Research Article

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The solutions of nonlinear fractional partial differential equations by using a novel technique

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Abstract: In this article, the solutions of higher nonlinear partial differential equations (PDEs) with the Caputo operator are presented. The fractional PDEs are modern tools to model various phenomena more accurately. The residual power series method (RPSM) is used for the solution analysis of fractional partial differential equations (FPDEs), which has direct implementation for the solutions of fractional partial differential equations. In this work, the solutions to a few nonlinear FPDEs are handled by the proposed technique. The general and particular schemes of RPSM are constructed and implemented successfully. The fractional solutions of PDEs have provided many useful dynamics of the targeted problems. The RPSM results for both integer and fractional-order FPDEs are further explained and elaborated by using graphs and tables. It is observed that the higher accuracy of RPSM is achieved with fewer calculations. Graphs and tables for fractional-order solutions are presented, which confirm the convergence phenomena of fractional solutions toward integer order solutions of each problem. The suggested method can be extended to the solutions of other nonlinear fractional partial differential equations.

Keywords: fractional partial differential equations, Caputo derivative operator, residual power series method, analytical technique

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1 Introduction

The generalization of the derivative and integral to arbitrary orders is known as fractional calculus (FC). It is an extension of ordinary calculus. Many physical phenomena are accurately modeled when compared to ordinary calculus. FC has gained prominence in recent decades due to its numerous applications in fields such as electrodynamics [1], tuberculosis [2], immunogenic tumor dynamics [3], cholera infection model [4], hepatitis B virus [5], pine wilt disease [6], diabetes [7], and so on. In FC, fractional partial differential equations (FPDEs) are regarded as the most precise techniques for developing mathematical models in applied mathematics, such as damping laws, rheology, diffusion, electrostatics, electrodynamics, fluid flows, and so on [8–11]. Several methods are used to solve FPDEs and system of FPDEs, for example, Laplace Adomin decomposition method [12], Laplace variational homotopy perturbation method [13], Chebyshev wavelet method [14], Elzaki transform method [15], natural transform decomposition method [16], finite element method [17], finite difference method [18], q-homotopy analysis method [19], matrix approximation technique [20], modified predictor–corrector method [21], and corrected Fourier series and fractional complex transformation [22–24]. In the same context, Erturk *et al.* have implemented fractional Lagrangian approach to describe the motion of a beam on a nanowire [25].

The residual power series method (RPSM) is an analytical procedure and is used to solve FPDEs and system FPDEs. Arqub introduced the RPSM in ref. [26], which was utilized to solve first- and second-order fuzzy differential equations. RPSM is a simple and accurate tool for mathematicians to obtain the series form solutions to a variety of differential equations. RPSM is based on the power series expansion and can be applied efficiently without the use of discretization, perturbation, or linearization [18,27]. RPSM is based on residual error concepts, which derive power series coefficients in a chain of algebraic equations with one or two variables. Furthermore, RPSM is a simple and accurate tool for solving various fractional differential equations and FPDEs as well. In the literature, RPSM was used to find the solution to Lane Emeden equations [28]. It has been usefully applied to other equations [29], including the fractional foam drainage equation [30], fractional diffusion equations [31], fuzzy type differential equations [26], Boussinesq–Burgers equations [32], fractional-order Burger types equations [33], the nonlinear KdV-Burgers equation [18], initial value problems of higher order [34], nonlinear coupled Boussinesq–Burgers equations [35], system of Fredhlom integral equations [36], system of multipantograph differential equations [37], the Lane–Emden equations [38], the Whitham–Broer–Kaup equations [39], fractional model of vibration equation [40], system evolutionary having two-component equation [41], the black-Scholes European option pricing equations [42], the nonlinear gas dynamics equations [43], the Schrödinger equations in one dimension space [44], Benney-Lin equation arising in falling film problems [45], and time fractional heat like PDEs [46]. In ref. [47], the researchers looked at the analysis and modeling of FPDEs for use in the reaction-diffusion model [48] and analytic approximation solutions of diffusion equations arising in oil pollution.

In the current research task, the analytical solutions of fractional-order time and space PDEs are calculated by using RPSM. The generalized RPSM algorithms are successfully constructed for the solution of all problems. The RPSM implementation is completed for a few numerical examples of the targeted problems. Graphs and tables are used to display the RPSM results. Despite the fact that only a few terms are used in the solution, RPSM provides a higher level of accuracy. The proposed method requires fewer calculations and does not rely on additional parameters or discratization. For fractional-order and integer order problems, the exact and RPSM solutions are in close contact. Furthermore, the RPSM methodology and accuracy have allowed it to be used for other fractional problems as well.

Finally, this article is structured as follows: Section 1 is devoted to some basic introductions related to the current work. Basic definitions are defined in Section 2. Section 3 discusses the specifics of the RPSM formulation for fractional partial differential equations. Section 4 contains some numerical examples with solutions and their graphs as well. Section 5 presents the Conclusion.

2 Basic definitions

2.1 Definitions

The fractional-order Caputo derivative of order $\delta > 0$ is defined by ref. [16]

$$D_{\Im}^{\delta}\mu(\zeta,\Im) = \frac{1}{\Gamma(m-\delta)}\int_{0}^{\Im}(\Im-\Im_o)^{m-\delta-1}\frac{\partial^m\mu(\zeta,\Im)}{\partial\Im^m}\mathrm{d}\Im, \quad \text{if } m-1<\delta\le m,$$

if $\delta = m$, for $m \in N$, then,

$$D_{\Im}^{\delta}\mu(\zeta,\Im) = \frac{\partial^m\mu(\zeta,\Im)}{\partial\Im^m}.$$

2.2 Definitions

The fractional power series (FPS) is represented by ref. [18]

$$\sum_{m=0}^{\infty}P_m(\Im-\Im_o)^{m\delta} = P_o + P_1(\Im-\Im_o)^{\delta} + P_2(\Im-\Im_o)^{2\delta} + \ldots, \quad m-1<\delta\le m,\ \Im\le\Im_o,$$

where P_m are the constant coefficients of the series. If $\Im_o = 0$, then FPS become fractional Maclaurin series.

2.3 Theorem

The FPS expansion for $\mu(\zeta,\Im)$ at $\Im = \Im_o$ can be expressed as ref. [18]:

$$\mu(\zeta,\Im) = \sum_{m=0}^{\infty}P_m(\zeta)(\Im-\Im_o)^{m\delta}.$$

If $D_{\Im}^{m\delta}\mu(\zeta,\Im)$ are continuous for $m = 0, 1, \ldots$, then

$$P_m(\zeta) = \frac{D_{\Im}^{m\delta}\mu(\zeta,\Im)}{\Gamma(1+m\delta)},$$

where $\zeta \in I$, $\Im_o \le \Im < \Im + R$.

The FPS expansion of $\mu(\zeta,\Im)$ can be expressed as follows:

$$\mu(\zeta,\Im)=\sum_{m=0}^{\infty}\frac{D_{\Im}^{m\delta}\mu(\zeta,\Im)}{\Gamma(1+m\delta)}(\Im-\Im_o)^{m\delta},$$

which is called generalization Taylor expansion. If $\delta = 1$, then it will be converted into Taylor series.

2.4 Definitions

The fractional derivatives $D^{\delta}(\Im)^{m\delta}$ is describe as suggested in ref. [18]

$$D_{\Im}^{\delta}(\Im)^{m\delta}=\frac{\Gamma(m\delta+1)}{\Gamma(m\delta-\delta+1)}(\Im)^{m\delta-\delta},$$

where, $0<\delta\le 1$ and $m = 1, 2, \ldots$.

