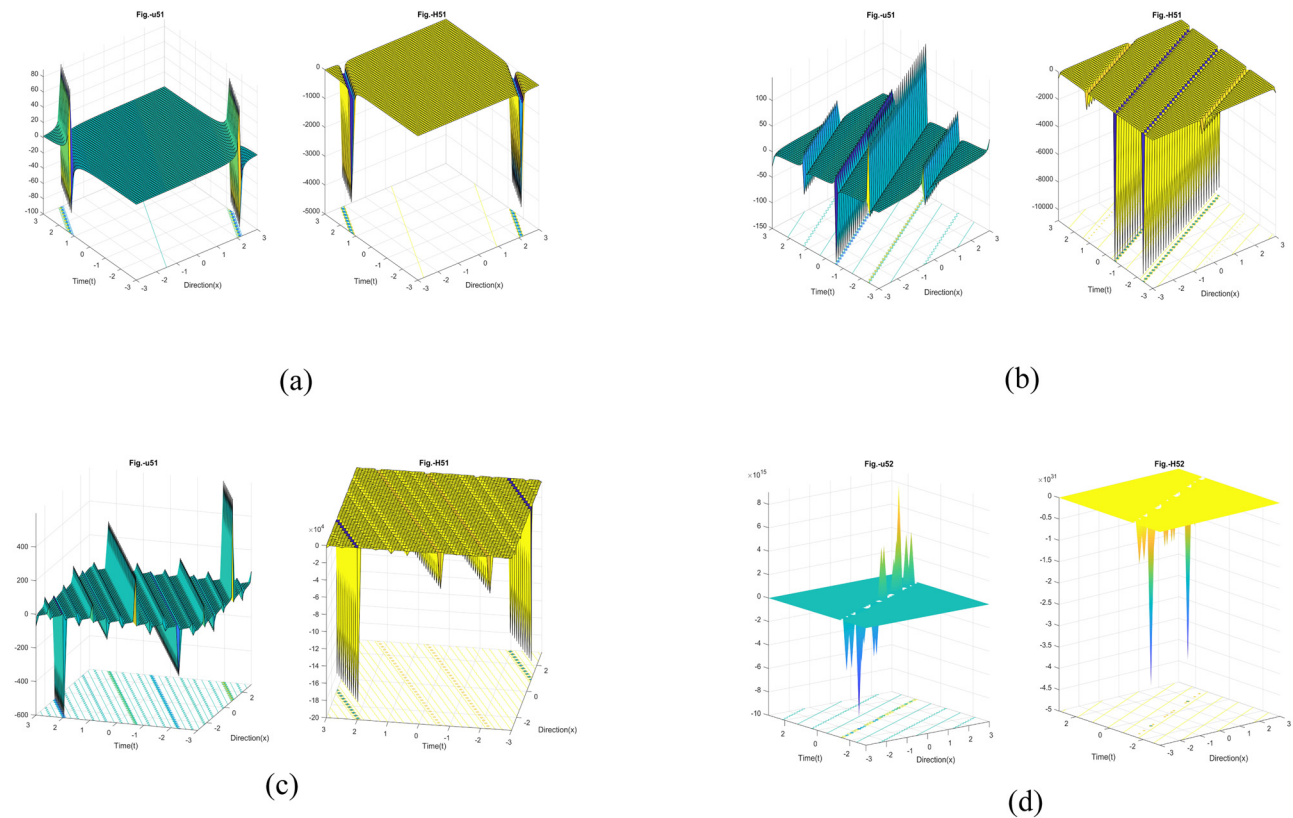


Figure 5: Plot of $u_{51}(x, t)$, $H_{51}(x, t)$, and $u_{52}(x, t)$, $H_{52}(x, t)$ for various values of λ and σ . (a) $\lambda = \sigma = 0.5$ and $l = m = n = 1$; (b) $\lambda = \sigma = l = m = n = 1$; (c) $\lambda = \sigma = m = 1$ and $l = n = 3$; and (d) $\lambda = \sigma = l = m = n = 1$.

σ as $\lambda, \sigma \in \mathbb{R} - (-0.7, 0.7)$. Furthermore, both solutions $u_{13}(x, t)$ and $H_{13}(x, t)$ represent steep shape kink for $-0.7 \leq \lambda$ and $\sigma \leq 0.7$, but $H_{13}(x, t)$ is a compacton soliton shown in Figure 2a for $\lambda = 0.8$ and $\sigma = 0.8$. Alternatively, for the lower value of λ and/or the upper value of σ , as earlier interval, the same solution displays the kink shape soliton asserted in Figure 2b, since the velocity c depends much on λ than σ .