3 RPSM methodology

To understand the RPSM procedure [1], we will take to solve time and space fractional partial differential equations (FPDEs).

$$D_{\Im}^{\delta}\mu(\zeta,\Im)+\mu(\zeta,\Im)D_{\zeta}^{\rho}\mu(\zeta,\Im)=H(\zeta),\quad 0<\delta\le 1 \quad\text{and}\quad 0<\rho\le 1, \tag{1}$$

having initial condition

$$\mu(\zeta,0)=f(\zeta), \tag{2}$$

and the truncated series of Eq. (1) is

$$\mu_k(\zeta,\Im)=f(\zeta)+\sum_{m=1}^{k}f_m\frac{(\Im)^{m\delta}}{\Gamma(1+m\delta)}. \tag{3}$$

Residual function of Eq. (1) is given by

$$\text{Res}_{\mu}(\zeta,\Im)=D_{\Im}^{\delta}\mu(\zeta,\Im)+\mu(\zeta,\Im)D_{\zeta}^{\rho}\mu(\zeta,\Im)=H(\zeta), \tag{4}$$

and the kth-residual function is defined as follows:

$$\text{Res}_{\mu,k}(\zeta,\Im)=D_{\Im}^{\delta}\mu_k(\zeta,\Im)+\mu_k(\zeta,\Im)D_{\zeta}^{\rho}\mu_k(\zeta,\Im)-H(\zeta). \tag{5}$$

For first approximation putting $k = 1$, Eqs. (3) and (5) can be simplified as follows:

$$\mu_1(\zeta,\Im)=f(\zeta)+f_1(\zeta)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}, \tag{6}$$

$$\text{Res}_{\mu,1}(\zeta,\Im)=D_{\Im}^{\delta}\mu_1(\zeta,\Im)+\mu_1(\zeta,\Im)D_{\zeta}^{\rho}\mu_1(\zeta,\Im)-H(\zeta). \tag{7}$$

By putting Eq. (14) in Eq. (15), we obtain,

$$\text{Res}_{\mu,1}(\zeta,\Im)=f_1(\zeta)+f(\zeta)D_{\zeta}^{\rho}f(\zeta)-H(\zeta), \tag{8}$$

using $\text{Res}_{\mu,1}(\zeta,0)$=0, to obtained,

$$f_1(\zeta)=H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta). \tag{9}$$

first RPSM approximation is expressed as follows:

$$\mu_1(\zeta,\Im)=f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}. \tag{10}$$

For $k = 2$, Eqs. (3) and (5) can be written as follows:

$$\mu_2(\zeta,\Im)=f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+f_2(\zeta)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}, \tag{11}$$

$$\text{Res}_{\mu,2}(\zeta,\Im)=D_{\Im}^{\delta}\mu_2(\zeta,\Im)+\mu_2(\zeta,\Im)D_{\zeta}^{\rho}\mu_2(\zeta,\Im)-H(\zeta), \tag{12}$$

and by putting Eq. (11) in Eq. (12), we obtain

$$\begin{aligned}\text{Res}_{\mu,2}(\zeta,\Im)&\\ = D_{\Im}^{\delta}&\left(f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+f_2(\zeta)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\right)\\ &+\left(f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+f_2(\zeta)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\right)\\ &\times D_{\zeta}^{\rho}\left(f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+f_2(\zeta)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\right)-H(\zeta).\end{aligned} \tag{13}$$

By using $D_{\Im}^{\delta}$ in Eq. (13) and then using $D_{\Im}^{\delta}\text{Res}_{\mu,2}(\zeta,0)=0$, we obtain

$$f_2(\zeta)=-f(\zeta)D_{\zeta}^{\rho}f(\zeta)-[H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta)]D_{\zeta}^{\rho}f(\zeta)-f(\zeta)[D_{\zeta}^{\rho}H(\zeta)-D_{\zeta}^{\rho}(f(\zeta)D_{\zeta}^{\rho}f(\zeta))]. \tag{14}$$

Second RPSM approximation is expressed as follows:

$$\begin{aligned}\mu_2(\zeta,\Im)=&f(\zeta)+(H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta))\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\\ &+(-f(\zeta)D_{\zeta}^{\rho}f(\zeta)-[H(\zeta)-f(\zeta)D_{\zeta}^{\rho}f(\zeta)]\\ &\times D_{\zeta}^{\rho}f(\zeta)-f(\zeta)[D_{\zeta}^{\rho}H(\zeta)\\ &-D_{\zeta}^{\rho}(f(\zeta)D_{\zeta}^{\rho}f(\zeta))])\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}.\end{aligned} \tag{15}$$

In general, $f_m(\zeta)$ for $k \geq 3$ is calculated by putting Eq. (3) in Eq. (5) and then computing $D_{\mathfrak{I}}^{(k-1)\delta} of \text{Res}_{\mu,k}(\zeta, \mathfrak{I})$. Therefore, $f_m(\zeta)$ are calculated by using the following algebraic equations.

$$D_{\mathfrak{I}}^{(k-1)\delta}\text{Res}_{\mu,k}(\zeta, 0) = 0, \quad k = 1, 2, \ldots. \tag{16}$$

4 Numerical examples

4.1 Example

Take a look at the fractional partial differential equation [49],

$$D_{\mathfrak{I}}^{\delta}\mu(\zeta, \mathfrak{I}) + \mu(\zeta, \mathfrak{I})D_{\zeta}^{\rho}\mu(\zeta, \mathfrak{I}) = \zeta, \quad 0 < \delta \leq 1 \quad \text{and} \quad 0 < \rho \leq 1, \tag{17}$$

having initial condition

$$\mu(\zeta, 0) = 1, \tag{18}$$

and the truncated series of Eq. (17) is

$$\mu_k(\zeta, \mathfrak{I}) = 1 + \sum_{m=1}^{k} C_m \frac{(\mathfrak{I})^{m\delta}}{\Gamma(1 + m\delta)}, \tag{19}$$

and the residual function for Eq. (17) is define as follows:

$$\text{Res}_{\mu,k}(\zeta, \mathfrak{I}) = D_{\mathfrak{I}}^{\delta}\mu_k(\zeta, \mathfrak{I}) + \mu_k(\zeta, \mathfrak{I})D_{\zeta}^{\rho}\mu_k(\zeta, \mathfrak{I}) - \zeta. \tag{20}$$

For $k = 1$, Eqs. (19) and (20) can be simplified as follows:

$$\mu_1(\zeta, \mathfrak{I}) = 1 + C_1\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)}, \tag{21}$$

$$\text{Res}_{\mu,1}(\zeta, \mathfrak{I}) = D_{\mathfrak{I}}^{\delta}\mu_1(\zeta, \mathfrak{I}) + \mu_1(\zeta, \mathfrak{I})D_{\zeta}^{\rho}\mu_1(\zeta, \mathfrak{I}) - \zeta. \tag{22}$$

By putting Eq. (21) in Eq. (22), we obtain

$$\text{Res}_{\mu,1}(\zeta, \mathfrak{I}) = C_1 - \zeta, \tag{23}$$

and by using $\text{Res}_{\mu,1}(\zeta, 0) = 0$, we obtain

$$C_1 = \zeta. \tag{24}$$

First RPSM approximation is expressed as follows:

$$\mu_1(\zeta, \mathfrak{I}) = 1 + \zeta\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)}. \tag{25}$$

For $k = 2$, Eqs. (19) and (20) can be written as follows:

$$\mu_2(\zeta, \mathfrak{I}) = 1 + \zeta\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} + C_2\frac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)}, \tag{26}$$

$$\text{Res}_{\mu,2}(\zeta, \mathfrak{I}) = D_{\mathfrak{I}}^{\delta}\mu_2(\zeta, \mathfrak{I}) + \mu_2(\zeta, \mathfrak{I})D_{\zeta}^{\rho}\mu_2(\zeta, \mathfrak{I}) - \zeta. \tag{27}$$

By putting Eq. (26) in Eq. (27), we obtain

$$\begin{aligned}\text{Res}_{\mu,2}(\zeta, \mathfrak{I}) &= \zeta + C_2\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} + \left(1 + \zeta\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} + C_2\frac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)}\right) \\ &\quad \times \left(\frac{\zeta^{(1-\rho)}\mathfrak{I}^{\delta}}{\Gamma(2 - \rho)\Gamma(1 + \delta)}\right) - \zeta.\end{aligned} \tag{28}$$

By using $D_{\mathfrak{I}}^{\delta}$ in Eq. (28) and then using $D_{\mathfrak{I}}^{\delta}\text{Res}_{\mu,2}(\zeta, 0) = 0$, we have

$$C_2 = -\frac{\zeta^{(1-\rho)}}{\Gamma(2 - \rho)}. \tag{29}$$

Second RPSM approximation is expressed as follows:

$$\mu_2(\zeta, \mathfrak{I}) = 1 + \zeta\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} - \left(\frac{\zeta^{(1-\rho)}}{\Gamma(2 - \rho)}\right)\frac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)}. \tag{30}$$

Similarly, for $k = 3$, Eqs. (19) and (20) can be simplified as follows:

$$\begin{aligned}\mu_3(\zeta, \mathfrak{I}) = 1 &+ \zeta\frac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} - \left(\frac{\zeta^{(1-\rho)}}{\Gamma(2 - \rho)}\right)\frac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)} \\ &+ C_3\frac{(\mathfrak{I})^{3\delta}}{\Gamma(1 + 3\delta)},\end{aligned} \tag{31}$$

$$\text{Res}_{\mu,3}(\zeta, \mathfrak{I}) = D_{\mathfrak{I}}^{\delta}\mu_3(\zeta, \mathfrak{I}) + \mu_3(\zeta, \mathfrak{I})D_{\zeta}^{\rho}\mu_3(\zeta, \mathfrak{I}) - \zeta. \tag{32}$$

By putting Eq. (31) in Eq. (32), we obtain

$$\text{Res}_{\mu,3}(\zeta, \mathfrak{I}) = \begin{cases}\zeta - \dfrac{\zeta^{(1-\rho)}\mathfrak{I}^{\delta}}{\Gamma(2 - \rho)\Gamma(1 + \delta)} + C_3\dfrac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)} \\ + \left(1 + \zeta\dfrac{(\mathfrak{I})^{\delta}}{\Gamma(1 + \delta)} - \left(\dfrac{\zeta^{(1-\rho)}}{\Gamma(2 - \rho)}\right)\dfrac{(\mathfrak{I})^{2\delta}}{\Gamma(1 + 2\delta)} + C_3\dfrac{(\mathfrak{I})^{3\delta}}{\Gamma(1 + 3\delta)}\right) \\ \left(\dfrac{\zeta^{(1-\rho)}\mathfrak{I}^{\delta}}{\Gamma(2 - \rho)\Gamma(1 + \delta)} - \dfrac{\zeta^{(1-2\rho)}\mathfrak{I}^{2\delta}}{\Gamma(2 - 2\rho)\Gamma(1 + 2\delta)}\right) \\ - \zeta.\end{cases} \tag{33}$$

By applying $D_{\mathfrak{I}}^{2\delta}$ both sides of Eq. (33) and then using $D_{\mathfrak{I}}^{2\delta}\text{Res}_{\mu,3}(\zeta, 0) = 0$, we have

$$C_3 = -\frac{2\zeta^{(2-\rho)}}{\Gamma(2 - \rho)}. \tag{34}$$

Third RPSM approximation is expressed as follows:

$$\mu_3(\zeta,\Im) = 1 + \zeta\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} - \left(\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)} - \left(\frac{2\zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)}. \tag{35}$$

By putting $\delta = \rho = 1$, we obtain the closed form solution of the problem:

$$\mu(\zeta,\Im) = \zeta\tanh(\Im) + \operatorname{sech}(\Im). \tag{36}$$

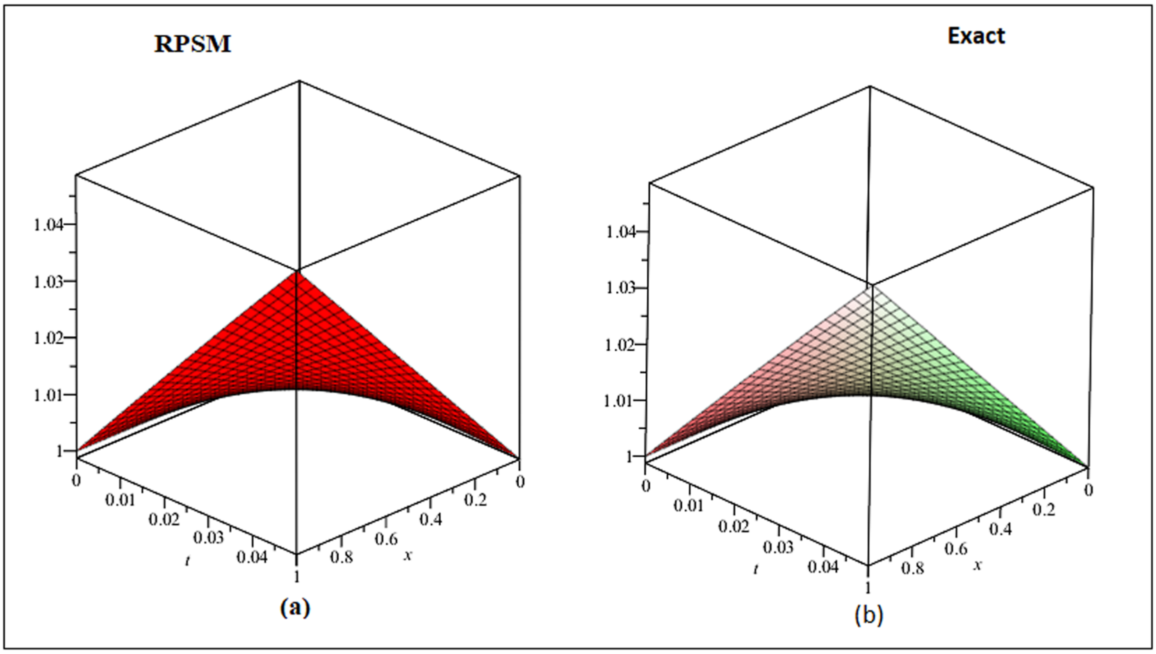

Figure 1: 3D solution graph, at δ and $\rho = 1$ of Example 4.1.