For very small values of λ and σ , as $\lambda = \sigma = 0.001$, the exact solution $u_{22}(x, t)$ portrays a single spike-type singular soliton illustrated in Figure 3a. By ascending the values of the free parameters as well as increasing the speed of wave c , the same solution turns out to be breather-type soliton as displayed in Figure 3b for $\lambda = 0.5$ and $\sigma = 0.5$. Moreover, for the negative values of σ , the wave just overturns for $\lambda = 0.5$ and $\sigma = -0.5$ as shown in

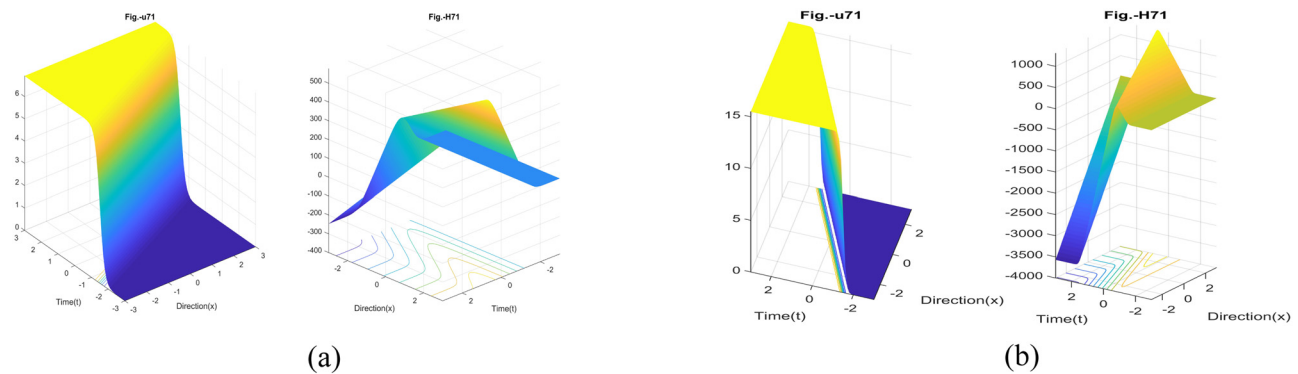


Figure 6: Plot of $u_{71}(x, t)$ and $H_{71}(x, t)$ for various values of λ and σ . (a) $\lambda = \sigma = l = n = 1$ and $m = 4$ and (b) $\lambda = \sigma = l = n = 1$ and $m = 8$.

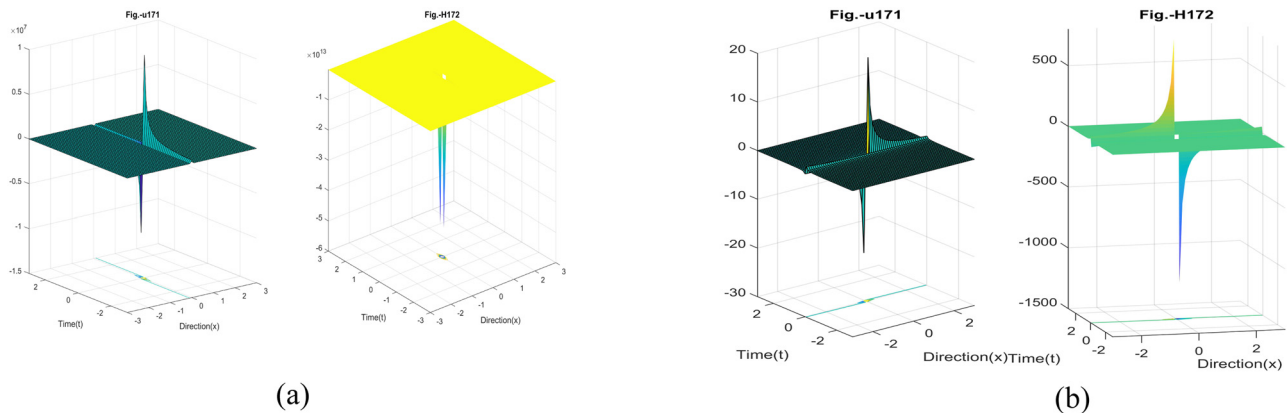


Figure 7: Plot of $u_{17_1}(x, t)$ and $H_{17_1}(x, t)$ for various values of λ and σ . (a) $\lambda = \sigma = 0.001$ and (b) $\lambda = \sigma = 5$.

Figure 1c. Moreover, by increasing the values gradually, like $\lambda = 3.5$ and $\sigma = 3.5$, the same solution changed into ideal kink shape soliton as sketched in Figure 3d.