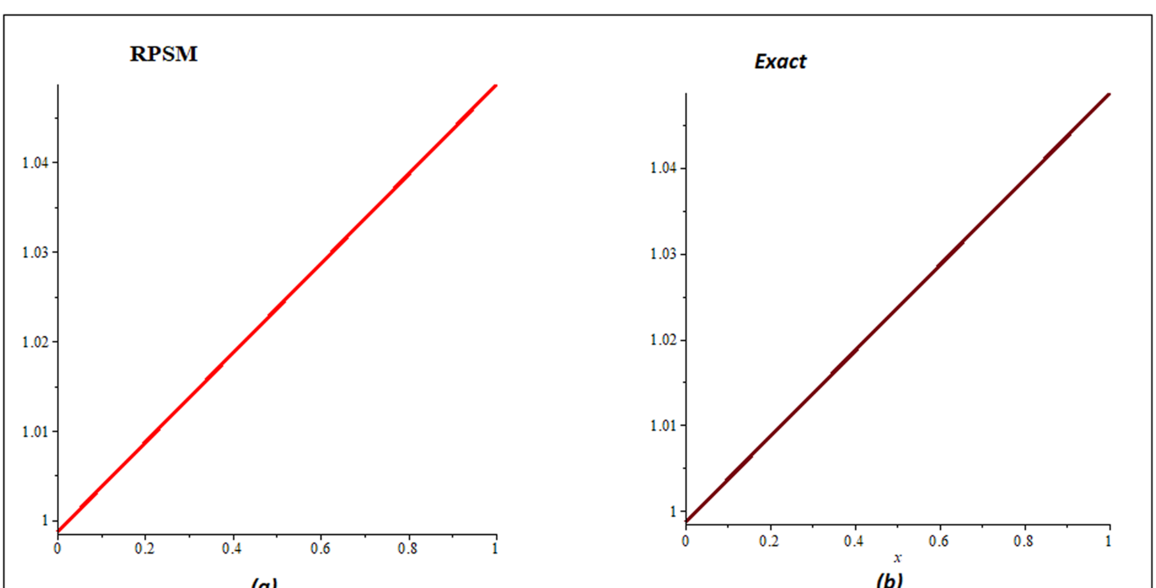

Figure 2: 2D solution graph, at δ and $\rho = 1$ of Example 4.1.

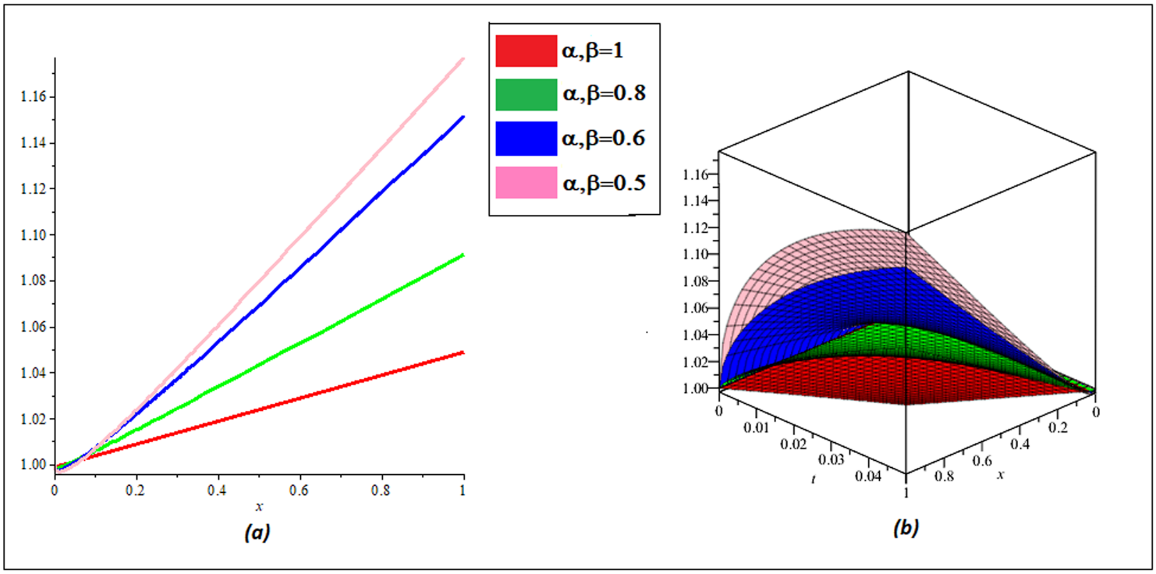

Figure 3: 2D and 3D plots exact and RPSM solutions of Example 4.1 for different values of δ.

Table 1: AE of Example 1, at $\Im = 0.05$ and δ, $\rho = 1$

ζ	Exact–RPSM $K = 1$	Exact–RPSM $K = 2$	Exact–RPSM $K = 3$
0.1	$1.252861704 \times 10^{-3}$	2.861704×10^{-6}	1.305×10^{-6}
0.2	$1.257024208 \times 10^{-3}$	7.024208×10^{-6}	1.309×10^{-6}
0.3	$1.261186712 \times 10^{-3}$	1.1186712×10^{-5}	1.313×10^{-6}
0.4	$1.265349216 \times 10^{-3}$	1.5349216×10^{-5}	1.318×10^{-6}
0.5	$1.269511720 \times 10^{-3}$	1.9511720×10^{-5}	1.321×10^{-6}
0.6	$1.273674224 \times 10^{-3}$	2.3674224×10^{-5}	1.326×10^{-6}
0.7	$1.277836728 \times 10^{-3}$	2.7836728×10^{-5}	1.330×10^{-6}
0.8	$1.281999232 \times 10^{-3}$	3.1999232×10^{-5}	1.334×10^{-6}
0.9	$1.286161736 \times 10^{-3}$	3.6161736×10^{-5}	1.338×10^{-6}
1	$1.29032424 \times 10^{-3}$	4.0324240×10^{-5}	1.343×10^{-6}

4.2 Example

Take a look at the fractional partial differential equation of the form [49],

$$D_{\Im}^{\delta}\mu(\zeta,\Im) + \mu(\zeta,\Im)D_{\zeta}^{\rho}\mu(\zeta,\Im) = 1, \quad 0 < \delta \le 1 \text{ and } 0 < \rho \le 1 \tag{37}$$

having initial condition

$$\mu(\zeta,0) = -\zeta, \tag{38}$$

and the truncated series of Eq. (37) is expressed as follows:

$$\mu_k(\zeta,\Im) = -\zeta + \sum_{m=1}^{k} C_m \frac{(\Im)^{m\delta}}{\Gamma(1+m\delta)}. \tag{39}$$

The residual function for Eq. (37) is define as follows:

$$\mathrm{Res}_{\mu,k}(\zeta,\Im) = D_{\Im}^{\delta}\mu_k(\zeta,\Im) + \mu_k(\zeta,\Im)D_{\zeta}^{\rho}\mu_k(\zeta,\Im) - 1. \tag{40}$$

For $k = 1$, Eqs. (39) and (40) can be simplified as follows:

$$\mu_1(\zeta,\Im) = -\zeta + C_1\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}, \tag{41}$$

$$\mathrm{Res}_{\mu,1}(\zeta,\Im) = D_{\Im}^{\delta}\mu_1(\zeta,\Im) + \mu_1(\zeta,\Im)D_{\zeta}^{\rho}\mu_1(\zeta,\Im) - 1. \tag{42}$$

By putting Eq. (41) in Eq. (46), we obtain

$$\mathrm{Res}_{\mu,1}(\zeta,\Im) = C_1 + \left(-\zeta + C_1\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\right)\left(-\frac{\zeta^{1-\rho}}{\Gamma(2-\rho)}\right) - 1, \tag{43}$$

and using $\mathrm{Res}_{\mu,1}(\zeta,0) = 0$, we obtain

$$C_1 = \frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}. \tag{44}$$

First RPSM approximation defined as follows:

$$\mu_1(\zeta,\Im) = -\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}. \tag{45}$$

For $k = 2$, Eqs. (39) and (40) can be written as follows:

$$\mu_2(\zeta, \Im) = -\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} + C_2\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}, \tag{46}$$

$$\mathrm{Res}_{\mu,2}(\zeta, \Im) = D_{\Im}^{\delta}\mu_2(\zeta, \Im) + \mu_2(\zeta, \Im)D_{\zeta}^{\rho}\mu_2(\zeta, \Im) - 1. \tag{47}$$

By putting Eq. (46) in Eq. (40), we obtain

$$\begin{aligned}\mathrm{Res}_{\mu,2}(\zeta, \Im) = {}& \frac{(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)} + C_2\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & + \Bigg(-\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & + C_2\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\Bigg)\Bigg(-\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)} \\ & - \frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(3-2\rho)}\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\Bigg) - 1,\end{aligned} \tag{48}$$

By substituting $D_{\Im}^{\delta}$ into Eq. (48) and then using $D_{\Im}^{\delta}\mathrm{Res}_{\mu,2}(\zeta, 0) = 0$, we obtain

$$C_2 = -\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\right). \tag{49}$$

Second RPSM approximation is expressed as follows:

$$\begin{aligned}\mu_2(\zeta, \Im) = {}& -\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & - \Bigg(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} \\ & + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\Bigg)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}.\end{aligned} \tag{50}$$

Similarly, for $k = 3$, Eqs. (39) and (40) can be simplified as follows:

$$\begin{aligned}\mu_3(\zeta, \Im) = {}& -\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & - \Bigg(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} \\ & + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\Bigg) \\ & \times \frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)} + C_3\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)},\end{aligned} \tag{51}$$

$$\mathrm{Res}_{\mu,3}(\zeta, \Im) = D_{\Im}^{\delta}\mu_3(\zeta, \Im) + \mu_3(\zeta, \Im)D_{\zeta}^{\rho}\mu_3(\zeta, \Im) - 1. \tag{52}$$

By putting Eq. (51) in Eq. (52), we obtain

$$\begin{aligned}\mathrm{Res}_{\mu,3}(\zeta, \Im) = {}& \frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)} \\ & - \left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\right) \\ & \times \frac{(\Im)^{\delta}}{\Gamma(1+\delta)} + C_3\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)} \\ & + \Bigg(-\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & - \left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\right) \\ & \times \frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)} + C_3\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)}\Bigg) \\ & \Bigg(-\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)} + \left(-\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(3-2\rho)\Gamma(2-\rho)}\right) \times \frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & - \Bigg(\frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(4-3\rho)\Gamma(3-2\rho)} + \frac{\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(4-3\rho)\Gamma(2-\rho)\Gamma(2-\rho)} \\ & - \frac{\zeta^{(1-2\rho)}}{\Gamma(2-2\rho)}\Bigg)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\Bigg) - 1.\end{aligned}$$

By applying $D_{\Im}^{2\delta}$ both sides of Eq. (53) and then using $D_{\Im}^{2\delta}\mathrm{Res}_{\mu,3}(\zeta, 0) = 0$, we have,

$$\begin{aligned}C_3 = -\Bigg[& \frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(4-3\rho)}}{\Gamma(4-3\rho)\Gamma(3-2\rho)} + \frac{\Gamma(4-2\rho)\zeta^{(4-3\rho)}}{\Gamma(4-3\rho)\Gamma(2-\rho)^2} \\ & - 2\left(\frac{\Gamma(2-\rho)\Gamma(3-\rho)\zeta^{(2-2\rho)} - \Gamma(3-\rho)\zeta^{(4-3\rho)}}{\Gamma(2-\rho)^2\Gamma(3-2\rho)}\right) \\ & + \frac{\Gamma(3-\rho)\zeta^{(4-3\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)} + \frac{\zeta^{(4-3\rho)}}{\Gamma(2-\rho)^3} - \frac{\Gamma(2-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^3}\Bigg].\end{aligned}$$

Third RPSM approximation is expressed as follows:

$$\begin{aligned}\mu_3(\zeta, \Im) = {}& -\zeta + \left(\frac{\Gamma(2-\rho) - \zeta^{(2-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} \\ & - \left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}}{\Gamma(3-2\rho)} + \frac{\zeta^{(3-2\rho)} - \Gamma(2-\rho)\zeta^{(1-\rho)}}{\Gamma(2-\rho)\Gamma(2-\rho)}\right) \\ & \times \frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)} - \Bigg[\frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(4-3\rho)}}{\Gamma(4-3\rho)\Gamma(3-2\rho)} \\ & + \frac{\Gamma(4-2\rho)\zeta^{(4-3\rho)}}{\Gamma(4-3\rho)\Gamma(2-\rho)^2} \\ & - 2\left(\frac{\Gamma(2-\rho)\Gamma(3-\rho)\zeta^{(2-2\rho)} - \Gamma(3-\rho)\zeta^{(4-3\rho)}}{\Gamma(2-\rho)^2\Gamma(3-2\rho)}\right) \\ & + \frac{\Gamma(3-\rho)\zeta^{(4-3\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)} + \frac{\zeta^{(4-3\rho)}}{\Gamma(2-\rho)^3} \\ & - \frac{\Gamma(2-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^3}\Bigg]\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)}.\end{aligned}$$

By putting $\delta = \rho = 1$, we obtain the closed form solution of the problem:

$$\mu(\zeta, \Im) = \frac{2\zeta - 2\Im + \Im^2}{2(\Im - 1)}. \tag{53}$$

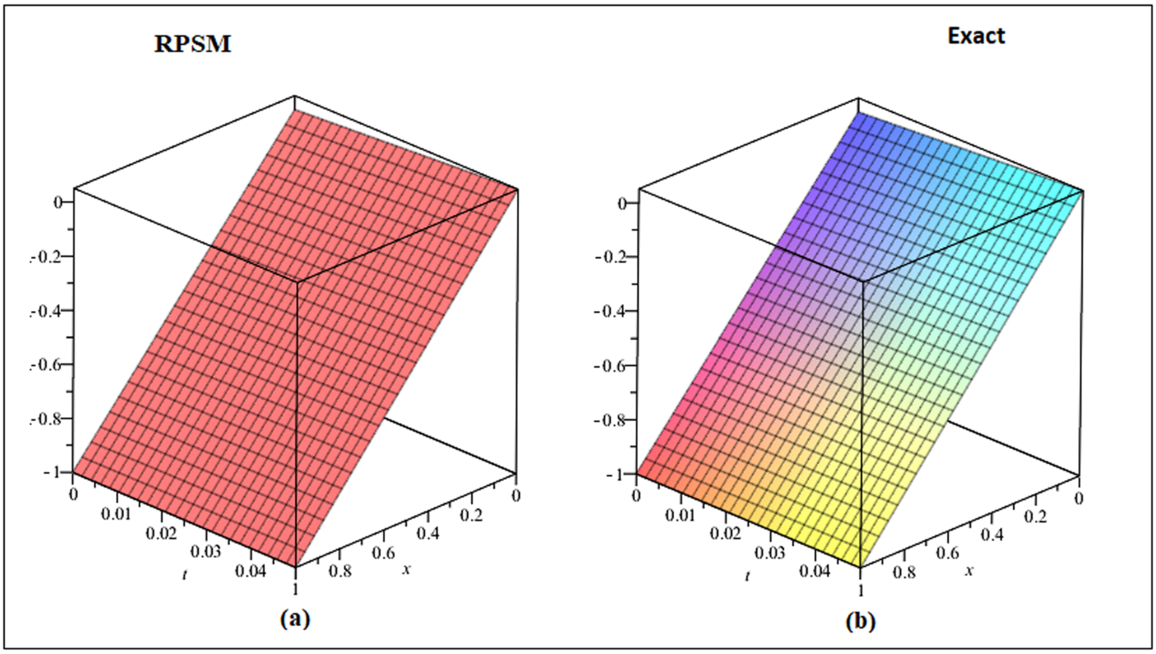

Figure 4: 3D (a) RPSM solution and (b) exact solution graph, at δ and $\rho = 1$ of Example 4.2.