The solution $u_{3_1}(x, t)$ represents an oblique plane for $-0.4 < \lambda, \sigma < 0.4$, and when increasing the values of the parameters, it demonstrates singular soliton as traced in Figure 4a–c for $\lambda = \sigma = 0.5$; $\lambda = 1$ and $\sigma = 0.5$; and $\lambda = 3$ and $\sigma = 3$, respectively. On the other hand, solution $H_{3_1}(x, t)$ portrays steep soliton as shown in Figure 4d for $\lambda = 0.3$ and $\sigma = 0.3$ and a special soliton as marked in Figure 4e for $\lambda = 10$ and $\sigma = 1$.

4.2 Wave profile description through the NAE method

Manafian *et al.* [39] investigated the Boussinesq equation and established only 15 solutions. On the other hand, we have derived 30 solutions in this article, which is twice as many as the solutions developed in ref. [39]. Of these 30 solutions, some of our solutions coincide with those found by Manafian *et al.*, indicating that the solutions we have ascertained are valid, while the others are new.

The exact solutions $u_{5_1}(x, t)$ and $H_{5_1}(x, t)$ represent two symmetrical solitons directed in opposite directions as sketched in Figure 5a for $\lambda = \sigma = 0.5$; $l = m = n = 1$, respectively. Considering larger values of λ and σ as well as speed of the wave c , like $\lambda = \sigma = l = m = n = 1$, the solutions yield solitons as shown in Figure 5b. For decreasing the negative value of $(m^2 - 4ln)$ and $\lambda = \sigma = m = 1$ and $l = n = 3$, the solution $u_{5_1}(x, t)$ shows irregular periodic soliton and $H_{5_1}(x, t)$ as portrayed in Figure 5c. On the other hand, another solution $u_{5_2}(x, t)$ and $H_{5_2}(x, t)$ display unequal spike-type singular wave asserted in Figure 5d for $\lambda = \sigma = l = m = n = 1$.

For the values of free parameters $\lambda = \sigma = l = n = 1$ and $m = 4$, solutions $u_{7_1}(x, t)$ and $H_{7_1}(x, t)$ depict standard kink soliton and compacton as illustrated in Figure 6a. By increasing the positive value of $m^2 - 4ln$, as $\lambda = \sigma = l = n = 1$ and $m = 8$, the same solutions yield kink and smooth kink soliton as marked out in Figure 6b, respectively.

Choosing the infinitesimal value of the parameters, for instance $\lambda = \sigma = 0.001$, the rational function solution $u_{17_1}(x, t)$ represents singular soliton with descending spike and $H_{17_1}(x, t)$ depicts singular soliton with two equal spikes as traced in Figure 7a. Moreover, for the higher value of the velocity c corresponds to $\lambda = \sigma = 5$, both solutions are representing same shape of wave as portrayed in Figure 7b.

From the earlier discussion, it is clear that the obtained solutions of the variant Boussinesq equation through the IBSEF and the NAE methods provide a variety of functional waves, such as kink, singular kink, breather-type, symmetric, compacton, bell-shape soliton, and irregular periodic wave. The wave profiles depend on the positive and negative values of the free parameters and disclose the dynamic behavior of the studied phenomenon. It has been established that wave profiles are strongly interrelated with free parameters, especially with wave velocity c and wave number λ compared to other parameters.

5 Conclusion

In this study, we successfully conducted two significant approaches, namely the IBSEF method and the new auxiliary method to the variant Boussinesq equation, and achieved some standard and advanced solutions and some broad-ranging exact solutions. Setting certain

values of the associated parameters, broad-spectrum solutions are able to recover the known solutions and provide the optimal solutions for other values from which the standard functional waves, such as kink, singular kink, breather-type, symmetric, compacton, bell-shape soliton, and irregular periodic wave can be extracted. Furthermore, through some of the documented solutions, we have shown how wave numbers and velocities affect wave profiles. The solutions attained in this study are illustrative and have rich structured wave profiles (Figures 1–7) that might help describe the unidirectional wave propagation in nonlinear media with dispersion relativistic one-particle theory, the long waves on the surface of the liquid layer under gravity action, *etc.*

Funding information: This research was supported by National Natural Science Foundation of China (No. 71601072), Key Scientific Research Project of Higher Education Institutions in Henan Province of China (No. 20B110006), and the Fundamental Research Funds for the Universities of Henan Province.

Author contributions: All authors have accepted responsibility for the entire content of this manuscript and approved its submission.

Conflict of interest: The authors state no conflict of interest.

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