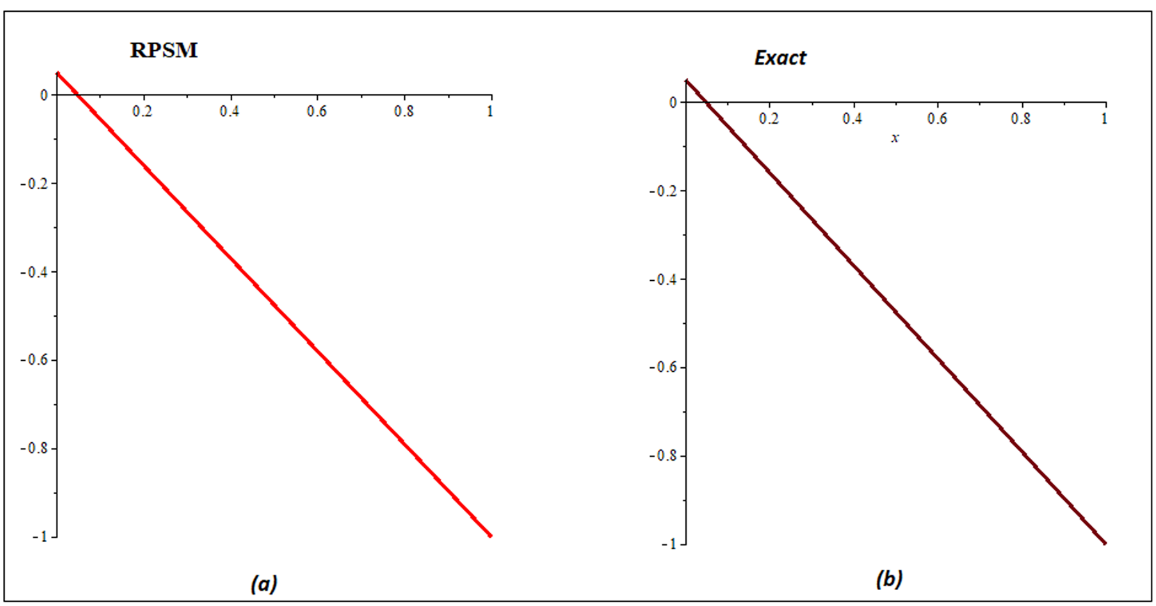

Figure 5: 2D (a) RPSM solution (b) exact solution graph, at δ and $\rho = 1$ of Example 4.2.

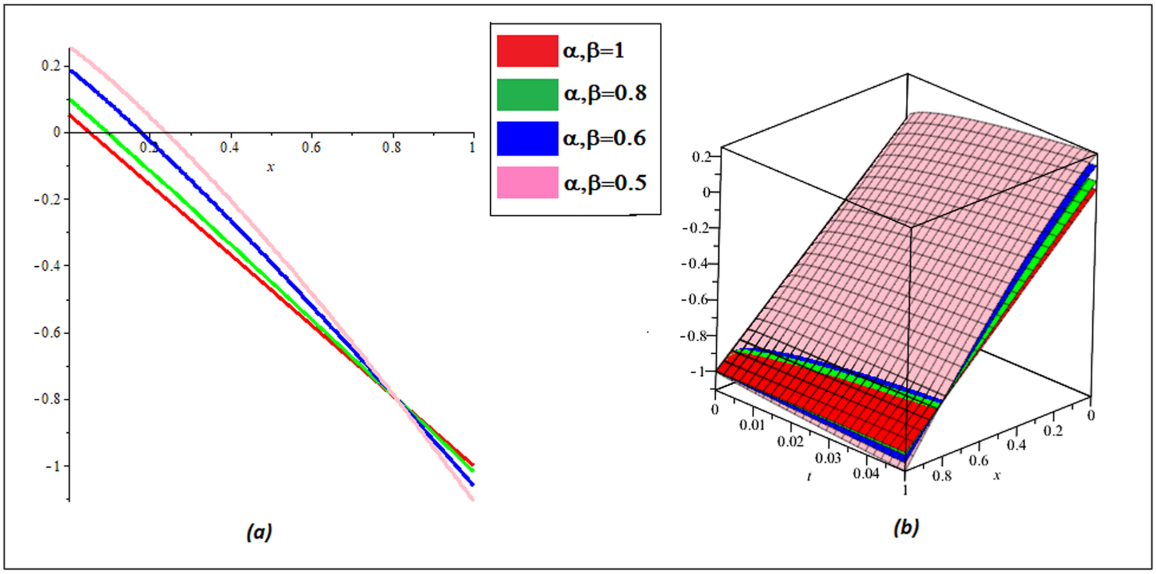

Figure 6: 2D and 3D plots for RPSM solution of Example 4.2 for different values of δ.

Table 2: AE comparison of RPSM and ADM [49] for Example 4.2

t	ζ	RPSM	ADM
0.01	0.3	2.0×10^{-9}	2.9798×10^{-9}
	0.6	1.0×10^{-9}	6.0101×10^{-9}
	0.9	4.0×10^{-9}	9.04041×10^{-9}
0.05	0.3	1.31580×10^{-6}	1.80921×10^{-6}
	0.6	6.5790×10^{-7}	3.78289×10^{-6}
	0.9	2.63160×10^{-6}	5.75658×10^{-6}
0.1	0.3	2.222220×10^{-5}	2.77778×10^{-5}
	0.6	1.111120×10^{-5}	6.11111×10^{-5}
	0.9	4.444450×10^{-5}	9.44444×10^{-5}

4.3 Example

Let us consider the fractional partial differential equation of the form [49],

$$D_{\Im}^{\delta}\mu(\zeta,\Im) + \mu(\zeta,\Im)D_{\zeta}^{\rho}\mu(\zeta,\Im) = 0, \quad 0 < \delta \le 1 \text{ and } 0 < \rho \le 1, \tag{54}$$

having initial condition

$$\mu(\zeta,0) = \zeta + 1. \tag{55}$$

The truncated series of Eq. (54) is expressed as follows:

$$\mu_k(\zeta,\Im) = \zeta + 1 + \sum_{m=1}^{k} C_m \frac{(\Im)^{m\delta}}{\Gamma(1+m\delta)}. \tag{56}$$

The residual function for Eq. (54) is define as follows:

$$\mathrm{Res}_{\mu,k}(\zeta,\Im) = D_{\Im}^{\delta}\mu_k(\zeta,\Im) + \mu_k(\zeta,\Im)D_{\zeta}^{\rho}\mu_k(\zeta,\Im). \tag{57}$$

For $k = 1$, Eqs. (56) and (57) can be simplified as follows:

$$\mu_1(\zeta,\Im) = \zeta + 1 + C_1\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}, \tag{58}$$

$$\mathrm{Res}_{\mu,1}(\zeta,\Im) = D_{\Im}^{\delta}\mu_1(\zeta,\Im) + \mu_1(\zeta,\Im)D_{\zeta}^{\rho}\mu_1(\zeta,\Im). \tag{59}$$

By putting Eq. (58) in Eq. (59), we obtain

$$\mathrm{Res}_{\mu,1}(\zeta,\Im) = C_1 + \left(\zeta + 1 + C_1\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\right)\left(\frac{\zeta^{1-\rho}}{\Gamma(2-\rho)}\right), \tag{60}$$

and by using $\mathrm{Res}_{\mu,1}(\zeta,0) = 0$, we obtain

$$C_1 = -\left(\frac{\zeta^{(2-\rho)} + \zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right). \tag{61}$$

First RPSM approximation is defined as follows:

$$\mu_1(\zeta,\Im) = \zeta + 1 - \left(\frac{\zeta^{(2-\rho)} + \zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}. \tag{62}$$

For $k = 2$, Eqs. (56) and (57) can be written as follows:

$$\mu_2(\zeta,\Im) = \zeta + 1 - \left(\frac{\zeta^{(2-\rho)} + \zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)} + C_2\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}, \tag{63}$$

$$\mathrm{Res}_{\mu,2}(\zeta,\Im) = D_{\Im}^{\delta}\mu_2(\zeta,\Im) + \mu_2(\zeta,\Im)D_{\zeta}^{\rho}\mu_2(\zeta,\Im). \tag{64}$$

By putting Eq. (63) in Eq. (64), we obtain

$$\begin{aligned}&\mathrm{Res}_{\mu,2}(\zeta,\Im)\\ &= -\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)+C_2\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\\ &+\left(\zeta+1-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+C_2\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\right)\\ &\times\left(\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)}-\left(\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}+\frac{\zeta^{(1-2\rho)}}{\Gamma(2-2\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\right).\end{aligned}\tag{65}$$

By using $D_{\Im}^{\delta}$ into Eq. (65) and then using $D_{\Im}^{\delta}\mathrm{Res}_{\mu,2}(\zeta,0)=0$, we obtain,

$$C_2=\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}.\tag{66}$$

Second RPSM approximation is expressed as follows:

$$\mu_2(\zeta,\Im)=\begin{cases}\zeta+1-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\\ +\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right.\\ \quad\left.+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\times\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}.\end{cases}$$

Similarly, for $k=3$, Eqs. (56) and (57) can be simplified as follows:

$$\begin{aligned}\mu_3(\zeta,\Im)&=\zeta+1-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\\ &+\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right.\\ &\left.+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\times\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}+C_3\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)},\end{aligned}\tag{67}$$

$$\mathrm{Res}_{\mu,3}(\zeta,\Im)=D_{\Im}^{\delta}\mu_3(\zeta,\Im)+\mu_3(\zeta,\Im)D_{\zeta}^{\rho}\mu_3(\zeta,\Im).\tag{68}$$

By putting Eq. (67) in Eq. (68), we obtain

$$\begin{aligned}\mathrm{Res}_{\mu,3}(\zeta,\Im)=&-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\\ &+\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\\ &\times\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+C_3\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\\ &+\left[\zeta+1-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\right.\\ &+\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right.\\ &\left.\left.+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}+C_3\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)}\right]\\ &\times\left[\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)}-\left(\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right.\right.\\ &\left.+\frac{\zeta^{(1-2\rho)}}{\Gamma(2-2\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}\\ &+\left(\frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)\Gamma(4-3\rho)}\right.\\ &+\frac{\Gamma(3-\rho)\zeta^{(2-3\rho)}}{\Gamma(2-\rho)\Gamma(3-3\rho)}+\frac{\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)^2\Gamma(4-3\rho)}\\ &\left.\left.+\frac{\Gamma(3-2\rho)\zeta^{(2-3\rho)}}{\Gamma(2-\rho)^2\Gamma(3-3\rho)}\right)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\right],\end{aligned}$$

and applying $D_{\Im}^{2\delta}$ and then using $D_{\Im}^{2\delta}\mathrm{Res}_{\mu,3}(\zeta,0)=0$, we obtain

$$\mu_3(\zeta,\Im)=\begin{cases}\zeta+1-\left(\frac{\zeta^{(2-\rho)}+\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\frac{(\Im)^{\delta}}{\Gamma(1+\delta)}+\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\frac{(\Im)^{2\delta}}{\Gamma(1+2\delta)}\\ -\left[(\zeta+1)\left(\frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)\Gamma(4-3\rho)}+\frac{\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)^2\Gamma(4-3\rho)}\right)+\left(-\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right)\right.\\ \times\left(\frac{-\zeta^{(2-\rho)}-\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)+\left(\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)}+\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}+\frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right)\\ \left.+\left(\frac{-\zeta^{(2-\rho)}-\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\left(-\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right)\right]\frac{(\Im)^{3\delta}}{\Gamma(1+3\delta)}.\end{cases}\tag{69}$$

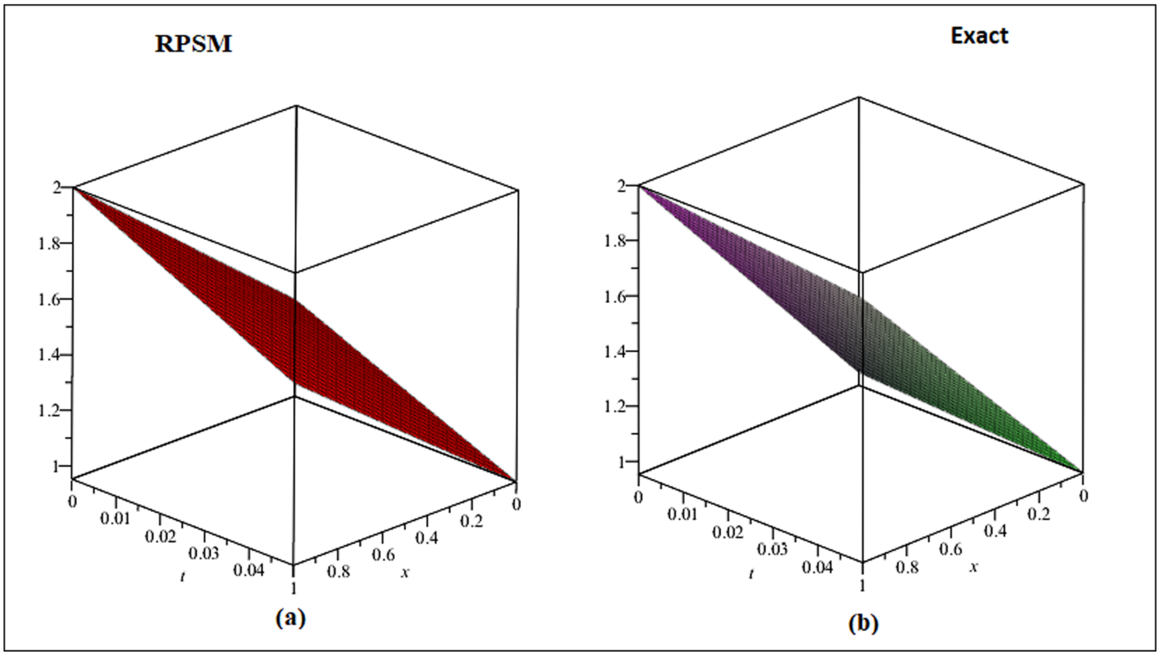

Figure 7: 3D solution comparison graph, at δ and $\rho = 1$ of Example 4.3.

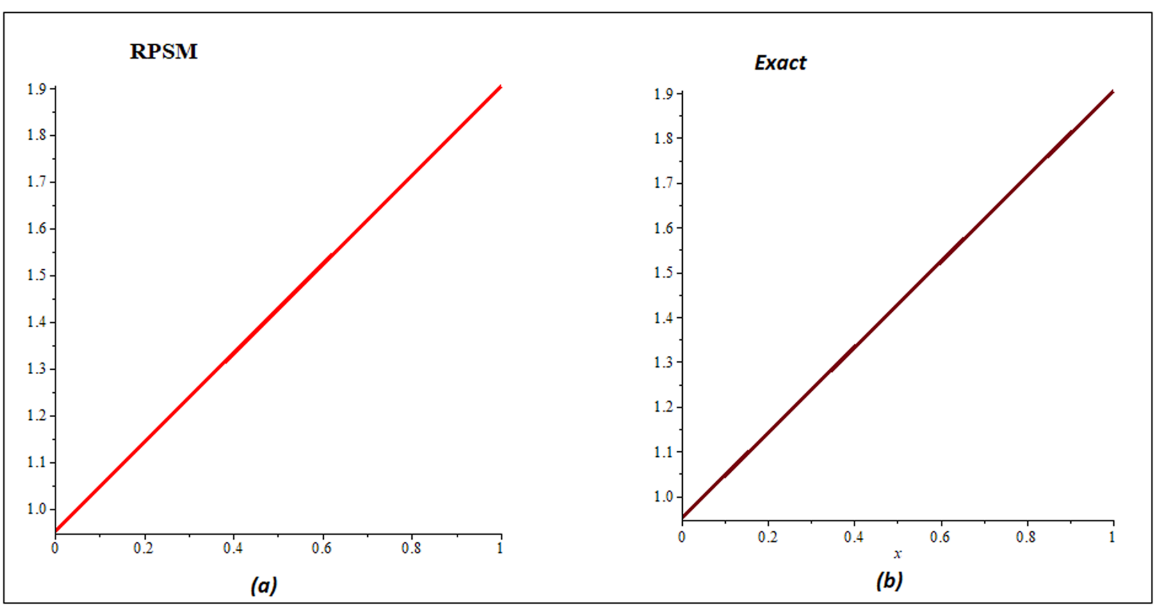

Figure 8: 2D solution comparison graph, at δ and $\rho = 1$ of Example 4.3.

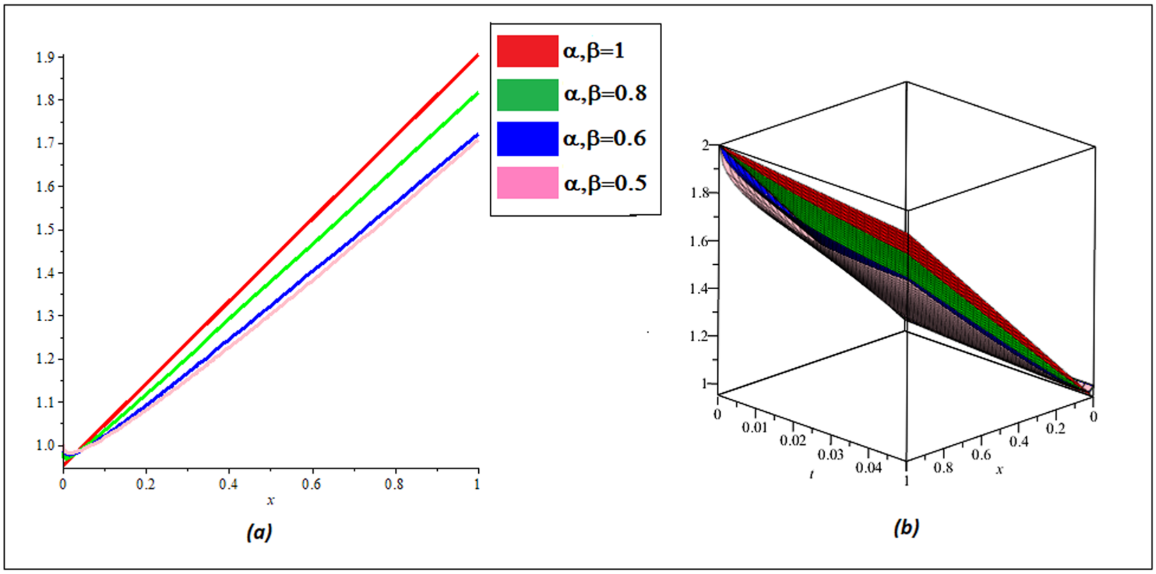

Figure 9: 2D and 3D plots for RPSM solution of Example 4.3 for different values of δ.

Table 3: AE of Example 4.3, at $\Im = 0.05$ and $\delta, \rho = 1$

ζ	Exact–RPSM $K = 1$	Exact–RPSM $K = 2$	Exact–RPSM $K = 3$
0.1	2.6190476×10^{-3}	1.3095236×10^{-4}	6.5476×10^{-6}
0.2	2.8571428×10^{-3}	1.4285712×10^{-4}	7.1428×10^{-6}
0.3	3.0952380×10^{-3}	1.5476188×10^{-4}	7.7380×10^{-6}
0.4	3.3333332×10^{-3}	1.6666664×10^{-4}	8.3332×10^{-6}
0.5	3.5714284×10^{-3}	1.7857140×10^{-4}	8.9284×10^{-6}
0.6	3.8095236×10^{-3}	1.9047616×10^{-4}	9.5236×10^{-6}
0.7	4.0476188×10^{-3}	2.0238092×10^{-4}	1.01188×10^{-5}
0.8	4.2857140×10^{-3}	2.1428568×10^{-4}	1.07140×10^{-5}
0.9	4.5238092×10^{-3}	2.2619044×10^{-4}	1.13092×10^{-5}
1	4.7619044×10^{-3}	2.380952×10^{-4}	1.19044×10^{-5}

$$C_3 = -\Bigg[(\zeta+1)\left(\frac{\Gamma(3-\rho)\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)\Gamma(4-3\rho)} + \frac{\Gamma(4-2\rho)\zeta^{(3-3\rho)}}{\Gamma(2-\rho)^2\Gamma(4-3\rho)}\right) + \left(-\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right) \times \left(\frac{-\zeta^{(2-\rho)}-\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right) + \left(\frac{\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right) \times \left(\frac{\Gamma(3-\rho)\zeta^{(3-2\rho)} + \Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)} + \frac{\zeta^{(3-2\rho)}+\zeta^{(2-2\rho)}}{\Gamma(2-\rho)^2}\right) + \left(\frac{-\zeta^{(2-\rho)}-\zeta^{(1-\rho)}}{\Gamma(2-\rho)}\right)\left(-\frac{\Gamma(3-\rho)\zeta^{(2-2\rho)}}{\Gamma(2-\rho)\Gamma(3-2\rho)}\right)\Bigg].$$

Third RPSM approximation is expressed as follows:

By substituting $\delta = \rho = 1$ in Eq. (69), we obtain

$$\mu(\zeta, \Im) = \frac{1+\zeta}{1+\Im}, \tag{70}$$

which is the closed form solution of Example 4.3.

5 Results and discussion

Figures 1 and 2 depict the 3D and 2D comparison plots of the exact and RPSM solutions, respectively, while Figure 3 depicts the 2D and 3D plots of Example 4.1 for various fractional-order δ. Figures 4 and 5 show the 3D and 2D comparison plots of the exact and RPSM-solution, respectively, while Figure 6 shows the 2D and 3D plots of Example 4.2 for different fractional-order δ. Figures 7 and Figure 8 show the 3D and 2D comparison plots of the Exact and RPSM solutions, respectively, while Figure 9 shows the 2D and 3D plots of Example 4.3 for different fractional-order δ. Tables 1, 2, and 3 are the absolute error (AE) tables for different spaces of Examples 4.1–4.3, respectively.

6 Conclusion

The solutions of nonlinear systems are generally very difficult to investigate. In the present work, the solutions of some illustrative nonlinear problems of fractional partial differential equations are handled in a very simple and straightforward procedure. The fractional and integer order solutions to the targeted problems are analyzed by using graphs and tables. It is noted that RPSM has a direct and simple implementation to obtain solutions to the problems. The graphs and tables have confirmed the sufficient degree of accuracy of RPSM. In Figures 3, 6, and 9, the solutions at various fractional-orders of time and space are shown. The RPSM solutions are compared with adomian decomposition method (ADM) solutions, and the dominant accuracy of RPSM is confirmed over ADM. It is also investigated that the integer order solutions are in close contact with the exact solutions of each problem. The fractional solutions are obtained and found to be in strong agreement with the actual solutions to the problems. The proposed technique is very effective for nonlinear cases and thus supports a better way for solving FPDEs that arise frequently in science and engineering.

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